Fundamental groups of manifolds with S^1-category 2

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Abstract

A closed topological n-manifold M^n is of S^1-category 2 if it can be covered by two open subsets W_1,W_2 such that the inclusions $W_i \to M^n$ factor homotopically through maps $W_i \to S^1 \to M^n$. We show that the fundamental group of such an n-manifold is a cyclic group or a free product of two cyclic groups with nontrivial amalgamation. In particular, if $n = 3$, the fundamental group is cyclic. [1] [2]

1 Introduction

The concept of the A-*category* of a manifold was introduced by Clapp and Puppe [1]. For a closed, connected 3-manifold M it is defined as follows: Let A be a closed connected k-manifold, $0 \leq k \leq 2$. A subset B in the 3-manifold M is A-*contractible* if there are maps $\varphi : B \longrightarrow A$ and $\alpha : A \longrightarrow M$ such that the inclusion map $i : B \longrightarrow M$ is homotopic to $\alpha \cdot \varphi$. The A-*category* $cat_A(M)$ of M is the smallest number of sets, open and A-contractible needed to cover M. Note that $2 \leq cat_A(M) \leq 4$. Endowing M with a (essentially unique) differential structure, an A-*function* on M is a smooth function $M \longrightarrow R$ whose critical set is a finite disjoint union of components each diffeomorphic to A. The invariant $crit_A(M)$ of M is the minimum number of components of the critical set over all A-functions on M..

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If A is a point, then $crit_{point}(M) = crit(M)$ has been calculated by Takens ([10]). He shows that $crit(M) = 2$ if and only if $M = S^3$ and $crit(M) = 3$ if and only if M is a connected sum of S^2-bundles over S^1. A related invariant of a more geometrical nature is $C(M)$, which is the smallest number of open 3-cells needed to cover M. Hempel-McMillan [6] (see also [4]) showed that in fact $C(M) = crit(M)$. Finally, $cat_{\text{point}}(M) = cat(M)$, is the Lusternik-Schnirelmann category of M, and in [2] it is shown that $cat(M) = 2$ if and only if $\pi_1(M) = 1$ and $cat(M) = 3$ if and only if $\pi_1(M)$ is a non-trivial free group (of finite rank). Hence, modulo the Poincarè conjecture, the three invariants $crit(M)$, $C(M)$, and $cat(M)$ coincide for closed 3-manifolds.

If $A = S^1$, then $crit_{S^1}(M)$ has been studied in ([8]). A smooth function $M \longrightarrow S^1$ whose critical set is a finite link in M is called a *round* function and Khimshiashvili and Siersma [8] show that round functions exist on all (orientable) 3-manifolds. Furthermore they show that $crit_{S^1}(M) = 2$ if and only if M is a lens space. A related invariant of a more geometrical nature is $T(M)$, which is the smallest number of open solid 3-tori needed to cover M. In [3]) it is shown that in fact (for orientable 3-manifolds) $T(M) = crit_{S^1}(M)$.

In this paper we show that for a closed 3-manifold M we have $cat_{S^1}(M) = 2$ if and only if $\pi_1(M)$ is cyclic.

By results of Olum [9] this implies that M is homotopy equivalent to a lens space. Therefore, modulo the conjecture that homotopy lens spaces are lens spaces, $crit_{S^1}(M) = 2$ if and only if $T(M) = 2$ if and only if $cat_{S^1}(M) = 2$.

The case that $cat_{S^1}(M) = 3$ seems to be difficult and one is lead to conjecture that the three invariants $crit_{S^1}(M)$, $T(M)$, and $cat_{S^1}(M)$ coincide for closed 3-manifolds.

The paper is organized as follows: For a closed topological n-manifold M^n we assume that $cat_{S^1}(M^n) = 2$. As a starting point we show in section 2 that then M can be constructed from two compact S^1-contractible submanifolds that intersect along their boundaries, and we prove some basic properties of S^1-contractible submanifolds and intersection numbers of their boundary surfaces with closed curves. In section 3 we show that all closed 2-manifolds with negative Euler characteristic have $cat_{S^1}(M^2) = 3$. Section 4 is devoted to the proof of the

Main Theorem: *Suppose M^n is closed, $n \geq 3$ and $\mathrm{cat}_{S^1} M^n = 2$. Then $\pi_1 M^n = A *_C B$ with A, B and C cyclic non-trivial or $\pi_1 M^n = 1$.*

Finally, in section 5 we apply the Main Theorem to infer that if $\mathrm{cat}_{S^1} M^3 = 2$ then $\pi_1(M)$ is cyclic.

2 Preliminaries

A subspace W of the manifold M^n is S^1-contractible (in M^n) if there exist maps $f : W \to S^1$, $\alpha : S^1 \to M^n$ such that the inclusion $\iota : W \to M^n$ is homotopic to αf. If $H : W \times I \to M^n$ is a homotopy between ι and αf, and $* \in W$, we have a commutative diagram

$$\begin{array}{ccc}\pi_1(W,*) & \xrightarrow{\iota_*} & \pi_1(M^n,*) \\ {\scriptstyle f_*}\downarrow & & \approx\downarrow{\scriptstyle \gamma_\#} \\ \pi_1(S^1,f(*)) & \xrightarrow{\alpha_*} & \pi_1(M^n,\alpha f(*)),\end{array}$$

where $\gamma = H \mid_{\{*\}\times I}$ is the trace of the homotopy. Hence $\operatorname{im}\iota_*$ is cyclic.

Notice that a subset of an S^1-contractible set is also S^1-contractible.

$\operatorname{cat}_{S^1} M$ is the smallest m such that there exist m open S^1-contractible subsets of M whose union is M.

It is easy to show that cat_{S^1} is a homotopy type invariant.

We first note that for the case that $\operatorname{cat}_{S^1} M^n = 2$ we can choose compact S^1-contractible submanifolds that intersect along their boundaries:

Lemma 1. *If U_0 and U_1 are open subsets of the closed manifold M^n whose union is M^n then there exist compact n-submanifolds W_0, W_1 such that $W_0 \cup W_1 = M^n$, $W_0 \cap W_1 = \partial W_0 = \partial W_1$ and $W_i \subset U_i$ $(i = 0, 1)$.*

Proof. Let $g : M^n \to [0, 1]$ be a map such that $g(M^n - U_i) = \{i\}, (i = 0, 1)$. For ϵ with $0 < \epsilon < 1/2$ there is an $\epsilon-$approximation f of g such that $f^{-1}(1/2)$ is an $(n-1)-$submanifold of M (see [7], Theorem 1.1). Let $W_0 = f^{-1}([1/2, 1])$ and $W_1 = f^{-1}([0, 1/2])$. These submanifolds satisfy the conclusion of the lemma. □

Corollary 1. *Suppose $\operatorname{cat}_{S^1} M^n = 2$ where M^n is a closed n-manifold. Then there exist S^1-contractible compact n-submanifolds W_0, W_1 such that $W_0 \cup W_1 = M^n$ and $W_0 \cap W_1 = \partial W_0 = \partial W_1$.*

Lemma 2. *If W^n is S^1-contractible in M^n and every loop in W^n is nullhomotopic in M^n, then W^n is contractible in M^n.*

Proof. The inclusion $W \to M$ is homotopic to a composition $W \xrightarrow{\tilde{f}} A \xrightarrow{\tilde{\alpha}} M$ where $p : A \to S^1$ is the covering space of S^1 corresponding to $f_*(\pi_1 W, *) \subset \pi_1(S^1, f(*))$ and $\tilde{f}$ is a lift of f, $\tilde{\alpha} = \alpha p$. If $A \approx R^1$, then W is contractible in M; if not, α must be null homotopic and, again, W is contractible in M. □

We think of S^1 as the space of complex numbers with modulus 1. If $\alpha : S^1 \to M$ and $m \in \mathbb{Z}$, we define α^m by $\alpha^m(z) = \alpha(z^m)$. Clearly, if $\beta \simeq \alpha$ then $\beta^m \simeq \alpha^m$ where $\simeq$ means "is homotopic in M^n to". If F is a compact $(n-1)$-submanifold of M^n with empty boundary and $\alpha : S^1 \to M$ is a loop, we define the intersection number $\alpha \cdot F = \min\{|\beta^{-1}(F)| : \beta \simeq \alpha\}$.

Lemma 3. *Let M^n be a closed n-manifold and let W_0^n and W_1^n be compact nonempty n-submanifolds of M^n such that $W_0^n \cup W_1^n = M^n$ and $W_0^n \cap W_1^n = \partial W_0^n = \partial W_1^n$ and let $\alpha : S^1 \to M$ be a loop. If $\alpha^m \cdot (W_0 \cap W_1) = 0$ and $m \neq 0$, then $\alpha \cdot (W_0 \cap W_1) = 0$.*

Proof. We may assume $m > 0$. Write $F = W_0^n \cap W_1^n$. The number $\alpha \cdot F$ is finite and we may assume that α is in general position with respect to F so that $|\alpha^{-1}(F)| = \alpha \cdot F = p$ say. Suppose $p > 0$. Since $\alpha^m \cdot F = 0$ there exists a loop β, homotopic to α^m, such that $\beta(S^1) \cap F = \emptyset$.

There is a homotopy $\varphi : S^1 \times I \to M$ with $\varphi|_{S^1\times\{0\}} = \alpha^m$ and $\varphi|_{S^1\times\{1\}} = \beta$. Using transversality of maps between topological manifolds (for example Theorem 1.1 of [7]) we may assume that φ is in general position with respect to F. Then $S = \varphi^{-1}(F)$ consists of simple closed curves in $int(S^1 \times I)$ and arcs, with the endpoints of each arc in $S_1 \times \{0\}$. Each arc of S splits off a disk from $S^1 \times I$. Since $p > 0$ there is an innermost such disk D such that $\partial D = a \cup b$, where a is an arc of S and b is an arc on $S^1 \times 0$ and $D \cap S - a$ is empty or consists of simple closed curves only. Then $\varphi \mid D$ defines a homotopy rel ∂ of the restriction of α to b to a map from b into F, contradicting the fact that $|\alpha^{-1}(F)| = \alpha \cdot F$. Hence $0 = p = \alpha \cdot F$. $\square$

Now consider again the case that $\mathrm{cat}_{S^1} M^n = 2$. Recall that we can write $M^n = W_0^n \cup W_1^n$ as a union of two compact submanifolds with $W_0^n \cap W_1^n = \partial W_0^n = \partial W_1^n$ such that for $i = 0, 1$ we have homotopy commutative diagrams

$$\begin{tikzcd} W_i^n \arrow[rr] \arrow[dr, "f_i"'] & & M \\ & S^1 \arrow[ur, "\alpha_i"'] & \end{tikzcd}$$

Proposition 1. *For $i = 0, 1$, we can take α_i so that $\alpha_i(S^1)$ does not intersect $W_0^n \cap W_1^n$.*

Proof. If every loop in W_i^n is nullhomotopic in M^n then, by lemma 2, W_i^n is contractible in M^n and therefore we can take as α_i a constant map with image in $\mathrm{int}(W_0^n)$ or $\mathrm{int}(W_1^n)$. If there is a loop γ in W_i^n that is not nullhomotopic in M^n, then $\gamma \simeq \alpha_i f_i \gamma \simeq \alpha_i^m$ for some $m \neq 0$. Hence $0 = \gamma \cdot (W_0^n \cap W_1^n) = \alpha_i^m \cdot (W_0^n \cap W_1^n)$ and, by Lemma 3, $\alpha_i \cdot (W_0^n \cap W_1^n) = 0$. Therefore, we can take as α_i a loop such that $\alpha_i(S^1) \cap W_0^n \cap W_1^n = \emptyset$. $\square$

Lemma 4. *Suppose $n > 2$. Then every component of $W_0^n \cap W_1^n$ is separating.*

Proof. Such a component C is S^1-contractible and so the inclusion induced homomorphism factors as

$$H_{n-1}(C; \mathbb{Z}_2) \to H_{n-1}(S^1; \mathbb{Z}_2) \to H_{n-1}(M^n; \mathbb{Z}_2).$$

Hence C bounds in M^n and so C is separating. $\square$

3 2-manifolds.

Note that for in a closed 2-manifold M, disks, annuli, and Möbius bands are S^1-contractible.

Since

$S^2 =$ (disk)$\cup$(disk)
$P^2 =$(Möbius band)$\cup$(disk)
$T^2 =$(annulus)$\cup$(annulus)
$K^2 =$(Klein bottle)=(annulus)$\cup$(annulus)

we have $\mathrm{cat}_{S^1}(S^2) = \mathrm{cat}_{S^1}(P^2) = \mathrm{cat}_{S^1}(T^2) = \mathrm{cat}_{S^1}(K^2) = 2$.

We will see that all other closed 2-manifolds have cat_{S^1} equal to 3.

Proposition 2. *Let M^2 be a closed 2-manifold. Suppose there is a compact 1-submanifold of M^2, with empty boundary, such that, for every component X of its complement,* $\mathrm{im}(\pi_1 X \to \pi_1 M^2)$ *is cyclic. Then $\chi(M^2) \geq 0$.*

Proof. Let F be a compact 1-submanifold of M^2, with a minimal number of components, having the property of the statement. We claim that every component X of $M^2 - F$ has nonnegative Euler characteristic. For, if $\chi(X) < 0$ then $\partial\overline{X} \to \overline{X}$ is π_1-injective, $\pi_1\overline{X}$ is not cyclic and $\mathrm{im}(\pi_1\overline{X} \to \pi_1 M^2)$ is cyclic. These three properties imply that $\partial\overline{X} \to M^2 - X$ is not π_1-injective and, therefore, some component C of $\partial\overline{X}$ bounds a 2-disk D in $M^2 - X$. But then $\mathrm{im}(\pi_1(\overline{X} \cup D) \to \pi_1 M^2)$ is cyclic and $F - C$ is a compact 1-submanifold having the property of the statement, contradicting our minimality assumption. Hence $\chi(\overline{X}) = \chi(X) \geq 0$ for every component X of $M - F$. Therefore $\chi(M^2) = \sum \chi(\overline{X}) - \chi(F) = \sum \chi(\overline{X}) \geq 0$, where in the sum X runs over the components of $M^2 - F$. $\square$

Corollary 2. *If $\mathrm{cat}_{S^1} M^2 = 2$, then $\chi(M^2) \geq 0$.*

Proof. By Corollary 1, there are S^1-contractible submanifolds W_0, W_1 such that $W_0 \cup W_1 = M^2$, $W_0 \cap W_1 = \partial W_0 = \partial W_1$. Every component X of $M^2 - W_0 \cap W_1$ is S^1-contractible and so $\mathrm{im}(\pi_1 X \to \pi_1 M^2)$ is cyclic. By Prop. 1, $\chi(M^2) \geq 0$. $\square$

4 n-manifolds.

In this section we prove the Main Theorem :

Suppose M^n is closed, $n \geq 3$ and $\mathrm{cat}_{S^1} M^n = 2$. Then $\pi_1 M^n = A *_C B$ with A, B and C cyclic non-trivial or $\pi_1 M^n = 1$.

Suppose $\mathrm{cat}_{S^1} M^n = 2$. Recall that we can write $M^n = W_0^n \cup W_1^n$, where W_0^n and W_1^n are S^1-contractible compact n-submanifolds with $W_0^n \cap W_1^n = \partial W_0^n = \partial W_1^n$.

We first consider the case that W_i is connected:

Theorem 1. *If W_0 and W_1 are connected, then $\pi_1 M^n$ is cyclic.*

Proof. By Lemma 4, $W_0^n \cap W_1^n$ is connected. Let $A = \mathrm{im}(\pi_1 W_0^n \to \pi_1 M^n)$, $B = \mathrm{im}(\pi_1 W_1^n \to \pi_1 M^n)$ and $C = \mathrm{im}(\pi_1(W_0^n \cap W_1^n) \to \pi_1 M^n)$. Since W_0^n, W_1^n and $W_0^n \cap W_1^n$ are S^1-contractible A, B and C are cyclic.

We have natural homomorphisms $\pi_1 W_0^n \to A \to A *_C B$ and similarly for $\pi_1 W_1^n$ and $\pi_1(W_0^n \cap W_1^n)$. We also have a natural homomorphism $\psi : A *_C B \to$

$\pi_1(M)$. By Van Kampen's theorem and the universal property of $A *_C B$, we have the following commutative diagram with a homomorphism φ.

$$\begin{tikzcd}
& \pi_1(M^n) \arrow[d, dotted, shift left, "\varphi"] & \\
\pi_1(W_0^n) \arrow[ur] \arrow[r] & A *_C B \arrow[u, shift left, "\psi"] & \pi_1(W_1^n) \arrow[ul] \arrow[l] \\
& \pi_1(W_0^n \cap W_1^n) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Since $\psi\varphi$ and $\varphi\psi$ are the identity on $A \cup B$ we have $\psi\varphi = id$ and $\varphi\psi = id$. Hence $\pi_1 M^n = A *_C B$ and $H_1 M^n = A \oplus_C B := (A \oplus B)/\{(c, -c) : c \in C\}$.

Observe that this implies that $A = \mathrm{im}(H_1(W_0^n) \to H_1(M^n))$, $B = \mathrm{im}(H_1(W_1^n) \to H_1(M^n))$ and $C = \mathrm{im}(H_1(W_0^n \cap W_1^n) \to H_1(M^n))$.
Case (i): M^n is orientable.

We show that $A = C = B$ (and so $\pi_1 M^n$ is cyclic).

We have

$$\begin{tikzcd}
H^n(W_i^n) \arrow[d, equal] & H^n(M^n) \arrow[l] \arrow[d, phantom, "\wr"] & H^n(M^n, W_i^n) \arrow[l, "\approx"'] \arrow[d, phantom, "\wr\ \text{excision}"] & H^{n-1}(W_i^n) \arrow[l, "0"'] & & H^n(M^n) \arrow[ll, "0"'] \arrow[dl] \\
0 & \mathbb{Z} & H^n\left(W_{1-i}^n, \partial W_{1-i}^n\right) \arrow[d, phantom, "\wr\ \text{Poincaré duality}"] & & H^{n-1}\left(S^1\right) = 0 \arrow[ul] & \\
& & H_0\left(W_{1-i}^n\right) = \mathbb{Z} & & &
\end{tikzcd}$$

Hence $0 = H^{n-1}(W_i^n) = H_1(W_i^n, \partial W_i^n)$, so $H_1(\partial W_i^n) \to H_1(W_i^n)$ is onto. Therefore $C = \mathrm{im}(H_1(\partial W_0) \to H_1(M)) = \mathrm{im}(H_1(W_0) \to H_1(M)) = A$ and similarly $C = B$.
Case (ii): M^n is nonorientable.

By a similar proof as in case (i) taking $\mathbb{Z}_2$ coefficients, we obtain that C has odd index in A and in B. Hence $\mathrm{coker}(H_1 W_0^n \to H_1 M^n) = B/C$ is a finite cyclic group of odd order. Since the subgroup of $H_1 M^n$ consisting of all orientation-preserving loops has index two in $H_1 M^n$ it follows that $\mathrm{im}(H_1 W_0^n \to H_1 M^n)$ contains an orientation-reversing loop and hence W_0^n (and similarly W_1^n) is non orientable. Therefore for the orientable two-fold covering $p : \tilde{M}^n \to M^n$ the lift $\tilde{W}_i = p^{-1}(W_i^n)$ is connected. We may assume that α_i is an embedding. Since an orientation reversing loop is not null-homotopic in M it follows that $\tilde{S}^1 = p^{-1}(S^1)$ is homeomorphic to S^1, α_i lifts to an embedding $\tilde{\alpha}_i$, f_i lifts to $\tilde{f}_i$ and we obtain the following diagram

$$\begin{tikzcd}
\tilde{W}_i \arrow[dr, "\tilde{f}_i"] \arrow[dd, "p"'] & & \tilde{M}^n \arrow[dd, "p"] \\
& \tilde{S}^1 \arrow[ur, "\tilde{\alpha}_i"] \arrow[dd, "p"'] & \\
W_i^n \arrow[dr, "f_i"] & & M^n \\
& S^1 \arrow[ur, "\alpha_i"] &
\end{tikzcd}$$

Then $\tilde{\alpha}_i \tilde{f}_i$ is homotopic to the inclusion $\tilde{i} : \tilde{W}_i \to \tilde{M}^n$ and $\mathrm{cat}_{S^1} \tilde{M}^n = 2$ and by case (i) $\pi_1 p^{-1}(W_i) \to \pi_1 \tilde{M}^n$ is surjective.

Hence $\mathrm{im}(\pi_1 W_i^n \to \pi_1 M^n)$ contains $\mathrm{im}(\pi_1 \tilde{M}^n \to \pi_1 M^n)$, the index 2 subgroup of orientation preserving loops, and since W_i^n is nonorientable, $\mathrm{im}(\pi_1 W_i^n \to \pi_1 M^n) = \pi_1(M^n)$. Therefore $\pi_1 M^n$ is cyclic. $\square$

We now consider the case that W_0 or W_1 is not connected.

By Proposition 1 we can assume $\alpha_i(S^1)$ does not intersect $W_0^n \cap W_1^n$. Now depending on whether $\alpha_i(S^1)$ is in W_i or W_{1-i} we will prove the following

Theorem 2. (a) *If $\alpha_0(S^1) \subset W_1$ or $\alpha_1(S^1) \subset W_0$, then $\pi_1(M^n)$ is cyclic.*

(b) *If $\alpha_i(S^1) \subset W_i$ $(i = 0, 1)$ and F^{n-1} is any component of $W_0 \cap W_1$ separating $\alpha_0(S^1)$ from $\alpha_1(S^1)$, then $\pi_1 M^n = A_0 *_C A_1$*

with $C = \mathrm{im}(\pi_1 F^{n-1} \to \pi_1 M^n)$ cyclic, and $A_i = \mathrm{im}(\pi_1 X_i \to \pi_1 M^n)$ cyclic $(i = 0, 1)$, where X_i is the component of $M^n - F^{n-1}$ containing $\alpha_i(S^1)$.

Write $F = W_0 \cap W_1$. To study $\pi_1 M$ we now attach 2-cells to F, W_0 and W_1 along loops that are nullhomotopic in M, obtaining spaces $\hat{F}$, $\hat{W}_0$, $\hat{W}_1$ such that the fundamental group of their components are cyclic and $\hat{F}$ is π_1-injective in $\hat{W}_i$ $(i = 0, 1)$. The new space $\hat{M}$ will have the same fundamental group as M.

Let $W_i^1, W_i^2, \ldots$ be the component of W_i. If $W_0^j \cap W_1^k \neq \emptyset$ write $F_{jk} = W_0^j \cap W_1^k$. Notice that the image of $\pi_1 W_i^j \longrightarrow \pi_1 M$ and $\pi_1 F_{jk} \longrightarrow \pi_1 M$ are cyclic. Let $K_{jk} = \ker\left(\pi_1 F_{jk} \longrightarrow \pi_1 M\right)$, $K_i^j = \ker\left(\pi_1 W_i^j \longrightarrow \pi_1 M\right)$. For every jk, attach 2-cells to F_{jk} along a collection of loops whose normal closure in $\pi_1 F_{jk}$ is K_{jk}. Let E_{jk} be the union of these 2-cells. For every i and every j attach 2-cells to W_i^j along a collection of loops whose normal closure in $\pi_1 W_i^j$ is K_i^j. Let A_i^j be the union of these 2-cells.

Let $\widehat{F}_{jk} = F_{jk} \cup E_{jk}$, $\widehat{W}_0^j = W_0^j \cup A_0^j \cup \left(\cup_k E_{jk}\right)$, $\widehat{W}_1^j = W_1^j \cup A_1^j \cup \left(\cup_j E_{jk}\right)$, $\widehat{W}_i = \cup_j \widehat{W}_i^J$ and $\widehat{M} = \widehat{W}_0 \cup \widehat{W}_1$.

Note:

- $\pi_1 \widehat{F}_{jk}$ is cyclic for every jk

- $\pi_1 \widehat{W}_i^j$ is cyclic for every i and every j.
- The inclusion $M \longrightarrow \widehat{M}$ induces an isomorphism on fundamental group.
- The inclusions $\widehat{F}_{jk} \longrightarrow \widehat{W}_0^j$, $\widehat{F}_{jk} \longrightarrow \widehat{W}_1^j$ are π_1-injective.

If Y is a union of subspaces of M which are components of W_0 or of W_1 we write $\widehat{Y} = \cup \left\{ \widehat{W}_i^j : W_i^j \text{ is a component of } W_0 \text{ or of } W_1 \text{ contained in } Y \right\}$. Observe that if Y is connected then $\pi_1 \widehat{Y} \to \pi_1 \widehat{M}$ is injective (use, for example, [5, Lemma 2.2]) and we have a commutative diagram with $\pi_1 \widehat{Y} \to \pi_1 Y$ surjective:

$$\begin{tikzcd}
\pi_1 Y \arrow[r] \arrow[d] & \pi_1 M \arrow[d, "\approx"'] \\
\pi_1 \widehat{Y} \arrow[r] & \pi_1 \widehat{M}
\end{tikzcd}$$

Hence we can identify the image of $\pi_1 Y$ in $\pi_1 M$ with $\pi_1 \hat{Y}$.

Lemma 5. *Let β be loops in $\widehat{F}_{jk}$ that are homotopic in $\widehat{W}_0^j$ or in $\widehat{W}_1^k$. Then they are homotopic in $\widehat{F}_{jk}$.*

Proof. Since the fundamental groups of $\widehat{F}_{jk}$, $\widehat{W}_0^j$ and $\widehat{W}_1^k$ are abelian, the inclusions $\widehat{F}_{jk} \longrightarrow \widehat{W}_0^j$ and $\widehat{F}_{jk} \longrightarrow \widehat{W}_1^k$ are H_1-injective. Hence β and γ are homologous, and therefore homotopic, in $\widehat{F}_{jk}$ □

Recall that $F = W_0 \cap W_1$. In the following lemma we will use the graph G of (M, F) which is defined as follows. The vertices (resp. edges) of G are in one-to-one correspondence with the closures of the components of $M - F$ (resp. with the components of F). The endpoints of an edge e of G corresponding to components F' of F correspond to W_0' and W_1', components of W_0 and W_1, where $F' \subset W_i'$ $(i = 0, 1)$.

If $n > 2$, the graph G is a tree because of Lemma 2.

An example, in the form of a schematic diagram of $\widehat{M}$, is shown in Figure 1. The graph G of (M, F) is obtained by collapsing each $\widehat{W}_i^j$ to a point.

Lemma 6. *Let β and γ be loops in different components of $M - F$ that are homotopic in M. Let $p : [0, 1] \longrightarrow M$ be a map, with $p(0) \in \operatorname{im} \beta$, $p(1) \in \operatorname{im} \gamma$, such that $p^{-1}(F)$ has minimal cardinality m. Write $p^{-1}(F) = \{t_1, \ldots, t_m\}$ where $t_1 < t_2 < \cdots < t_m$. Then there is a sequence of loops $\beta_0, \beta_1, \ldots, \beta_m$ such that*

1) $\beta_0 = \beta$ and $\beta_{m+1} = \gamma$

2) $\operatorname{im} \beta_j$ is contained in the component of F where $p(t_j)$ lies $(j = 1, \ldots, m)$ and

3) β_j is homotopic to β_{j+1} in $\widehat{W}_0$ or in $\widehat{W}_1$ $(j = 0, 1, \ldots, m)$

Proof. Let $\varphi : S^1 \times I \longrightarrow M$ be a homotopy between β and γ in M. By general position (transversality of maps between topological manifolds e.g. Theorem 1.1 of [KS]) we may assume that $S = \varphi^{-1}(F)$ is a collection of simple closed curves in $int(S^1 \times I)$.

Let $D_1, \ldots, D_t$ be 2-disks embedded in $S^1 \times I$ such that $\partial D_1, \ldots, \partial D_t$ are components of S and all components of $S - \cup_{j=1}^{m} D_j$ are not null-homotopic in $S^1 \times I$. Since the inclusion of $\widehat{F}$ in $\widehat{M}$ is π_1-injective we can define a homotopy $\widehat{\varphi} : S^1 \times I \longrightarrow \widehat{M}$ such that $\widehat{\varphi}$ coincides with φ on $S^1 \times I - \cup_{j=1}^{m} \operatorname{int} D_j$ and $\widehat{\varphi}\left(\cup_{j=1}^{m} D_j\right) \subset \widehat{F}$. If the components of $S - \cup_{j=1}^{m} D_j$ are suitably indexed as $s_1, s_2, \ldots, s_{r-1}$, and $s_0 = S^1 \times \{0\}\,,\ s_r = S^1 \times \{1\}\,,$ then $\varphi|_{s_i}$ $(i = 0, \ldots, r)$ defines a loop β_i' in M with β_i' homotopic to β_{i+1}' (i=0,...,r) in $\widehat{W}_0$ or $\widehat{W}_1$.

The sequence of loops $\beta_0', \beta_1', \ldots, \beta_r'$ has the following properties

a) The first one is β and the last one is γ

b) Their images are contained in F, except the first one and the last one

c) Each loop in the sequence is homotopic to the next one in $\widehat{W}_0$ or in $\widehat{W}_1$.

Now let $\beta_0, \beta_1, \ldots, \beta_s$ be a sequence of loops satisfying a), b) and c), such that s is minimal. We claim that $s = m + 1$ and that 2) holds.

Let G be the graph of (M, F). Consider the path Δ in G associated to the sequence $(\beta_0, \beta_1, \ldots, \beta_s)$, that is, the sequence of edges $(e_1, \ldots, e_s)$ such that, for $0 < i < s$, $\operatorname{im} \beta_i$ is contained in the component of F associated to e_i. The loop β_0 (resp. β_s) is homotopic to β_1 (resp. β_{s-1}) in the component of $\widehat{W}_0$ or in $\widehat{W}_1$ containing the component associated to u (resp. v) where u (resp. v) is a vertex of e_1 (resp. e_{s-1}). Δ is a path from u to v in G. Suppose Δ is not a simple path. Then $e_i = e_{i+1}$ for some i and, by Lemma 5, β_i is homotopic to β_{i+1} in the component of F associated to e_i; then, we omit β_{i+1} in the sequence $(\beta_0, \beta_1, \ldots, \beta_s)$ we still have a sequence satisfying a), b) and c) contradicting the minimality of s. Hence Δ is a simple path in G from u to v.

The map p also defines a path $(e_1', \ldots, e_s')$ of minimal length from u to v; the component associated to e_j' is the components of F containing $p\,(t_j)$. This path is also simple and, since G is a tree, we must have $e_j' = e_j$ for all j. Hence $s = s' = m + 1$ and the component of F containing $\operatorname{im} \beta_j$ is the one to which $p\,(t_j)$ belongs $(j = 1, \ldots, m)$. □

In the following we wish to prove that in some cases the monomorphism $\pi_1 \widehat{F'} \longrightarrow \pi_1 \widehat{W'}$ is surjective, where F' is a component of F and W' is a component of W_0 or of W_1 containing F'. To do so it suffices to show that every loop in W' is homotopic in $\widehat{W'}$ to a loop in F'; this implies that every element of $\pi_1 \widehat{W'}$ is conjugate to an element of the image of $\pi_1 \widehat{F'} \longrightarrow \pi_1 \widehat{W'}$ but, since $\pi_1 \widehat{W'}$ is abelian, this image must be $\pi_1 \widehat{W'}$.

Now, recall that, by Prop. 3, we may assume that the images of α_0 and α_1 do not intersect F.

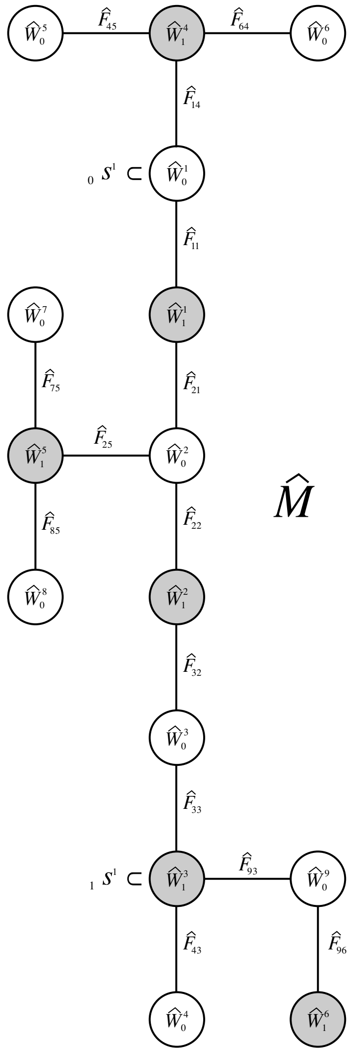

Figure 1: A schematic diagram of $\widehat{M}$

Lemma 7. *Let W_i^q be a component of W_i which does not contain $\alpha_i\left(S^1\right)$. for $i = 0, 1$ and let F'_{jk} be the component of ∂W_i^q separating $\operatorname{int} W_i^q$ from $\alpha_i\left(S^1\right)$. Then $\pi_1 \widehat{F}'_{jk} \longrightarrow \pi_1 \widehat{W}_i^q$ is an isomorphism.*

Proof. Since $\pi_1 \widehat{F}'_{jk} \longrightarrow \pi_1 \widehat{W}_i^q$ is injective we only need to prove surjectivity. Let β be a loop in W_i^q. Then β is homotopic in M to a power of α_i. A map $p : [0,1] \longrightarrow M$ with $p(0) \in \operatorname{im}\beta$, $p(1) \in \operatorname{im}\alpha_i$ and $\left|p^{-1}(F)\right|$ minimal is such that $p(t_1) \in F'_{jk}$ where $p^{-1}(F) = \{t_1, \ldots, t_m\}$ and $t_1 < t_2 < \cdots < t_m$.

By Lemma 6, there is a sequence $(\beta, \beta_1, \ldots, \beta_{m+1})$ where β_{m+1} is a power of α_i, β is homotopic to β_1 in $\widehat{W}_i^q$ and $\operatorname{im}\beta_1 \subset F'_{jk}$. Hence $\pi_1\widehat{F}'_{jk} \longrightarrow \pi_1\widehat{W}_i^q$ is surjective. $\square$

In the next lemma we refer to the graph G of (M, F).

Lemma 8. *There is an n-submanifold Q^n of M^n with the following properties:*

$(*)$ *Q^n is a union of components of W_0 and W_1 and the sub-graph G_Q of G corresponding to $(Q^n, Q^n \cap F)$ is linear and connected;*

$(*)$ *$\alpha_i\left(S^1\right)$, $(i = 1, 2)$ lies in a component of W_0 or W_1 corresponding to a vertice of degree 1 in G_Q;*

$(*)$ *inclusion induces an isomorphism $\pi_1\widehat{Q}^n \cong \pi_1\widehat{M}^n$.*

For example, for the manifold pair (M, F) represented in Figure 1, $\widehat{Q} = \widehat{W}_0^1 \cup \widehat{W}_1^1 \cup \widehat{W}_0^2 \cup \widehat{W}_1^2 \cup \widehat{W}_0^3 \cup \widehat{W}_1^3$.

Proof. Recalling that G is a finite tree, let W^p be a component of W_0 or W_1 corresponding to a vertex of degree 1 in G and let $Q_1^n = \overline{M^n - W^p}$. If W^p does not contain $\alpha_i\left(S^1\right)$ for $i = 1, 2$ then by Lemma 7, $\pi_1\widehat{F}'_{jk} \longrightarrow \pi_1\widehat{W}_i^p$ is an isomorphism, where $F'_{jk} = W_i^p \cap Q_1^n$. By Van Kampen's Theorem inclusion induces an isomorphism $\pi_1\widehat{Q_1}^n \cong \pi_1\widehat{M}^n$. We now obtain Q^n by cutting off from M^n all those components of W_0 and W_1 corresponding to vertices of degree 1 which do not contain $\alpha_i\left(S^1\right)$ for $i = 1, 2$ and repeating this process inductively. $\square$

Corollary 3. *If $\alpha_0\left(S^1\right)$ and $\alpha_1\left(S^1\right)$ are contained in the same component of $M - F$ then $\pi_1 M$ is cyclic.*

Proof. By Lemma 8, $\pi_1 M \approx \pi_1\widehat{Q^n}$ where now Q^n is equal to the component W^p of W_0 or of W_1 containing $\alpha_0(S^1)$ and $\alpha_1(S^1)$. Hence $\pi_1\widehat{Q}^n \cong \pi_1\widehat{W}^p$ is cyclic and the result follows. $\square$

Proof of Theorem 2(a).

Suppose $\alpha_1\left(S^1\right) \subset W_0$. We may assume $\alpha_1\left(S^1\right) \subset \operatorname{int} W_0$ and let $f'_1 = f_0\alpha_1 f_1$. Then

$$\begin{tikzcd}
W_1 \arrow[rr] \arrow[dr, "f'_1"'] & & M \\
& S^1 \arrow[ur, "\alpha_0"'] &
\end{tikzcd}$$

is also homotopy commutative and we can take $\alpha'_1 = \alpha_0$ instead of α_1. By Corollary 3, $\pi_1 M$ is cyclic.

Similarly, if $\alpha_0\left(S^1\right) \subset W_1$ then $\pi_1 M$ is cyclic.

Proof of Theorem 2(b).

Assume $\alpha_i\left(S^1\right) \subset W_i$ $(i = 1, 2)$. Let Q^n be as in Lemma 8 and let W_0^p (resp. W_1^p) $(p = 1, \ldots, s)$ be the components of $W_0 \cap Q^n$ (resp. $W_1 \cap Q^n$) indexed such that $\operatorname{int} W_0^1 \supset \alpha_0\left(S^1\right)$, $\operatorname{int} W_1^s \supset \alpha_1\left(S^1\right)$ and $W_0^p \cap W_1^q \neq \emptyset$ iff $p = q$ or $p = q+1$. Write $F_{q,q} = W_0^q \cap W_1^q$ and $F_{q+1,q} = W_0^{q+1} \cap W_1^q$.

Claim 1. $\pi_1 \widehat{F}_{q+1,q} \longrightarrow \pi_1 \widehat{W}_0^{q+1}$ and $\pi_1 \widehat{F}_{q+1,q} \longrightarrow \pi_1 \widehat{W}_1^q$ are isomorphisms.

To see this, let β be any loop in W_0^{q+1}. Then β is homotopic, in M, to a loop in W_0^1 (namely a power of α_0). By Lemma 6, β is then homotopic, in $\widehat{W}_0^{q+1}$, to a loop in $\widehat{F}_{q+1,q}$. Hence $\pi_1 \widehat{F}_{q+1,q} \longrightarrow \pi_1 \widehat{W}_0^{q+1}$ is an isomorphism.

In a similar way, using the fact that any loop in W_1^q is homotopic, in M, to a loop in W_1^s, we see that $\pi_1 \widehat{F}_{q+1,q} \longrightarrow \pi_1 \widehat{W}_1^q$ is an isomorphism.

Claim 2. If $1 < q < s$ then $\pi_1 \widehat{F}_{q,q} \longrightarrow \pi_1 \widehat{W}_0^q$ and $\pi_1 \widehat{F}_{q,q} \longrightarrow \pi_1 \widehat{W}_1^q$ are isomorphisms.

To see this let β be any loop in W_0^q. Then β is homotopic in M, to a loop in W_0^1 and therefore, by Lemma 6, β is homotopic in $\widehat{W}_0^q$ to a loop γ in $F_{q,q-1}$. Let δ be a loop in $\operatorname{int} W_1^{q-1}$ homotopic to γ in W_1^{q-1}. Then δ is homotopic in M to a loop in W_1^s and therefore, using Lemma 6, δ is homotopic in $\widehat{W}_1^{q-1}$ to a loop δ_1 in $F_{q,q-1}$ and δ_1 is homotopic in $\widehat{W}_0^q$ to a loop δ_2 in $F_{q,q}$. By Lemma 5 γ is homotopic to δ_1 in $\widehat{F}_{q,q-1}$. Hence, in $\widehat{W}_0^q$, $\beta \simeq \gamma \simeq \delta_1 \simeq \delta_2$. Therefore $\pi_1 \widehat{F}_{q,q} \longrightarrow \pi_1 \widehat{W}_0^q$ is an isomorphism.

Similarly we show that if β is any loop in W_1^q, then, in $\widehat{W}_1^q$ we have $\beta \simeq \gamma \simeq \delta_1 \simeq \delta_2$, where now γ and δ_1 are loops in $F_{q+1,q}$ and δ_2 is a loop in $F_{q,q}$. Therefore $\pi_1 \widehat{F}_{q,q} \longrightarrow \pi_1 \widehat{W}_1^q$ is an isomorphism.

Now let F' be any component of $W_0 \cap W_1$ separating $\alpha_0\left(S^1\right)$ from $\alpha_1\left(S^1\right)$, that is, $F' = F_{q,q}$ or $F' = F_{q+1,q}$ for some q. Let X_i be the closure of the component of $M - F'$ containing $\alpha_i\left(S^1\right)$. The argument in the proof of Lemma 8 shows that the inclusion of $\widehat{W}_0^1$ in $\widehat{X}_0$ and the inclusion of $\widehat{W}_0^s$ in $\widehat{X}_1$ induce isomorphisms of fundamental groups. Hence $\pi_1 \widehat{X}_0$, $\pi_1 \widehat{X}_1$ and $\pi_1 \widehat{F}'$ are cyclic and therefore A_0, A_1 and C are cyclic (see the remark before Lemma 5).
Since by Van Kampen's Theorem we have $\pi_1 \widehat{M} = \pi_1 \widehat{X}_0 *_{\pi_1 \widehat{F}'} \pi_1 \widehat{X}_1$ it follows that $\pi_1 M = A_0 *_C A_1$.

This completes the proof of Theorem 3.

To complete the proof of the Main Theorem it remains to show that if $\pi_1 M^n$ is not trivial, then the amalgamating subgroup C is non-trivial.

Lemma 9. *Let W^0 and W^1 be disjoint compact n-submanifolds of M^n where W^0 is S^1-contractible in M^n and W^1 is connected and contractible in M. Let $T = D^{n-1} \times [0,1]$ be a tube in M^n such that $W^i \cap T = D^{n-1} \times \{i\}$ (i=0,1). Then $W^0 \cup T \cup W^1$ is S^1-contractible in M.*

Proof. Let $a = \{0\} \times [0,1]$ be the core of T, $p = (0,0)$ and $q = (0,1)$ so $\partial a = \{p, q\}$. Then $W^0 \cup T \cup W^1$ deformation retracts to $W^0 \cup a \cup W^1$ in M so it suffices to show that $W^0 \cup a \cup W^1$ is S^1-contractible in M. Since it is easy to see that $W^0 \cup a$ is S^1-contractible in M, it suffices to show that the diagram below is homotopy commutative

$$\begin{tikzcd} W^0 \cup a \cup W^1 \arrow[rr, "\text{inclusion}"] \arrow[dr, "r"'] & & M \\ & W^0 \cup a \arrow[ur, "\text{inclusion}"'] & \end{tikzcd}$$

where r is the retraction with $r\left(W^1\right) = q$ and the other two maps are inclusions.

To construct the homotopy $H : (W^0 \cup a \cup W^1) \times I \longrightarrow M$ we note that since W^1 is contractible in M there is a map $H : W^1 \times \left[0, \frac{1}{2}\right] \longrightarrow M$ such that $H\left(x, 0\right) = x$ and $H\left(W^1 \times \left\{\frac{1}{2}\right\}\right)$ is a point. Extend H to $W^1 \times [0,1]$ by defining $H\left(x,t\right) = H\left(q, 1-t\right)$ for $\frac{1}{2} \leq t \leqq 1$. Since $H|_{q \times [0,1]}$ defines a nullhomotopic loop of the form $\gamma \cdot \gamma^{-1}$ we can extend H to $\left(a \cup W^1\right) \times [0,1]$ in such way that $H\left(p,t\right) = p$ for t$\in [0,1]$ and $H\left(x,1\right) = x$ if $x \in a$. Finally, extend H to $\left(W^0 \cup a \cup W^1\right) \times [0,1]$ by defining $H\left(x,t\right) = x$ for $x \in W^0$, $t \in [0,1]$. □

We denote the number of components of a submanifold W of M^n by $|W|$.

Corollary 4. *Suppose that M^n admits a decomposition $M^n = W_0 \cup W_1$ where W_0 and W_1 are S^1-contractible submanifolds of M^n with $W_0 \cap W_1 = \partial W_0 = \partial W_1$ and such that $|W_0| + |W_1| = c$ is minimal. If $|W_0| > 1$ (resp. $|W_1| > 1$) then no component of W_0 (resp. W_1) is contractible in M^n.*

Proof. Suppose, say, that $|W_0| > 1$ and W_0 has a contractible (in M^n) component W_0^1. Let $T = D^{n-1} \times [0,1]$ be a tube in M^n joining $W_0 - W_0^1$ to W_0^1 i.e. $T \cap (W_0 - W_0^1) = D^{n-1} \times \{0\}$ and $T \cap W_0^1 = D^{n-1} \times \{1\}$. Then by Lemma 9, $W_0 \cup T = (W_0 - W_0^1) \cup T \cup W_0^1$ is S^1-contractible and, as a submanifold of W_1, $\mathrm{cl}\left(W_1 - T\right)$ is S^1-contractible. This contradicts the minimality of c since $|W_0 \cup T| + |\mathrm{cl}\left(W_1 - T\right)| = c - 1$. □

We now finish the proof of the Main Theorem.

We express M^n as the union of two S^1-contractible submanifolds W_0, W_1 with $W_0 \cap W_1 = \partial W_0 = \partial W_1$ such that $|W_0| + |W_1| = c$ is minimal.

If $c = 2$ then $\pi_1 M$ is cyclic by Theorem 1. Hence we can assume $c > 2$. By Proposition 1 and Theorem 2 we can assume that $\alpha_i\left(S^1\right) \subset \operatorname{int} W_i^1$ $(i = 0, 1)$ where W_i^1 is a component of W_i. Furthermore for a component F' of ∂W_0^1 separating $\alpha_0\left(S^1\right)$ from $\alpha_1\left(S^1\right)$ and the closures X_i of the components of $M-F'$ containing $\alpha_i\left(S^1\right)$ $(i=0,1)$ we have $\pi_1 M = A_0 *_C A_1$ where $C = \operatorname{im}(\pi_1 F' \longrightarrow \pi_1 M)$ and $A_i = \operatorname{im}(\pi_1 X_i \longrightarrow \pi_1 M)$ are cyclic $(i=0,1)$.

We now show that C is not trivial.

Suppose, on the contrary, that C is trivial. If W_0^2 (resp. W_1^2) is a component of W_0 (resp W_1) contained in X_1 (resp. X_0) then every loop in W_0^2 (resp. W_1^2) is homotopic to a loop in W_0^1 (resp. W_1^1) and therefore, by Lemma 6, to a loop in F'. By assumption this loop is null homotopic in M^n and so, by Lemma 2, W_0^2 (resp. W_1^2) is contractible in M, which is impossible by Corollary 4. Hence there are no components of W_0 (resp. W_1) contained in X_1 (resp. X_0) and so $X_1 = W_1^1$, $X_0 = W_0^1$ and $c=2$, a contradiction.

5 Closed 3-manifolds.

If the fundamental group of a closed 3-manifold M^3 is cyclic, then, by results of Olum [9], M^3 is homotopy equivalent to a lens space $L(p,q)$ including S^3 and $S^1 \times S^2$, or $S^1 \tilde{\times} S^2$. Since these spaces can be expressed as the union of two solid tori or two solid Klein bottles and since cat_{S^1} is a homotopy-type invariant it follows that $\operatorname{cat}_{S^1} M^3 = 2$.

This shows sufficiency for the following

Theorem 3. *Let M^3 be a closed 3-manifold. Then $\operatorname{cat}_{S^1} M^3 = 2$ if and only if $\pi_1 M^3$ is cyclic.*

Proof. By the Main Theorem, if $\pi_1 M^3$ is not cyclic then $\pi_1 M^n = A *_C B$ is a non-trivial free product with amalgamation, with A, B and C cyclic. Hence $\pi_1 M^n$ is infinite with center $C \neq 1$ and so $\pi_1 M^n$ is not a non-trivial free product and it follows that every 2−sphere in M is homotopically trivial. Hence the prime decomposition of M shows that $\pi_1 M^n = \pi_1 M'$ where M' is a closed irreducible 3-manifold.

First assume that M is orientable or non-orientable but P^2-irreducible. Then Waldhausen's proof of Satz 1.2 [11], applies to show that M' contains a closed surface, different from S^2 or P^2, with fundamental group isomorphic to a subgroup of C, which is impossible. Hence $\pi_1 M^3$ is cyclic.

If M' is non-orientable and contains a 2−sided P^2 then $i_* \pi_1 P^2 \cong \mathbf{Z}_2$ is conjugate to a subgroup of A,B, or C and it follows that A,B and C are finite cyclic, hence $H_1(M')$ is finite, a contradiction, since M' is closed and non-orientable. □

Question: Suppose that M^n is closed and $\operatorname{cat}_{S^1} M^n = 2$. Is $\pi_1 M^n$ cyclic if $n > 3$?

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