CHERN CLASSES OF SINGULAR VARIETIES, REVISITED

PAOLO ALUFFI

Abstract. We introduce a notion of 'proChow group' of algebraic varieties, reproducing the notion of Chow group for complete varieties, and functorial with respect to arbitrary morphisms. We construct a natural transformation from the functor of constructible functions to the proChow functor, extending MacPherson's natural transformation. We illustrate the result by giving very short proofs of (generalizations of) two well-known facts on Chern-Schwartz-MacPherson classes.

1. Introduction

Let X be a variety over an algebraically closed field of characteristic 0. Equivalent notions of *total Chern class* of X were given independently by Marie-Hélène Schwartz ([9]) and by Robert MacPherson ([8]) for compact complex algebraic varieties, in homology; the definition was later extended to complete algebraic varieties on an algebraically closed field of characteristic 0, in the Chow group A_*X of X. We call this class *Chern-Schwartz-MacPherson* (CSM) *class* of X, denoted $c_{\mathrm{SM}}(X)$.

The CSM class agrees with the ordinary Chern class of the tangent bundle if X is nonsingular: $c_{\mathrm{SM}}(X) = c(TX) \cap [X]$ in this case. It also satisfies a remarkable functorial property: it is defined as the value $c_*(1\!\!1_X)$ taken by a natural transformation c_* of the constructible function functor F to the Chow functor A_*, on the constant function $1\!\!1_X$. Alexandre Grothendieck and Pierre Deligne had conjectured the existence of this natural transformation; MacPherson constructed it explicitly in [8], by using other important invariants introduced in the same article.

In this note we propose an alternative construction of Chern-Schwartz-MacPherson classes, in an 'enriched' Chow group $\widehat{\mathsf{A}}_*X$ (the *proChow* group) obtained by taking appropriate limits of ordinary Chow groups. The proChow group is a covariant functor with respect to every morphism (while the Chow group is only functorial with respect to *proper* morphisms); we prove that a corresponding *CSM transformation* $\mathsf{F} \rightsquigarrow \widehat{\mathsf{A}}_*$ is natural with respect to arbitrry morphisms. If X is complete, the proChow group of X is canonically isomorphic to the ordinary Chow group, and its *proCSM class* equals the Chern-Schwartz-MacPherson class.

Our definition is direct, without reference to auxiliary invariants such as Chern-Mather classes or the local Euler obstruction. To illustrate its use, we give very condensed proofs of two known results on CSM classes: the *product formula* of Kwieciński ([7]), and the Ehlers-Barthel-Brasselet-Fieseler formula for the CSM classes of toric varieties ([3]).

2. the proChow functor

We work over an algebraically closed field k of characteristic 0.

Let $\mathscr{S}$ be a category of k-varieties. For U in $\mathscr{S}$, let $\mathscr{S}_U$ be the category whose objects are the $\mathscr{S}$-morphisms $i : U \to Z^i$ from U to *complete* varieties in $\mathscr{S}$, and whose morphisms

$j \mapsto i$ are the commutative diagrams of k-varieties

$$\begin{array}{ccc} & \overset{j}{\nearrow} & Z^j \\ U & & \downarrow \pi \\ & \underset{i}{\searrow} & Z^i \end{array}$$

where π is a *proper* morphism. We assume that the following conditions on $\mathscr{S}$ are verified:

- For all U in $\mathscr{S}$ and every pair of objects i, j of $\mathscr{S}_U$, there is an object k in $\mathscr{S}_U$ such that $k \to i$ and $k \to j$, and $k : U \hookrightarrow Z^k$ is a *closure* (that is, k is an open embedding, and $\overline{U} = Z^k$);
- If U is nonsingular, one may choose the closure Z^k as above and *good:* that is, Z^k is nonsingular, and the complement $Z^k \smallsetminus U$ is a divisor with normal crossings and nonsingular components.

For example, these conditions are satisfied for the category of *all* k-varieties (in characteristic zero, by resolution of singularities).

Definition 2.1. *The* proChow group *of U (with respect to $\mathscr{S}$) is the limit* $\widehat{\mathsf{A}}_*^{\mathscr{S}} U := \varprojlim_i \mathsf{A}_* Z^i$.

Concretely, an element $\rho \in \widehat{\mathsf{A}}_*^{\mathscr{S}} U$ consists of the choice of an element ρ^i in the (conventional) Chow group $\mathsf{A}_* Z^i$ for every i in $\mathscr{S}_U$, subject to the condition of compatibility $\pi_* \rho^j = \rho^i$ for every $\pi : j \to i$. We say that ρ^i is the *component of ρ in* $\mathsf{A}_* Z^i$.

We will omit the upper index $\mathscr{S}$ when no ambiguity seems likely; the reader will note that the proChow group does depend on the chosen category $\mathscr{S}$. The following facts are however independent of $\mathscr{S}$ (if $\mathscr{S}$ satisfies the conditions of 'cofinality' specified above), and immediately verified:

Lemma 2.2. *With notation as above:*

- *If U is complete, there is a canonical isomorphism* $\widehat{\mathsf{A}}_* U \cong \mathsf{A}_* U$.
- *In order to specify an element of* $\widehat{\mathsf{A}}_* U$, *it suffices to choose a compatible set of* $\rho^i \in \mathsf{A}_* Z^i$ *for all closures* $i : U \to Z^i$ *in* $\mathscr{S}$.
- *If, further, U is nonsingular, it suffices to choose a compatible set of* $\rho^i \in \mathsf{A}_* Z^i$ *for every* good *closure* $i : U \to Z^i$ *in* $\mathscr{S}$.

Every subscheme B of U determines a distinguished element $[\overline{B}]$ of $\widehat{\mathsf{A}}_* U$: for every closure $j : U \to Z^j$, choose the class $[\overline{B}] \in \mathsf{A}_* Z^j$ of the closure of B in Z^j; this choice is clearly compatible. If U is complete, $[\overline{B}] \in \widehat{\mathsf{A}}_* U \cong \mathsf{A}_* U$ is the ordinary 'fundamental class' of $\overline{B}$.

The proChow group $\widehat{\mathsf{A}}_* = \widehat{\mathsf{A}}_*^{\mathscr{S}}$ is a functor $\mathscr{S} \rightsquigarrow$ Abelian Groups: if $f : X \to Y$ is a morphism in $\mathscr{S}$, then $j \to j \circ f$ induces a functor $\mathscr{S}_Y \to \mathscr{S}_X$, and hence a homomorphism $f_* : \widehat{\mathsf{A}}_* X \to \widehat{\mathsf{A}}_* Y$. Concretely, for $\rho \in \widehat{\mathsf{A}}_* X$ and $j : Y \to Z^j$ in $\mathscr{S}_Y$, the component of $f_* \rho$ dans $\mathsf{A}_* Z^j$ is simply equal to the component of ρ. If f is proper and X and Y are complete, then $f_* : \widehat{\mathsf{A}}_* X \cong \mathsf{A}_* X \to \mathsf{A}_* Y \cong \widehat{\mathsf{A}}_* Y$ is the ordinary proper push-forward of Chow groups. Note however that while A_* is only functorial with respect to proper morphisms, the proChow $\widehat{\mathsf{A}}_*$ is functorial with respect to all morphisms in $\mathscr{S}$.

3. proCSM classes proCSM

With $\mathscr{S}$ as in §2, and X in $\mathscr{S}$, we define the group of *$\mathscr{S}$-constructible functions* $\mathsf{F}^{\mathscr{S}}(X)$ as the group of finite $\mathbb{Z}$-linear combination of caracteristic functions $\mathbb{1}_U$ (where $\mathbb{1}_U(p) = 1$ if $p \in U$, and 0 if $p \in X \smallsetminus U$) where U are *nonsingular* locally closed subvarieties of X, such that the inclusions $U \subset X$ are morphisms of $\mathscr{S}$.

We now pose a further condition on $\mathscr{S}$. We require that the conventional push-forward of constructible functions (defined by taking the fiberwise Euler characteristic, see [8] for the complex case) preserve $\mathscr{S}$-constructibility: that is, that it defines a push-forward $f_* : \mathsf{F}^{\mathscr{S}}(X) \to \mathsf{F}^{\mathscr{S}}(Y)$ for every morphism $f : X \to Y$ dans $\mathscr{S}$. We also require that $1\!\!1_X$ be $\mathscr{S}$-constructible for every X in $\mathscr{S}$.

Under these hypotheses, $\mathsf{F}^{\mathscr{S}}$ defines (in characteristic zero!) a covariant functor $\mathscr{S} \rightsquigarrow$ Abelian Groups, and every X in $\mathscr{S}$ determines a distinguished element $1\!\!1_X \in \mathsf{F}^{\mathscr{S}}(X)$. We will usually omit the upper index $\mathscr{S}$.

Next, we define a homomorphism $\mathsf{F}(X) \to \widehat{\mathsf{A}}_*X$, $\alpha \mapsto \{\alpha\}$, and a distinguished element $\{X\} := \{1\!\!1_X\} \in \widehat{\mathsf{A}}_*X$. We begin with the nonsingular case:

Definition 3.1. *Let U be nonsingular, in $\mathscr{S}$. The* proCSM class *of U in $\widehat{\mathsf{A}}_*U$, denoted $\{U\}$, is the element of the proChow group determined by $c(\Omega^1_{\overline{U}}(\log D)^\vee) \cap [\overline{U}] \in \mathsf{A}_*\overline{U}$ for any good closure $\overline{U}$ of U in $\mathscr{S}_U$, where $D = \overline{U} \smallsetminus U$ is the corresponding normal crossing divisor, and $\Omega^1_{\overline{U}}(\log D)^\vee$ denotes the dual of the bundle of differential forms with logarithmic poles along D.*

This choice is compatible in the sense of §2, as we will see in Theorem 3.3; thus it defines an element of $\widehat{\mathsf{A}}_*U$, by Lemma 2.2.

Now let X be an arbitrary (that is, not necessarily nonsingular) in $\mathscr{S}$, and let $\alpha \in F(X)$ be a constructible function on X. Let $\alpha = \sum_U m_U 1\!\!1_U$, with U nonsingular, locally closed, $i_U : U \subset X$ in $\mathscr{S}$, and $m_U \in \mathbb{Z}$.

Definition 3.2. *The* proCSM class of α *is the sum $\{\alpha\} = \sum_U m_U i_{U*}\{U\} \in \widehat{\mathsf{A}}_*X$. The* proCSM class of X *is the class $\{X\} := \{1\!\!1_X\}$.*

We can now state and proof the main result of this note.

Theorem 3.3. *With notation as above:*

1. *The classes specified in Definition 3.1 are compatible: that is, if $i : U \to \overline{U}^i$ and $j : U \to \overline{U}^j$ are good closures of U in $\mathscr{S}_U$, with complements D^i, D^j, and $\pi : \overline{U}^j \to \overline{U}^i$ is a morphism such that $i = \pi \circ j$, then $\pi_*\left(c(\Omega^1_{\overline{U}^j}(\log D^j)^\vee) \cap [\overline{U}^j]\right) = c(\Omega^1_{\overline{U}^i}(\log D^i)^\vee) \cap [\overline{U}^i]$.*
2. *The class in Definition 3.2 does not depend on the choices: that is, if $\alpha = \sum_U m_U 1\!\!1_U = \sum_V n_V 1\!\!1_V$ are two realizations of α as finite linear combinations of characteristic functions of nonsingular locally closed subvarieties of X, then $\sum_U m_U i_{U*}\{U\} = \sum_V n_V i_{V*}\{V\}$.*
3. *The homomorphism $\mathsf{F}(X) \to \widehat{\mathsf{A}}_*X$, $\alpha \mapsto \{\alpha\}$ given in Definition 3.2 gives a* natural *transformation $\mathsf{F} \rightsquigarrow \widehat{\mathsf{A}}_*$; that is, $f_*\{\alpha\} = \{f_*(\alpha)\}$ for every morphism $f : X \to Y$ in $\mathscr{S}$.*
4. *If X is* complete, *then the proCSM class of X is the ordinary Chern-Schwartz-MacPherson class: $\{X\} = c_{\mathrm{SM}}(X) \in \mathsf{A}_*X \cong \widehat{\mathsf{A}}_*X$.*

One can prove the first three points independently of Macpherson's result in [8]; this is done in [1]. In the particular case of complete varieties and proper morphisms, the third point gives a natural transformation as prescribed by the (Chow version of) the Grothendieck-Deligne conjecture; the equality of proCSM classes and CSM classes for complete varieties follows then from the uniqueness of this natural transformation (which is an immediate consequence of resolution of singularities.)

We present here a proof that uses MacPherson's theorem (that is, the fact that MacPherson's transformation c_* is natural), and that leads to a quicker argument.

Proof. – If U is nonsingular, $\overline{U}$ is a good closure of U, and $D = \overline{U} \smallsetminus U$, then

$$(\dagger) \qquad c(\Omega^1_{\overline{U}}(\log D)^\vee) \cap [\overline{U}] = c_*(1\!\!1_U) \in A_*\overline{U} \quad .$$

This follows easily from the fact that c_* is natural, and from an explicit Chern class computation; see for example Proposition 15.3 in [6], or Theorem 1 in [2].

(*1*) follows from (†), from the fact that c_* is natural, and from the definition of push-forward of constructible functions:

$$\pi_*\left(c(\Omega^1_{\overline{U}^i}(\log D^i)^\vee) \cap [\overline{U}^i]\right) = \pi_* c_*(1\!\!1_U) = c_*\pi_*(1\!\!1_U) = c_*(1\!\!1_U) = c(\Omega^1_{\overline{U}^j}(\log D^j)^\vee) \cap [\overline{U}^j] \quad .$$

The proof of the other points is streamlined by the following alternative version of Definition 3.2:

Lemma 3.4. *For every $z : X \to Z$ in $\mathscr{S}_X$, and every $\alpha \in \mathsf{F}(X)$, the component of $\{\alpha\}$ in A_*Z is $c_*(z_*(\alpha))$.*

To prove the lemma, use (†) to write the component of $\{U\}$ in A_*Z, for U nonsingular and $z_U : U \to Z$ in $\mathscr{S}$, as $c_*(z_{U*}(1\!\!1_U))$. If $\alpha \in \mathsf{F}(X)$, then $\alpha = \sum_U m_U 1\!\!1_U$, with U nonsingular and the inclusion $U \subset X$ in $\mathscr{S}$; thus, for every $z : X \to Z$, the component of $\sum_U m_U\{U\}$ in A_*Z is $\sum_U m_U c_*(z_* 1\!\!1_U) = c_*(z_*(\sum_U m_U 1\!\!1_U)) = c_*(z_*(\alpha))$, as stated.

(*2*) follows immediately, since $c_*(z_*(\alpha))$ only depends from α, and not from the decomposition $\alpha = \sum_* m_U 1\!\!1_U$. On the other hand, if X is itself complete, then taking $Z = X$, $z = \mathrm{id}_X$, and $\alpha = 1\!\!1_X$ in Lemma 3.4, one gets (*4*).

Finally, Lemma 3.4 implies (*3*). Indeed, let $z : Y \to Z$ be any object of $\mathscr{S}_Y$; then $w \circ f$ is an object of $\mathscr{S}_X$ and, by the definition of push-forward of proChow groups, the component in A_*Z of the push-forward $f_*\{\alpha\}$ is simply equal to the component of $\{\alpha\}$ in A_*Z. By Lemma 3.4, this component is $c_*((w \circ f)_*\alpha) = c_*(z_*(f_*(\alpha)))$, and once more by Lemma 3.4 this is equal to the component of $\{f_*(\alpha)\}$ in A_*Z, yielding (*3*). □

4. Examples

We illustrate the formalism presented in §2 and §3 by giving condensed proofs (valid in the proCSM context) of two known results on Chern-Schwartz-MacPherson classes.

We will use different categories, as permitted by the constructions given in the previous sections. Denoting by $\widehat{\mathsf{A}}_*$ the proChow functor obtained with $\mathscr{S} =$ the category of all k-varieties, and by F the functor of constructible functions on this category, note that for every other category $\mathscr{S}$ one has canonical homomorphisms $\mathsf{F}^{\mathscr{S}}(X) \to \mathsf{F}(X)$, $\widehat{\mathsf{A}}_*(X) \to \widehat{\mathsf{A}}^{\mathscr{S}}_*(X)$, compatible with the corresponding proCSM natural transformations.

The first result is the *product formula* of Michał Kwieciński. Let $\mathscr{S}$ be the category of *products* $X \times Y$ (technically, of pairs (X, Y)), where X and Y are k-varieties, and where morphisms $X_1 \times Y_1 \to X_2 \times Y_2$ consist of pairs (f, g), where $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ are morphisms. The conditions specified in §2 and §3 are clearly satisfied (in characteristic 0), and one therefore has a proChow functor $\widehat{\mathsf{A}}^\times_*$ and a functor $\mathsf{F}^\times$ of $\mathscr{S}$-constructible functions. The group $\mathsf{F}^\times(X \times Y)$ consists of functions $\alpha \otimes \beta$ defined by $\alpha \otimes \beta(x, y) = \alpha(x)\beta(y)$, where $\alpha \in \mathsf{F}(X)$, $\beta \in \mathsf{F}(Y)$ are (ordinary) constructible functions. We'll denote the corresponding proCSM class by $\{\alpha \otimes \beta\}^\times$.

Further, there is an evident canonical homomorphism

$$\widehat{\mathsf{A}}_*(X) \otimes \widehat{\mathsf{A}}_*(Y) \xrightarrow{\otimes} \widehat{\mathsf{A}}^\times_*(X \times Y) \quad ,$$

$(\alpha, \beta) \mapsto \alpha \otimes \beta$, induced by the exterior products for ordinary Chow groups ([4], §1.10).

Theorem 4.1. *Let X and Y be two varieties, $\alpha \in \mathsf{F}(X)$, $\beta \in \mathsf{F}(Y)$ and $\alpha \otimes \beta \in \mathsf{F}^{\times}(X \times Y)$ as above. Then $\{\alpha \otimes \beta\}^{\times} = \{\alpha\} \otimes \{\beta\}$.*

Proof. – By bilinearity, the statement follows from the case $\alpha = 1\!\!1_U$, $\beta = 1\!\!1_V$ for U, V nonsingular subvarieties of X, Y resp.; that is, it suffices to verify that for U, V nonsingular, and for good closures $\overline{U}$, $\overline{V}$ of U, V, with complements $D = \overline{U} \smallsetminus U$, $E = \overline{V} \smallsetminus V$,

$$c(\Omega^1_{\overline{U} \times \overline{V}}(\log(D+E))^\vee) \cap [\overline{U} \times \overline{V}] = \left(c(\Omega^1_{\overline{U}}(\log D)^\vee) \cap [\overline{U}]\right) \otimes \left(c(\Omega^1_{\overline{V}}(\log E)^\vee) \cap [\overline{V}]\right) \quad ,$$

and that follows immediately from the standard computation of Chern classes for differential forms with logarithmic poles. □

The particular case in which X and Y are complete reproduces Kwieciński's theorem ([7]), since in that case all the proChow groups in the statement are isomorphic to the conventional Chow group (by Lemma 2.2), and the proCSM classes are equal to the Chern-Schwartz-MacPherson classes (by Theorem 3.3).

Our second example is the formula of Fritz Ehlers for the Chern-Schwartz-MacPherson class of a toric variety; see [5], p. 113, et [3] for the proof for conventional CSM classes. For a statement and proof in the more general proChow case, let $\mathscr{S}$ be the category of toric k-varieties, with T-equivariant morphisms. The corresponding functor and proCSM classes will be denoted $\widehat{\mathsf{A}}^{\mathsf{T}}$ and $\{X\}^{\mathsf{T}}$, respectively; the *fundamental class* of $B \subset X$ in $\widehat{\mathsf{A}}^{\mathsf{T}}_*(X)$ will be denoted $[\overline{B}]$.

Theorem 4.2. *Let X be a toric variety. Then $\{X\}^{\mathsf{T}} = \sum_{B \in X/T} [\overline{B}] \in \widehat{\mathsf{A}}^{\mathsf{T}}_*(X)$, where the sum is over the (finite) set of T-orbits.*

Proof. – Since X is the union of T-orbits B, we have $\{X\}^{\mathsf{T}} = \sum \{B\}^{\mathsf{T}}$, and consequently it suffices to prove that if B is the open orbit in the toric subvariety $\overline{B} \subset X$, then $\{B\}^{\mathsf{T}} = [\overline{B}] \in \widehat{\mathsf{A}}^{\mathsf{T}}_*(B)$; this is equivalent to proving that if $\overline{B}$ is a *good* (toric) closure of B, and $D = \overline{B} \smallsetminus B$, $c(\Omega^1_{\overline{B}}(\log D)^\vee) \cap [\overline{B}] = [\overline{B}] \in \mathsf{A}_*(\overline{B})$: and this is true because $\Omega^1_{\overline{B}}(\log D)$ is trivial ([5], Proposition, p. 87). □

In the particular case in which X is a *complete* toric variety, this reproduces Ehlers' formula.

Theorem 4.2 admits (with the same proof) a generalization to toral embeddings that are not necessarily normal.

References

[1] P. Aluffi. Limits of Chow groups, and a new construction of Chern-Schwartz-MacPherson classes. *Pure Appl. Math. Q.*, 2(4):915–941, 2006.

[2] P. Aluffi. Differential forms with logarithmic poles and Chern-Schwartz-MacPherson classes of singular varieties. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(7): 619–624, 1999.

[3] G. Barthel, J.-P. Brasselet, K.-H. Fieseler. Classes de Chern des variétés toriques singulières. *C. R. Acad. Sci. Paris Sér. I Math.*, 315(2): 187–192, 1992.

[4] W. Fulton. *Intersection theory.* Springer-Verlag, Berlin, 1984.

[5] W. Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[6] M. Goresky, W. Pardon. Chern classes of automorphic vector bundles. *Invent. Math.*, 147(3): 561–612, 2002.

[7] M. Kwieciński. Formule du produit pour les classes caractéristiques de Chern-Schwartz-MacPherson et homologie d'intersection. *C. R. Acad. Sci. Paris Sér. I Math.*, 314(8): 625–628, 1992.

[8] R. D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100: 423–432, 1974.

[9] M.-H. Schwartz. Classes caractéristiques définies par une stratification d'une variété analytique complexe. *C. R. Acad. Sci. Paris*, 260 : 3262–3264, 3535–3537, 1965.

Math Dept., Florida State University, Tallahassee FL 32306, U.S.A.

Max-Planck-Institut für Mathematik, Postfach 7280, D-53072 Bonn, Deutschland
E-mail address: aluffi@math.fsu.edu