LIMITS OF TRANSLATES OF PLANE CURVES, I

PAOLO ALUFFI, CAREL FABER

Abstract. We classify all possible limits of families of translates of a fixed, arbitrary complex plane curve. We do this by giving a set-theoretic description of the projective normal cone (PNC) of a subscheme, determined by the curve, of the $\mathbb{P}^8$ of 3×3 matrices. In a sequel to this paper we determine the multiplicities of the components of the PNC. The knowledge of the PNC as a cycle is essential in our computation of the degree of the PGL(3)-orbit closure of an arbitrary plane curve, performed in [4].

1. Introduction

In this paper we determine the possible *limits* of a fixed, arbitrary complex plane curve $\mathscr{C}$, obtained by applying to it a family of translations $\alpha(t)$ centered at a singular transformation of the plane. In other words, we describe the curves in the boundary of the PGL(3)-orbit closure of a given curve $\mathscr{C}$.

Our main motivation for this work comes from enumerative geometry. In [4] we have determined the *degree* of the PGL(3)-orbit closure of an arbitrary (possibly singular, reducible, non-reduced) plane curve; several characteristic numbers of families of plane curves, the degrees of certain maps to moduli spaces of plane curves, and isotrivial versions of the Gromov-Witten invariants of the plane, are particular cases of this invariant. A description of the limits of a curve, and in fact a more refined type of information is an essential ingredient of our approach. This information is obtained in this paper and in its sequel [5]; the results were announced and used in [4].

The set-up is as follows. Consider the natural action of PGL(3) on the projective space of plane curves of a fixed degree. The orbit closure of a curve $\mathscr{C}$ is dominated by the closure $\widetilde{\mathbb{P}}^8$ of the graph of the rational map c from the $\mathbb{P}^8$ of 3×3 matrices to the $\mathbb{P}^N$ of plane curves of degree d, associating to $\varphi \in \mathrm{PGL}(3)$ the translate of $\mathscr{C}$ by φ. The boundary of the orbit consists of limits of $\mathscr{C}$ and plays an important role in the study of the orbit closure.

Our computation of the degree of the orbit closure of $\mathscr{C}$ hinges on the study of $\widetilde{\mathbb{P}}^8$, and especially of the scheme-theoretic inverse image in $\widetilde{\mathbb{P}}^8$ of the base scheme $\mathscr{S}$ of c. Viewing $\widetilde{\mathbb{P}}^8$ as the blow-up of $\mathbb{P}^8$ along $\mathscr{S}$, this inverse image is the exceptional divisor, and may be identified with the projective normal cone (PNC) of $\mathscr{S}$ in $\mathbb{P}^8$. A description of the PNC leads to a description of the limits of $\mathscr{C}$: the image of the PNC in $\mathbb{P}^N$ is contained in the set of limits, and the complement, if nonempty, consists of easily identified 'stars' (that is, unions of concurrent lines).

This paper is devoted to a set-theoretic description of the PNC for an arbitrary curve. This suffices for the determination of the limits, but does not suffice for the enumerative applications in [4]; these applications require the full knowledge of

the PNC *as a cycle,* that is, the determination of the multiplicities of its different components. We obtain this additional information in [5].

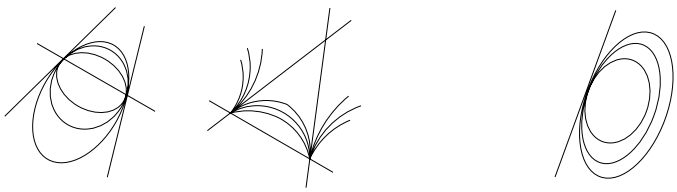
The final result of our analysis (including multiplicities) is reviewed in §2 of [4]. The proofs of the facts stated there are given in the present article and its sequel. The main theorem of this paper (Theorem 2.4, in §2.5) gives a precise set-theoretic description of the PNC, relying upon five types of families and limits identified in §2.3. In this introduction we confine ourselves to formulating a weaker version, focusing on the determination of limits.

The limits of a curve $\mathscr{C}$ are necessarily curves with *small linear orbit,* that is, curves with infinite stabilizer. Such curves are classified in §1 of [3]; for these curves, the limits can be determined using the results in [2] (see also §4). The following statement reduces the computation of the limits of an arbitrary curve $\mathscr{C}$ to the case of curves with small orbit.

Theorem 1.1. *Let $\mathscr{X}$ be a limit of a plane curve $\mathscr{C}$ of degree d, obtained by applying to it a $\mathbb{C}((t))$-valued point of* PGL(3) *with singular center. Then $\mathscr{X}$ is in the orbit closure of a star (reproducing projectively the d-tuple cut out on $\mathscr{C}$ by a line meeting it properly), or of curves with small orbit determined by the following features of $\mathscr{C}$:*

- I: *The linear components of the support $\mathscr{C}'$ of $\mathscr{C}$;*
- II: *The nonlinear components of $\mathscr{C}'$;*
- III: *The points at which the tangent cone of $\mathscr{C}$ is supported on at least* 3 *lines;*
- IV: *The Newton polygons of $\mathscr{C}$ at the singularities and inflection points of $\mathscr{C}'$;*
- V: *The Puiseux expansions of formal branches of $\mathscr{C}$ at the singularities of $\mathscr{C}'$.*

The limits corresponding to these features may be described as follows. In cases I and III they are 'fans', that is, unions of a star and a general line; in case II, they are supported on the union of a nonsingular conic and a tangent line; in case IV, they are supported on the union of the coordinate triangle and several curves from a pencil $y^c = \rho\, x^{c-b} z^b$, with $b < c$ coprime positive integers; and in case V they are supported on unions of quadritangent conics and the distinguished tangent line. The following picture illustrates the limits in cases IV and V:



IV V

A more precise description of the limits is given in §2.3, with reference to the classification of these curves obtained in §1 of [3].

The proof of Theorem 1.1 (or rather of its more precise form given in Theorem 2.4) is by an explicit reduction process, and goes along the following lines. The stars mentioned in the statement are obtained by families of translations $\alpha(t)$ ('germs') centered at an element $\alpha(0) \notin \mathscr{S}$. To analyze germs centered at points of $\mathscr{S}$, we introduce a notion of equivalence of germs (Definition 3.1), such that equivalent germs

lead to the same limit. We then prove that every germ centered at a point of $\mathscr{S}$ is essentially equivalent to one with matrix representation

$$\begin{pmatrix} 1 & 0 & 0 \\ q(t) & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with $0 \le b \le c$ and q, r, and s polynomials. Here coordinates are chosen so that the point $p = (1 : 0 : 0)$ belongs to $\mathscr{C}$. Studying the limits obtained by applying such germs to $\mathscr{C}$, we identify five specific types of families (the *marker germs* listed in §2.3), reflecting the features of $\mathscr{C}$ at p listed in Theorem 1.1, and with the stated kind of limit. We prove that unless the germ is of one of these types, the corresponding limit is already accounted for (for example, it is in the closure of the set of stars mentioned in the statement).

In terms of the graph of the rational map c mentioned above, we prove that every component of the PNC is hit at a general point by the lift in $\widetilde{\mathbb{P}}^8$ of one of the five distinguished types of germs. This yields our set-theoretic description of the PNC. In fact, the lifts intersect the corresponding components transversally, and this will be important in our determination of the multiplicities of the components in [5].

The procedure underlying the proof of Theorem 2.4 may be applied to any given plane curve, producing a list of its limits; in practice, one needs to find the marker germs for the curve (cf. §2.3), determining corresponding components of the PNC. The two examples that follow illustrate this process, and show that components of all types may already occur on curves of degree 4. Simple translations are used to bring the marker germs provided by §2.3 into the form given here.

Example 1.1. Consider the reducible quartic $\mathscr{C}_1$ given by the equation

$$(y+z)(xy^2 + xyz + xz^2 + y^2z + yz^2) = 0\,.$$

It consists of an irreducible cubic with a node at $(1 : 0 : 0)$ and a line through the node and the inflection point $(0 : 1 : -1)$. The other inflection points are $(0 : 1 : 0)$ and $(0 : 0 : 1)$. According to Theorem 2.4 and to the list in §2.3, the PNC for $\mathscr{C}_1$ has one component of type I, one component of type II, one component of type III, corresponding to the triple point $(1 : 0 : 0)$, and four components of type IV: one for each of the inflection points $(0 : 1 : 0)$ and $(0 : 0 : 1)$, one for the node $(0 : 1 : -1)$ and the tangent line $x = y + z$ to the cubic at that point, and one for the triple point $(1 : 0 : 0)$ and the two lines in the tangent cone $y^2 + yz + z^2 = 0$ to the cubic at that point. A list of representative marker germs is as follows:

$$\mathrm{I}: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & t \end{pmatrix}; \quad \mathrm{II}: \begin{pmatrix} 2 & 0 & 0 \\ -3 & t & 0 \\ 6 & 0 & t^2 \end{pmatrix}; \quad \mathrm{III}: \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix};$$

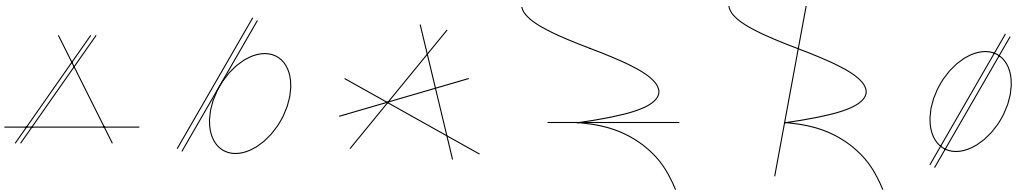
and, for type IV:

$$\begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ -t & 0 & t^3 \end{pmatrix}, \quad \begin{pmatrix} t & 0 & 0 \\ -t & t^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ t & -1 & t^3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho t & 0 \\ 0 & t & t^2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 t & 0 \\ 0 & t & t^2 \end{pmatrix}$$

(where ρ is a primitive third root of unity). The latter two marker germs have the same center and lead to projectively equivalent limits, hence they contribute the same component of the PNC. Generators of the ideals of the corresponding limits of $\mathscr{C}_1$ are

$$xy^2z, \quad x^2(8y^2-9xz), \quad x(y+z)(y^2+yz+z^2), \quad y(y^2z+x^3), \quad z(yz^2+x^3),$$
$$x(y^2z-x^3), \quad y^2(y^2-(\rho+2)xz), \quad \text{and} \quad y^2(y^2-(\rho^2+2)xz),$$

respectively: a triangle with one line doubled, a conic with a double tangent line, a fan with star centered at $(1:0:0)$, a cuspidal cubic with its cuspidal tangent (two limits), a cuspidal cubic with the line through the cusp and the inflection point, and finally a conic with a double transversal line (two limits). Visually:



According to Theorem 1.1, all limits of $\mathscr{C}_1$ (other than stars of lines) are projectively equivalent to one of these curves, or to limits of them (cf. §4). □

Example 1.2. Consider the irreducible quartic $\mathscr{C}_2$ given by the equation

$$(y^2-xz)^2 = y^3z.$$

It has a ramphoid cusp at $(1:0:0)$, an ordinary cusp at $(0:0:1)$, and an ordinary inflection point at $(3^3 5{:}-2^6 3^2{:}-2^{12})$; there are no other singular or inflection points. The PNC for $\mathscr{C}_2$ has one component of type II, two components of type IV, corresponding to the inflection point and the ordinary cusp, and one component of type V, corresponding to the ramphoid cusp. (Note that there is no component of type IV corresponding to the ramphoid cusp.) Representative marker germs for the latter two components are

$$\text{IV}: \begin{pmatrix} 0 & t^3 & 0 \\ t^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \text{V}: \begin{pmatrix} 1 & 0 & 0 \\ t^4 & t^5 & 0 \\ t^8 & 2t^9 & t^{10} \end{pmatrix}$$

and generators of the ideals of the corresponding limits of $\mathscr{C}_2$ are

$$z(y^2z-x^3) \quad \text{and} \quad (y^2-xz+x^2)(y^2-xz-x^2),$$

respectively: a cuspidal cubic with its inflectional tangent and a pair of quadritangent conics. The connected component of the stabilizer of the latter limit is the additive group. The germ with entries 1, t, and t^2 on the diagonal and zeroes elsewhere leads to the limit $(y^2-xz)^2$, a double conic; its orbit is too small to produce an additional component of type IV. □

Necessary preliminary considerations, and the full statement of the main theorem, are found in §2. The determination of the limits by successive reductions of a given family of curves, proving the result, is worked out in §3. In §4 we summarize the more straightforward situation for curves with small orbits.

Caporaso and Sernesi use our determination of the limits in [8] (Theorem 5.2.1). Hacking [11] and Hassett [13] study the limits of the family of nonsingular plane curves of a given degree, by methods different from ours: they allow the plane to degenerate together with the curve. It would be interesting to compare their results to ours. Harris and Morrison ([12], p. 138) pose the *flat completion problem* for embedded families of curves, asking for the determination of all curves in $\mathbb{P}^n$ that can arise as flat limits of a family of embedded stable curves over the punctured disc. The present article solves the isotrivial form of this problem, for plane curves.

In principle, a solution of the isotrivial flat completion problem for plane curves can already be found in the marvelous article [10] by Aldo Ghizzetti, dating back to the 1930s. However, Ghizzetti's results do not lead to a description of the PNC, which is necessary for our application in [4], and which is the main result of this paper and of its sequel.

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2. Set-theoretic description of the PNC

2.1. **Limits of translates.** We work over $\mathbb{C}$. We choose homogeneous coordinates $(x : y : z)$ in $\mathbb{P}^2$, and identify PGL(3) with the open set of nonsingular matrices in the space $\mathbb{P}^8$ parametrizing 3×3 matrices. We act with PGL(3) on the space $\mathbb{P}^N = \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(d))$ of degree-d plane curves as follows: if $F(x, y, z)$ is a generator of the homogeneous ideal of a plane curve $\mathscr{C}$, and $\alpha \in \mathrm{PGL}(3)$, we denote by $\mathscr{C} \circ \alpha$ the curve with ideal $(F \circ \alpha) = (F(\alpha(x, y, z)))$.

An *isotrivial* family of plane curves over the punctured disk will be a family $\mathscr{C} \circ \alpha(t)$, where $\alpha(t)$ is a $\mathbb{C}((t))$-valued point of PGL(3). We may and will view $\alpha(t)$ as a 3×3 matrix with entries in $\mathbb{C}[[t]]$ such that $\alpha(0) \neq 0$ and $\det \alpha(t) \not\equiv 0$; we will only consider matrices $\alpha(t)$ such that $\det \alpha(0) = 0$.

These arcs of matrices will be called *germs,* and viewed as germs of curves in $\mathbb{P}^8$. The flat limit of a family $\mathscr{C} \circ \alpha(t)$ as $t \to 0$:

$$\lim_{t \to 0} \mathscr{C} \circ \alpha(t)$$

may be computed concretely by clearing common powers of t in the expanded expression $F(\alpha(t))$ and then setting $t = 0$.

By truncating $\alpha(t)$ at a sufficiently high power of t (which does not affect the limit) we may assume $\alpha(t)$ is polynomial, hence parametrizing an algebraic curve near $\alpha(0)$.

If $\mathscr{C}$ is a smooth curve of degree ≥ 4, the isotriviality condition considered above is equivalent to the requirement that the curves $\mathscr{C} \circ \alpha(t)$ be abstractly isomorphic for general $t \neq 0$, cf. [6], p. 56. For any $\mathscr{C}$ and any germ $\alpha(t)$ in $\mathbb{P}^8$, isotriviality may be forced, if necessary, by adding to $\alpha(t)$ high powers of t, ensuring $\det \alpha(t) \not\equiv 0$ without

changing the limit of the family; thus we may and will assume henceforth that the families $\mathscr{C} \circ \alpha(t)$ are isotrivial.

Our goal is the determination of all possible limits as above for a given, arbitrary plane curve $\mathscr{C}$.

2.2. **The Projective Normal Cone.** The set of all translates $\mathscr{C} \circ \alpha$ is the *linear orbit* of $\mathscr{C}$, which we denote by $\mathscr{O}_{\mathscr{C}}$; the complement of $\mathscr{O}_{\mathscr{C}}$ in its closure $\overline{\mathscr{O}_{\mathscr{C}}}$ is the *boundary* of the orbit of $\mathscr{C}$. By the *limits of* $\mathscr{C}$ we will mean the limits of families $\mathscr{C} \circ \alpha(t)$ with $\alpha(0) \notin \mathrm{PGL}(3)$. It is not hard to see that if $\dim \mathscr{O}_{\mathscr{C}} < 8$ (the orbit is *small,* in the terminology of [2] and [3]) then there are germs centered at a singular matrix and with limit equal to $\mathscr{C}$; in this case, the whole orbit closure $\overline{\mathscr{O}_{\mathscr{C}}}$ consists of limits of $\mathscr{C}$. In any case, the boundary is a subset of the set of limits of $\mathscr{C}$, and in fact equality holds if and only if $\dim \mathscr{O}_{\mathscr{C}} = 8$.

The set of limit curves is itself a union of orbits of plane curves; our goal is a description of representative elements of these orbits; in particular, this will yield a description of the boundary of $\mathscr{O}_{\mathscr{C}}$. In this section we relate the set of limits of $\mathscr{C}$ to the *projective normal cone* mentioned in the introduction.

Points of $\mathbb{P}^8$, that is, 3×3 matrices, may be viewed as rational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. The kernel of a singular matrix $\alpha \in \mathbb{P}^8$ determines a line of $\mathbb{P}^2$ (if $\operatorname{rk} \alpha = 1$) or a point (if $\operatorname{rk} \alpha = 2$); $\ker \alpha$ will denote this locus. Likewise, the image of α is a point of $\mathbb{P}^2$ if $\operatorname{rk} \alpha = 1$, or a line if $\operatorname{rk} \alpha = 2$.

The action map $\alpha \mapsto \mathscr{C} \circ \alpha$ for $\alpha \in \mathrm{PGL}(3)$ defines a rational map

$$c : \mathbb{P}^8 \dashrightarrow \mathbb{P}^N \quad .$$

We denote by $\mathscr{S}$ the base scheme of this rational map. The closure of the graph of c may be identified with the blow-up $\widetilde{\mathbb{P}}^8$ of $\mathbb{P}^8$ along $\mathscr{S}$. The support of $\mathscr{S}$ consists of the matrices α such that (with notation as above) $F(\alpha(x, y, z)) \equiv 0$; that is, matrices whose image is contained in $\mathscr{C}$.

We denote by E the exceptional divisor of this blow-up; this is the projective normal cone (PNC) of $\mathscr{S}$ in $\mathbb{P}^8$. We have the following commutative diagram:

$$\begin{array}{ccccc} E & \hookrightarrow & \widetilde{\mathbb{P}}^8 & \hookrightarrow & \mathbb{P}^8 \times \mathbb{P}^N \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathscr{S} & \hookrightarrow & \mathbb{P}^8 & \overset{c}{\dashrightarrow} & \mathbb{P}^N \end{array}$$

Lemma 2.1. *As a subset of* $\mathbb{P}^8 \times \mathbb{P}^N$*, the support of* E *is*

$$|E| = \{(\alpha, \mathscr{X}) \in \mathbb{P}^8 \times \mathbb{P}^N : \mathscr{X} \text{ is a limit of } \mathscr{C} \circ \alpha(t) \\ \text{for some germ } \alpha(t) \text{ centered at } \alpha \in \mathscr{S} \text{ and not contained in } \mathscr{S}\} \quad .$$

Proof. Let $\pi : \widetilde{\mathbb{P}}^8 \to \mathbb{P}^8$ denote the blow-up map; that is, the restriction of the projection on the first factor of $\mathbb{P}^8 \times \mathbb{P}^N$.

Any curve germ $\alpha(t)$ in $\mathbb{P}^8$ centered at $\alpha \in \mathscr{S}$ and not contained in $\mathscr{S}$ lifts to a germ in $\widetilde{\mathbb{P}}^8$ centered at a point of E; this yields the $\supset$ inclusion.

For the other inclusion, let $\tilde{\alpha}(t)$ be a germ centered at a point $\tilde{\alpha}$ of E, and not contained in E; such a germ may be obtained (for example) by successively intersecting $\widetilde{\mathbb{P}}^8$ in $\mathbb{P}^8 \times \mathbb{P}^N$ with general divisors of type $(1,1)$ through $\tilde{\alpha}$. As π is 1-to-1 in the complement of E, $\tilde{\alpha}(t)$ is the lift of a (unique) curve germ $\alpha(t)$ in $\mathbb{P}^8$ centered at a point of $\mathscr{S}$, and not contained in $\mathscr{S}$. □

Corollary 2.2. *The set of limits of $\mathscr{C}$ consists of the image of the PNC in $\mathbb{P}^N$, and of limits of families $\mathscr{C} \circ \alpha(t)$ with $\alpha = \alpha(0)$ a singular matrix whose image is not contained in $\mathscr{C}$.*

In the latter case: if α has rank 1, the limit consists of a multiple line supported on $\ker \alpha$; if α has rank 2, the limit consists of a star of lines through $\ker \alpha$, reproducing projectively the tuple of points cut out by $\mathscr{C}$ on the image of α.

Proof. By Lemma 2.1, the PNC dominates the set of limits of families $\mathscr{C} \circ \alpha(t)$ for which $\alpha(t)$ is centered at a point of indeterminacy of c. This gives the first statement.

To verify the second assertion, assume that $\alpha(t)$ is centered at a singular matrix α at which c *is* defined; α is then a rank-1 or rank-2 matrix such that $F(\alpha(x,y,z)) \not\equiv 0$. After a coordinate change we may assume without loss of generality that

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $F(x,0,0)$, resp. $F(x,y,0)$ are not identically zero. These are then the forms defining the limits of the corresponding families, and the descriptions given in the statement are immediately verified in these cases. □

Remark 2.3. Denote by R the proper transform in $\widetilde{\mathbb{P}}^8$ of the set of singular matrices in $\mathbb{P}^8$. Corollary 2.2 asserts that the set of limits of $\mathscr{C}$ is the image of the union of the PNC and R. A more explicit description of the image of R has eluded us; for a smooth curve $\mathscr{C}$ of degree ≥ 5 these 'star limits' have two moduli. It would be interesting to obtain a classification of curves $\mathscr{C}$ with smaller 'star-moduli'.

The image of the *intersection* of R and the PNC will play an important role in this paper. Curves in the image of this locus will be called 'rank-2 limits'; we note that the set of rank-2 limits has dimension ≤ 6.

Corollary 2.2 translates the problem of finding the limits for (isotrivial) families of plane curves $\mathscr{C} \circ \alpha(t)$ into the problem of describing the PNC for the curve $\mathscr{C}$. Each component of the PNC is a 7-dimensional irreducible subvariety of $\widetilde{\mathbb{P}}^8 \subset \mathbb{P}^8 \times \mathbb{P}^N$. We will describe it by listing representative points of the component.

More precisely, note that PGL(3) acts on $\mathbb{P}^8$ by right multiplication, and that this action lifts to a right action of PGL(3) on $\widetilde{\mathbb{P}}^8$. Each component of the PNC is a union of orbits of this action. For each component, we will list germs $\alpha(t)$ lifting on $\widetilde{\mathbb{P}}^8$ to germs $\tilde{\alpha}(t)$ so that the union of the orbits of the centers $\tilde{\alpha}(0)$ is dense in that component.

2.3. **Marker germs.** We will now list germs determining the components of the PNC in the sense explained above. We will call such a germ a *marker germ,* as the center of its lift to $\widetilde{\mathbb{P}}^8$ (the corresponding *marker center*) 'marks' a component of the PNC.

The marker germs and the corresponding centers depend on different features of $\mathscr{C}$, as described in general terms in Theorem 1.1; this leads to the description in terms of 'types' used there.

The first two types depend on global features of $\mathscr{C}$; the latter three depend on features of special points of $\mathscr{C}$ (inflection points and singularities of the support of $\mathscr{C}$). The terminology employed here matches the one in §2 of [4]; for example, a *fan* is the union of a star and a general line. In four of the five types, $\alpha = \alpha(0)$ is a rank-1 matrix and the line $\ker \alpha$ plays an important role; we will call this 'the kernel line'.

Type I. Assume $\mathscr{C}$ contains a line, defined by a linear polynomial L. Write a generator of the ideal of $\mathscr{C}$ as

$$F(x, y, z) = L(x, y, z)^m G(x, y, z)$$

with L not a factor of G. Type I limits are obtained by germs

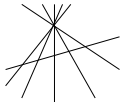
$$\alpha(t) = \alpha(0) + t\beta(t) \quad ,$$

where $\alpha(0)$ has rank 2 and image the line defined by L.

As we are assuming (cf. §2.1) that $\det \alpha(t) \not\equiv 0$, the image of $\beta(t)$ is not contained in $\operatorname{im} \alpha(0)$, so that the limit $\lim_{t\to 0} L \circ \beta(t)$ is a well-defined line ℓ. The limit $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ consists of the m-fold line ℓ, and a star of lines through the point $\ker \alpha(0)$. This star reproduces projectively the tuple cut out on L by the curve defined by G.

The limit is in general a fan, and degenerates to a star if the m-fold line ℓ contains the point $\ker \alpha(0)$. Fans and stars are studied in [3]; they are items (3) and (5) in the classification of curves with small linear orbits, in §1 of loc. cit:





For types II—V we choose coordinates so that $p = (1 : 0 : 0)$ is a point of $\mathscr{C}$; for types II, IV, and V we further require that $z = 0$ is a chosen component ℓ of the tangent cone to $\mathscr{C}$ at p.

Type II. Assume that p is a nonsingular, non-inflectional point of the support $\mathscr{C}'$ of $\mathscr{C}$, contained in a nonlinear component, with tangent line $z = 0$. Let

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix} \quad .$$

Then the ideal of $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is generated by

$$x^{d-2S}(y^2 + \rho xz)^S \quad ,$$

where S is the multiplicity of the component in $\mathscr{C}$, and $\rho \neq 0$; that is, the limit consists of a (possibly multiple) nonsingular conic tangent to the kernel line, union (possibly) a multiple of the kernel line. Such curves are items (6) and (7) in the classification of curves with small orbit:



The extra kernel line is present precisely when $\mathscr{C}$ is not itself a multiple nonsingular conic.

Type III. Assume that p is a singular point of $\mathscr{C}'$ of multiplicity m in $\mathscr{C}$, with tangent cone supported on at least three lines. Let

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} \quad .$$

Then $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is a fan consisting of a star centered at $(1:0:0)$ and projectively equivalent to the tangent cone to $\mathscr{C}$ at p, and of a residual $(d-m)$-fold line supported on the kernel line $x = 0$.

Type IV. Assume that p is a singular or inflection point of the support of $\mathscr{C}$. Germs of type IV are determined by the choice of the line ℓ in the tangent cone to $\mathscr{C}$ at p, and by the choice of a side of a corresponding Newton polygon, with slope strictly between -1 and 0. This procedure is explained in more detail in §2.4.

Let $b < c$ be relatively prime positive integers such that $-b/c$ is the slope of the chosen side. Let

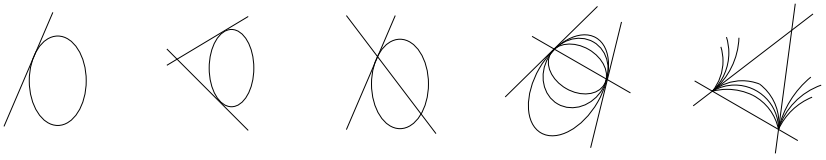
$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \quad .$$

Then the ideal of $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is generated by a polynomial of the form

$$x^{\overline{q}} y^r z^q \prod_{j=1}^{S} (y^c + \rho_j x^{c-b} z^b) \quad ,$$

with $\rho_j \neq 0$. The number S of 'cuspidal' factors in the limit curve is the number of segments cut out by the integer lattice on the selected side of the Newton polygon. The germ listed above contributes a component of the PNC unless $b/c = 1/2$ and the limit curve is supported on a conic union (possibly) the kernel line.

The limit curves arising in this way are items (7) through (11) in the classification in [3]:



They are studied enumeratively in [2].

Type V. Assume p is a singular point of the support of $\mathscr{C}$. Germs of type V are determined by the choice of the line ℓ in the tangent cone to $\mathscr{C}$ at p, the choice of a formal branch $z = f(y) = \gamma_{\lambda_0} y^{\lambda_0} + \dots$ for $\mathscr{C}$ at p tangent to ℓ, and the choice of a certain 'characteristic' rational number $C > \lambda_0$ (assuming these choices can be made). This procedure is also explained in more detail in §2.4.

For $a < b < c$ positive integers such that $\frac{c}{a} = C$ and $\frac{b}{a} = \frac{C-\lambda_0}{2} + 1$, let

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix}$$

where $\underline{\cdots}$ denotes the truncation modulo t^c. The integer a is chosen to be the minimal one for which all entries in this germ are polynomials. Then $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is given by

$$x^{d-2S} \prod_{i=1}^{S} \left(zx - \frac{\lambda_0(\lambda_0-1)}{2}\gamma_{\lambda_0} y^2 - \frac{\lambda_0 + C}{2}\gamma_{\frac{\lambda_0+C}{2}} yx - \gamma_C^{(i)} x^2 \right) \quad ,$$

where S and $\gamma_C^{(i)}$ are defined in §2.4. These curves consist of (at least two) 'quadritangent' conics—that is, nonsingular conics meeting at exactly one point—and (possibly) a multiple kernel line:



They are item (12) in the classification in [3], and are studied enumeratively in [2], §4.1.

2.4. **Details for types IV and V.** *Type IV:* Let $p = \operatorname{im}\alpha(0)$ be a singular or inflection point of the support of $\mathscr{C}$; choose a line in the tangent cone to $\mathscr{C}$ at p, and choose coordinates $(x : y : z)$ as before, so that $p = (1 : 0 : 0)$ and the selected line in the tangent cone has equation $z = 0$. The *Newton polygon* for $\mathscr{C}$ in the chosen coordinates is the boundary of the convex hull of the union of the positive quadrants with origin at the points (j, k) for which the coefficient of $x^i y^j z^k$ in the generator F for the ideal of $\mathscr{C}$ is nonzero (see [7], p. 380). The part of the Newton polygon consisting of line segments with slope strictly between -1 and 0 does not depend on the choice of coordinates fixing the flag $z = 0$, $p = (1 : 0 : 0)$.

The limit curves are then obtained by choosing a side of the polygon with slope strictly between -1 and 0, and setting to 0 the coefficients of the monomials in F *not* on that side. These curves are studied in [2]; typically, they consist of a union of cuspidal curves. The kernel line is part of the distinguished triangle of such a curve, and in fact it must be one of the distinguished tangents.

Type V: Let $p = \operatorname{im}\alpha(0)$ be a singular point of the support of $\mathscr{C}$, and let m be the multiplicity of $\mathscr{C}$ at p. Again choose a line in the tangent cone to $\mathscr{C}$ at p, and choose coordinates $(x : y : z)$ so that $p = (1 : 0 : 0)$ and $z = 0$ is the selected line.

We may describe $\mathscr{C}$ near p as the union of m 'formal branches', cf. §3.9; those that are tangent to the line $z = 0$ (but not equal to it) may be written

$$z = f(y) = \sum_{i\geq 0} \gamma_{\lambda_i} y^{\lambda_i}$$

with $\lambda_i \in \mathbb{Q}$, $1 < \lambda_0 < \lambda_1 < \ldots$, and $\gamma_{\lambda_0} \neq 0$.

The choices made above determine a finite set of rational numbers, which we call the 'characteristics' for $\mathscr{C}$ (w.r.t. p and the line $z = 0$): these are the numbers C for which there exist two branches $\mathscr{B}$, $\mathscr{B}'$ tangent to $z = 0$ that agree modulo y^C, differ at y^C, and have $\lambda_0 < C$.

Let S be the number of branches that agree with $\mathscr{B}$ (and $\mathscr{B}'$) modulo y^C. The initial exponents λ_0 and the coefficients γ_{λ_0}, $\gamma_{\frac{\lambda_0+C}{2}}$ for these S branches agree. Let $\gamma_C^{(1)}, \ldots, \gamma_C^{(S)}$ be the coefficients of y^C in these branches (so that at least two of these numbers are distinct, by the choice of C). Then the limit is defined by

$$x^{d-2S} \prod_{i=1}^{S} \left(zx - \frac{\lambda_0(\lambda_0 - 1)}{2} \gamma_{\lambda_0} y^2 - \frac{\lambda_0 + C}{2} \gamma_{\frac{\lambda_0+C}{2}} yx - \gamma_C^{(i)} x^2 \right) \quad .$$

This is a union of quadritangent conics with (possibly) a multiple of the distinguished tangent, which must be supported on the kernel line.

2.5. **The main theorem, and the structure of its proof.** Simple dimension counts show that, for each type as listed in §2.3, the union of the orbits of the marker centers is a set of dimension 7 in $\widetilde{\mathbb{P}}^8 \subset \mathbb{P}^8 \times \mathbb{P}^N$; hence it is a dense set in a component of the PNC. In fact, marker centers of type I, III, IV, and V have 7-dimensional orbit, so the corresponding components of the PNC are the orbit closures of these points.

Type II marker centers are points $(\alpha, \mathscr{X}) \in \mathbb{P}^8 \times \mathbb{P}^N$, where α is a rank-1 matrix whose image is a general point of a nonlinear component of $\mathscr{C}$. The support of $\mathscr{X}$ contains a conic tangent to the kernel line; this gives a 1-parameter family of 6-dimensional orbits in $\mathbb{P}^8 \times \mathbb{P}^N$, accounting for a component of the PNC.

We can now formulate a more precise version of Theorem 1.1:

Theorem 2.4 (Main theorem)**.** *Let $\mathscr{C} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arbitrary plane curve. The marker germs listed in §2.3 determine components of the PNC for $\mathscr{C}$, as explained above. Conversely, all components of the PNC are determined by the marker germs of type I–V listed in §2.3.*

By the considerations in §2.2, this statement implies Theorem 1.1.

The first part of Theorem 2.4 has been established above. In order to prove the second part, we will define a simple notion of 'equivalence' of germs (Definition 3.1), such that, in particular, equivalent germs $\alpha(t)$ lead to the same component of the PNC. We will show that any given germ $\alpha(t)$ centered at a point of $\mathscr{S}$ either is equivalent (after a parameter change, if necessary) to one of the marker germs, or its lift in $\widetilde{\mathbb{P}}^8$ meets the PNC at a point of R (cf. Remark 2.3) or of the boundary of the orbit of a marker center. In the latter cases, the center of the lift varies in a locus of dimension < 7, hence such germs do not contribute components to the PNC. The following lemma allows us to identify easily limits in the intersection of R and the PNC.

Lemma 2.5. *Assume that $\alpha(0)$ has rank 1. If $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is a star with center on $\ker \alpha(0)$, then it is a rank-2 limit.*

Proof. Assume $\mathscr{X} = \lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is a star with center on $\ker \alpha(0)$. We may choose coordinates so that $x = 0$ is the kernel line, and the generator for the ideal of $\mathscr{X}$ is

a polynomial in x, y only. If

$$\alpha(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{pmatrix} \quad ,$$

then $\mathscr{X} = \lim_{t\to 0} \mathscr{C} \circ \beta(t)$ for

$$\beta(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & 0 \\ a_{21}(t) & a_{22}(t) & 0 \\ a_{31}(t) & a_{32}(t) & 0 \end{pmatrix} \quad .$$

Since $\alpha(0)$ has rank 1 and kernel line $x = 0$,

$$\alpha(0) = \begin{pmatrix} a_{11}(0) & 0 & 0 \\ a_{21}(0) & 0 & 0 \\ a_{31}(0) & 0 & 0 \end{pmatrix} = \beta(0) \quad .$$

Now $\beta(t)$ is contained in the rank-2 locus, verifying the assertion. □

A limit $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ as in this lemma will be called a 'kernel star'.

The rest of this paper consists of the successive reductions bringing a given germ $\alpha(t)$ centered at a point of $\mathscr{S}$ into one of the forms given in §2.3, or establishing that it does not contribute a component of the PNC. This will conclude the proof of Theorem 2.4.

3. Proof of the main theorem

3.1. **Germs centered at points of $\mathscr{S}$.** In this section we show that, for a given curve $\mathscr{C}$, any germ $\alpha(t)$ contributing to the PNC is 'equivalent' (up to a coordinate and parameter change, if necessary) to a marker germ as listed in §2.3. As noted in §2, we may assume that $\det \alpha(t) \not\equiv 0$, and that the image of $\alpha(0)$ is contained in $\mathscr{C}$.

Observe that if the center $\alpha(0)$ has rank 2 and is a point of $\mathscr{S}$, then $\alpha(t)$ is already of the form given in §2.3, Type I; it is easy to verify that the limit is then as stated there. This identifies completely components of type I. Thus, we will assume in most of what follows that $\alpha(0)$ has rank 1, and its image is a point of $\mathscr{C}$.

3.2. **Equivalence of germs.** We begin by making precise the notion of 'equivalence' of germs.

Definition 3.1. Two germs $\alpha(t)$, $\beta(t)$ are *equivalent* if $\beta(t\nu(t)) \equiv \alpha(t) \circ m(t)$, with $\nu(t)$ a unit in $\mathbb{C}[[t]]$, and $m(t)$ a germ such that $m(0) = I$ (the identity).

For example: if $n(t)$ is a $\mathbb{C}[[t]]$-valued point of PGL(3), then $\alpha(t) \circ n(t)$ is equivalent to $\alpha(t) \circ n(0)$. We will frequently encounter this situation.

The proof of the following lemma is essentially immediate, and left to the reader.

Lemma 3.2. *Let $\mathscr{C}$ be any plane curve, with defining homogeneous ideal $(F(x, y, z))$. If $\alpha(t)$, $\beta(t)$ are equivalent germs, then the initial terms in $F \circ \alpha(t)$, $F \circ \beta(t)$ coincide up to a nonzero multiplicative constant; in particular, the limits $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$, $\lim_{t\to 0} \mathscr{C} \circ \beta(t)$ are equal.*

If α and β are equivalent germs, note that $\alpha(0) = \beta(0)$; by Lemma 3.2 it follows that, for every curve $\mathscr{C}$, α and β lift to germs in $\widetilde{\mathbb{P}}^8$ centered at the same point.

3.3. **Summary of the argument.** The general plan for the rest of this section is as follows: we will show that every contributing $\alpha(t)$ centered at a rank-1 matrix is equivalent (in suitable coordinates, and possibly up to a parameter change) to one of the form

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix} \quad ,$$

where $b \le c$ resp. $a < b \le c$ are positive integers, $z = f(y)$ is a formal branch for $\mathscr{C}$ at $(1:0:0)$, and $\underline{\cdots}$ denotes the truncation modulo t^c (cf. §2.3 and §2.4).

The main theorem will follow from further analyses of these forms, identifying which do *not* contribute components to the PNC, and leading to the restrictions explained in §2.3 and §2.4. Specifically, the germ on the left leads to components of type II, III, and IV (§3.7); the one on the right will require a subtler study, performed in §3.10–§3.15, leading to the definition of 'characteristics' and to the description given in §2.4 (cf. Proposition 3.22).

3.4. **Linear algebra.** We begin with some elementary linear algebra over $\mathbb{C}[[t]]$: we will show that every germ as specified in §3.1 is equivalent to one which, up to a parameter change, has matrix representation

$$\begin{pmatrix} 1 & 0 & 0 \\ q(t) & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

in suitable coordinates, with $1 \le b \le c$ and q, r, and s polynomials satisfying certain conditions.

In the following considerations we will deal with 3×3 matrices with entries in $\mathbb{C}[[t]]$, that is, $\mathbb{C}[[t]]$-valued points of $\mathrm{Hom}(V, W)$, for V, W 3-dimensional complex vector spaces with chosen bases.

Every such matrix $\alpha(t)$ determines a germ in $\mathbb{P}^8$. A generator F of the ideal of $\mathscr{C}$ will be viewed as an element of $\mathrm{Sym}^d W^*$, for $d = \deg \mathscr{C}$; the composition $F \circ \alpha(t)$, a $\mathbb{C}[[t]]$-valued point of $\mathrm{Sym}^d V^*$, generates the ideal of $\mathscr{C} \circ \alpha(t)$.

Matrices of the form

$$\lambda(t) = \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}$$

will play an important role in our study. We will call them '1-PS', as they correspond to 1-parameter subgroups of PGL(3), that is, homomorphisms λ from $\mathbb{C}^*$ to PGL(3).

We will say that two matrices $\alpha(t)$, $\beta(t)$ are equivalent if the corresponding germs are equivalent in the sense of Definition 3.1.

The following lemma will allow us to simplify matrix expressions of germs up to equivalence. Define the degree of the zero polynomial to be $-\infty$.

Lemma 3.3. *Let*

$$h_1(t) = \begin{pmatrix} u_1 & b_1 & c_1 \\ a_2 & u_2 & c_2 \\ a_3 & b_3 & u_3 \end{pmatrix}$$

be a matrix with entries in $\mathbb{C}[[t]]$*, such that* $h_1(0) = I$*, and let* $a \leq b \leq c$ *be integers. Then* $h_1(t)$ *can be written as a product* $h_1(t) = h(t) \cdot j(t)$*, with*

$$h(t) = \begin{pmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ r & s & 1 \end{pmatrix} \quad , \quad j(t) = \begin{pmatrix} v_1 & e_1 & f_1 \\ d_2 & v_2 & f_2 \\ d_3 & e_3 & v_3 \end{pmatrix}$$

where q*,* r*,* s *are* polynomials*, satisfying*

(1) $h(0) = j(0) = I$*;*
(2) $\deg(q) < b - a$*,* $\deg(r) < c - a$*,* $\deg(s) < c - b$*;*
(3) $d_2 \equiv 0 \pmod{t^{b-a}}$*,* $d_3 \equiv 0 \pmod{t^{c-a}}$*,* $e_3 \equiv 0 \pmod{t^{c-b}}$*.*

Proof. Necessarily $v_1 = u_1, e_1 = b_1$ and $f_1 = c_1$. Use division with remainder to write

$$v_1^{-1} a_2 = D_2 t^{b-a} + q$$

with $\deg(q) < b - a$, and let $d_2 = v_1 D_2 t^{b-a}$ (so that $qv_1 + d_2 = a_2$). This defines q and d_2, and uniquely determines v_2 and f_2. (Note that $q(0) = d_2(0) = f_2(0) = 0$ and that $v_2(0) = 1$.)

Similarly, we let r be the remainder of

$$(v_1 v_2 - e_1 d_2)^{-1}(v_2 a_3 - d_2 b_3)$$

after division by t^{c-a}; and s be the remainder of

$$(v_1 v_2 - e_1 d_2)^{-1}(v_1 b_3 - e_1 a_3)$$

after division by t^{c-b}.

Then $\deg(r) < c - a$, $\deg(s) < c - b$ and $r(0) = s(0) = 0$; moreover, we have

$$v_1 r + d_2 s \equiv a_3 \pmod{t^{c-a}}, \qquad e_1 r + v_2 s \equiv b_3 \pmod{t^{c-b}},$$

so we take $d_3 = a_3 - v_1 r - d_2 s$ and $e_3 = b_3 - e_1 r - v_2 s$. This defines r, s, d_3 and e_3, and uniquely determines v_3. □

Corollary 3.4. *Let* $h_1(t)$ *be a matrix with entries in* $\mathbb{C}[[t]]$*, such that* $h_1(0) = I$*, and let* $a \leq b \leq c$ *be integers. Then there exists a constant invertible matrix* L *such that the product*

$$h_1(t) \cdot \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}$$

is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ r & s & 1 \end{pmatrix} \cdot \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \cdot L$$

where q*,* r*,* s *are polynomials such that* $\deg(q) < b-a$*,* $\deg(r) < c-a$*,* $\deg(s) < c-b$*, and* $q(0) = r(0) = s(0) = 0$*.*

Proof. With notation as in Lemma 3.3 we have

$$j(t)\cdot\begin{pmatrix} t^a & 0 & 0\\ 0 & t^b & 0\\ 0 & 0 & t^c\end{pmatrix} = \begin{pmatrix} v_1t^a & e_1t^b & f_1t^c\\ d_2t^a & v_2t^b & f_2t^c\\ d_3t^a & e_3t^b & v_3t^c\end{pmatrix} = \begin{pmatrix} t^a & 0 & 0\\ 0 & t^b & 0\\ 0 & 0 & t^c\end{pmatrix}\cdot\ell(t) \quad ,$$

with

$$\ell(t) = \begin{pmatrix} v_1 & e_1t^{b-a} & f_1t^{c-a}\\ d_2t^{a-b} & v_2 & f_2t^{c-b}\\ d_3t^{a-c} & e_3t^{b-c} & v_3\end{pmatrix} \quad .$$

By (3) in Lemma 3.3, $\ell(t)$ has entries in $\mathbb{C}[[t]]$ and is invertible; in fact, $L = \ell(0)$ is lower triangular, with 1's on the diagonal.

Therefore Lemma 3.3 gives

$$h_1(t)\cdot\begin{pmatrix} t^a & 0 & 0\\ 0 & t^b & 0\\ 0 & 0 & t^c\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0\\ q & 1 & 0\\ r & s & 1\end{pmatrix}\cdot\begin{pmatrix} t^a & 0 & 0\\ 0 & t^b & 0\\ 0 & 0 & t^c\end{pmatrix}\cdot\ell(t) \quad ,$$

which is equivalent to

$$\begin{pmatrix} 1 & 0 & 0\\ q & 1 & 0\\ r & s & 1\end{pmatrix}\cdot\begin{pmatrix} t^a & 0 & 0\\ 0 & t^b & 0\\ 0 & 0 & t^c\end{pmatrix}\cdot L \quad ,$$

as needed. □

The gist of this result is that, up to equivalence, matrices 'to the left of a 1-PS' and centered at the identity may be assumed to be lower triangular, and to have polynomial entries, with controlled degrees.

3.5. **Key reduction.** The following statement is an easy consequence of Corollary 3.4, and is the starting point of our classification of components of the PNC into 'types'.

We denote by v the 'valuation' of a power series or polynomial, that is, its order of vanishing at 0; we define $v(0)$ to be $+\infty$.

Proposition 3.5. *Let $\alpha(t)$ be a 3×3 matrix with entries in $\mathbb{C}[[t]]$, such that $\alpha(0)\neq 0$ and $\det\alpha(t)\not\equiv 0$. Then there exist constant invertible matrices H, M such that $\alpha(t)$ is equivalent to*

$$\beta(t) = H\cdot\begin{pmatrix} 1 & 0 & 0\\ q & 1 & 0\\ r & s & 1\end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0\\ 0 & t^b & 0\\ 0 & 0 & t^c\end{pmatrix}\cdot M \quad ,$$

with

- *$b\leq c$ nonnegative integers, q, r, s polynomials;*
- *$\deg(q) < b$, $\deg(r) < c$, $\deg(s) < c-b$;*
- *$q(0) = r(0) = s(0) = 0$.*

If, further, $b = c$ and q, r are not both zero, then we may assume that $v(q) < v(r)$.

Finally, if $q(t)\not\equiv 0$ then we may choose $q(t) = t^a$, with $a = v(q) < b$ (and thus $a < v(r)$ if $b = c$).

Proof. By standard diagonalization of matrices over Euclidean domains, every $\alpha(t)$ as in the statement can be written as a product

$$h_0(t) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \cdot k(t) \quad ,$$

where $b \leq c$ are nonnegative integers, and $h_0(t)$, $k(t)$ are invertible (over $\mathbb{C}[[t]]$). Letting $H = h_0(0)$, $h_1(t) = H^{-1} \cdot h_0(t)$, and $K = k(0)$, this shows that $\alpha(t)$ is equivalent to

$$H \cdot h_1(t) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \cdot K$$

with $h_1(0) = I$, and K constant and invertible.

By Corollary 3.4, this matrix is equivalent to

$$\beta(t) = H \cdot \begin{pmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ r & s & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \cdot L \cdot K$$

with L invertible, and q, r, s polynomials satisfying the needed conditions. Letting $M = L \cdot K$ gives the statement in the case $b < c$.

If $b = c$, then the condition that $\deg s < c - b = 0$ forces $s \equiv 0$. Conjugating by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

interchanges q and r; so we may assume $v(q) \leq v(r)$ (if q and r are not both 0) up to conjugation. Conjugating by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u & 1 \end{pmatrix}$$

replaces r by $uq + r$, allowing us to obtain $v(q) < v(r)$. That is, there is an invertible matrix C such that $\alpha(t)$ is equivalent to

$$(H \cdot C^{-1}) \cdot \begin{pmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ r & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^b \end{pmatrix} \cdot (C \cdot M) \quad ,$$

with $v(q) < v(r)$. Absorbing C^{-1} into H and C into M completes the proof of the case $b = c$.

If $q(t) \not\equiv 0$ has valuation a, then we can extract its a-th root as a power series. It follows that there exists a unit $\nu(t) \in \mathbb{C}[[t]]$ such that $q(t\nu(t)) = t^a$. Therefore,

$$\beta(t\nu(t)) = H \cdot \begin{pmatrix} 1 & 0 & 0 \\ t^a & 1 & 0 \\ r(t\nu(t)) & s(t\nu(t)) & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu(t)^b & 0 \\ 0 & 0 & \nu(t)^c \end{pmatrix} \cdot M \quad .$$

Another application of Corollary 3.4 allows us to truncate the power series $r(t\nu(t))$, $s(t\nu(t))$ to obtain polynomials $\underline{r}$, $\underline{s}$ satisfying the same conditions as r, s, at the price

of multiplying to the right of the 1-PS by a constant invertible matrix $\underline{K}$: that is, $\beta(t\nu(t))$ (and hence $\alpha(t)$) is equivalent to

$$H \cdot \begin{pmatrix} 1 & 0 & 0 \\ t^a & 1 & 0 \\ \underline{r} & \underline{s} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \cdot \left[\underline{K} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu(0)^b & 0 \\ 0 & 0 & \nu(0)^c \end{pmatrix} \cdot M \right] \quad .$$

Renaming $r = \underline{r}$, $s = \underline{s}$, and absorbing the factors on the right into M completes the proof of Proposition 3.5. □

3.6. **Coordinate choice.** The matrices H, M appearing in Proposition 3.5 may be omitted by changing the bases of W and V accordingly. Switching back to considering germs in $\mathbb{P}^8$, we have proved that an arbitrary germ α is equivalent to one that, with a suitable choice of coordinates, may be written as

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ q(t) & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with $0 \le b \le c$ and q, r, and s polynomials satisfying the conditions listed in Proposition 3.5. We may in fact assume that $b > 0$, since we are already reduced to the case in which $\alpha(0)$ is a rank-1 matrix. If $b > 0$, then the center $\alpha(0)$ is the matrix

$$\alpha(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with image the point $(1 : 0 : 0)$ and kernel the line $x = 0$.

Further, if $(1 : 0 : 0)$ is not a point of the curve $\mathscr{C}$ then $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is simply a multiple kernel line with ideal $(x^{\deg \mathscr{C}})$ (cf. Corollary 2.2). Thus we may assume that $p = (1 : 0 : 0)$ is a point of $\mathscr{C}$. *In what follows, we will assume that α is a germ in the standard form given above.*

It will now be convenient to switch to affine coordinates centered at the point $(1 : 0 : 0)$. We write

$$F(1 : y : z) = F_m(y, z) + F_{m+1}(y, z) + \cdots + F_d(y, z) \quad ,$$

with $d = \deg \mathscr{C}$, F_i homogeneous of degree i, and $F_m \neq 0$. Thus, $F_m(y, z)$ generates the ideal of the *tangent cone* of $\mathscr{C}$ at p.

3.7. **Components of type III and IV.** We first consider the case in which $q = r = s = 0$, that is, in which $\alpha(t)$ is itself a 1-PS:

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}$$

with $1 \le b \le c$. Also, we may assume that b and c are coprime: this only amounts to a reparametrization of the germ by $t \mapsto t^{1/gcd(b,c)}$; the new germ is not equivalent to the old one in terms of Definition 3.1, but clearly achieves the same limit.

Germs with $b = c$ $(= 1)$ lead to components of type III, cf. §2.3 (also cf. [4], §2, Fact 4(i)):

Proposition 3.6. *If $q = r = s = 0$ and $b = c$, then $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is a fan consisting of a star projectively equivalent to the tangent cone to $\mathscr{C}$ at p, and of a residual $(d-m)$-fold line supported on* $\ker \alpha$.

Proof. The composition $F \circ \alpha(t)$ is

$$F(x : t^b y : t^b z) = t^{bm} x^{d-m} F_m(y,z) + t^{b(m+1)} x^{d-(m+1)} F_{m+1}(y,z) + \cdots + t^{dm} F_d(y,z) \quad .$$

By definition of limit, $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ has ideal $(x^{d-m} F_m(y,z))$, proving the assertion. □

The case $b < c$ corresponds to the germs of type II and type IV in §2.3. We have to prove that contributing germs of this type are precisely those satisfying the further restrictions specified there: specifically, $-b/c$ must be a slope of one of the Newton polygons for $\mathscr{C}$ at the point. We first show that $z = 0$ must be a component of the tangent cone:

Lemma 3.7. *If $q = r = s = 0$ and $b < c$, and $z = 0$ is not contained in the tangent cone to $\mathscr{C}$ at p, then $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is a rank-2 limit.*

Proof. The condition regarding $z = 0$ translates into $F_m(1,0) \neq 0$. Applying $\alpha(t)$ to F, we find:

$$F(x : t^b y : t^c z) = t^{bm} x^{d-m} F_m(y, t^{c-b} z) + t^{b(m+1)} x^{d-(m+1)} F_{m+1}(y, t^{c-b} z) + \cdots$$

Since $F_m(1,0) \neq 0$, the dominant term on the right-hand-side is $x^{d-m} y^m$. This proves the assertion, by Lemma 2.5. □

Components of the PNC that arise due to 1-PS with $b < c$ may be described in terms of the *Newton polygon* for $\mathscr{C}$ at $(0,0)$ relative to the line $z = 0$, which we may now assume to be part of the tangent cone to $\mathscr{C}$ at p. The Newton polygon for $\mathscr{C}$ in the chosen coordinates is the boundary of the convex hull of the union of the positive quadrants with origin at the points (j,k) for which the coefficient of $x^i y^j z^k$ in the equation for $\mathscr{C}$ is nonzero (see [7], p. 380). The part of the Newton polygon consisting of line segments with slope strictly between -1 and 0 does not depend on the choice of coordinates fixing the flag $z = 0$, $p = (0,0)$.

Proposition 3.8. *Assume $q = r = s = 0$ and $b < c$.*

- *If $-b/c$ is not a slope of the Newton polygon for $\mathscr{C}$, then the limit $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is supported on (at most) three lines; these curves do not contribute components to the PNC.*
- *If $-b/c$ is a slope of a side of the Newton polygon for $\mathscr{C}$, then the ideal of the limit $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is generated by the polynomial obtained by setting to* 0 *the coefficients of the monomials in F* not *on that side. Such polynomials are of the form*

$$G = x^{\overline{q}} y^r z^q \prod_{j=1}^{S} (y^c + \rho_j x^{c-b} z^b) \quad .$$

Proof. For the first assertion, simply note that under the stated hypotheses only one monomial in F is dominant in $F \circ \alpha(t)$; hence, the limit is supported on the union of the coordinate axes. A simple dimension count shows that such limits may span

at most a 6-dimensional locus in $\mathbb{P}^8 \times \mathbb{P}^N$, and it follows that such germs do not contribute a component to the PNC.

For the second assertion, note that the dominant terms in $F \circ \alpha(t)$ are precisely those on the side of the Newton polygon with slope equal to $-b/c$. It is immediate that the resulting polynomial can be factored as stated. □

If the point $p = (1 : 0 : 0)$ is a singular or an inflection point of the support of $\mathscr{C}$, and $b/c \neq 1/2$, we find the type IV germs of §2.3; also cf. [4], §2, Fact 4(ii). The number S of 'cuspidal' factors in G is the number of segments cut out by the integer lattice on the selected side of the Newton polygon. If $b/c = 1/2$, then a dimension count shows that the corresponding limit will contribute a component to the PNC (of type IV) unless it is supported on a conic union (possibly) the kernel line.

If p is a *nonsingular, non-inflectional* point of the support of $\mathscr{C}$, then the Newton polygon consists of a single side with slope $-1/2$; these are the type II germs of §2.3. Also cf. [4], Fact 2(ii).

3.8. **Type V components—outline.** Having dealt with the 1-PS case in the previous section, we may now assume that

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ q(t) & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with the conditions listed in Proposition 3.5, and further *such that q, r, and s do not all vanish identically.*

Our final task is to show that contributing germs of this kind must in fact be of the form specified in §2.3 and §2.4. We will show that a germ $\alpha(t)$ as above leads to a rank-2 limit (and hence does not contribute a component to the PNC) unless $\alpha(t)$ and certain formal branches (cf. [7] and [9], Chapter 6 and 7) of the curve are closely related.

3.9. **Formal branches.** We will in fact express the limit of a curve $\mathscr{C}$ in terms of 'limits' of its formal branches. For lack of a reference we sketch a development of this notion in this subsection.

Choose affine coordinates $(y, z) = (1 : y : z)$ so that $p = (0, 0)$, and let $\Phi(y, z) = F(1 : y : z)$ be the generator for the ideal of $\mathscr{C}$ in these coordinates. Decompose $\Phi(y, z)$ in $\mathbb{C}[[y, z]]$:

$$\Phi(y, z) = \Phi_1(y, z) \cdot \cdots \cdot \Phi_r(y, z)$$

with $\Phi_i(y, z)$ irreducible power series. These define the *irreducible branches* of $\mathscr{C}$ at p. Each Φ_i has a unique tangent line at p; if this tangent line is *not* $y = 0$, by the Weierstrass preparation theorem we may write (up to a unit in $\mathbb{C}[[y, z]]$) Φ_i as a monic polynomial in z with coefficients in $\mathbb{C}[[y]]$, of degree equal to the multiplicity m_i of the branch at p (cf. for example [9], §6.7). If Φ_i *is* tangent to $y = 0$, we may likewise write it as a polynomial in y with coefficients in $\mathbb{C}[[z]]$; *mutatis mutandis,* the discussion which follows applies to this case as well.

Concentrating on the first case, let

$$\Phi_i(y, z) \in \mathbb{C}[[y]][z]$$

be a monic polynomial of degree m_i, defining an irreducible branch of $\mathscr{C}$ at p, not tangent to $y = 0$. Then Φ_i splits (uniquely) as a product of linear factors over the ring $\mathbb{C}[[y^*]]$ of power series with *rational nonnegative* exponents:

$$\Phi_i(y,z) = \prod_{j=1}^{m_i} (z - f_{ij}(y)) \quad ,$$

with each $f_{ij}(y)$ of the form

$$f(y) = \sum_{k\geq 0} \gamma_{\lambda_k} y^{\lambda_k}$$

with $\lambda_k \in \mathbb{Q}$, $1 \leq \lambda_0 < \lambda_1 < \dots$, and $\gamma_{\lambda_k} \neq 0$. We call each such $z = f(y)$ a *formal branch* of $\mathscr{C}$ at p. The branch is *tangent to* $z = 0$ if the dominating exponent λ_0 is > 1. The terms $z - f_{ij}(y)$ in this decomposition are the Puiseux series for $\mathscr{C}$ at p.

Summarizing: if $\mathscr{C}$ has multiplicity m at p then $\mathscr{C}$ splits into m formal branches at p. In §3.10 and ff. we will need to determine $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ as a union of 'limits' of the individual formal branches at p. The difficulty here resides in the fact that we cannot perform an arbitrary 'change of variable' in a power series with fractional exponents. In the case in which we will need to do this, however, $\alpha(t)$ will have the following special form:

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with $a < b \leq c$ positive integers and $r(t)$, $s(t)$ polynomials (satisfying certain restrictions, which are immaterial here). The difficulty we mentioned may be circumvented by the following *ad hoc* definition.

Definition 3.9. The *limit* of a formal branch $z = f(y)$, along a germ $\alpha(t)$ as above, is defined by the dominant term in

$$(r(t) + s(t)t^b y + t^c z) - f(t^a) - f'(t^a)t^b y - f''(t^a)t^{2b}\frac{y^2}{2} - \cdots$$

where $f'(y) = \sum \gamma_{\lambda_k}\lambda_k y^{\lambda_k - 1}$ etc.

By 'dominant term' we mean the coefficient of the lowest power of t after cancellations. This coefficient is a polynomial in y and z, giving the limit of the branch according to our definition.

This definition behaves as expected: that is, the limit of $\mathscr{C}$ is the union of the limits of its individual formal branches. This fact will be used several times in the rest of the paper, and may be formalized as follows.

Lemma 3.10. *Let $\Phi(y,z) \in \mathbb{C}[[y]][z]$ be a monic polynomial,*

$$\Phi(y,z) = \prod_i (z - f_i(y))$$

a decomposition over $\mathbb{C}[[y^*]]$, and let $\alpha(t)$ be as above. Then the dominant term in $\Phi \circ \alpha(t)$ is the product of the limits of the branches $z = f_i(y)$ along α, defined as in Definition 3.9.

The proof of this statement is essentially straightforward, hence we leave it to the interested reader.

3.10. **Marker germs, type V—Proposition 3.11.** We can now begin the analysis sketched in §3.8.

Let m be the multiplicity of $\mathscr{C}$ at $p = (0,0)$. For simplicity, we assume that no branches of $\mathscr{C}$ are tangent to the line $y = 0$, leaving to the reader the necessary adjustments in the presence of such branches. We write the generator F for the ideal of $\mathscr{C}$ as a product of formal branches

$$F = \prod_{i=1}^{m} (z - f_i(y))$$

where each f_i is expressed as a power series with fractional exponents. Among these branches, we will especially focus on the ones that are tangent to the line $z = 0$, which may be written explicitly as

$$z = f(y) = \sum_{i \geq 0} \gamma_{\lambda_i} y^{\lambda_i}$$

with $\lambda_i \in \mathbb{Q}$, $1 < \lambda_0 < \lambda_1 < \ldots$, and $\gamma_{\lambda_i} \neq 0$.

Notation. For $C \in \mathbb{Q}$, we will denote by $f_{(C)}(y)$ the finite sum ('truncation')

$$f_{(C)}(y) = \sum_{\lambda_i < C} \gamma_{\lambda_i} y^{\lambda_i} \quad .$$

For (an understood) $c \in \mathbb{Z}$, we will also write $\underline{g(t)}$ for the truncation of $g(t)$ modulo t^c, so that $\underline{f(t^a)} = f_{(C)}(t^a)$ when $C = \frac{c}{a}$. Note that for all $b > a$ the truncation $\underline{f'(t^a)t^b}$ is determined by b and $\underline{f(t^a)}$ (and hence by $f_{(C)}(y)$ and a, b).

Proposition 3.11. *Let $\alpha(t)$ be as specified in §3.8, and assume that $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is not a rank-2 limit. Then $\mathscr{C}$ has a formal branch $z = f(y)$, tangent to $z = 0$, such that α is equivalent to a germ*

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix} \quad ,$$

with $a < b < c$ positive integers. Further, it is necessary that $\frac{c}{a} \leq \lambda_0 + 2(\frac{b}{a} - 1)$.

The proof of this key reduction requires the analysis of several cases. We will first show that under the hypothesis that $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is not a rank-2 limit we may assume that $q(t) \not\equiv 0$, and this will allow us to replace it with a power of t; next, we will deal with the $b = c$ case; and finally we will see that if $b < c$ and $\alpha(t)$ is not in the stated form, then the limit of *every* branch of $\mathscr{C}$ is a $(0:0:1)$-star. This will imply that the limit of $\mathscr{C}$ is a kernel star in this case, proving the assertion by Lemma 2.5.

We will indicate the precise statements filling this outline, leaving details in the proofs—which essentially amount to case-by-case analyses—to the reader.

3.11. **Reduction to $q \neq 0$.** The first remark is that, under the assumptions that q, r, and s do not all vanish, we may in fact assume that $q(t)$ is not identically zero.

Lemma 3.12. *If $\alpha(t)$ is as in §3.8, and $q = 0$, then $\lim_{t\to 0} \mathcal{C} \circ \alpha(t)$ is a rank-2 limit.*

Proof. Left to the reader. Assume $q = 0$, and study the action of $\alpha(t)$ on individual monomials $x^A y^B z^C$ in the generator F of the ideal of $\mathscr{C}$:

$$m_{ABC} := x^A y^B (r(t)x + s(t)t^b y + t^c z)^C t^{bB} \quad .$$

Identifying dominant terms in m_{ABC} in the various possibilities for the vanishing of r and s always leads to kernel stars, which are rank-2 limits by Lemma 2.5. □

3.12. **Reduction to $b < c$.** By Lemma 3.12 and the last part of Proposition 3.5 we may replace $\alpha(t)$ with an equivalent germ

$$\begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with $a < b \leq c$, and $r(t)$, $s(t)$ polynomials of degree $< c$, $< (c - b)$ respectively and vanishing at $t = 0$.

Next, we have to show that if $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is not a rank-2 limit then $b < c$ and $r(t)$, $s(t)$ are as stated in Proposition 3.11.

Lemma 3.13. *Let $\alpha(t)$ be as above. If $b = c$, then $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is a rank-2 limit.*

Proof. Decompose $F(1 : y : z)$ in $\mathbb{C}[[y, z]]$: write $F(1 : y : z) = G(y, z) \cdot H(y, z)$, where $G(y, z)$ collects the branches that are *not* tangent to $z = 0$.

If $b = c$, then $s = 0$ necessarily:

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r(t) & 0 & t^b \end{pmatrix} \quad ,$$

and further $a < v(r)$ (cf. Proposition 3.5).

Using this fact, the reader will verify that the limits of the branches collected in G are supported on the kernel line $x = 0$. The limit of each (formal) branch collected in $H(y, z)$ may be computed as in Definition 3.9, and is found to be given by a homogeneous equation in x and z only: that is, a $(0 : 1 : 0)$-star.

It follows that the limit of $\mathscr{C}$ is again a kernel star, hence a rank-2 limit by Lemma 2.5. □

By Lemma 3.13, we may now assume that α is parametrized by

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ r(t) & s(t)t^b & t^c \end{pmatrix}$$

with the usual conditions on $r(t)$ and $s(t)$, and further $a < b < c$.

The limit of $\mathscr{C}$ under α is analyzed by studying limits of formal branches.

The behavior of branches that are not tangent to $z = 0$ is given by the following statement.

Lemma 3.14. *Under these assumptions on α, the limits of branches that are not tangent to the line $z = 0$ are necessarily $(0 : 0 : 1)$-stars. Further, if $a < v(r)$ then the limit of such a branch is the kernel line $x = 0$.*

Proof. This is a straightforward application of the definition of limit of formal branches (Definition 3.9). □

3.13. **End of the proof of Proposition 3.11.** To conclude the proof of Proposition 3.11, we only need to consider formal branches that are tangent to $z = 0$:

$$z = f(y) = \sum_{i \geq 0} \gamma_{\lambda_i} y^{\lambda_i}$$

with $1 < \lambda_0 < \lambda_1 < \dots$

Lemma 3.15. *Under the same assumptions on α as in Lemma 3.14, the limit of $z = f(y)$ by α is a $(0 : 0 : 1)$-star unless*

- $r(t) \equiv f(t^a) \pmod{t^c}$;
- $s(t) \equiv f'(t^a) \pmod{t^{c-b}}$.

Proof. The limit of the branch is given by the dominant terms in

$$r(t) + s(t)t^b y + t^c z = f(t^a) + f'(t^a)t^b y + \dots$$

If $r(t) \not\equiv f(t^a) \pmod{t^c}$, then the weight of the branch is necessarily $< c$, so the ideal of the limit is generated by a polynomial in x and y, as needed. The same reasoning applies if $s(t) \not\equiv f'(t^a) \pmod{t^{c-b}}$. □

To verify the condition on $\frac{c}{a}$ stated in Proposition 3.11, note that the limit of the formal branch $z = f(y)$ is now given by the dominant term in

$$r(t) + s(t)t^b y + t^c z = f(t^a) + f'(t^a)t^b y + \frac{f''(t^a)t^{2b}y^2}{2} + \dots :$$

the dominant weight will be less than c (causing the limit to be a $(0 : 0 : 1)$-star) if $c > 2b + v(f''(t^a)) = 2b + a(\lambda_0 - 2)$. The stated condition follows at once, completing the proof of Proposition 3.11.

3.14. **Characterization of type V germs.** In the following, we will replace t by a power of t in the germ obtained in Proposition 3.11, if necessary, in order to ensure that the exponents appearing in its expression are relatively prime integers; the resulting germ determines the same component of the PNC.

In order to complete the characterization of type V germs given in §2.3, we need to determine the possible triples $a < b < c$ yielding germs contributing components of the PNC. This determination is best performed in terms of $B = \frac{b}{a}$ and $C = \frac{c}{a}$. Let

$$z = f(y) = \sum_{i \geq 0} \gamma_{\lambda_i} y^{\lambda_i}$$

with $\lambda_i \in \mathbb{Q}$, $1 < \lambda_0 < \lambda_1 < \dots$, and $\gamma_{\lambda_i} \neq 0$, be a formal branch tangent to $z = 0$. Every choice of such a branch and of a rational number $C = \frac{c}{a} > 1$ determines a truncation

$$f_{(C)}(y) = \sum_{\lambda_i < C} \gamma_{\lambda_i} y^{\lambda_i} \quad .$$

With this notation, the truncation $\underline{f(t^a)}$ equals $f_{(C)}(t^a)$.

The choice of a rational number $B = \frac{b}{a}$ satisfying $1 < B < C$ and $B \geq \frac{C-\lambda_0}{2} + 1$ determines now a germ as prescribed by Proposition 3.11:

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix}$$

(choosing the smallest positive integer a for which the entries of this matrix have integer exponents). Observe that the truncation $\underline{f(t^a)} = f_{(C)}(t^a)$ is identically 0 if and only if $C \leq \lambda_0$. Also observe that $\underline{f'(t^a)t^b}$ is determined by $f_{(C)}(t^a)$, as it equals the truncation to t^c of $(f_{(C)})'(t^a)t^b$.

Proposition 3.16. *If $C \leq \lambda_0$ or $B \neq \frac{C-\lambda_0}{2} + 1$, then $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is a rank-2 limit.*

We deal with the different cases separately.

Lemma 3.17. *If $C \leq \lambda_0$, then $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is a $(0:1:0)$-star.*

Proof. If $C = \frac{c}{a} \leq \lambda_0$, then $f_{(C)}(y) = 0$, so

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \quad .$$

The statement follows by computing the limit of individual formal branches, using Definition 3.9. □

By Lemma 2.5, the limits obtained in Lemma 3.17 are rank-2 limits, so the first part of Proposition 3.16 is proved. As for the second part, the limit of a branch tangent to $z = 0$ depends on whether the branch truncates to $f_{(C)}(y)$ or not. These cases are studied in the next two lemmas. Recall that, by our choice, $B \geq \frac{C-\lambda_0}{2} + 1$.

Lemma 3.18. *Assume $C > \lambda_0$, and let $z = g(y)$ be a formal branch tangent to $z = 0$, such that $g_{(C)}(y) \neq f_{(C)}(y)$. Then the limit of the branch is supported on a kernel line.*

Proof. The limit of the branch is determined by the dominant terms in

$$\underline{f(t^a)} + \underline{f'(t^a)t^b}y + t^c z = g(t^a) + g'(t^a)t^b y + \dots .$$

As the truncations $g_{(C)}$ and $f_{(C)}$ do not agree, the dominant term is independent of z. Under our hypotheses on B and C, it is found to be independent of y as well, as needed. □

Lemma 3.19. *Assume $C > \lambda_0$, and let $z = g(y)$ be a formal branch tangent to $z = 0$, such that $g_{(C)}(y) = f_{(C)}(y)$. Denote by $\gamma_C^{(g)}$ the coefficient of y^C in $g(y)$.*

- *If $B > \frac{C-\lambda_0}{2} + 1$, then the limit of the branch $z = g(y)$ by $\alpha(t)$ is the line*

$$z = (C - B + 1)\gamma_{C-B+1}y + \gamma_C^{(g)} \quad .$$

- *If $B = \frac{C-\lambda_0}{2} + 1$, then the limit of the branch $z = g(y)$ by $\alpha(t)$ is the conic*

$$z = \frac{\lambda_0(\lambda_0 - 1)}{2}\gamma_{\lambda_0} y^2 + \frac{\lambda_0 + C}{2}\gamma_{\frac{\lambda_0+C}{2}} y + \gamma_C^{(g)} \quad .$$

Proof. Rewrite the expansion whose dominant terms give the limit of the branch as:

$$t^c z = (g(t^a) - \underline{f(t^a)}) + (g'(t^a)t^b - \underline{f'(t^a)t^b})y + \frac{g''(t^a)}{2}t^{2b}y^2 + \dots$$

The dominant term has weight $c = Ca$ by our choices; if $B > \frac{C-\lambda_0}{2} + 1$ then the weight of the coefficient of y^2 exceeds c, so it does not survive the limiting process, and the limit is a line. If $B = \frac{C-\lambda_0}{2} + 1$, the term in y^2 is dominant, and the limit is a conic.

The explicit expressions given in the statement are obtained by reading the coefficients of the dominant terms. □

We can now complete the proof of Proposition 3.16:

Lemma 3.20. *Assume $C > \lambda_0$. If $B > \frac{C-\lambda_0}{2} + 1$, then the limit $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ is a rank-2 limit.*

Proof. We will show that the limit is necessarily a kernel star, which gives the statement by Lemma 2.5.

As $B > 1$, the coefficient γ_{C-B+1} is determined by the truncation $f_{(C)}$, and in particular it is the same for all formal branches with that truncation. Since $B > \frac{C-\lambda_0}{2} + 1$, by Lemma 3.19 the branches considered there contribute lines through the fixed point $(0 : 1 : (C-B+1)\gamma_{C-B+1})$. We are done if we check that all other branches contribute a kernel line $x = 0$: and this is implied by Lemma 3.14 for branches that are not tangent to $z = 0$ (note $a < v(r)$ for the germs we are considering), and by Lemma 3.18 for formal branches $z = g(y)$ tangent to $z = 0$ but whose truncation $g_{(C)}$ does not agree with $f_{(C)}$. □

3.15. **Quadritangent conics.** We are ready to complete the proof of Theorem 2.4, by determining the limits of the last contributing germs. These have been reduced to the form listed as type V in §2.3 (up to a coordinate change, and replacing t by a power of t):

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^b & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^b} & t^c \end{pmatrix}$$

for some branch $z = f(y) = \gamma_{\lambda_0} y^{\lambda_0} + \dots$ of $\mathscr{C}$ tangent to $z = 0$ at $p = (0, 0)$, and further satisfying $C > \lambda_0$ and $B = \frac{C-\lambda_0}{2} + 1$ for $B = \frac{b}{a}$, $C = \frac{c}{a}$. Type V components of the PNC will arise depending on the limit $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$, which we now determine.

Lemma 3.21. *If $C > \lambda_0$ and $B = \frac{C-\lambda_0}{2} + 1$, then the limit $\lim_{t\to 0} \mathscr{C} \circ \alpha(t)$ consists of a union of quadritangent conics, with distinguished tangent equal to the kernel line $x = 0$, and of a multiple of the distinguished tangent line.*

Proof. Both γ_{λ_0} and $\gamma_{\frac{\lambda_0+C}{2}}$ are determined by the truncation $f_{(C)}$ (since $C > \lambda_0$); hence the equations of the conics

$$z = \frac{\lambda_0(\lambda_0-1)}{2}\gamma_{\lambda_0}y^2 + \frac{\lambda_0+C}{2}\gamma_{\frac{\lambda_0+C}{2}}y + \gamma_C$$

contributed (according to Lemma 3.19) by different branches with truncation $f_{(C)}$ may only differ in the coefficient γ_C.

It is immediately verified that all such conics are tangent to the kernel line $x = 0$, at the point $(0 : 0 : 1)$, and that any two distinct such conics meet only at the point $(0 : 0 : 1)$; thus they are necessarily quadritangent.

Finally, the branches that do not truncate to $f_{(C)}(y)$ must contribute kernel lines, by Lemmas 3.14 and 3.18. □

The degenerate case in which only one conic arises corresponds to germs not contributing components of the projective normal cone, by dimension considerations. A component is present as soon as there are two or more conics, that is, as soon as two branches contribute distinct conics to the limit.

This leads to the description given in §2.3. We say that a rational number C is 'characteristic' for $\mathscr{C}$ (with respect to $z = 0$) if at least two formal branches of $\mathscr{C}$ (tangent to $z = 0$) have the same nonzero truncation, but different coefficients for y^C.

Proposition 3.22. *The set of characteristic rationals is finite.*

The limit $\lim_{t\to 0}\mathscr{C}\circ\alpha(t)$ *obtained in Lemma 3.21 determines a component of the projective normal cone precisely when* C *is characteristic.*

Proof. If $C \gg 0$, then branches with the same truncation must in fact be identical, hence they cannot differ at y^C, hence C is not characteristic. Since the set of exponents of any branch is discrete, the first assertion follows.

The second assertion follows from Lemma 3.21: if $C > \lambda_0$ and $B = \frac{C-\lambda_0}{2} + 1$, then the limit is a union of a multiple kernel line and conics with equation

$$z = \frac{\lambda_0(\lambda_0-1)}{2}\gamma_{\lambda_0}y^2 + \frac{\lambda_0+C}{2}\gamma_{\frac{\lambda_0+C}{2}}y + \gamma_C \quad :$$

these conics are different precisely when the coefficients γ_C are different, and the statement follows. □

Proposition 3.22 leads to the procedure giving components of type V explained in §2.4 (also cf. [4], §2, Fact 5), concluding the proof of Theorem 2.4.

4. Boundaries of orbits

We have now completed the set-theoretic description of the PNC determined by an arbitrary plane curve $\mathscr{C}$. As we have argued in §2, this yields in particular a description of the boundary of $\mathscr{O}_{\mathscr{C}}$. In this section we include a few remarks aimed at making this description more explicit.

If $\dim \mathscr{O}_{\mathscr{C}} = 8$, then the boundary of $\mathscr{O}_{\mathscr{C}}$ consists of the image of the union of the PNC and of the proper transform R of the complement of PGL(3) in $\mathbb{P}^8$ (cf. Remark 2.3). Curves in the image of R are stars (Corollary 2.2). Curves in the image of the components of the PNC belong to the orbit closures of the limits of the marker

germs listed in §2.3. We have proved that this list is exhaustive; therefore, the boundary of a given curve $\mathscr{C}$ may be determined (up to stars) by identifying the marker germs for $\mathscr{C}$, and taking the union of the orbit closures of the (finitely many) corresponding limits.

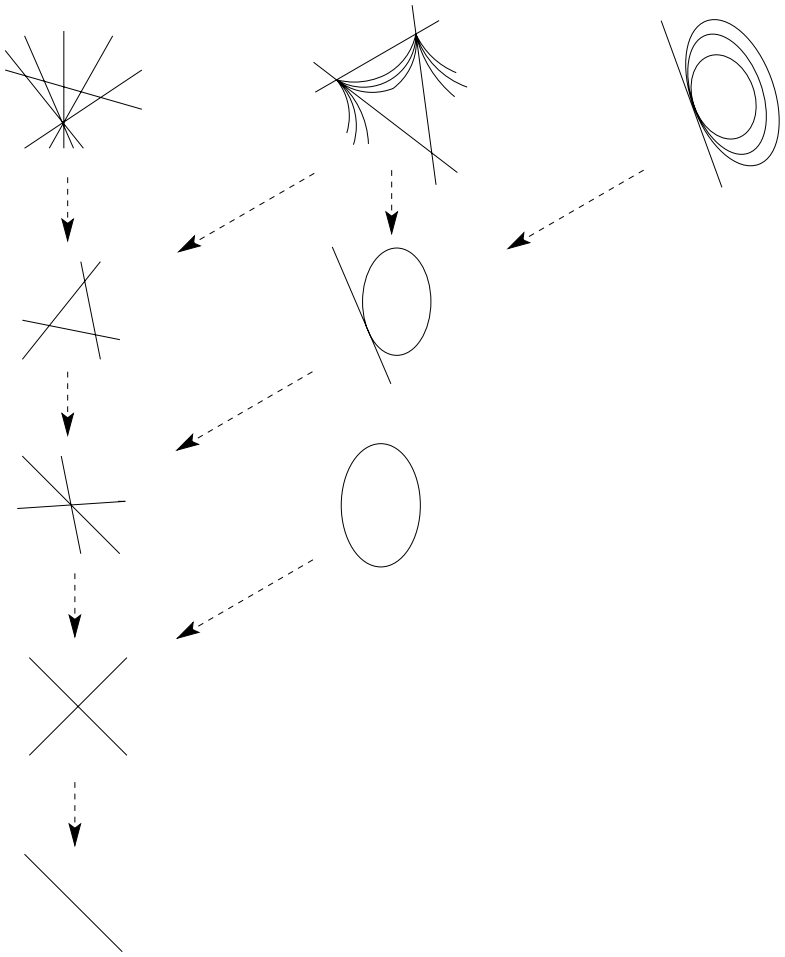
This reduces the determination of the curves in the boundary of the orbit of a given curve to the determination for curves with *small* orbit (i.e., of dimension ≤ 7). We note that, for a curve $\mathscr{C}$ with small orbit, some components of the PNC will in fact dominate $\overline{\mathscr{O}_{\mathscr{C}}}$: indeed, in this case $\mathscr{C}$ has positive dimensional stabilizer in PGL(3); the limit of a germ centered at a singular matrix and otherwise contained in the stabilizer is $\mathscr{C}$ itself. This germ can be chosen to be equivalent to a marker germ, identifying a component of the PNC which dominates the orbit closure.

As mentioned in the introduction, the boundary for a curve with small orbit may be determined by a direct method. Indeed, for such a curve we have constructed in [2] explicit sequences of blow-ups at nonsingular centers which resolve the indeterminacies of the basic rational map, and hence dominate the corresponding graph $\widetilde{\mathbb{P}}^8$. The boundary of the curve may be determined by studying the image in $\mathbb{P}^N$ of the various exceptional divisors of these blow-ups.

The result may be summarized by indicating which types of curves with small orbits are in the boundary of a given curve with small orbit. The diagram shown below expresses part of this relation in terms of representative pictures for curves with small orbits from [3]. The five rows display curves with orbits of dimension 7, 6, 5, 4, 2 respectively. Arrows indicate specialization: for example, the picture indicates that the boundary of the orbit of the union of a conic and a tangent line contains stars, but not single conics. Stars with more than three lines are not displayed, to avoid cluttering the picture; the three kinds of curves displayed in the top row all degenerate to such stars, the only exceptions being the special cases of the second diagram given by the union of a conic and a transversal line, and by a single cuspidal cubic.

The situation illustrated here is precisely what one would expect from naive considerations; it is confirmed by the study of the blow-ups mentioned above. Slightly more refined phenomena (for example, involving multiplicities of the components) are not represented in the picture, but can be easily established by applying the results of this paper or by analyzing the blow-ups of [2].

We close by pointing out one such phenomenon. In general, the union of a set of quadritangent conics and a tangent line can specialize to the union of a conic and a tangent line in two ways: (i) by type II germs aimed at a general point of one of the conic components, and (ii) by a suitable type IV germ aimed at the tangency point. The multiplicity of the conic in the limit is then the multiplicity of the selected component in case (i), and the sum of the multiplicities of all conic components in case (ii). If the curve consists solely of quadritangent conics, it degenerates to a multiple conic in case (ii), a possibility not explicitly drawn here.



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Dept. of Mathematics, Florida State University, Tallahassee FL 32306, U.S.A.
E-mail address: aluffi@math.fsu.edu

Inst. för Matematik, Kungliga Tekniska Högskolan, S-100 44 Stockholm, Sweden

Dept. of Mathematics, The Johns Hopkins University, Baltimore MD 21218, U.S.A.
E-mail address: faber@math.kth.se, faber@math.jhu.edu