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Convolution and Fourier restriction estimates for measures on curves in $\mathbb{R}^2$

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Abstract

We study convolution and Fourier restriction estimates for some degenerate curves in $\mathbb{R}^2$.

Notational comment: This note concerns certain operators defined on functions on $\mathbb{R}^2$. Thus f will always denote an appropriate function on $\mathbb{R}^2$, L^p will usually mean the L^p space constructed with Lebesgue measure m_2 on $\mathbb{R}^2$, and $\|\cdot\|_p$ stands for the norm in L^p.

The following two theorems are well known and are prototypical for many important results in harmonic analysis:

Theorem 1. *Suppose $a < b$ and write $Tf(x) = \int_a^b f\big(x - (t, t^2)\big)\, dt$. Then there is a constant C such that*

$$\|Tf\|_3 \leq C\ \|f\|_{\frac{3}{2}}.$$

Theorem 2. *If $1 \leq p < \frac{4}{3}$ and $\frac{1}{q} = 3(1 - \frac{1}{p})$, there is a constant $C = C(p)$ such that the estimate*

$$\Big(\int_a^b |\widehat{f}(t, t^2)|^q dt\Big)^{\frac{1}{q}} \leq C(p)\ \|f\|_p$$

holds.

It is natural to wonder what happens to Theorems 1 and 2 when the curve (t, t^2) is replaced by a general $\big(t, \phi(t)\big)$. Since the curvature of the parabola is key to the proofs of Theorems 1 and 2, a reasonable starting point for generalization is the hypothesis $\phi'' \geq \delta > 0$. And it has been known for some time that this hypothesis is sufficient to ensure that the analogs of Theorems 1 and 2 hold. The next step is to investigate this situation when ϕ'' is allowed to vanish, and it is easy to see that one can no longer expect the exact analogs of Theorems 1 and 2 to hold. There are then two possibilities. The first is to "dampen" the measure dt by introducing a factor $\omega(t)$ which is small when $\phi''(t)$ is small and then to attempt to obtain the exact analogs of Theorems 1 and 2 with dt replaced by $\omega(t)\ dt$. The second is to keep the dt and then to see how the conclusions of Theorems 1 and 2 change.

Concerning the first approach, there are the following results:

Theorem 1a. ([4]) *Suppose* $\phi'' > 0$ *and* $\phi^{(3)} \geq 0$ *on* (a, b). *Define the operator* T *by* $Tf(x) = \int_a^b f\big(x - (t, \phi(t))\big)\ \phi''(t)^{1/3} dt$. *Then*

$$\|Tf\|_3 \leq 12^{\frac{1}{3}}\ \|f\|_{\frac{3}{2}}$$

whenever $f = \chi_E$ *and* E *is a Borel subset of* $\mathbb{R}^2$.

Theorem 2a. ([6], [5]) *If* $1 \leq p < \frac{4}{3}$ *and* $\frac{1}{q} = 3(1 - \frac{1}{p})$, *there is a constant* $C = C(p)$ *such that the estimate*

$$\Big(\int_a^b |\widehat{f}\big(t, \phi(t)\big)|^q \phi''(t)^{\frac{1}{3}} dt\Big)^{\frac{1}{q}} \leq C(p)\ \|f\|_p$$

holds whenever ϕ *is as in Theorem 1a.*

Here are some comments on Theorems 1a and 2a: **(a)** The measure $\phi''(t)^{1/3}dt$ appearing in Theorems 1a and 2a is the so called affine arclength measure. Its relevance to problems like these was advocated by Drury ([3]). It is the optimal choice of a measure on the graph of ϕ for these convolution and restriction problems.

(b) The weakness of Theorem 1a is that the estimate it provides is not a strong type $(3/2, 3)$ estimate like the one in Theorem 1, but (equivalent to) a weak type $(3/2, 3)$ estimate. Whether the strong type estimate always holds is, to my knowledge, an open problem.

(c) The most interesting feature of Theorems 1a and 2a is that the estimates they contain are uniform in the sense that they are independent both of ϕ (subject to the monotonicity hypotheses imposed on ϕ) and of the length of the interval (a, b).

The purpose of this note, however, is to explore the approach second-mentioned above (keep the dt) and, in particular, to prove Theorems 1b and 2b below. In what follows, $d\lambda$ will refer to the measure on $\mathbb{R}^2$ corresponding to dt on the graph of a strictly convex function ϕ. We will be interested in estimates on λ of the form

$$(1) \qquad \lambda(P) \leq c\ m_2(P)^{\alpha},$$

to hold for all parallelograms $P \subset \mathbb{R}^2$ and where $0 < \alpha \le 1/3$.

Theorem 1b. *Define the operator T by $Tf(x) = \int_a^b f\big(x - (t, \phi(t))\big)\, dt$ and suppose that (1) holds. Then the estimate*

$$\|Tf\|_q \le 2^{1+\alpha}\ c\ \|f\|_p \tag{2}$$

holds for $(\frac{1}{p}, \frac{1}{q}) = (2\alpha, \alpha)$ whenever $f = \chi_E$ and E is a Borel subset of $\mathbb{R}^2$. On the other hand, if $\frac{1}{p} - \frac{1}{q} = \alpha$ and if (2) holds (for $f = \chi_E$) without the $2^{1+\alpha}$, then (1) holds with c replaced by $\widetilde{C}(p) \cdot c$.

Theorem 2b. *Suppose that (1) holds. Then the estimate*

$$\Big(\int_a^b |\widehat{f}\big(t, \phi(t)\big)|^q dt\Big)^{\frac{1}{q}} \le C(p) \cdot c^{\frac{1}{q}}\ \|f\|_p \tag{3}$$

holds whenever $\frac{1}{q} = \frac{1}{\alpha}(1 - \frac{1}{p})$ and $1 \le p < 1 + \alpha$. If, on the other hand, (3) holds without the $C(p)$ for some p and q satisfying $\frac{1}{q} = \frac{1}{\alpha}(1 - \frac{1}{p})$, then (1) holds with c replaced by $\widetilde{C}(p) \cdot c$.

Here are some remarks on these results: **(d)** If an estimate

$$\|Tf\|_q \le C(p, q)\ \|f\|_p \tag{4}$$

is to hold even for $f = \chi_E$, then the point $(1/p, 1/q)$ must lie in the triangle $\mathcal{T}$ which is the closed convex hull of the points $(0, 0)$, $(1, 1)$, and $(2/3, 1/3)$. This is just a consequence of the fact that the dimension of the graph of ϕ is one. The point $(1/p, 1/q) = (2\alpha, \alpha)$ is a point of intersection of the line $1/p - 1/q = \alpha$ with the boundary of this triangle. If (4) holds for $f = \chi_E$ when $(1/p, 1/q) = (2\alpha, \alpha)$ and if $\alpha < 1/3$, then standard arguments show that (4) must hold, for all measurable f, whenever $1/p - 1/q = \alpha$ and $(1/p, 1/q)$ is in the interior of the triangle $\mathcal{T}$. Thus the following are equivalent: (1); (4) for $f = \chi_E$ and $(1/p, 1/q) = (2\alpha, \alpha)$; (4) for all $(1/p, 1/q)$ in the interior of $\mathcal{T}$ satisfying $1/p - 1/q = \alpha$; (3) for any $(1/p, 1/q)$ with $1/q = (1 - 1/p)/\alpha$; and (3) for all $(1/p, 1/q)$ with $1/q = (1 - 1/p)/\alpha$ with $1 \le p < 1 + \alpha$. The first three of these equivalences can be viewed as a partial response to an old problem of Stein [7], pp. 122-123: "characterize (if possible, in terms of the size of the measure $d\mu$, whatever that means) the condition of $f \to f * d\mu$ yielding a bounded operator from an L^p space to an L^r space".

(e) Theorem 1b is sharp up to the fact that (2) is proven only for $f = \chi_E$.

(f) Although the relation $1/q = (1 - 1/p)/\alpha$ in Theorem 2c is also sharp, the range $1 \le p < 1 + \alpha$ may not be best possible if $\alpha < 1/3$. The fact that the dimension of the graph of ϕ is one can be used to show that if the restriction estimate (3) holds, then $p \leq 4/3$. Also, when $\alpha \leq 1/4$ the condition $p \leq 1/(1-\alpha)$ is necessary to keep $q \geq 1$ in $1/q = (1-1/p)/\alpha$. When $\alpha = 1/3$, $1+\alpha = 4/3$, but when $\alpha < 1/3$ we have $1+\alpha < \min\big(4/3, 1/(1-\alpha)\big)$. If $\phi(t) = t^{1/\alpha - 1}$ on $(0,\infty)$ then it will be shown below (Theorem 3) that, for this ϕ, (3) actually holds whenever $1/q = (1-1/p)/\alpha$ and $1 \leq p < \min\big(4/3, 1/(1-\alpha)\big)$. It is conceivable that the condition $1 \leq p < 1+\alpha$ in Theorem 2b can be replaced by $1 \leq p < \min\big(4/3, 1/(1-\alpha)\big)$, but we have no idea how to prove this.

The remainder of this note consists of the proofs of Theorems 1b and 2b and of the statement and proof of Theorem 3.

Proof of Theorem 1b: Suppose that ϕ is a strictly convex function defined on (a,b) for which (1) holds and assume (for the moment) the following result:

Lemma. *If (1) holds and A is a Borel subset of (a,b), then*

$$m_1(A)^{\frac{1}{\alpha}-1} \leq c^{\frac{1}{\alpha}} 2^{1+\frac{1}{\alpha}} \int_a^b \chi_A(s)\ |\phi'(t) - \phi'(s)|\ ds$$

for any $t \in (a,b)$.

Modifying the proof from [4], we begin by observing that
(5)

$$\int_{\mathbb{R}^2} \big(T\chi_E(x)\big)^{\frac{1}{\alpha}} dx = \int_{\mathbb{R}^2} \chi_E(x) \int_a^b \Big(\int_a^b \chi_E\big(x+(t-s, \phi(t)-\phi(s)\ )\big)\ ds\Big)^{\frac{1}{\alpha}-1} dt\ dx.$$

Applying the lemma to the inside integral,

$$\int_a^b \Big(\int_a^b \chi_E\big(x + (t-s, \phi(t)-\phi(s)\ )\big)\ ds\Big)^{\frac{1}{\alpha}-1} dt \leq$$

$$c^{\frac{1}{\alpha}} 2^{1+\frac{1}{\alpha}} \int_a^b \int_a^b \chi_E\big(x + (t-s, \phi(t)-\phi(s)\ )\big) |\phi'(t) - \phi'(s)|\ ds\ dt.$$

The fact that the Jacobian of the mapping

$$(t,s) \to \big(t-s, \phi(t)-\phi(s)\big)$$

is $|\phi'(t) - \phi'(s)|$ yields

$$\int_a^b \int_a^b \chi_E\big(x + (t-s, \phi(t)-\phi(s)\ )\big) |\phi'(t) - \phi'(s)|\ ds\ dt \leq m_2(E).$$

(The strict convexity of ϕ assures that this mapping is one-to-one.) It follows that

$$\int_a^b \Big(\int_a^b \chi_E\big(x + (t-s, \phi(t)-\phi(s)\ )\big)\ ds\Big)^{\frac{1}{\alpha}-1} dt \leq c^{\frac{1}{\alpha}} 2^{1+\frac{1}{\alpha}} m_2(E).$$

Thus, from (5),

$$\int_{\mathbb{R}^2} \big(T\chi_E(x)\big)^{\frac{1}{\alpha}} dx \le c^{\frac{1}{\alpha}} 2^{1+\frac{1}{\alpha}} m_2(E)^2$$

as desired.

The reverse implication of Theorem 1b follows easily by taking $f = \chi_{P+P}$ in (2). Thus the proof of Theorem 1b will be complete after the proof of the lemma.

Proof of the lemma: Let I be an interval such that

$$m_1(I) = m_1(A),\ t \in I \subseteq (a, b),$$

and

$$\int_I |\phi'(t) - \phi'(s)|\ ds \le \int_A |\phi'(t) - \phi'(s)|\ ds.$$

(To see that I exists, let $J(t,s)$ be an interval with endpoints t and s, define $\tau : A \to (a,b)$ by $\tau(s) = t \pm m_1\big(A \cap J(t,s)\big)$ with the choice of $+$ or $-$ depending on whether $s > t$ or $s < t$. Note that τ is a measure-preserving map of A onto some interval I containing t, and observe that $|\tau(s) - t| \le |s - t|$ implies that $|\phi'\big(\tau(s)\big) - \phi'(t)| \le |\phi'(s) - \phi'(t)|$.) Suppose that $\widetilde{a}$ and $\widetilde{b}$ are the endpoints of I. Then either $\widetilde{b} - t \ge m_1(A)/2$ or $t - \widetilde{a} \ge m_1(A)/2$. In the first case let P be the parallelogram with one side the tangent to the graph of ϕ at $\big(t, \phi(t)\big)$, with one side vertical through $\big(\widetilde{b}, \phi(\widetilde{b})\big)$, and with the points $\big(t, \phi(t)\big)$, $\big(\widetilde{b}, \phi(\widetilde{b})\big)$ as two of its vertices. Then

$$\frac{m_2(P)}{2} \le \int_t^{\widetilde{b}} \big(\phi(\widetilde{b}) - \phi'(t)(\widetilde{b} - v) - \phi(v)\big)\ dv = \int_t^{\widetilde{b}} \int_v^{\widetilde{b}} \big(\phi'(s) - \phi'(t)\big)\ ds\ dv =$$

$$\int_t^{\widetilde{b}} \big(\phi'(s) - \phi'(t)\big)(s - t)\ ds \le m_1(A) \int_{\widetilde{a}}^{\widetilde{b}} |\phi'(t) - \phi'(s)|\ ds.$$

So, recalling (1),

$$\Big(\frac{m_1(A)}{2c}\Big)^{\frac{1}{\alpha}} \le \Big(\frac{\widetilde{b} - t}{c}\Big)^{\frac{1}{\alpha}} = \Big(\frac{\lambda(P)}{c}\Big)^{\frac{1}{\alpha}} \le m_2(P) \le 2\, m_1(A) \int_{\widetilde{a}}^{\widetilde{b}} |\phi'(t) - \phi'(s)|\ ds.$$

Thus

$$\frac{m_1(A)^{\frac{1}{\alpha} - 1}}{(2c)^{\frac{1}{\alpha}}} \le 2 \int_A |\phi'(t) - \phi'(s)|\ ds,$$

completing the proof if $\widetilde{b} - t \ge m_1(A)/2$. The proof if $t - \widetilde{a} \ge m_1(A)/2$ is analogous.

Proof of Theorem 2b: Assume that ϕ is such that (1) holds for some $\alpha \in (0, 1/3)$. The proof relies on Drury's idea [2] of establishing the dual estimate

$$\|\widehat{gd\lambda}\|_s \leq C(c,r)\ \|g\|_{L^r(\lambda)} \ \text{if } 1 - \frac{1}{r} = \frac{1}{\alpha s},\ 1 \leq r < \frac{1+\alpha}{\alpha} \tag{6}$$

by using induction on r. (We recall that $d\lambda$ is dt on the curve $\gamma(t) = \big(t, \phi(t)\big)$. The function g is defined on the curve γ.) To state the inductive hypothesis precisely, we recall Drury's notion of "offspring curves": suppose γ's domain of definition is the interval I. Let $h_0 = 0$ and suppose that $h_1, \ldots, h_N$ are nonnegative numbers. Define the (possibly empty) interval I_0 by $I_0 \doteq \cap_n (I - h_n)$. Then the curve $\Gamma(t) = \sum_n \gamma(t + h_n)$ defined on I_0 is called an offspring curve of γ. Our inductive hypothesis on $r_0 \geq 1$ is that if $d\Lambda$ is dt on some offspring curve Γ, then

$$\|\widehat{gd\Lambda}\|_s \leq C(r) c^{1-\frac{1}{r}}\ \|g\|_{L^r(\lambda)} \ \text{if } 1 - \frac{1}{r} = \frac{1}{\alpha s},\ 1 \leq r \leq r_0. \tag{7}$$

Here c is the constant in (1). As the induction may be started with $r_0 = 1$, assume that (7) holds for some r_0. We adopt the convention that $\int\int \cdots\ dt\ dh$ means $\int_0^\infty \int_{I_h} \cdots\ dt\ dh$ where $I_h = I_0 \cap (I_0 - h)$. Drury's idea is to observe that

$$\|\widehat{gd\Lambda}\|_s^2 \leq 2 \cdot \Big\| \int_{t_1 < t_2} g\big(\Gamma(t_1)\big) g\big(\Gamma(t_2)\big) e^{-2\pi i \left(\Gamma(t_1) + \Gamma(t_2)\right) \cdot \xi}\ dt_1\ dt_2 \Big\|_{L^{s/2}(\xi)} \tag{8}$$

and then to proceed by interpolating two estimates. The first of these estimates, which follows from (7) and in which s_0 is the s corresponding to $r = r_0$ in (7), is

$$\Big\| \int\int F\big(\Gamma(t) + \Gamma(t+h)\big) e^{-2\pi i \left(\Gamma(t) + \Gamma(t+h)\right) \cdot \xi}\ dt\ dh \Big\|_{L^{s_0}(\xi)} \leq$$

$$C(r_0) c^{1-\frac{1}{r_0}} \int \Big(\int |F\big(\Gamma(t) + \Gamma(t+h)\big)|^{r_0}\ dt \Big)^{\frac{1}{r_0}}\ dh, \tag{9}$$

where F is an appropriate function on $\mathbb{R}^2$. The second, a consequence of the Plancherel theorem, is

$$\Big\| \int\int F\big(\Gamma(t) + \Gamma(t+h)\big) e^{-2\pi i \left(\Gamma(t) + \Gamma(t+h)\right) \cdot \xi}\ dt\ dh \Big\|_{L^2(\xi)} \leq$$

$$\Big(\int\int \left| F\big(\Gamma(t) + \Gamma(t+h)\big) J(t, t+h)^{-\frac{1}{2}} \right|^2 dt\ dh \Big)^{\frac{1}{2}}, \tag{10}$$

where $J(t,t+h)$ is the Jacobian of the mapping $(t,h)\mapsto \Gamma(t)+\Gamma(t+h)$. (The strict convexity of ϕ again assures that this mapping is one-to-one.) As in [1] we regard both (9) and (10) as estimates of the form

$$\Big\| \int\int F\big(\Gamma(t)+\Gamma(t+h)\big)e^{-2\pi i\left(\Gamma(t)+\Gamma(t+h)\right)\cdot\xi}\,dt\,dh\Big\|_{L^d(\xi)} \leq$$

$$C(a)c^{\frac{1}{a}-\frac{1}{b}}\Big(\int\Big(\int\big|F\big(\Gamma(t)+\Gamma(t+h)\big)J(t,t+h)^{\beta}\big|^b\,dt\Big)^{\frac{a}{b}}\,dh\Big)^{\frac{1}{a}}. \tag{11}$$

Thus (9) is the estimate (11) at the endpoint

$$(\frac{1}{a},\frac{1}{b},\beta,\frac{1}{d}) = \big(1,\frac{1}{r_0},0,\alpha(1-\frac{1}{r_0})\big)$$

while (10) gives the endpoint

$$(\frac{1}{a},\frac{1}{b},\beta,\frac{1}{d}) = (\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}).$$

Interpolating, it follows that (11) holds if

$$1\leq a\leq 2,\ (1-\frac{2}{r_0})\frac{1}{a}+\frac{1}{b}=1-\frac{1}{r_0},\ \frac{1-\alpha}{a}+\frac{\alpha}{b}+\frac{1}{d}=1,\ \beta=\frac{1}{a}-1. \tag{12}$$

With (8) in mind we will take $d=s/2$. For appropriate $E\subset\Gamma$, we put $F\big(\Gamma(t)+\Gamma(t+h)\big)=\chi_E\big(\Gamma(t)\big)\chi_E\big(\Gamma(t+h)\big)$ to deduce from (11) that if a, b, and β satisfy (12) with $d=s/2$, then

$$\|\widehat{\chi_E d\Lambda}\|_s^2 \leq C(a)\cdot c^{\frac{1}{a}-\frac{1}{b}}\Big(\int\Big(\int\big[\chi_E\big(\Gamma(t)\big)\chi_E\big(\Gamma(t+h)\big)J(t,t+h)^{\beta}\big]^b\,dt\Big)^{\frac{a}{b}}\,dh\Big)^{\frac{1}{a}}. \tag{13}$$

To estimate the right hand side of (13) we will need the inequality

$$J(t,t+h)\geq c^{-\frac{1}{\alpha}}\,h^{\frac{1}{\alpha}-2}. \tag{14}$$

Since $\Gamma(t)$ is an offspring curve of $\gamma(t)=\big(t,\phi(t)\big)$, a calculation shows that (14) will follow from

$$\phi'(t+h)-\phi'(t)\geq c^{-\frac{1}{\alpha}}\,h^{\frac{1}{\alpha}-2}, \tag{15}$$

and this can be deduced from (1): let P be the parallelogram with $\big(t,\phi(t)\big)$ and $\big(t+h,\phi(t+h)\big)$ as two of its vertices, with one side vertical through

$\big(t,\phi(t)\big)$ and with one side the tangent at $\big(t+h,\phi(t+h)\big)$ to the graph of ϕ. Then

$$m_2(P) \le 2\int_t^{t+h} \big(\phi(t)+\phi'(t+h)(s-t)-\phi(s)\big)\,ds = 2\int_t^{t+h}\int_t^s \big(\phi'(t+h)-\phi'(u)\big)\,du\,ds \le$$

$$2\int_t^{t+h}\int_t^s \big(\phi'(t+h)-\phi'(t)\big)\,du\,ds = h^2\big(\phi'(t+h)-\phi'(t)\big).$$

Thus it follows from (1) that

$$h = \lambda(P) \le c\; m_2(P)^\alpha \le c\Big(h^2\big(\phi'(t+h)-\phi'(t)\big)\Big)^\alpha,$$

and this gives (15). Using (14) and recalling that $\beta = 1/a - 1$, the right hand side of (13) is bounded by

$$(16)\quad C(a)\; c^{\frac{1}{a}-\frac{1}{b}+\frac{1}{\alpha}\frac{a-1}{a}} \Big(\int \Big(\int \chi_E\big(\Gamma(t)\big)\chi_E\big(\Gamma(t+h)\big)\,dt\Big)^{\frac{a}{b}} h^{(\frac{1}{\alpha}-2)(1-a)}\,dh\Big)^{\frac{1}{a}}.$$

Continuing to follow Drury, we write

$$\Phi(h) = \int \chi_E\big(\Gamma(t)\big)\chi_E\big(\Gamma(t+h)\big)\,dt.$$

Abusing notation by identifying a set $E\subset\Gamma$ with its preimage $\Gamma^{-1}(E)\subset\mathbb{R}$, we have

$$\|\Phi\|_{L^1(\mathbb{R})} = m_1(E)^2,\;\; \|\Phi\|_{L^\infty(\mathbb{R})} \le m_1(E).$$

Set $1/\tau = (1/\alpha-2)(a-1)$, so that

$$(17)\qquad |h|^{(\frac{1}{\alpha}-2)(1-a)} \in L^{\tau,\infty}(\mathbb{R}),$$

and let τ' be the index conjugate to τ. If $a\tau'/b \ge 1$, we may estimate

$$\|\Phi\|_{L^{\frac{a\tau'}{b},1}(\mathbb{R})} \lesssim m_1(E)^{\frac{b}{a\tau'}\cdot 2+(1-\frac{b}{a\tau'})\cdot 1} = m_1(E)^{1+\frac{b}{a\tau'}},$$

where the implicit constant depends only on a, b, and α. Thus

$$\Big(\|\Phi^{\frac{a}{b}}\|_{L^{\tau',1}(\mathbb{R})}\Big)^{\frac{1}{a}} \lesssim m_1(E)^{\frac{1}{b}+\frac{1}{a\tau'}}.$$

With (17) it follows from (16) and (13) that

(18)

$$\|\widehat{\chi_E d\Lambda}\|_s^2 \le C(a)\cdot c^{\frac{1}{a}-\frac{1}{b}+\frac{1}{\alpha}\frac{a-1}{a}} m_1(E)^{\frac{1}{b}+\frac{1}{a\tau'}} = C(a)\cdot c^{\frac{1}{a}-\frac{1}{b}+\frac{1}{\alpha}\frac{a-1}{a}} m_1(E)^{\frac{1}{b}+2-\frac{1}{\alpha}+\frac{1-\alpha}{\alpha}\frac{1}{a}}.$$

If we define r by

$$\frac{2}{r}=\frac{1}{b}+2-\frac{1}{\alpha}+\frac{1-\alpha}{\alpha}\frac{1}{a}, \tag{19}$$

then we have

$$\frac{1}{a}-\frac{1}{b}+\frac{1}{\alpha}\frac{a-1}{a}=2-\frac{2}{r}.$$

Also, the equations

$$\frac{1-\alpha}{a}+\frac{\alpha}{b}+\frac{1}{d}=1,\quad \frac{1}{d}=\frac{2}{s},$$

the first coming from (12), show that

$$1-\frac{1}{r}=\frac{1}{\alpha s}.$$

Thus, formally, (18) gives an estimate (7) for $g=\chi_E$. We want to conclude that (7) implies

$$\|\widehat{\chi_E d\Lambda}\|_s \leq C(r)\cdot c^{1-\frac{1}{r}}\ m_1(E)^{\frac{1}{r}} \text{ if } 1-\frac{1}{r}=\frac{1}{\alpha s},\ 1\leq r<r_1 \tag{20}$$

where

$$\frac{2}{r_1}=\frac{\alpha}{1-\alpha}+\frac{1}{r_0}\frac{1-3\alpha}{1-\alpha}. \tag{21}$$

From this (6) will follow by iteration and interpolation.

To deduce (20) from (18), we begin by noting that the argument which yields (18) runs only if the parameter τ satisfies the inequalites $0<1/\tau<1$ and $a\tau'/b>1$. The first of these inequalities imposes the restriction $1<a<(1-\alpha)/(1-2\alpha)$. For small enough $\delta>0$ and $(1-\alpha)/(1-2\alpha)-\delta<a<(1-\alpha)/(1-2\alpha)$, it follows that $a\tau'/b>1$. The conjunction of

$$(1-\frac{2}{r_0})\frac{1}{a}+\frac{1}{b}=1-\frac{1}{r_0}$$

from (12) and equation (19) yields

$$\frac{2}{r}=3-\frac{1}{\alpha}-\frac{1}{r_0}+\frac{1}{a}(\frac{1-\alpha}{\alpha}-1+\frac{2}{r_0}).$$

This gives $r_1-\delta'<r<r_1$ if r_1 is defined by (21) and if $(1-\alpha)/(1-2\alpha)-\delta<a<(1-\alpha)/(1-2\alpha)$. Thus (20) holds for these values of r and so, by interpolation, for $1\leq r<r_1$ as claimed. This completes the proof that (1) implies (3).

If, on the other hand, (4) holds for p, q with $1/q = (1-1/p)/\alpha$ then (1) follows from the (easily verified) existence of f with $\chi_P \le \widehat{f}$ and $\|f\|_p \le C(p)\ m_2(P)^{1-1/p}$. Thus Theorem 2b is proved.

Theorem 3. *Suppose* $0 < \alpha < 1/3$ *and let* $\phi(t) = t^{1/\alpha - 1}$ *on* $(0,\infty)$. *Then the estimate*

$$\Big(\int_0^\infty |\widehat{f}\big(t,\phi(t)\big)|^q dt\Big)^{\frac{1}{q}} \le C(p)\ \|f\|_p$$

holds whenever $\frac{1}{q} = \frac{1}{\alpha}(1-\frac{1}{p})$ *and* $1 \le p < \min\big(\frac{4}{3}, \frac{1}{1-\alpha}\big)$.

Proof of Theorem 3: We consider first the case $1/4 \le \alpha$ and will again prove the dual estimate, for indicator functions,

$$\|\widehat{\chi_F d\lambda}\|_s \le C(r)\ m_1(F)^{\frac{1}{r}} \ \text{ if } \ 1 - \frac{1}{r} = \frac{1}{\alpha s},\ s > 4. \tag{22}$$

Here we identify $F \subset \gamma$ with $\gamma^{-1}(F) \subset \mathbb{R}$. (The range $s > 4$ corresponds to $1 \le p < 4/3$.) We will actually establish the inequality

$$\|(\chi_F d\lambda) * (\chi_F d\lambda)^{\sim}\|_{L^{d,\infty}(\mathbb{R}^2)} \le \ C(r)\ m_1(F)^{\frac{2}{r}} \tag{23}$$

for r, s, and d with $1 - 1/r = 1/(\alpha s)$, $1/d + 2/s = 1$, and with $s > 4$. Then Hunt's generalization of the Hausdorff-Young theorem will imply (22):

$$\Big\|\big(\widehat{\chi_F d\lambda}\big)^2\Big\|^{\frac{1}{2}}_{L^{\frac{s}{2},\infty}(\mathbb{R}^2)} \le$$

$$C(r)\ \|(\chi_F d\lambda) * (\chi_F d\lambda)^{\sim}\|^{\frac{1}{2}}_{L^{d,\infty}(\mathbb{R}^2)} \le \ C(r)\ m_1(F)^{\frac{1}{r}}.$$

Now (23) will follow from the inequality
(24)

$$\int_0^\infty \int_0^\infty \chi_E\big(t-u, \phi(t)-\phi(u)\big)\chi_F(u)\chi_F(t)\ du\ dt \le C(r)\ m_2(E)^{\frac{2}{s}} m_1(F)^{\frac{2}{r}}.$$

Rewriting the left hand side of (24) as

$$\int_0^\infty \int_0^\infty \chi_E\big(t-u, \phi(t)-\phi(u)\big)\frac{1}{|\phi'(t)-\phi'(u)|}|\phi'(t)-\phi'(u)|\ \chi_F(u)\chi_F(t)\ du\ dt$$

and bearing in mind that the change of variables $(s,t) \mapsto \big(t-s, \phi(t)-\phi(s)\big)$ has Jacobian with absolute value $|\phi'(t)-\phi'(s)|$, it is enough to prove the inequality

$$\eta\Big(\int\int_{\{u\in F,\ t\in F: \frac{1}{|\phi'(t)-\phi'(u)|}\ge\eta\}} |\phi'(t)-\phi'(u)|\ du\ dt\Big)^{\frac{s-2}{s}} \le C(r)\ m_1(F)^{\frac{2}{r}} \tag{25}$$

for $\eta > 0$. A homogeneity argument (recall that ϕ' is homogeneous of degree $1/\alpha - 2$) shows that it suffices to establish (25) for $\eta = 1$. With $\rho = 1/\alpha - 2$ this will follow from

$$\int\int_{\{u\in F,\ t\in F: |t^\rho - u^\rho| \le 1\}} dt\ du \le C(r)\ m_1(F)^{\frac{2s}{r(s-2)}}. \tag{26}$$

The fact that $2s/\big(r(s-2)\big) < 2$ implies that the part of this integral over $\{0 \le u,\ t \le 10\}$ is controlled by $m_1(F)^{2s/(r(s-2))}$, and so we consider the integral over

$$\{1 \le u \le t,\ t^\rho \le 1 + u^\rho\} \subseteq \{1 \le u \le t \le u + \kappa u^{1-\rho}\}$$

for some $\kappa > 0$. Thus, pretending that $\kappa = 1$, (26) will follow from

$$\Big| \int_1^\infty h(u) Tg(u)\ du \Big| \le C(r)\ \|h\|_{L^c} \|g\|_{L^c} \tag{27}$$

if

$$Tg(u) = \int_u^{u+u^{1-\rho}} g(t)\ dt,$$

$L^c = L^c\big((1,\infty)\big)$, and $c = \frac{r(s-2)}{s}$. We will majorize T by $\sum_j T_j$ where

$$T_j g(u) = \chi_{[0,2^j]}(u) \big(\chi_{[-2^{(1-\rho)j},0]} * g(u)\big).$$

Thus, if c' is the index conjugate to c,

$$\|T_j g\|_{L^{c'}} \le\ 2^{j(\frac{1}{c'} - \frac{1}{c})} \|\chi_{[-2^{(1-\rho)j},0]} * g\|_{L^c} \le 2^{j(\frac{1}{c'} - \frac{1}{c})} 2^{j(1-\rho)} \|g\|_{L^c},$$

and (27) will follow if

$$\frac{1}{c'} - \frac{1}{c} + 1 - \rho < 0. \tag{28}$$

Recalling that $\rho = 1/\alpha - 2$, that $c = r(s-2)/s$, that $1/r = 1 - 1/(\alpha s)$, and finally that $s > 4$, a little algebra gives (28).

The case corresponding to $0 < \alpha < 1/4$ is similar: it is enough to prove (22) for r, s with $1 - 1/r = 1/(\alpha s)$ and $(r, s) = (\infty, 1/\alpha)$. The estimate corresponding to (25) in this case is

$$\eta \Big(\int\int_{\{u\in F,\ t\in F: \frac{1}{|\phi'(t) - \phi'(u)|} \ge \eta\}} |\phi'(t) - \phi'(u)|\ du\ dt \Big)^{1-2\alpha} \le C(\alpha)$$

which follows (via another homogeneity argument) from

$$\int_1^\infty \int_u^{u+u^{1-\rho}} dt\ du < \infty$$

where, again, $\rho = 1/\alpha - 2$. This completes the proof of Theorem 3.

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