MIXED FINITE ELEMENT FORMULATION AND ERROR ESTIMATES BASED ON PROPER ORTHOGONAL DECOMPOSITION FOR THE NON-STATIONARY NAVIER–STOKES EQUATIONS*

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Abstract. In this paper, proper orthogonal decomposition (POD) is used to reduce the formulation of mixed finite element (MFE) for the non-stationary Navier–Stokes equations and error estimates between a reference solution and POD solution of reduced MFE formulation are derived. The basic idea of this reduction technique is that ensembles of data are first compiled from transient solutions computed equation system derived with usual MFE method for the non-stationary Navier–Stokes equations or from physics system trajectories via drawing samples of experiments and interpolation (or date assimilation), and then the basis functions of usual MFE method are substituted with the POD basis functions to reconstruct the elements of the ensemble so as to derive the optimizing reduced MFE formulation based POD technique for the Navier–Stokes equations since there are few basis functions in the POD basis ensemble. It is shown by considering results obtained for numerical simulations of cavity flows that the error between POD solution of reduced MFE formulation and the reference solution is consistent with theoretical results. Moreover, it is also shown that this result validates feasibility and efficiency of POD method.

Key words. mixed finite element method, proper orthogonal decomposition, the non-stationary Navier–Stokes equations, error estimate

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PII.

1. Introduction. Mixed finite element (MFE) methods are one of the important approaches for solving system of partial differential equations, for example, the non-stationary Navier–Stokes equations (see [1], [2], and [3]). However, fully discrete system of MFE solutions for the non-stationary Navier–Stokes equations is of many degrees of freedom. Thus, the important problem is how to simplify the computational load and save time–consuming calculations and resource demands in the actual computational process in a way that guarantees a sufficiently accurate numerical solution. Proper orthogonal decomposition (POD), also known

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as Karhunen–Loève expansions in signal analysis and pattern recognition (see [4]), or principal component analysis in statistics (see [5]), or the method of empirical orthogonal functions in geophysical fluid dynamics (see [6], [7]) or meteorology (see [8]), is a technique offering adequate approximate for representing fluid flow with reduced number of degrees of freedom, i.e., with lower dimensional models (see [9]) so as to alleviate the computational load and provide CPU and memory requirements savings, and has found widespread applications in problems related to the approximation of large–scale models. Although the basic properties of POD method are well established and studies have been conducted to evaluate the suitability of this technique for various fluid flows (see [10], [11], and [12]), its applicability and limitations of optimizing reduced MFE formulation for the Navier–Stokes equations are not well documented.

The POD method mainly provides a useful tool for efficiently approximating a large amount of data. The method essentially provides an orthogonal basis for representing the given data in a certain least squares optimal sense, that is, it provides a way to find optimal lower dimensional approximations of the given data. In addition to being optimal in a least squares sense, POD has the property that it uses a modal decomposition that is completely data dependent and does not assume any prior knowledge of the process used to generate the data. This property is advantageous in situations where a priori knowledge of the underlying process is insufficient to warrant a certain choice of basis. Combined with the Galerkin projection procedure, POD provides a powerful method for generating lower dimensional models of dynamical systems that have a very large or even infinite dimensional phase space. In many cases, the behavior of a dynamic system is governed by characteristics or related structures, even though the ensemble is formed by a large number of different instantaneous solutions. POD method can capture these temporal and spatial structures by applying a statistical analysis to the ensemble of data. In fluid dynamics, Lumley first employed the POD technique to capture the large eddy coherent structures in a turbulent boundary layer (see [13]); this technique was further extended in [14], where a link between the turbulent structure and dynamics of a chaotic system was investigated. In Holmes et al [9], the overall properties of POD are reviewed and extended to widen the applicability of the method. The method of snapshots was introduced by Sirovich [15], and is widely used in applications to reduce the order of POD eigenvalue problem. Examples of these are optimal flow control problems [16–18] and turbulence [9, 13, 14, 19, 20]. In many applications of POD, the method is used to generate basis functions for a reduced order model (ROM), which can simplify and provide quicker assessment of the major features of the fluid dynamics for the purpose of flow control demonstrated by Kurdila et al [18] or design optimization shown by Ly et al [17]. This application is used in a variety of other physical applications, such as in [17], which demonstrates an effective use of POD for a chemical vapor deposition reactor. Some reduced order finite difference models and MFE formulations and error estimates for the upper tropical pacific ocean model based on POD (see, [21–25]). And finite difference scheme based on POD for the non-stationary Navier–Stokes equations (see [26]). However, to the best of our knowledge, there are no published results to address that POD is used to reduce the formulation of MFE for the nonlinear non-stationary Navier–Stokes equations and error estimates between reference solution and POD solution reduced MFE formulation.

In this paper, POD is used to reduce the formulation of MFE for the nonstationary Navier–Stokes equations and the error estimates between reference solution and POD solution of optimizing reduced MFE formulation are derived. It is shown by considering results obtained for numerical simulations of cavity flows that the error between POD solution optimizing reduced MFE formulation based POD technique and reference solution is consistent with theoretical results. Moreover, it is also shown that this validates the feasibility and efficiency of POD method. Though Kunisch and Volkwein have presented some Galerkin proper orthogonal decomposition methods for parabolic problems and a general equation in fluid dynamics in [27] and [28], our method is different from their approaches, whose methods consist of Galerkin projection approaches where the original variables are substituted for linear combination of POD basis and the error estimates of the velocity field therein are only derived, their POD basis being generated with the solution of the physical system at all time instances. Especially, the velocity field is only approximated in [28], while velocity and pressure fields are all simultaneously approximated in our present method. While the SVD approach combined with POD technology is used to treat the Burgers equation in [29] and the cavity flow problem in [12], the error estimates have not completely been derived, especially, a reduced formulation of MFE for Navier-Stokes has not yet been derived. Therefore, our method improves upon existing methods and our POD basis is only generated with the solution of the physical system at the time instances which are useful and of interest for us.

2. MFE approximation for the nonstationary Navier–Stokes equations and Snapshots Generate. Let $\Omega \subset R^2$ be a bounded, connected and polygonal domain. Consider the following nonstationary Navier–Stokes equations.

Problem (I) Find $u = (u_1, u_2)$, p such that for $T > 0$,

$$\begin{cases} u_t - \nu\triangle u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega\times(0,T), \\ \text{div} u = 0 & \text{in } \Omega\times(0,T), \\ u(x,t) = \varphi(x,t) & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = \varphi(x,0) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where u represents the velocity vector, p the pressure, $f = (f_1, f_2)$ the given body force, $\varphi(x,t)$ the given vector function and ν the constant inverse Reynolds number.

The Sobolev spaces along with their properties used in this context are standard (see [30]). For example, for a bounded domain Ω, we denote by $H^m(\Omega)$ $(m \geq 0)$ and $L^2(\Omega) = H^0(\Omega)$ the usual Sobolev spaces equipped with the semi–norm and the norm, respectively,

$$|v|_{m,\Omega} = \left\{\sum_{|\alpha|=m}\int_\Omega |D^\alpha v|^2 \mathrm{d}x\mathrm{d}y\right\}^{1/2} \quad \text{and} \quad \|v\|_{m,\Omega} = \left\{\sum_{i=0}^m |v|_{i,\Omega}^2\right\}^{1/2} \qquad \forall v\in H^m(\Omega),$$

where $\alpha = (\alpha_1, \alpha_2)$, α_1 and α_2 are two nonnegative integers, and $|\alpha| = \alpha_1 + \alpha_2$. Especially, the subspace $H_0^1(\Omega)$ of $H^1(\Omega)$ is denoted by

$$H_0^1(\Omega) = \{v \in H^1(\Omega); u|_{\partial\Omega} = 0\}.$$

Note that $\|\cdot\|_1$ is equivalent to $|\cdot|_1$ in $H_0^1(\Omega)$. Let $L_0^2(\Omega) = \left\{q \in L^2(\Omega); \int_\Omega q\mathrm{d}x\mathrm{d}y = 0\right\}$, which is the subspace of $L^2(\Omega)$. It is necessary to introduce the Sobolev's spaces dependent on time t in order to discuss the generalized solution for Problem (I). Let Φ be a Hilbert space. For all $T > 0$ and integer $n \geq 0$, define

$$H^n(0,T;\Phi) = \left\{v \in \Phi;\ \int_0^T \sum_{0\leq i\leq n} \left(\frac{\mathrm{d}^i}{\mathrm{d}t^i}\|v\|_\Phi\right)^2 \mathrm{d}t < \infty\right\},$$

which is endowed with the norm

$$\|v\|_{H^n(\Phi)} = \left[\sum_{0\leq i\leq n} \int_0^T \left(\frac{\mathrm{d}^i}{\mathrm{d}t^i}\|v\|_\Phi\right)^2 \mathrm{d}t\right]^{\frac{1}{2}},$$

where $\|\cdot\|_\Phi$ is the norm of space Φ. Especially, if $n = 0$,

$$\|v\|_{L^2(\Phi)} = \left(\int_0^T \|v\|_\Phi^2 \mathrm{d}t\right)^{\frac{1}{2}}.$$

And define

$$L^\infty(0,T;\Phi) = \left\{v \in \Phi;\ \operatorname*{ess\,sup}_{0\leq t\leq T} \|v\|_\Phi < \infty\right\},$$

which is endowed with the norm

$$\|v\|_{L^\infty(\Phi)} = \operatorname*{ess\,sup}_{0\leq t\leq T} \|v\|_\Phi.$$

The variational formulation for the problem (I) is written as:

Problem (II) Find $(u,p) \in H^1(0,T;H^1(\Omega)^2) \times L^2(0,T;M)$, $u(x,t)|_{\partial\Omega} = \varphi(x,t)$ such that for all $t \in (0,T)$,

$$\left\{\begin{array}{l} (u_t,v) + a(u,v) + a_1(u,u,v) - b(p,v) = (f,v) \quad \forall v \in X, \\ b(q,u) = 0 \qquad \forall q \in M, \\ u(x,0) = \varphi(x,0) \quad \text{in } \Omega, \end{array}\right. \tag{2.2}$$

where $X = H_0^1(\Omega)^2$, $M = L_0(\Omega) = \{q \in L^2(\Omega);\ \int_\Omega q\mathrm{d}x\mathrm{d}y = 0\}$, $a(u,v) = \nu\int_\Omega \nabla u \cdot \nabla v\mathrm{d}x\mathrm{d}y$, $a_1(u,v,w) = \dfrac{1}{2}\displaystyle\int_\Omega \sum_{i,j=1}^2 [u_i\frac{\partial v_j}{\partial x_i}w_j - u_i\frac{\partial w_j}{\partial x_i}v_j]\mathrm{d}x\mathrm{d}y$, $u,v,w \in X$, $b(q,v) = \int_\Omega q\ \mathrm{div}v\mathrm{d}x\mathrm{d}y$.

Throughout the paper, C indicates a positive constant, and it is possibly different at different occurrences, which is independent of the mesh parameters h, but may depend on Ω, the Reynolds number, and other parameters introduced in this paper.

The following property for trilinear form $a_1(\cdot,\cdot,\cdot)$ is often used (see [1], [2], or [3]).

$$
\begin{aligned}
&a_1(u,v,w) = -a_1(u,w,v), \quad a(u,v,v) = 0 \quad \forall u,v,w \in X,\\
&|a_1(u,v,w)| \le C\|u\|_0^{1/2}|u|_1^{1/2}(|v|_1^{1/2}\|v\|_0^{1/2}|w|_1 + |v|_1\|w\|_0^{1/2}|w|_1^{1/2}) \quad \forall u,v,w \in X,
\end{aligned} \tag{2.3}
$$

where C is independent of $u,\ v$, and w. The bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ have the following properties

$$
a(v,v) \ge \nu|v|_1^2 \quad \forall v \in H_0^1(\Omega)^2, \tag{2.4}
$$

$$
|a(u,v)| \le |u|_1|v|_1 \quad \forall u,\ v \in H_0^1(\Omega)^2, \tag{2.5}
$$

and

$$
\sup_{v\in H_0^1(\Omega)^2} \frac{b(q,v)}{|v|_1} \ge \beta\|q\|_0 \quad \forall q \in L_0^2(\Omega), \tag{2.6}
$$

where β is a constant. Define

$$
N = \sup_{u,v,w\in X} \frac{a_1(u,v,w)}{|u|_1\cdot|v|_1\cdot|w|_1}; \quad \|f\|_{-1} = \sup_{v\in X}\frac{(f,v)}{|v|_1}. \tag{2.7}
$$

The following result is classical (see [1], [2], or [3]).

THEOREM 2.1. *If $f \in H^{-1}(\Omega)^2$, then the problem* (II) *has at least a solution which, in addition, is unique provided that $\nu^{-2}N\|f\|_{L^2(H^{-1})} < 1$, and there is the following prior estimate:*

$$
\|\nabla u\|_{L^2(L^2)} \le \nu^{-1}\|f\|_{L^2(H^{-1})} \equiv R, \quad \|u\|_0 \le \nu^{-1/2}\|f\|_{L^2(H^{-1})} = R\nu^{-1/2}.
$$

Let $\{\Im_h\}$ be a uniformly regular family of triangulation of $\bar{\Omega}$ (see [31] or [32]), indexed by a parameter $h = \max_{K\in\Im_h}\{h_K;\ h_K = \mathrm{diam}(K)\}$, i.e., there is a constant C, independent of h, such that $h \le Ch_K\ \forall K \in \Im_h$.

We introduce the following finite element spaces X_h and M_h of X and M, respectively. Let $X_h \subset X$ (which is at least the piecewise polynomial vector space of mth degree, where $m > 0$ is integer) and $M_h \subset M$ (which is the piecewise polynomial space of $(m-1)$th degree). Write $\hat{X}_h = X_h \times M_h$.

We assume that (X_h, M_h) satisfies the following approximate properties: $\forall v \in H^{m+1}(\Omega)^2 \cap X$ and $\forall q \in M \cap H^m(\Omega)$,

$$
\inf_{v_h\in X_h}\|\nabla(v - v_h)\|_0 \le Ch^m|v|_{m+1}, \quad \inf_{q_h\in M_h}\|q - q_h\|_0 \le Ch^m|q|_m, \tag{2.8}
$$

together the so–called discrete LBB(Ladyzhenskaya-Brezzi-Babushka) condition, i.e.,

$$
\sup_{v_h\in X_h}\frac{b(q_h,v_h)}{\|\nabla v_h\|_0} \ge \beta\|q_h\|_0 \quad \forall q_h \in M_h, \tag{2.9}
$$

where β is a constant independent of h.

There are many spaces X_h and M_h satisfying the discrete LBB conditions (see [32]). Here, we provide some examples as follows.

Example 2.1. The first order finite element space $X_h \times M_h$ can be taken as Bernardi–Fortin–Raugel's element (see [32]), i.e.,

$$\left\{ \begin{array}{l} X_h = \{v_h \in X \cap C^0(\bar{\Omega})^2;\ v_{h|K} \in P_K,\quad \forall K \in \Im_h\}, \\ M_h = \{\varphi_h \in M;\ \varphi_{h|K} \in P_0(K),\quad \forall K \in \Im_h\}, \end{array} \right. \tag{2.10}$$

where $P_K = P_1(K)^2 \oplus \text{span}\{\vec{n}_i \prod\limits_{j=1, j\neq i}^{3} \lambda_{Kj},\ i = 1,2,3\}$, $\vec{n}_i$ are the unit normal vector to side F_i opposite the vertex A_i of triangle K, λ_{Ki} are the barycenter coordinates corresponding to the vertex A_i $(i = 1, 2, 3)$ on K, and $P_m(K)$ is the space of piecewise polynomials of degree m on K.

Example 2.2. The first order finite element space $X_h \times M_h$ can also be taken as Mini's element, i.e.,

$$\left\{ \begin{array}{l} X_h = \{v_h \in X \cap C^0(\Omega)^2;\ v_h|_K \in P_K\ \ \forall K \in \Im_h\}, \\ M_h = \{q_h \in M \cap C^0(\Omega);\ q_h|_K \in P_1(K)\ \ \forall K \in \Im_h\}, \end{array} \right. \tag{2.11}$$

where $P_K = P_1(K)^2 \oplus \{\lambda_{K1}\lambda_{K2}\lambda_{K3}\}^2$.

Example 2.3. The second order finite element space $X_h \times M_h$ can be taken as

$$\left\{ \begin{array}{l} X_h = \{v_h \in X \cap C^0(\Omega)^2;\ v_h|_K \in P_K\ \ \forall K \in \Im_h\}, \\ M_h = \{q_h \in M \cap C^0(\Omega);\ q_h|_K \in P_1(K)\ \ \forall K \in \Im_h\}, \end{array} \right. \tag{2.12}$$

where $P_K = P_2(K)^2 \oplus \{\lambda_{K1}\lambda_{K2}\lambda_{K3}\}^2$.

Example 2.4. The third order finite element space $X_h \times M_h$ can be taken as

$$\left\{ \begin{array}{l} X_h = \{v_h \in X \cap C^0(\Omega)^2;\ v_h|_K \in P_K\ \ \forall K \in \Im_h\}, \\ M_h = \{q_h \in M \cap C^0(\Omega);\ q_h|_K \in P_2(K)\ \ \forall K \in \Im_h\}, \end{array} \right. \tag{2.13}$$

where $P_K = P_3(K)^2 \oplus \text{span}\{\lambda_{K1}\lambda_{K2}\lambda_{K3}\lambda_{Ki},\ i = 1,2,3\}^2$.

It has been proved (see [32]) that, for the finite element space $X_h \times M_h$ in Example 2.1–2.4, there exists an operator r_h: $X \to X_h$ such that, for any $v \in X$,

$$\begin{array}{l} b(q_h, v - r_h v) = 0\ \ \forall q_h \in M_h,\ \|\nabla r_h v\|_0 \le C\|\nabla v\|_0, \\ \|\nabla(v - r_h v)\|_0 \le Ch^k|v|_{k+1}\ \text{ if } v \in H^{k+1}(\Omega)^2,\quad k = 1,2,3. \end{array} \tag{2.14}$$

The spaces $X_h \times M_h$ used throughout next part in this paper mean those in Example 2.1–2.4, which are obviously satisfied discrete LBB condition (2.9).

In order to find the numerical solution for Problem (II), it is necessary to discretize Problem (II). We introduce a MFE approximation for the spatial variable and FDS for the time derivative. Let L be the positive integer, denote the time step increment by $k = T/L$ (T being the total time), $t^{(n)} = nk$, $0 \le n \le L$; $(u_h^n, p_h^n) \in X_h \times M_h$ the MFE approximation corresponding to $(u(t^{(n)}), p(t^{(n)})) \equiv (u^n, p^n)$. Then, the fully discrete MFE solution for the problem (I) may be written as:

Problem (III) Find $(u_h^n, p_h^n) \in X_h \times M_h$ such that $u_h^n|_{\partial\Omega} = \varphi_h(x, t_n)$ and satisfies

$$
\begin{cases}
(u_h^n, v) + ka(u_h^n, v) + ka_1(u_h^{n-1}, u_h^n, v) - kb(p_h^n, v) = k(f^n, v) + (u_h^{n-1}, v) \ \forall v \in X_h, \\
b(q, u_h^n) = 0 \quad \forall q \in M_h, \\
u_h^0 = \varphi_h(x, 0) \quad \text{in } \Omega,
\end{cases}
\tag{2.15}
$$

where $1 \le n \le L$, $\varphi_h(x, t_n) = r_h\varphi(x, t_n)|_{\partial\Omega}$ and $\varphi_h(x, 0) = r_h\varphi(x, t)|_{t=0}$.

Put $A(u_h, v_h) = (u_h^n, v_h) + ka(u_h^n, v_h) + ka_1(u_h^{n-1}, u_h^n, v_h)$. Since $A(u_h, u_h) = (u_h^n, u_h) + ka(u_h^n, u_h) + ka_1(u_h^{n-1}, u_h^n, u_h) = \|u_h\|_0 + k\nu\|\nabla u\|_0$, $A(\cdot, \cdot)$ is coercive in $X_h \times X_h$. And $kb(\cdot, \cdot)$ also satisfies the discrete LBB condition in $X_h \times M_h$, therefore, by MFE theory (see [1] or [32]), we can obtain the following result.

THEOREM 2.2. *Under the assumptions* (2.8)–(2.9)*, if* $k = O(h^2)$*,* $f \in (H^{-1}(\Omega))^2$ *satisfies* $\nu^{-2}N\|f\|_{L^2(H^{-1})} < 1$*, Problem* (III) *has a unique solution* $(u_h^n, p_h^n) \in X_h \times M_h$ *and satisfies*

$$
\begin{cases}
\|u_h^n\|_0^2 + k\nu \sum_{i=1}^{n} \|\nabla u_h^i\|_0^2 \le k\nu^{-1} \sum_{i=1}^{n} \|f^i\|_{-1}^2 + \|\varphi\|_0, \\
\|u^n - u_h^n\|_0 + k^{1/2} \sum_{i=1}^{n} \|\nabla(u^i - u_h^i)\|_0 + k^{1/2} \sum_{i=1}^{n} \|p^i - p_h^i\|_0 \le C(h^m + k),
\end{cases}
\tag{2.16}
$$

where $(u, p) \in [H_0^1(\Omega) \cap H^{m+1}(\Omega)]^2 \times [H^m(\Omega) \cap M]$ *is the exact solution for the problem* (I) *and* C *is the constant dependent on* $|u^n|_{m+1}$ *and* $|p^n|_m$*.*

If ν and the time step increment k are given, by solving Problem (III), we can obtain solution ensemble $\{u_{1h}^n,\ u_{2h}^n,\ p_h^n\}_{n=1}^L$ for Problem (III). And then we choose ℓ (for example, $\ell = 5$, 20, or 30, in general, $\ell \ll L$) instantaneous solutions $U_i(x, y) = (u_{1h}^i, u_{2h}^i, p_h^i)$ $(i = 1, 2, \cdots, \ell)$ (which are useful and of interest for us) from the L group of solutions $(u_{1h}^n,\ u_{2h}^n,\ p_h^n)$ $(1 \le n \le L)$ for Problem (III), which are known as snapshots.

3. Optimizing reduced MFE formulation based POD technique for the Navier–Stokes equations. The POD method has received much attention in recent years as a tool to analyze complex physical systems. In this section, we use POD technique to deal with the snapshots in Section 2 and produce an optimal representation in an average sense.

Let $\hat{X} = X \times M$. For $U_i(x, y) = (u_{1h}^i, u_{2h}^i, p_h^i)$ $(i = 1, 2, \cdots, \ell)$ in Section 2, we set

$$
\mathcal{V} = \operatorname{span}\{U_1, U_2, \cdots, U_\ell\},
\tag{3.1}
$$

and refer to $\mathcal{V}$ as ensemble consisting of the snapshots $\{U_i\}_{i=1}^{\ell}$ at least one of which is supposed to be non-zero. Let $\{\psi_j\}_{j=1}^{l}$ denote an orthogonal basis of $\mathcal{V}$ with $l = \dim \mathcal{V}$. Then each member of the ensemble can expressed as

$$
U_i = \sum_{j=1}^{l} (U_i, \psi_j)_{\hat{X}} \psi_j \quad \text{for} \quad i = 1, 2, \cdots,\ \ell,
\tag{3.2}
$$

where $(U_i,\psi_j)_{\hat{X}} = (u_h^i,\psi_{uj})_X + (p_h^i,\psi_{pj})_0$, $(\cdot,\cdot)_0$ is L^2-inner production, and ψ_{uj} and ψ_{pj} are orthogonal basis corresponding to u and p, respectively.

Definition 3.1. *The method of POD consists in finding the orthogonal basis such that for every d $(1 \le d \le l)$ the mean square error between the elements U_i $(1 \le i \le \ell)$ and corresponding d–th partial sum of* (3.2) *is minimized on average:*

$$\min_{\{\psi_j\}_{j=1}^d} \frac{1}{\ell}\sum_{i=1}^{\ell} \|U_i - \sum_{j=1}^{d}(U_i,\psi_j)_{\hat{X}}\psi_j\|_{\hat{X}} \tag{3.3}$$

such that

$$(\psi_i,\psi_j)_{\hat{X}} = \delta_{ij} \quad \text{for } 1 \le i \le d, 1 \le j \le i, \tag{3.4}$$

where $\|U_i\|_{\hat{X}} = [\|\nabla u_{1h}^i\|_0^2 + \|\nabla u_{2h}^i\|_0^2 + \|p_h^i\|_0^2]^{\frac{1}{2}}$. A solution $\{\psi_j\}_{j=1}^d$ of (3.3) *and* (3.4) *is known as a POD basis of rank d.*

We introduce the correlation matrix $K = (K_{ij})_{\ell\times\ell} \in R^{\ell\times\ell}$ corresponding to the snapshots $\{U_i\}_{i=1}^{\ell}$ by

$$K_{ij} = \frac{1}{\ell}(U_i,U_j)_{\hat{X}}. \tag{3.5}$$

The matrix K is positive semi-definite and has rank l. The solution of (3.3) and (3.4) can be found in [10, 15, or 28], for example.

Proposition 3.2. *Let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l > 0$ denote the positive eigenvalues of K and v_1, v_2, $\cdots$, v_l the associated eigenvectors. Then a POD basis of rank $d \le l$ is given by*

$$\psi_i = \frac{1}{\sqrt{\lambda_i}} v_i^T (U_1,U_2,\cdots,U_\ell)^T = \frac{1}{\sqrt{\lambda_i}}\sum_{j=1}^{\ell}(v_i)_j U_j, \tag{3.6}$$

where $(v_i)_j$ denotes the j-th component of the eigenvector v_i. Furthermore, the following error formula holds

$$\frac{1}{\ell}\sum_{i=1}^{\ell}\|U_i - \sum_{j=1}^{d}(U_i,\psi_j)_{\hat{X}}\psi_j\|_{\hat{X}} = \sum_{j=d+1}^{l}\lambda_j. \tag{3.7}$$

Let $\mathcal{V}^d = \text{span}\{\psi_1,\psi_2,\cdots,\psi_d\}$ and $X^d \times M^d = \mathcal{V}^d$ with $X^d \subset X$ and $M^d \subset M$. Set the Ritz-projection P^d: $X \to X^d$ and L^2-projection ρ^d: $M \to M^d$ denoted by, respectively,

$$a(P^d u, v_d) = a(u, v_d) \quad \forall v_d \in X^d \tag{3.8}$$

and

$$(\rho^d p, q_d)_0 = (p, q_d)_0 \quad \forall q_d \in M^d, \tag{3.9}$$

where $u \in X$ and $p \in M$. Due to (3.8) and (3.9) the linear operators P^d and ρ^d are well-defined and bounded:

$$\|\nabla(P^d u)\|_0 \le \|\nabla u\|_0, \quad \|\rho^d p\|_0 \le \|p\|_0 \quad \forall u \in X \text{ and } p \in M. \tag{3.10}$$

LEMMA 3.2. *For every d $(1 \le d \le l)$ the projection operators P^d and ρ^d satisfy respectively*

$$\frac{1}{\ell}\sum_{i=1}^{\ell}\|\nabla(u_h^i - P^d u_h^i)\|_0 \le \sum_{j=d+1}^{l}\lambda_j \tag{3.11}$$

and

$$\frac{1}{\ell}\sum_{i=1}^{\ell}\|\nabla(p_h^i - \rho^d p_h^i)\|_0 \le \sum_{j=d+1}^{l}\lambda_j. \tag{3.12}$$

Proof. For any $u \in X$ we deduce from (3.8) that

$$\begin{aligned}
\nu\|\nabla(u_h^i - P^d u_h^i)\|_0^2 &= a(u_h^i - P^d u_h^i, u_h^i - P^d u_h^i)\\
&= a(u_h^i - P^d u_h^i, u_h^i - v_d)\\
&\le \nu\|\nabla(u_h^i - P^d u_h^i)\|_0\|\nabla(u_h^i - v_d)\|_0 \ \ \forall v_d \in X^d.
\end{aligned}$$

Furthermore, we obtain that

$$\|\nabla(u_h^i - P^d u_h^i)\|_0 \le \|\nabla(u_h^i - v_d)\|_0 \ \ \forall v_d \in X^d. \tag{3.13}$$

Taking $v_d = \sum_{j=1}^{d}(u_h^i, \psi_{uj})_X\psi_{uj}$ (where ψ_{uj} is the component of ψ_j corresponding to u) in (3.13), we can obtain (3.11) from (3.7).

Using Hölder inequality and (3.9) can yield

$$\begin{aligned}
\|p_h^i - \rho^d p_h^i\|_0^2 &= (p_h^i - \rho^d p_h^i, p_h^i - \rho^d p_h^i)\\
&= (p_h^i - \rho^d p_h^i, p_h^i - q_d)\\
&\le \|p_h^i - \rho^d p_h^i\|_0\|p_h^i - q_d\|_0 \ \ \forall q_d \in M^d,
\end{aligned}$$

consequently,

$$\|p_h^i - \rho^d p_h^i\|_0 \le \|p_h^i - q_d\|_0 \ \ \forall q_d \in M^d. \tag{3.14}$$

Taking $q_d = \sum_{j=1}^{d}(p_h^i, \psi_{pj})_0\psi_{pj}$ (where ψ_{pj} is the component of ψ_j corresponding to p) in (3.14), from (3.7) we can obtain (3.12), which completes the proof of Lemma 3.2. □

Thus, using $\mathcal{V}^d = X^d \times M^d$, we can obtain the reduced formulation for Problem (III) as follows.

Problem (IV) Find $(u_d^n, p_d^n) \in \mathcal{V}^d$ such that $u_d^n|_{\partial\Omega} = \varphi_h^n$ and satisfies

$$\begin{cases}
(u_d^n, v_d) + ka(u_d^n, v_d) + ka_1(u_d^{n-1}, u_d^n, v_d) - kb(p_d^n, v_d)\\
\quad = k(f^n, v_d) + (u_d^{n-1}, v_d) \quad \forall v_d \in X^d,\\
b(q_d, u_d^n) = 0 \qquad \forall q_d \in M^d,\\
u_d^0 = u_h^0,
\end{cases} \tag{3.15}$$

where $1 \leq n \leq L$.

Remark 3.3. Problem (IV) is an optimizing reduced MFE formulation based on POD technique for Problem (III), since it only includes $3d$ freedom degree while Problem (III) includes $3N_p + N_K \approx 5N_p$ (where N_p is the number of the vertex in $\Im_h$ and N_K the number of the element in $\Im_h$) and $3d \ll 5N_p$ (see examples in Section 5). When one computes actual problems, one may obtain the ensemble of snapshots from physical system trajectories by drawing samples from experiments and interpolation (or data assimilation). For example, for weather forecast, one can use previous weather prediction results to construct the ensemble of snapshots, then restructure the POD basis for the ensemble of snapshots by above (3.3)–(3.6), and finally combine it with a Galerkin projection to derive an optimizing reduced order dynamical system, i.e., one needs only to solve the above Problem (IV) which has only few degrees of freedom, but it is unnecessary to solve Problem (III). Thus, the forecast of future weather change can be quickly simulated, which is of major importance for actual real-life applications.

4. Existence and error analysis of solution of the optimizing reduced MFE formulation based POD technique for the Navier–Stokes equations. This section is devoted to discussing the existence and error estimates for Problem (IV).

We see from (3.6) that $\mathcal{V}^d = X^d \times M^d \subset \mathcal{V} \subset X_h \times M_h \subset \hat{X}$, where $X_h \times M_h$ is one of those spaces in Example 2.1–2.4. Therefore, we have in the following result.

LEMMA 4.1. *There exists also an operator r_d: $X_h \to X^d$ such that,for all $v_h \in X_h$,*

$$b(q_d, u_h - r_d u_h) = 0 \;\; \forall q_d \in M_d, \;\; \|\nabla r_d u_h\|_0 \leq c\|\nabla u_h\|_0, \tag{4.1}$$

and, for every d $(1 \leq d \leq l)$,

$$\frac{1}{\ell}\sum_{i=1}^{\ell}\|\nabla(u_h^i - r_d u_h^i)\|_0 \leq C\sum_{j=d+1}^{l}\lambda_j. \tag{4.2}$$

Proof. We use the Mini's and the second finite element as examples. Define r_d as follows

$$r_d v_h|_K = P^d v_h|_K + \gamma\lambda_{K1}\lambda_{K2}\lambda_{K3} \;\; \forall v_h \in X_h \text{ and } K \in \Im_h, \tag{4.3}$$

where $\gamma = \int_K (v_h - P^d v_h)\mathrm{d}x / \int_K \lambda_{K1}\lambda_{K2}\lambda_{K3}\mathrm{d}x$. Then, using (3.10) and (3.11), by simply computing we educe (4.1). $\square$

Set $V^d \equiv \mathcal{V}^d|_X$ and

$$V = \{v \in X;\; b(q,v) = 0 \;\; \forall q \in M\}, \;\; V_h = \{v_h \in X_h;\; b(q_h, v_h) = 0 \;\; \forall q_h \in M_h\}.$$

using dual principle and equations (3.10) and (3.11), we deduce the following result (see [1, 32]).

LEMMA 4.2. *There exists an operator R_d: $V \cup V_h \to V^d$ such that, for all $v \in V \cup V_h$,*

$$(v - R_d v, v_d) = 0 \;\; \forall v_d \in V^l, \;\; \|\nabla R_d v\|_0 \leq C\|\nabla v\|_0,$$

and, for every d $(1 \le d \le l)$,

$$\frac{1}{\ell}\sum_{i=1}^{\ell}\|u_h^i - R_d u_h^i\|_{-1} \le \frac{C}{\ell}\sum_{i=1}^{\ell}\|\nabla(u_h^i - R_d u_h^i)\|_0 \le C\sum_{j=d+1}^{l}\lambda_j, \tag{4.4}$$

where $\|\cdot\|_{-1}$ denotes the normal of space $H^{-1}(\Omega)^2$. □

The following Discrete Gronwall Lemma is well–known and very useful in next analysis (see [1], [2], or [32]).

LEMMA 4.3 (GRONWALL LEMMA). *If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are three positive sequences, and $\{c_n\}$ is monotone, they satisfy*

$$a_n + b_n \le c_n + \bar\lambda\sum_{i=0}^{n-1} a_i, \quad \bar\lambda > 0, a_0 + b_0 \le c_0,$$

then

$$a_n + b_n \le c_n \exp(n\bar\lambda), \quad n \ge 0.$$ □

We have the following result for solution of Problem (V).

THEOREM 4.4. *Under the hypotheses of Theorem 2.2, Problem (V) has a unique solution $(u_d^n, p_d^n) \in X^d \times M^d$ and satisfies*

$$\|u_d^n\|_0^2 + k\nu\sum_{i=1}^{n}\|\nabla u_d^i\|_0^2 \le k\nu^{-1}\sum_{i=1}^{n}\|f^i\|_{-1}^2 + \|\varphi\|_0. \tag{4.5}$$

Proof. Using same technique as the proof of Theorem 2.2, we could prove that Problem (V) has a unique solution $(u_d^n, p_d^n) \in X^d \times M^d$ and satisfies (4.5). □

In the following theorem, error estimates of solution for Problem (IV) are derived.

THEOREM 4.5. *Under the hypotheses of Theorem 2.2, if $h^2 = O(k)$, $k = O(\ell^{-2})$, and $\max\limits_{1\le i\le \ell-1}\dfrac{t_{i+1}-t_i}{2k} \le \ell$, then the error between the solution (u_d^n, p_d^n) for Problem (V) and the solution (u_h^n, p_h^n) for Problem (IV) has the following error estimates, for $n = 1, 2, \cdots, L$,*

$$\begin{aligned}
&\|u_h^n - u_d^n\|_0 + \frac{1}{n}\sum_{i=1}^{n}\|\nabla(u_h^i - u_d^i)\|_0 + \frac{1}{n}\sum_{i=1}^{n}\|p_h^i - p_d^i\|_0 \\
&\quad\le C\left(\sum_{j=d+1}^{l}\lambda_j\right), \text{ if } n \in \{1, 2, \cdots, \ell\}; \\
&\|u_h^n - u_d^n\|_0 + \frac{1}{n}\sum_{i=1}^{n}\|\nabla(u_h^i - u_d^i)\|_0 + \frac{1}{n}\sum_{i=1}^{n}\|p_h^i - p_d^i\|_0 \\
&\quad\le C\left(k^{1/2} + \sum_{j=d+1}^{l}\lambda_j\right), \text{ if } n \notin \{1, 2, \cdots, \ell\}.
\end{aligned} \tag{4.6}$$

Proof. Subtracting Problem (IV) from Problem (III) taking $v_h = v_d \in X^d$ and $q_h = q_d \in M^d$ can yield

$$\begin{aligned}
&(u_h^n - u_d^n, v_d) + ka(u_h^n - u_d^n, v_d) - kb(p_h^n - p_d^n, v_d) + ka_1(u_h^{n-1}, u_h^n, v_d) \\
&-ka_1(u_d^{n-1}, u_d^n, v_d) = (u_h^{n-1} - u_d^{n-1}, v_d) \quad \forall v_d \in X^d,
\end{aligned} \tag{4.7}$$

$$b(q_d, u_h^n - u_d^n) = 0 \quad \forall q_d \in M^d, \tag{4.8}$$

$$u_h^0 - u_d^0 = 0. \tag{4.9}$$

Note that it follows, for $v_h^n \in X_h$, $\|v_h^n\|_0 \leq Ch\|\nabla(v_h^n)\|_0$ and $\|\nabla(v_h^n)\|_0 \leq Ch^{-1}\|v_h^n\|_0$ from the inverse estimate of finite element methods. We obtain, from (2.3), (2.7), Theorem 2.2, and Theorem 4.3, by Hölder inequality, that

$$\begin{aligned}
&|a_1(u_h^{n-1}, u_h^n, v_d) - a_1(u_d^{n-1}, u_d^n, v_d)| = |a_1(u_h^{n-1} - u_d^{n-1}, u_h^n, v_d) \\
&+a_1(u_d^{n-1}, u_h^n - u_d^n, v_d)| \leq Ch[\|\nabla(u_h^{n-1} - u_d^{n-1})\|_0 + \|\nabla(u_h^n - u_d^n)\|_0]\|\nabla v_d\|_0 \\
&\leq C[\|u_h^{n-1} - u_d^{n-1}\|_0 + \|u_h^n - u_d^n\|_0]\|\nabla v_d\|_0,
\end{aligned} \tag{4.10}$$

especially, if $v_d = P^d u_h^n - u_d^n$, then

$$\begin{aligned}
&|a_1(u_h^{n-1}, u_h^n, P^d u_h^n - u_d^n) - a_1(u_d^{n-1}, u_d^n, P^d u_h^n - u_d^n)| \\
&= |a_1(u_h^{n-1} - u_d^{n-1}, u_h^n, P^d u_h^n - u_d^n) + a_1(u_d^{n-1}, u_h^n - u_d^n, P^d u_h^n - u_d^n)| \\
&= |a_1(u_h^{n-1} - u_d^{n-1}, u_h^n, P^d u_h^n - u_h^n) + a_1(u_h^{n-1} - u_d^{n-1}, u_h^n, u_h^n - u_d^n) \\
&\quad +a_1(u_d^{n-1}, u_h^n - u_d^n, P u_h^n - u_h^n)| \\
&\leq C[\|u_h^{n-1} - u_d^{n-1}\|_0^2 + \|\nabla(u_h^n - P^d u_h^n)\|_0^2] + \varepsilon\|\nabla(u_h^n - u_d^n)\|_0^2,
\end{aligned} \tag{4.11}$$

where ε is a small positive constant which can be chosen arbitrarily.

Write $\bar{\partial}_t u_h^n = [u_h^n - u_h^{n-1}]/k$ and note that $\bar{\partial}_t u_d^n \in V^d$ and $\bar{\partial}_t R_d u_n^n \in V^d$. From Lemma 4.2, (4.7), and (4.10), we have that

$$\begin{aligned}
&\|\bar{\partial}_t u_h^n - \bar{\partial}_t u_d^n\|_{-1} \leq \|\bar{\partial}_t u_h^n - \bar{\partial}_t R_d u_h^n\|_{-1} + \|\bar{\partial}_t R_h u_h^n - \bar{\partial}_t u_d^n\|_{-1} \\
&\leq \|\bar{\partial}_t u_h^n - \bar{\partial}_t R_d u_h^n\|_{-1} + \sup_{v\in V} \frac{(\bar{\partial}_t R_d u_h^n - \bar{\partial}_t u_d^n, v)}{\|\nabla v\|_0} \\
&= \|\bar{\partial}_t u_h^n - \bar{\partial}_t R_d u_h^n\|_{-1} + \sup_{v\in V} \frac{(\bar{\partial}_t u_h^n - \bar{\partial}_t u_d^n, R_d v)}{\|\nabla v\|_0} \\
&= \|\bar{\partial}_t u_h^n - \bar{\partial}_t R_d u_h^n\|_{-1} + \sup_{v\in V} \frac{1}{\|\nabla v\|_0}[b(p_h^n - p_d^n, R_d v) - a(u_h^n - u_d^n, R_d v) \\
&\quad -a_1(u_h^{n-1}, u_h^n, R_d v) + a_1(u_d^{n-1}, u_d^n, R_d v)] \\
&= \|\bar{\partial}_t u_h^n - \bar{\partial}_t R_d u_h^n\|_{-1} + \sup_{v\in V} \frac{1}{\|\nabla v\|_0}[b(p_h^n - \rho^d p_h^n, R_d v) - a(u_h^n - u_d^n, R_d v) \\
&\quad -a_1(u_h^{n-1}, u_h^n, R_d v) + a_1(u_d^{n-1}, u_d^n, R_d v)] \\
&\leq \|\bar{\partial}_t u_h^n - \bar{\partial}_t R_d u_h^n\|_{-1} + C[\|p_h^n - \rho^d p_h^n\|_0 + \|u_h^{n-1} - u_d^{n-1}\|_0 + \|\nabla(u_h^n - u_d^n)\|_0].
\end{aligned} \tag{4.12}$$

By using (2.9), (4.7), (4.10), (4.12), and Lemma 4.1, we have that

$$\begin{aligned}
&\beta\|\rho^d p_h^n - p_d^n\|_0 \leq \sup_{v_h\in X_h} \frac{b(\rho^d p_h^n - p_d^n, v_h)}{\|\nabla v_h\|_0} = \sup_{v_h\in X_h} \frac{b(p_h^n - p_d^n, r_d v_h)}{\|\nabla v_h\|_0} \\
&= \sup_{v_h\in X_h} \frac{1}{\|\nabla v_h\|_0}[(\bar{\partial}_t u_h^n - \bar{\partial}_t u_d^n, r_d v_h) + a(u_h^n - u_d^n, r_d v) \\
&\quad +a_1(u_h^{n-1}, u_h^n, r_d v) - a_1(u_d^{n-1}, u_d^n, r_d v)] \\
&\leq C[\|\bar{\partial}_t u_h^n - \bar{\partial}_t u_d^n\|_{-1} + \|u_h^{n-1} - u_d^{n-1}\|_0 + \|\nabla(u_h^n - u_d^n)\|_0] \\
&\leq C[\|\bar{\partial}_t u_h^n - \bar{\partial}_t R_d u_h^n\|_{-1} + \|p_h^n - \rho^d p_h^n\|_0 + \|u_h^{n-1} - u_d^{n-1}\|_0 + \|\nabla(u_h^n - u_d^n)\|_0].
\end{aligned} \tag{4.13}$$

Thus, we obtain that

$$
\begin{aligned}
\|p_h^n - p_d^n\|_0 \le \|p_h^n - \rho^d p_h^n\|_0 + \|\rho^d p_h^n - p_d^n\|_0 \le C[\|u_h^{n-1} - u_d^{n-1}\|_0 \\
+\|\nabla(u_h^n - u_d^n)\|_0 + \|\bar{\partial}_t u_h^n - \bar{\partial}_t R_d u_h^n\|_{-1} + \|p_h^n - \rho^d p_h^n\|_0].
\end{aligned} \tag{4.14}
$$

Taking $v_d = P^d u_h^n - u_d^n$ in (4.7), it follows from (4.8) that

$$
\begin{aligned}
&(u_h^n - u_d^n, u_h^n - u_d^n) - (u_h^{n-1} - u_d^{n-1}, u_h^n - u_d^n) + ka(u_h^n - u_d^n, u_h^n - u_d^n) \\
&= (u_h^n - u_d^n, u_h^n - P^d u_h^n) + ka(u_h^n - P^d u_h^n, u_h^n - P^d u_h^n) \\
&\quad + kb(p_h^n - \rho^d p_h^n, u_h^n - u_d^n) + kb(p_h^n - p_d^n, u_h^n - P^d u_h^n) \\
&\quad - ka_1(u_h^{n-1}, u_h^n, P^d u_h^n - u_d^n) + ka_1(u_d^{n-1}, u_d^n, P^d u_h^n - u_d^n).
\end{aligned} \tag{4.15}
$$

Thus, noting that $a(a-b) = [a^2 - b^2 + (a-b)]/2$ (for $a \ge 0$ and $b \ge 0$), by (4.11), (4.14), Hölder inequality, Cauchy inequality, and Proposition 3.2, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\left[\|u_h^n - u_d^n\|_0^2 - \|u_h^{n-1} - u_d^{n-1}\|_0^2\right] + \nu k\|\nabla(u_h^n - u_d^n)\|_0^2 \\
&\le Ck\|u_h^{n-1} - u_d^{n-1}\|_0^2 + k\varepsilon_1\|\nabla(u_h^n - u_d^n)\|_0^2 + \varepsilon_2\|u_h^n - u_d^n\|_0^2 + k\varepsilon\|\nabla(u_h^n - u_d^n)\|_0^2 \\
&\quad + Ck[\|\nabla(u_h^n - P^d u_h^n)\|_0^2 + \|p_h^n - \rho^d p_h^n\|_0^2 + \|\bar{\partial}_t u_h^n - \bar{\partial}_t R_d u_h^n\|_{-1}^2],
\end{aligned} \tag{4.16}
$$

where ε_1 and ε_2 are two small positive constants which can be chosen arbitrarily. Taking $\varepsilon + \varepsilon_1 + C\varepsilon_2 = \nu/2$, if $h^2 = O(k)$ it follows from (4.16) and the inverse estimate of finite element methods that

$$
\begin{aligned}
&\left[\|u_h^n - u_d^n\|_0^2 - \|u_h^{n-1} - u_d^{n-1}\|_0^2\right] + \nu k\|\nabla(u_h^n - u_d^n)\|_0^2 \\
&\le Ck\|u_h^{n-1} - u_d^{n-1}\|_0^2 + Ck[\|\nabla(u_h^n - P^d u_h^n)\|_0^2 + \|p_h^n - \rho^d p_h^n\|_0^2 \\
&\quad + \|\nabla(u_h^n - R_d u_h^n)\|_0^2 + \|\nabla(u_h^{n-1} - R_d u_h^{n-1})\|_0^2], \quad 1 \le n \le L.
\end{aligned} \tag{4.17}
$$

When $n \in \{1, 2, \cdots, \ell\}$, summing (4.17) from $n = 1$ to $n = \ell$, and noting that $u_h^0 - u_d^0 = 0$ could yield that

$$
\begin{aligned}
&\frac{1}{\ell^2}\|u_h^\ell - u_d^\ell\|_0^2 + \frac{\nu k}{\ell^2}\sum_{i=1}^{\ell}\|\nabla(u_h^i - u_d^i)\|_0^2 \le \frac{Ck}{\ell^2}\sum_{i=1}^{\ell}\|u_h^{i-1} - u_d^{i-1}\|_0^2 \\
&+ Ck\left[\frac{1}{\ell}\sum_{i=1}^{\ell}\|\nabla(u_h^i - P^d u_h^i)\|_0 + \frac{1}{\ell}\sum_{i=1}^{\ell}\|p_h^i - \rho^d p_h^i\|_0\right. \\
&\left. + \frac{1}{\ell}\sum_{i=1}^{\ell}\|\nabla(u_h^i - R_d u_h^i)\|_0 + \frac{1}{\ell}\sum_{i=1}^{\ell}\|\nabla(u_h^{i-1} - R_d u_i^{n-1})\|_0\right]^2 \\
&\le \frac{Ck}{\ell^2}\sum_{i=1}^{\ell}\|u_h^{i-1} - u_d^{i-1}\|_0^2 + Ck\left(\sum_{j=d+1}^{l}\lambda_j\right)^2.
\end{aligned} \tag{4.18}
$$

Thus, we obtain that

$$
\|u_h^\ell - u_d^\ell\|_0^2 + k\sum_{i=1}^{\ell}\|\nabla(u_h^i - u_d^i)\|_0^2 \le Ck\sum_{s=0}^{\ell-1}\|u_h^s - u_d^s\|_0^2 + Ck\ell^2\left(\sum_{j=d+1}^{l}\lambda_j\right)^2. \tag{4.19}
$$

By using discrete Gronwall inequality and noting that $\ell k \leq C$, if $k = O(\ell^{-2})$, we obtain that

$$\|u_h^\ell - u_d^\ell\|_0^2 + k \sum_{i=1}^{\ell} \|\nabla(u_h^i - u_d^i)\|_0^2 \leq C \left(\sum_{j=d+1}^{l} \lambda_j \right)^2 . \tag{4.20}$$

Noting that $h\|\nabla(u_h^i - u_d^i)\|_0 \geq C\|u_h^i - u_d^i\|_0$ (inverse inequality) and $h^2 = O(k)$, we obtain that

$$\|u_h^i - u_d^i\|_0 + \frac{1}{\ell} \sum_{i=1}^{\ell} \|\nabla(u_h^i - u_d^i)\|_0 \leq C \sum_{j=d+1}^{l} \lambda_j, \quad 1 \leq i \leq \ell. \tag{4.21}$$

When $n \notin \{1, 2, \cdots, \ell\}$, we may as well let $t_n \in (t_{\ell-1}, t_\ell)$ and t_n be the nearest point to t_ℓ. Expanding u_h^n and p_h^n into Taylor series with respect to t_s can yield that

$$u_h^n = u_h^\ell - \eta k \frac{\partial u_h(\xi_1)}{\partial t}, \quad t_n \leq \xi_1 \leq t_\ell; \quad p_h^n = p_h^\ell - \eta k \frac{\partial p_h(\xi_2)}{\partial t}, \quad t_n \leq \xi_2 \leq t_\ell, \tag{4.22}$$

where $\eta < \ell$ (since $(t_{i+1} - t_i) \leq 2\ell k$) is the step number from t_n to t_ℓ. Summing (4.17) from 1 to ℓ, and noting that $u_h^0 - u_d^0 = 0$, from Lemma 4.1–4.2 and Proposition 3.2, we obtain that

$$\begin{aligned}
&\|u_h^n - u_d^n\|_0^2 + k \sum_{i=1}^{n} \|\nabla(u_h^i - u_d^i)\|_0^2 \leq Ck \sum_{i=0}^{\ell-1} \|u_h^i - u_d^i\|_0^2 + C\eta^2\ell^2k^3 \\
&+Ck \left[\sum_{i=1}^{\ell} \|\nabla(u_h^i - P^d u_h^i)\|_0 + \sum_{i=1}^{\ell} \|p_h^i - \rho^d p_h^i\|_0 + \sum_{i=1}^{\ell} \|\nabla(u_h^i - R_d u_h^i)\|_0 \right]^2 \\
&\leq Ck \sum_{i=0}^{\ell-1} \|u_h^i - u_d^i\|_0^2 + C\ell^2\eta^2k^3 + Ck\ell^2 \left[\sum_{j=d+1}^{l} \lambda_j \right]^2 .
\end{aligned} \tag{4.23}$$

If $k = O(\ell^{-2})$, by using discrete Gronwall inequality we obtain that

$$\|u_h^n - u_d^n\|_0 + \frac{1}{n} \sum_{i=1}^{n} \|\nabla(u_h^i - u_d^i)\|_0^2 \leq C \left[k^{1/2} + \sum_{j=d+1}^{l} \lambda_j \right] . \tag{4.24}$$

Combining (4.14) and (4.24) can yield (4.6). □

Combining Theorem 2.2 and Theorem 4.5 yields the following result.

THEOREM 4.6. *Under Theorem 2.2 and Theorem 4.5 hypotheses, the error estimate between the solutions for Problem* (II) *and the solutions for the reduced order basic Problem* (V) *is, for* $n = 1, 2, \cdots, L, m = 1, 2, 3,$

$$\begin{aligned}
&\|u^n - u_d^n\|_0 + \frac{1}{n} \sum_{i=1}^{n} \|\nabla(u^i - u_d^i)\|_0 + \frac{1}{n} \sum_{i=1}^{n} \|p^i - p_d^i\|_0 \\
&\leq C \left(h^m + k + \sum_{j=d+1}^{l} \lambda_j \right), \text{ if } n \in \{1, 2, \cdots, \ell\}; \\
&\|u^n - u_d^n\|_0 + \frac{1}{n} \sum_{i=1}^{n} \|\nabla(u^i - u_d^i)\|_0 + \frac{1}{n} \sum_{i=1}^{n} \|p^i - p_d^i\|_0 \\
&\leq C \left(h^m + k^{1/2} + \sum_{j=d+1}^{l} \lambda_j \right), \text{ if } n \notin \{1, 2, \cdots, \ell\}.
\end{aligned}$$

Remark 4.7. The conditions $k = O(\ell^{-2})$ and $\max\limits_{1\le i\le \ell-1} \dfrac{t_{i+1}-t_i}{2k} \le \ell$ in Theorem 4.5 show that enough snapshots must be taken and that the number of time steps between two snapshots cannot exceed twice the number of snapshots. Theorem 4.5 and Theorem 4.6 have presented the error estimates between the solution of the optimizing reduced MFE formulation Problem (IV) and the solution of usual MFE formulation Problem (III) and Problem (II), respectively. Since our methods employ some MFE solutions (u_h^n, p_h^n) $(n = 1, 2, \cdots, L)$ for Problem (III) as assistant analysis, the error estimates in Theorem 4.6 are correlated to the spatial grid scale h and time step size k. However, when one computes actual problems, one may obtain the ensemble of snapshots from physical system trajectories by drawing samples from experiments and interpolation (or data assimilation). For example, for weather forecast, one can use previous weather prediction results to construct the ensemble of snapshots. Thus, the assistant (u_h^n, p_h^n) $(n = 1, 2, \cdots, L)$ could be substituted with the interpolation functions of experimental and previous results, it is unnecessary to solve Problem (III), and it is only necessary to directly solve Problem (III) such that Theorem 4.5 is satisfied. Since Problem (IV) is only dependent on $d(d \ll l \le \ell \ll L)$ and is independent of the spatial grid scale h and time step size k, and, in general, $d(d \ll l \le \ell \ll L)$, i.e. it is only necessary to solve Problem (IV) with very few freedom degrees.

5. Some numerical experiments. In this section, we present some numerical examples of the physical model of cavity flows for Mini's element and different Reynolds numbers by the reduced formulation Problem (IV) validating the feasibility and efficiency of the POD method.

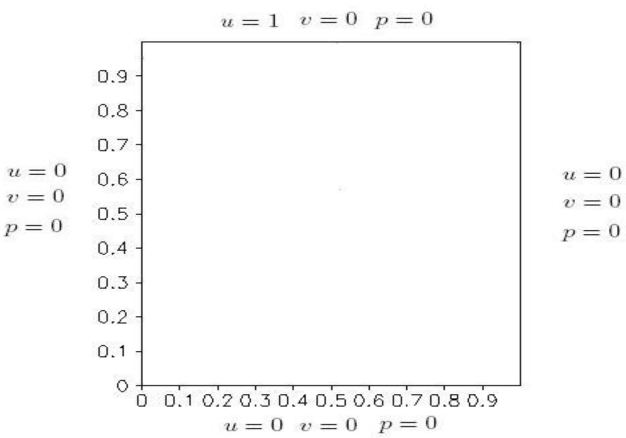

Figure 1. Physics model of the cavity flows: $t = 0$ i.e., $n = 0$ initial values on boundary

Let the side length of the cavity be 1 (see Figure 1). We first divide the cavity into $32 \times 32 = 1024$ small squares with side length $\triangle x = \triangle y = \frac{1}{32} = h$, and then link diagonal of square to divide each square into two triangles on same direction which consists of triangularization $\Im_h$. Take time step increment as $\triangle t = 0.001$. Except that u equal to 1 on upper boundary, other initial value, boundary values, and (f_1, f_2) are all taken as 0 (see Figure 1).

We obtain 20 values (i.e., snapshots) outputting at time $t = 10, 20, 30, \cdots, 200$ by solving usual MFE formulation, i.e., Problem (III). It is shown by computing that the maximal eigenvalues satisfy $\max\{\lambda_{u6}, \lambda_{v6}, \lambda_{p6}\} \leq 10^{-3}$. When $t = 200$, we obtain the solutions of the reduced formulation Problem (IV) based POD method of MEF depicted graphically in Figure 2 to Figure 5 on the right-hand side used 6 optimal POD bases if $Re = 750$ and also used 6 optimal POD bases if $Re = 1500$, but the solutions obtained with usual MFE formulation Problem (III) are depicted graphically in Figure 2 to Figure 5 on left-hand side (Since these figures are equal to solutions obtained with 20 bases, they are also referred to as the figures of the solution with full bases).

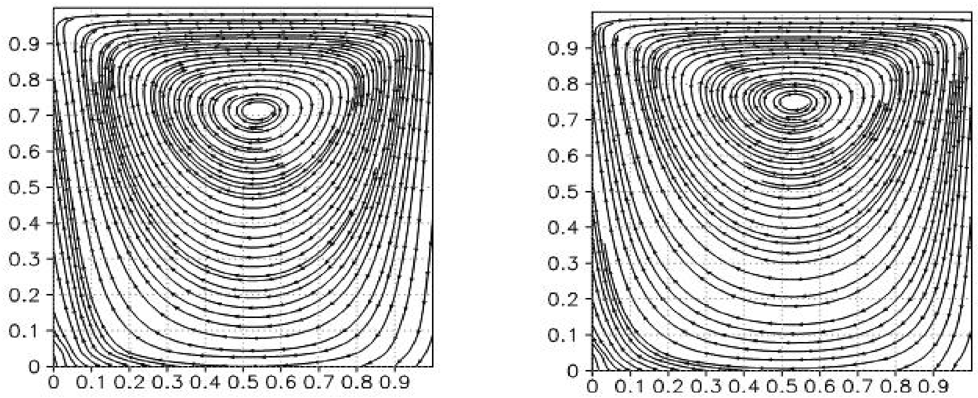

Figure 2. When Re=750, velocity stream line Figure for usual MFE solutions (on left-hand side figure) and $d = 5$ the solution of the reduced MFE formulation (on right-hand side figure)

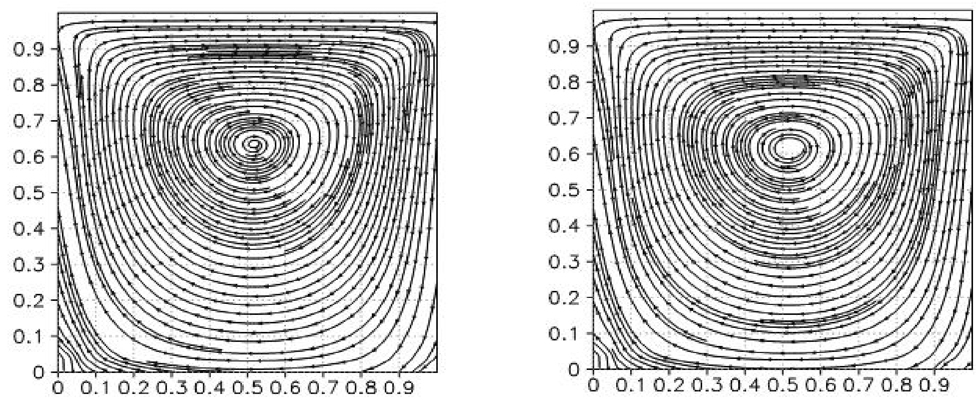

Figure 3. When Re=1500, velocity stream line Figure for usual MFE solution (on left-hand side figure) and $d = 5$ solution of the reduced MFE formulation (on right-hand side figure)

Figure 6 shows the errors between solutions obtained with different number of optimal POD bases and solutions obtained with full bases. Comparing the usual MFE formulation Problem (III) with the reduced MFE formulation Problem (IV) containing 6 optimal bases implementing 3000 times the numerical simulation computations, we find that for usual MFE formulation Problem (III) the required CPU time is 6 minutes, while for the reduced MFE formulation Problem (IV) with 6 optimal bases the corresponding time is only three seconds, i.e., the usual MFE formulation Problem (III) required CPU time is a factor of 120 larger than the reduced MFE formulation Problem (IV) with 6 optimal bases required CPU time, while the error between their solutions does not exceed 10^{-3}. It is also shown that finding the approximate solutions for the nonstationary Navier–Stokes equations with the reduced MFE formulation Problem (IV) is computationally very effective. And the results for numerical examples are consistent with those obtained for the theoretical case.

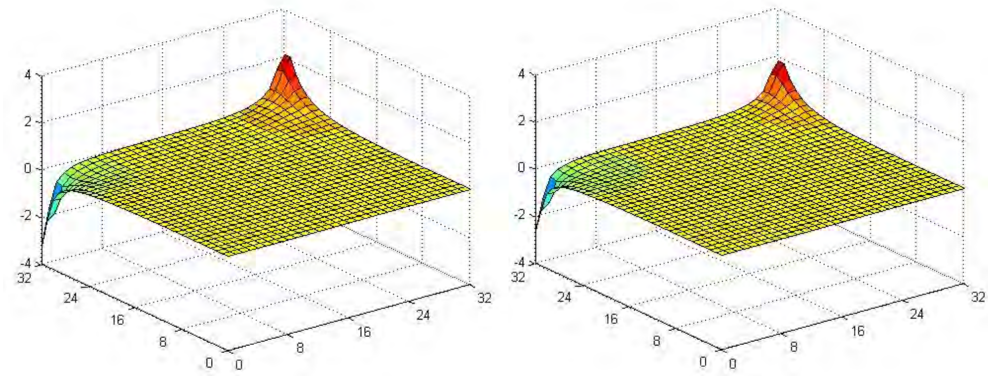

Figure 4. When Re=750, pressure Figure for usual MFE solution (on left-hand side figure) and $d = 5$ solution of reduced MFE formulation (on right-hand side Figure)

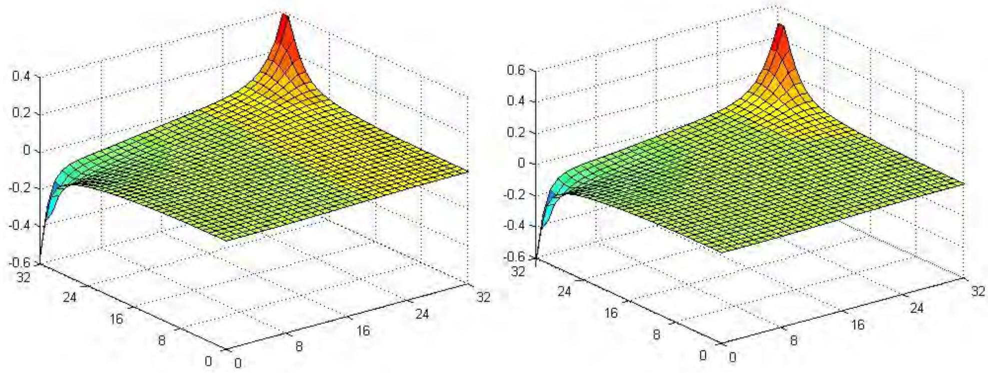

Figure 5. When Re=1500, pressure figure for usual MFE solutions (on left-hand side Figure) and $d = 5$ solution of reduced MFE formulation (on right-hand side Figure)

6. Conclusions. In this paper, we have employed the POD techniques to derive a reduced formulation for the nonstationary Navier–Stokes equations. We first reconstruct optimal orthogonal bases of ensembles of data which are compiled from transient solutions derived by using usual MFE equation system, while in actual applications, one may obtain the ensemble of snapshots from physical system trajectories by drawing samples from experiments and interpolation (or data assimilation). For example, for weather forecast, one can use previous weather prediction results to construct the ensemble of snapshots to restructure the POD basis for the ensemble of snapshots by methods of above Section 3. We have also combined the optimal orthogonal bases with a Galerkin projection procedure, thus yielding a new optimizing reduced MFE formulation of lower dimensional order and of high accuracy for the nonstationary Navier–Stokes equations. We have then proceeded to derive error estimates between our optimizing optimizing reduced MFE approximate solutions and the usual MFE approximate solutions, and have shown using numerical examples that the error between the optimizing reduced MFE approximate solution and the usual MFE solution is consistent with the theoretical error results, thus validating both feasibility and efficiency of our optimizing reduced MFE formulation. Future research work in this area will aim to extend the optimizing reduced MFE formulation, applying it to a realistic atmosphere quality forecast system and to more complicated PDEs. We have shown both by theoretical analysis as well as by numerical examples that the optimizing reduced MFE formulation presented herein has extensive perspective applications.

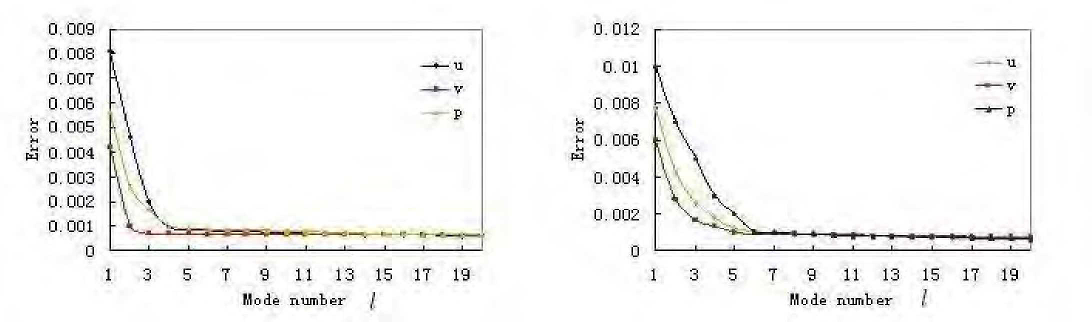

Figure 6. Error for Re=750 on left–hand side, error for Re=1500 on right-hand side

Though Kunisch and Volkwein have presented some Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics, i.e., for the nonstationary Navier–Stokes equations in [28], our method is different from their approaches, whose methods consist of Galerkin projection approaches where original variables are substituted for linear combination of POD basis and the error estimates of the velocity field therein are only derived, their POD basis is generated with the solutions of the physical system at all time instances, while our POD basis is generated with few solutions of the physical system which are useful and of interest for us. Especially, velocity field is only approximated in Reference [28], while velocity field and pressure are all synchronously approximated in our following method, and the error estimates of velocity field and pressure approximate solutions are also synchronously derived. Thus our method appears to be more optimal than that in [28].

REFERENCES

[1] V. Girault and P. A. Raviart, *Finite Element Methods for Navier–Stokes Equations, Theorem and Algorithms,* Springer–Verlag, 1986.

[2] J. G. Heywood, R. Rannacher, *Finite element approximation of the nonstationary Navier–Stokes problem, I. Regularity of solutions and second order estimates for spatial discretization,* SIAM J. Numer. Anal., 19 (1982), pp. 275–311.

[3] Z. H. Luo, *The third order estimate of mixed finite element for the Navier–Stokes problems,* Chinese Quarterly Journal of Mathematics, 10:3 (1995), pp. 9–12.

[4] K. Fukunaga, *Introduction to Statistical Recognition,* Academic Press, 1990.

[5] I. T. Jolliffe, Principal Component Analysis, Springer–Verlag, 2002.

[6] D. T. Crommelin, and A. J. Majda, *Strategies for model reduction: comparing different optimal bases,* J. Atmos. Sci., 61 (2004), pp. 2206–2217.

[7] A J. Majda, I. Timofeyev, and E. Vanden–Eijnden, *Systematic strategies for stochastic mode reduction in climate,* J. Atmos. Sci., 60 (2003), pp. 1705–1722.

[8] F. Selten, *Baroclinic empirical orthogonal functions as basis functions in an atmospheric model,* J. Atmos. Sci., 54 (1997), pp. 2100–2114.

[9] P. Holmes, J. L Lumley, and G. Berkooz, *Turbulence, Coherent Structures, Dynamical Systems and Symmetry,* Cambridge University Press, Cambridge, UK, 1996.

[10] G. Berkooz, P. Holmes, and J. L Lumley, *The proper orthogonal decomposition in analysis of turbulent flows,* Annual Review of Fluid Mechanics, 25 (1993), pp. 539–575.

[11] W. Cazemier, R. W. C. P. Verstappen, and A. E. P. Veldman, *Proper orthogonal decomposition and low–dimensional models for driven cavity flows,* Phys. Fluids, 10 (1998), pp. 1685–1699.

[12] D. Ahlman, F. Södelund, J. Jackson, A. Kurdila, and W. Shyy, *Proper orthogonal decomposition for time-dependent lid-driven cavity flows,* Numerical Heal Transfer Part B–Fundamentals, 42:4 (2002), pp. 285–306.

[13] J. L. Lumley, *Coherent Structures in Turbulence,* in Meyer R E (ed.), Transition and Turbulence, Academic Press, 1981, pp. 215-242.

[14] N. Aubry, P . Holmes, J. L. Lumley et al., *The dynamics of coherent structures in the wall region of a turbulent boundary layer,* Journal of Fluid Dynamics, 192 (1988), pp. 115-173.

[15] L. Sirovich, *Turbulence and the dynamics of coherent structures: Part I-III,* Quarterly of Applied Mathematics, 45:3 (1987), pp. 561–590.

[16] R. D. Roslin, M. D. Gunzburger, R. Nicolaides, et al., *A self-contained automated methodology for optimal flow control validated for transition delay,* AIAA Journal, 35 (1997), pp. 816–824.

[17] H. V. Ly and H. T. Tran, *Proper orthogonal decomposition for flow calculations and optimal control in a horizontal CVD reactor,* Quarterly of Applied Mathematics, 60 (2002), pp. 631–656.

[18] J. Ko, A. J. Kurdila, O. K. Redionitis, et al., *Synthetic Jets, Their Reduced Order Modeling and Applications to Flow Control,* AIAA Paper number 99-1000, 37 Aerospace Sciences Meeting & Exhibit, Reno, 1999.

[19] P. Moin and R. D. Moser, *Characteristic-eddy decomposition of turbulence in channel,* Journal of Fluid Mechanics, 200 (1989), pp. 417–509.

[20] M. Rajaee, S. K. F. Karlsson, and L. Sirovich, *Low dimensional description of free shear flow coherent structures and their dynamical behavior,* Journal of Fluid Mechanics, 258 (1994), pp. 1401–1402.

[21] Y. H. Cao, J. Zhu, Z. D. Luo, and I. M. Navon, *Reduced order modeling of the upper tropical pacific ocean model using proper orthogonal decomposition,* Computers & Mathematics with Applications, 52, Issues 8-9 (2006), pp. 1373–1386.

[22] Y. H. Cao, J. Zhu, I. M. Navon, and Z. D. Luo, *A reduced order approach to four-dimensional variational data assimilation using proper orthogonal decomposition,* International Journal for Numerical Methods in Fluids, 53 (2007), pp. 1571–1583.

[23] Z. D. Luo, J. Zhu, R. W. Wang, and I. M. Navon, *Proper orthogonal decomposition approach and error estimation of mixed finite element methods for the tropical Pacific Ocean reduced gravity model,* Computer Methods in Applied Mechanics and Engineering, (2007 in print)

[24] Z. D. Luo, J. Chen, J. Zhu, R. W. Wang, and I. M. Navon, *An optimizing reduced order FDS for the tropical Pacific Ocean reduced gravity model,* International Journal for Numerical Methods in Fluids, DoI:10. 1002/fld. 1452 (2007 in print).

[25] Ruiwen Wang, Jiang Zhu, Zhendong Luo, and Navon I M, *An equation–free reduced order modeling approach to tropic pacific simulation,* International Journal for Numerical Methods in Fluids, (2007 submitted).

[26] Z. D. Luo, R. W. Wang, J. Chen, and J. Zhu, *Finite difference scheme based on proper orthogonal decomposition for the nonstationary Navier-Stokes equations,* Science in China Series A: Math., (2007 in print).

[27] K. Kunisch, S. Volkwein, *Galerkin proper orthogonal decomposition methods for parabolic problems,* Numerische Mathematik, 90 (2001), pp. 177–148.

[28] K. Kunisch and S. Volkwein, *Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics,* SIAM J. Numer. Anal., 40:2 (2002), pp. 492–515.

[29] K. Kunisch, S. Volkwein, *Control of Burgers' equation by a reduced order approach using proper orthogonal decomposition,* Journal of Optimization Theory and Applications, 102 (1999), pp. 345–371.

[30] R. A. Adams, *Sobolev Space,* Academic Press, New York, 1975.

[31] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems,* North–Holland, Amsterdam, 1978.

[32] Z. D. Luo, *Mixed Finite Element Methods and Applications,* Chinese Science Press, Beijing, 2006.