

Uniformly dissipative approximations of stationary statistical properties of infinite dimensional dissipative complex systems

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Abstract

We show that a class of numerical schemes, uniformly dissipative approximations, which uniformly preserves the dissipativity of the continuous infinite dimensional dissipative complex (chaotic) systems possess highly desirable properties in terms of approximating stationary statistical properties. In particular, the stationary statistical properties of these uniformly dissipative schemes converge to those of the continuous in time dynamical system at vanishing mesh size. The idea is illustrated on the infinite Prandtl number model for convection and semi-discrete in time discretization although the general strategy works for a broad class of dissipative complex systems and fully discretized approximations. So far as we know, this is the first result on rigorous validation of numerical schemes for approximating stationary statistical properties of general infinite dimensional dissipative complex systems.

1 Introduction

Many dynamical systems arising in applications are dissipative complex systems in the sense that they possess a compact global attractor and the dynamics on the global attractor are complex/chaotic [37]. Well-known examples include the simple Lorenz 63 model, the Navier-Stokes equations at large Reynolds number or Grashoff number, the Boussinesq system for convection at large Rayleigh number, the Kuramoto-Sivashinsky equation at large spatial size, and many models for the atmosphere, ocean, weather and climate etc. The dynamics of these systems are typically very complex/chaotic with generic sensitive dependence on data. Therefore, it is hardly meaningful to discuss long time behavior of a single trajectory for this kind of complex system. Instead, we should study statistical properties of the system since

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they are physically much more relevant than single trajectories[31, 13, 29, 27]. If the system reaches some kind of stationary state, the objects that characterize the stationary statistical properties are the invariant measures or stationary statistical solutions of the system.

With a given complex system, analytical exact expressions for statistical properties are extremely rare just as exact solution formulas are rare for single trajectory. Therefore we naturally turn to numerical methods, especially with today's powerful computers and ever advancing computational technologies. The natural question then is what kind of numerical schemes would provide good approximations for the stationary statistical properties.

In terms of trajectory approximations, we are not aware of any effective long time integrator for dissipative complex/chaotic systems in general unless the long time dynamics is trivial or the trajectory under approximation is stable [19, 15]. It is not at all clear that those numerical methods that provide efficient and accurate approximations of the continuous complex dynamical system on a finite time interval are able to provide meaningful approximation for stationary statistical properties of the system since small errors (truncation and rounding) may accumulate and grow over a long time (think about the usual error estimates with a coefficient that grows exponentially in time). Here we forego the idea of long time fidel approximation of any single trajectory, but ask if it is possible to approximate the mean or statistical properties faithfully. The numerical study of stationary statistical properties of complex system still is a very challenging task since it involves long time integration (so that the statistical averaging is computed utilizing the time averaging under the assumption of ergodicity) and computation of large number of trajectories (if no ergodicity is assumed). As we will demonstrate in this paper, a class of numerical schemes, *uniformly dissipative schemes*, are able to asymptotically capture the stationary statistical properties of the continuous complex/chaotic dynamical system at vanishing mesh size. What we propose here is a class of schemes that preserves the dissipativity of the continuous in time system uniformly (in mesh size). Although these schemes may not approximate individual trajectory faithfully for long time due to accumulation of truncation and rounding errors and abundant instability/chaos, we will show that *stationary statistical properties characterized by the invariant measures (stationary statistical solutions) of the approximations converge to those of the continuous in time system*. This gives us a strong evidence that these uniformly dissipative schemes are the right schemes in investigating statistics.

We will illustrate the idea of uniform dissipativity and convergence of stationary statistical properties on the infinite Prandtl number model for convection and consider semi-discretization in time only although the methodology works for many more complex/chaotic dynamical systems [37] and fully discretized approximations. The choice of the infinite Prandtl number model is both for its physical significance (see the next section) and for the sake of simplicity in exposition.

The idea of uniformly dissipative approximation for dissipative dynamical system is a very natural one. Since the continuous in time dynamical system is dissipative (possess a global attractor), it is natural to consider numerical schemes that preserve the dissipativity in the sense that the solutions to the schemes should possess global attractors that are uniformly compact in some appropriate sense (say the union of the global attractors is pre-compact).

These uniformly dissipative schemes are usually implicit in some way (to ensure long time stability) and therefore have not been very popular in practice. What we shall demonstrate below is that these uniformly dissipative schemes enjoy a highly desirable property in terms of approximating stationary statistical properties: the stationary statistical properties of the schemes converge to those of the continuous in time dynamical systems. We hope that our work will stimulate further study, both analytical and numerical, on approximating statistical properties of dissipative systems.

Earlier works on long time behavior of numerical schemes for dissipative system mainly focused on the two dimensional incompressible Navier-Stokes system (see [15, 21, 24, 34, 38] among others) and the notion of long time stability. The uniform bound in the phase space and a finer/smaller space is called long time stability in these works (some of the authors only derived uniform bound/long time stability in the phase space without other bounds that are necessary for ensuring the dissipativity of the scheme). We prefer the term uniform dissipativity since long time stability could be misleading in the sense that it may imply the scheme is global in time stable in one single phase space only which is not sufficient to ensure the existence of the global attractor. Also, none of the authors discussed stationary statistical properties of their schemes. To the best of our knowledge, our work is the first in establishing the convergence of stationary statistical properties and therefore the usefulness of uniformly dissipative schemes in approximating stationary statistical properties.

The manuscript is organized as follows: we give an introduction in the first section; in section 2 we propose a semi-discrete (discrete in time) scheme for the infinite Prandtl number model and verify that it is uniformly dissipative and enjoys the property that the stationary statistical properties of the scheme converge to those of the continuous in time model; we then provide conclusion and remarks in the third section.

2 A uniformly dissipative scheme for the infinite Prandtl number model

2.1 The infinite Prandtl number model for convection

One of the fundamental systems in fluid dynamics is the *Boussinesq system for Rayleigh-Bénard convection* which is a model for convection, i.e., fluid motion induced by differential heating under Boussinesq approximation ([39, 14]). We assume that the fluids occupy the (non-dimensionalized) region $\Omega = [0, L_x] \times [0, L_y] \times [0, 1]$ with periodicity imposed in the horizontal directions for simplicity.

The Boussinesq system exhibits extremely rich phenomena (see for instance [14, 39], and the recent reviews [4, 35]). In fact, the Boussinesq system is considered a fundamental paradigm for nonlinear dynamics including instabilities and bifurcations, pattern formation, chaotic dynamics and fully developed turbulence [25]. On the other hand, we have very limited mathematical knowledge on the system. Therefore various physically relevant simplifications are highly desirable in order to make progress.

For fluids such as silicone oil or the earth's mantle, the Prandtl number is large, therefore, we may formally set the Prandtl number to infinity in the non-dimensional Boussinesq system, and we arrive at the following (see for instance [4, 6, 8, 16, 39] among others) **infinite Prandtl number model (non-dimensional)**

$$\nabla p = \Delta \mathbf{u} + Ra \mathbf{k}T, \quad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}|_{z=0,1} = 0, \quad (1)$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T, \quad T|_{z=0} = 1, T|_{z=1} = 0 \quad (2)$$

where \mathbf{u} is the Eulerian velocity of the fluid, p represents the kinematic pressure of the fluid, T is the temperature of the fluid, \mathbf{k} is a unit vector in the z direction, and Ra is the Rayleigh number measuring the ratio of differential heating over overall dissipation.

It is well-known that for complex systems such as convection system at large Rayleigh number where turbulent/chaotic behavior abound (see for instance [6, 14, 25, 4, 35]), statistical properties for such systems are much more important and physically relevant than single trajectories [31, 13, 27, 29].

Although there have been extensive works on heat transport in Rayleigh-Bénard convection [1, 4, 6, 7, 16, 35, 22, 23], basic statistical properties of the system such as the heat transport in the vertical direction quantified by the Nusselt number and the mean velocity field are not very well understood. On the other hand, the infinite Prandtl number model is much simpler than the Boussinesq system since the Navier-Stokes equations are replaced by the Stokes equations (and therefore the phase space is that of the temperature only). We also know that the statistics of the infinite Prandtl number model are close to those of the Boussinesq system at large Prandtl number [42, 43, 44]. Therefore it makes sense for us to study some fundamental statistical properties of convection utilizing the simple infinite Prandtl number model since we can generally expect to push to physically more interesting higher Rayleigh number without sacrificing accuracy with the currently available computing resource.

In our case of infinite Prandtl number convection at large Rayleigh number, even the computation on order one time scale (diffusive time scale) is a challenge since it in fact is a long time integration in disguise. To see this, we can rewrite the infinite Prandtl number model as

$$\frac{\partial T}{\partial t} + Ra A^{-1}(\mathbf{k}T) \cdot \nabla T = \Delta T, \quad T|_{z=0} = 1, T|_{z=1} = 0 \quad (3)$$

where A denotes the Stokes operator with viscosity one and the associated boundary conditions. It is then apparent that this is an advection dominated problem (large Péclet number) for large Rayleigh number Ra . We divide both sides of the equation by Ra and introduce the fast time scale $\tau = Ra t$ we may rewrite the infinite Prandtl number model in the following alternative form with an order one advection term

$$\frac{\partial T}{\partial \tau} + A^{-1}(\mathbf{k}T) \cdot \nabla T = \frac{1}{Ra} \Delta T. \quad (4)$$

It appears that the leading order dynamics at large Rayleigh number is the non-local advection equation $\frac{\partial T}{\partial \tau} + A^{-1}(\mathbf{k}T) \cdot \nabla T = 0$. However, this is only valid on order one time scale

for the fast time τ . What we are interested in is order one time scale for the diffusive time t which means long time for the fast time τ (of the order of Ra).

2.2 A semi-discrete in time scheme

In this subsection, we provide a specific semi-discrete in time convergent dissipative scheme for the infinite Prandtl number model. The scheme is semi-implicit and utilizes a background temperature profile. Indeed, consider a generic background temperature profile $\tau(z)$ which satisfies the non-homogeneous Dirichlet boundary condition of T . We introduce the perturbative temperature field $\theta = T - \tau$. The exact form of the background profile τ to be used will be specified later. It is easy to see that θ satisfies the following equation

$$\frac{\partial \theta}{\partial t} + Ra A^{-1}(\mathbf{k}\theta) \cdot \nabla \theta + Ra A^{-1}(\mathbf{k}\theta)_3 \tau'(z) = \Delta \theta + \tau''(z), \quad \theta|_{z=0,1} = 0, \quad (5)$$

and we are searching for solution in the space $H_{0,per}^1$ (the subspace of H^1 with zero trace in the z direction and periodic in the horizontal directions). Here $A^{-1}(\mathbf{k}\theta)_3$ represents the third component (vertical velocity) of $A^{-1}(\mathbf{k}\theta)$.

The *semi-implicit semi-discrete in time scheme* that we propose is given by,

$$\frac{\theta^{n+1} - \theta^n}{k} + Ra A^{-1}(\mathbf{k}\theta^n) \cdot \nabla \theta^{n+1} + Ra A^{-1}(\mathbf{k}\theta^{n+1})_3 \tau'(z) = \Delta \theta^{n+1} + \tau''(z) \quad (6)$$

where θ^n denotes the approximate solution at time kn where k is the time step. A more accurate notation would be θ_k^n to indicate the dependence on the time step k . However, we will suppress the k dependence in the notion for simplicity except in the convergence proof.

Notice that the scheme is linear although the PDE is nonlinear.

Also, we would arrive at a different scheme if we discretize in time first and then apply the translation/background profile (see concluding remarks).

The background temperature profile τ will be taken as a locally smoothed (mollified) version of the following piecewise linear function

$$\tau(z) = \begin{cases} 1 - \frac{z}{2\delta}, & 0 \leq z \leq \delta, \\ \frac{1}{2}, & \delta \leq z \leq 1 - \delta, \\ \frac{1-z}{2\delta} & 1 - \delta \leq z \leq 1. \end{cases} \quad (7)$$

The choice of the parameter δ will be specified later.

We would like to remark here that the typical choice of τ being the conduction state $1 - z$ is not a good one. In fact, the linearized equation is unstable in this case and hence we have to utilize the non-linear term (this is where the new background profile comes into picture) to stabilize the whole system. It is also worthwhile to point out that boundary conditions play an important role in the stabilization process. For instance if we choose $\tau = 1 - z$ (the pure conduction state) and utilize periodicity for the perturbative variables in all three directions (the so-called homogeneous Rayleigh-Bénard convection), the nonlinear system is not stable [5] (look at solutions that are z independent).

2.3 Wellposedness

The well-posedness of the discrete scheme follows from Lax-Milgram theorem [28].

Indeed, the weak formulation of the discrete scheme can be re-written into the form

$$B_n(\theta^{n+1}, \psi) = L_n(\psi) \quad (8)$$

where

$$\begin{aligned} B_n(\theta^{n+1}, \psi) &= \left(\frac{1}{k} \theta^{n+1} + Ra A^{-1}(\mathbf{k}\theta^n) \cdot \nabla \theta^{n+1} + Ra A^{-1}(\mathbf{k}\theta^{n+1})_3 \tau', \psi \right) + (\nabla \theta^{n+1}, \nabla \psi) \\ L_n(\psi) &= -(\tau'(z), \frac{\partial \psi}{\partial z}) + \left(\frac{1}{k} \theta^n, \psi \right). \end{aligned} \quad (10)$$

It is easy to see that B_n is a continuous bilinear form on $H_{0,per}^1 \times H_{0,per}^1$ and L_n is a continuous linear functional on $H_{0,per}^1$. We only need to verify the coercivity for B_n in order to show the solvability thanks to Lax-Milgram theorem.

For this purpose we notice that, thanks to the specific form of the background profile τ (7), the homogeneous boundary conditions for θ^{n+1} , $(A^{-1}(\mathbf{k}\theta^{n+1}))_3$, $\nabla(A^{-1}(\mathbf{k}\theta^{n+1}))_3$, elliptic regularity (for the Stokes operator) and Poincaré inequality, there exists a constant c_1 independent of k, n, Ra , such that,

$$Ra \left| \int_{\Omega} \tau'(z) (A^{-1}(\mathbf{k}\theta^{n+1}))_3 \theta^{n+1} \right| \leq c_1 \delta^2 Ra \|\nabla \theta^{n+1}\|^2 \leq \frac{1}{4} \|\nabla \theta^{n+1}\|^2 \quad (11)$$

provided that we choose¹

$$\delta = (4c_1 Ra)^{-\frac{1}{2}}. \quad (12)$$

Therefore

$$B_n(\theta^{n+1}, \theta^{n+1}) \geq \frac{1}{k} \|\theta^{n+1}\|^2 + \frac{3}{4} \|\nabla \theta^{n+1}\|^2 \quad (13)$$

which proves the coercivity. Here $\|\theta\|$ denotes the spatial L^2 norm of θ .

This ends the proof of the well-posedness of the discrete scheme.

2.4 Uniform dissipativity

Next, we prove the uniform dissipativity. Here and below, c_j s denote generic constants independent of k, n (but may depend on the Rayleigh number).

We first derive a uniform bound in the L^2 space. For this purpose we take the inner product of the scheme with $\psi = \theta^{n+1}$ and utilize the identity $(a-b, b) = \frac{1}{2}(|a|^2 - |b|^2 + |a-b|^2)$

¹The choice of τ or δ given here is not optimal. A near optimal choice would be $\delta \sim Ra^{-\frac{1}{3}}$ but the control on the linear destabilizing term is much longer [8, 11, 44]. We use the simple one since optimal bound is not our goal here.

together with the estimate on the destabilizing term (11), and we have

$$\begin{aligned}
& \frac{1}{2k}(\|\theta^{n+1}\|^2 - \|\theta^n\|^2 + \|\theta^{n+1} - \theta^n\|^2) + \|\nabla\theta^{n+1}\|^2 \\
& \leq \|\tau'\| \|\nabla\theta^{n+1}\| + Ra \int_{\Omega} \tau'(z) (A^{-1}(\mathbf{k}\theta^{n+1}))_3 \theta^{n+1} \\
& \leq \|\tau'\|^2 + \frac{1}{2} \|\nabla\theta^{n+1}\|^2.
\end{aligned}$$

Therefore, there exists a constant c_2 , such that

$$\frac{1}{k}(\|\theta^{n+1}\|^2 - \|\theta^n\|^2 + \|\theta^{n+1} - \theta^n\|^2) + \|\nabla\theta^{n+1}\|^2 \leq 2\|\tau'\|^2 \leq c_2 Ra^{\frac{1}{2}} \quad (14)$$

which further implies, thanks to Poincaré inequality,

$$(1+k)\|\theta^{n+1}\|^2 \leq \|\theta^n\|^2 + c_2 k Ra^{\frac{1}{2}}. \quad (15)$$

This further implies, with the help of a simple iteration,

$$\|\theta^{n+1}\|^2 \leq (1+k)^{-(n+1)} \|\theta_0\|^2 + c_2 Ra^{\frac{1}{2}}. \quad (16)$$

This is a uniform estimate in the L^2 space.

A byproduct of this estimate is that

$$\frac{1}{N} \sum_{n=0}^N \|\nabla\theta^{n+1}\|^2 \leq \frac{\|\theta_0\|^2}{kN} + c_2 Ra^{\frac{1}{2}} \quad (17)$$

which is a bound on the Nusselt number in this discretized case for large N , and a bound in $L^2(H^1)$ for the scheme.

We also have

$$\sum_{n=0}^N \|\theta^{n+1} - \theta^n\|^2 \leq \|\theta_0\|^2 + c_2 k N Ra^{\frac{1}{2}}. \quad (18)$$

In order to obtain uniform estimates in H^1 , we take the inner product of the scheme with $\psi = -\Delta\theta^{n+1}$ and we have

$$\begin{aligned}
& \frac{1}{2k}(\|\nabla\theta^{n+1}\|^2 - \|\nabla\theta^n\|^2 + \|\nabla(\theta^{n+1} - \theta^n)\|^2) + \|\Delta\theta^{n+1}\|^2 \\
& \leq \|\tau''\| \|\Delta\theta^{n+1}\| + Ra \|\tau'\| \|A^{-1}(\mathbf{k}\theta^{n+1})\|_{\infty} \|\Delta\theta^{n+1}\| + Ra \|A^{-1}(\mathbf{k}\theta^n)\|_{\infty} \|\nabla\theta^{n+1}\| \|\Delta\theta^{n+1}\| \\
& \leq \|\tau''\| \|\Delta\theta^{n+1}\| + c_3 Ra \|\tau'\| \|\theta^{n+1}\| \|\Delta\theta^{n+1}\| + c_4 Ra \|\theta^n\| \|\theta^{n+1}\|^{\frac{1}{2}} \|\Delta\theta^{n+1}\|^{\frac{3}{2}} \\
& \leq \frac{1}{2} \|\Delta\theta^{n+1}\|^2 + c_5
\end{aligned}$$

where we have applied the regularity result for the Stokes operator, the Sobolev imbedding of H^2 into L^{∞} , interpolation inequality, the uniform L^2 estimate (16) and Hölder type inequality.

This implies

$$(1+k)\|\nabla\theta^{n+1}\|^2 \leq \|\nabla\theta^n\|^2 + 2c_6k \quad (19)$$

which further implies, with the help of a simple iteration,

$$\|\nabla\theta^{n+1}\|^2 \leq (1+k)^{-n}\|\nabla\theta_1\|^2 + 2c_6. \quad (20)$$

This is the desired uniform estimates in the H^1 space, i.e., there is a uniform in k bounded absorbing ball in H^1 which attracts all solutions with L^2 initial data.

Uniform estimates in Sobolev spaces with more derivatives can be derived just as in the case of continuous in time system. Here we demonstrate that the H^2 norm of the solution is asymptotically uniformly bounded in time, i.e., there is a absorbing ball in H^2 which attracts all solutions with L^2 initial data uniformly for all k .

For this purpose we apply Δ to both sides of the scheme (6) and then multiply the scheme by $\Delta\theta^{n+1}$ and integrate over the domain. This leads to the following

$$\begin{aligned} & \frac{1}{2k}(\|\Delta\theta^{n+1}\|^2 - \|\Delta\theta^n\|^2 + \|\Delta(\theta^{n+1} - \theta^n)\|^2) + \|\nabla\Delta\theta^{n+1}\|^2 \\ \leq & \|\tau^{(4)}\|\|\Delta\theta^{n+1}\| + Ra(\|\Delta(A^{-1}(\mathbf{k}\theta^n))\|_{L^6}\|\nabla\theta^{n+1}\|_{L^3} + 2\|\nabla A^{-1}(\mathbf{k}\theta^n)\|_{L^\infty}\|\nabla^2\theta^{n+1}\|)\|\Delta\theta^{n+1}\| \\ & + Ra(\|\Delta(A^{-1}(\mathbf{k}\theta^{n+1}))\|\|\tau'\|_{L^\infty} + 2\|\nabla(A^{-1}(\mathbf{k}\theta^{n+1}))\|\|\nabla\tau'\|_{L^\infty} + \|A^{-1}(\mathbf{k}\theta^{n+1})\|\|\Delta\tau'\|_{L^\infty})\|\Delta\theta^{n+1}\| \\ \leq & c_7(\|\Delta\theta^{n+1}\| + \|\Delta\theta^{n+1}\|^2) \\ \leq & c_8(\|\Delta\theta^{n+1}\| + \|\nabla\Delta\theta^{n+1}\|\|\nabla\theta^{n+1}\|) \\ \leq & \frac{1}{2}\|\nabla\Delta\theta^{n+1}\|^2 + c_9 \end{aligned}$$

where we have applied the identify $\int A^{-1}(\mathbf{k}\theta^n)\nabla\Delta\theta^{n+1}\Delta\theta^{n+1} = 0$, Hölder's inequality, elliptic regularity, Sobolev imbedding, Cauchy-Schwarz, and interpolation inequality.

This leads to the inequality

$$(1+k)\|\Delta\theta^{n+1}\|^2 \leq \|\Delta\theta^n\|^2 + 2c_9k \quad (21)$$

which further implies, with the help of a simple iteration,

$$\|\Delta\theta^{n+1}\|^2 \leq (1+k)^{-n+1}\|\Delta\theta_2\|^2 + 2c_9. \quad (22)$$

This is the desired uniform estimates in the H^2 space, i.e., there is a uniform in k bounded absorbing ball in H^2 which attracts all solutions with L^2 initial data.

To summarize, we have the following lemma

Lemma 1 (Uniform bound/dissipativity) *There exists a constant c_9 independent of the time step k such that the scheme (6) possess an absorbing ball in H^1 and H^2 with radius $2\sqrt{c_9}$ which attracts all bounded sets in L^2 .*

2.5 Consistency and convergence

Here we check the consistency first since this is what we need in the sequel.

Multiplying the scheme (6) by $k(\theta^{n+1} - \theta^n)$ and integrating over the domain we have

$$\begin{aligned} \|\theta^{n+1} - \theta^n\|^2 &\leq k\left\{-\frac{1}{2}(\|\nabla\theta^{n+1}\|^2 - \|\nabla\theta^n\|^2 + \|\nabla(\theta^{n+1} - \theta^n)\|^2)\right. \\ &\quad \left. + \|\tau'\| \|\nabla(\theta^{n+1} - \theta^n)\| + c_{10}\|\theta^n\|_\infty \|\nabla\theta^{n+1}\| \|\theta^{n+1} - \theta^n\| + c_{11}\|\theta^{n+1}\|_\infty \|\tau'\| \|\theta^{n+1} - \theta^n\|\right\} \\ &\leq k(c_{12} + c_{13}\|\theta^{n+1} - \theta^n\|) \end{aligned}$$

where we have applied Cauchy-Schwarz inequality, Hölder's inequality, elliptic regularity and uniform bounds in H^1 (20).

This implies the following *consistency result*

$$\|\theta^{n+1} - \theta^n\| \leq c_{14}k^{\frac{1}{2}}. \quad (23)$$

provided that $\theta_0 \in H_{0,per}^1$.

If we assume $\theta_0 \in H_{0,per}^1 \cap H^2$, we may deduce from the scheme (6) and the uniform bound (22) the following stronger consistency result

$$\|\theta^{n+1} - \theta^n\| \leq c_{15}k. \quad (24)$$

Next, we show that the solutions to the scheme converge to the solution of the infinite Prandtl number model in $L^2(0, T^*, L^2)$ for any given time $T^* > 0$ as $k \rightarrow 0$.

For this purpose, we rewrite the scheme (6) as

$$\frac{\partial \tilde{\theta}_k(t)}{\partial t} + Ra A^{-1}(\mathbf{k}\theta_k(t)) \cdot \nabla \theta_k(t+k) + Ra A^{-1}(\mathbf{k}\theta_k(t+k))_3 \tau'(z) = \Delta \theta_k(t+k) + \tau''(z), \quad (25)$$

where

$$\theta_k(t) = \theta_k^n, \quad t \in [nk, (n+1)k), \quad (26)$$

$$\tilde{\theta}_k(t) = \theta_k^n + \frac{t - nk}{k}(\theta_k^{n+1} - \theta_k^n), \quad t \in [nk, (n+1)k). \quad (27)$$

The estimates (16, 17) imply that θ_k and $\tilde{\theta}_k$ are uniformly (in k) bounded in $L^\infty(0, T^*; L^2)$ and $L^2(0, T^*; H_{0,per}^1)$. Hence we have a subsequence, still denoted θ_k and $\tilde{\theta}_k$ and $\theta, \tilde{\theta} \in L^\infty(0, T^*; L^2) \cap L^2(0, T^*; H_{0,per}^1)$ such that

$$\theta_k \rightharpoonup \theta, \text{ weak } * \text{ in } L^\infty(0, T^*; L^2), \quad (28)$$

$$\theta_k \rightharpoonup \theta, \text{ weak in } L^2(0, T^*; H_{0,per}^1), \quad (29)$$

$$\tilde{\theta}_k \rightharpoonup \tilde{\theta}, \text{ weak } * \text{ in } L^\infty(0, T^*; L^2), \quad (30)$$

$$\tilde{\theta}_k \rightharpoonup \tilde{\theta}, \text{ weak in } L^2(0, T^*; H_{0,per}^1). \quad (31)$$

It is also easy to check, thanks to (18),

$$\int_0^{T^*-a} \|\theta_k(t+k) - \theta_k(t)\|^2 dt \leq c_{16}k, \quad (32)$$

$$\int_0^{T^*} \|\theta_k(t) - \tilde{\theta}_k(t)\|^2 dt \leq c_{17}k. \quad (33)$$

Therefore

$$\theta = \tilde{\theta}, \quad (34)$$

$$\theta_k(\cdot + k) \rightharpoonup \theta, \text{ weak } * \text{ in } L^\infty(0, T^*; L^2), \quad (35)$$

$$\theta_k(\cdot + k) \rightharpoonup \theta, \text{ weak in } L^2(0, T^*; H_{0,per}^1). \quad (36)$$

Further more, thanks to a compactness theorem due to Témam ([36] Ch. 13, Theorem 13.3), there exists a sub-subsequence of $\tilde{\theta}_k$ which convergence strongly in $L^q(0, T^*; L^2), \forall q \in [1, 2)$. Indeed, testing the scheme (25) against a test function v and integrating from t to $t+a$, we have

$$\begin{aligned} & |(\tilde{\theta}_k(t+a) - \tilde{\theta}_k(t), v)| \\ & \leq \int_t^{t+a} \{ \|\nabla \theta_k(s+k)\| \|\nabla v\| + \|\tau'\| \|\frac{\partial v}{\partial z}\| \\ & \quad + Ra \|A^{-1}(\mathbf{k}\theta_k(s))\|_{L^\infty} \|\theta_k(s+k)\| \|\nabla v\| + Ra \|A^{-1}(\mathbf{k}\theta_k(s+k))_3\|_{L^\infty} \|\tau'\| \|v\| \} ds \\ & \leq c_{18} \|\nabla v\| a^{\frac{1}{2}} \end{aligned} \quad (37)$$

where we have applied the regularity for the Stokes operator, Sobolev imbedding, Poincaré inequality, and the a priori estimates in $L^\infty(L^2)$ and $L^2(H^1)$ valid for L^2 initial data. Now set $v = \tilde{\theta}_k(t+a) - \tilde{\theta}_k(t)$ and utilize the $L^2(H^1)$ estimate on $\tilde{\theta}_k$, we have

$$\int_0^{T^*-a} \|\tilde{\theta}_k(t+a) - \tilde{\theta}_k(t)\|^2 \leq c_{19}a^{\frac{1}{2}}. \quad (38)$$

This implies the strong convergence by Témam's compactness theorem.

Combining this strong convergence in $L^q(L^2), q \in [1, 2)$ with the uniform $L^\infty(0, T^*; L^2)$ estimate, we conclude that the sub-subsequence in fact converges strongly in $L^q(0, T^*; L^2), \forall q \in [1, \infty)$. Hence we may summarize the a priori estimates as

$$\theta_k(\cdot), \theta_k(\cdot + k), \tilde{\theta}_k(\cdot), \tilde{\theta}_k(\cdot + k) \rightarrow \theta(\cdot), \text{ in } L^q(0, T^*; L^2), \forall q \in [1, \infty), \quad (39)$$

$$\theta_k(\cdot), \theta_k(\cdot + k), \tilde{\theta}_k(\cdot), \tilde{\theta}_k(\cdot + k) \rightharpoonup \theta(\cdot), \text{ weakly in } L^2(0, T^*; H_{0,per}^1). \quad (40)$$

Now for any $\phi \in H_{0,per}^1$ and $\psi \in C^1([0, T^*])$ with $\psi(T^*) = 0$, we can rewrite the scheme (25) in the following weak form

$$\begin{aligned} & \int_0^{T^*} \int_\Omega \{ -\tilde{\theta}_k(\mathbf{x}, t) \phi(\mathbf{x}) \psi'(t) + Ra A^{-1}(\mathbf{k}\theta_k(\mathbf{x}, t)) \cdot \nabla \theta_k(\mathbf{x}, t+k) \phi(\mathbf{x}) \psi(t) \\ & \quad + Ra A^{-1}(\mathbf{k}\theta_k(\mathbf{x}, t+k))_3 \tau'(z) \phi(\mathbf{x}) \psi(t) + \nabla \theta_k(\mathbf{x}, t+k) \cdot \nabla \phi(\mathbf{x}) \psi(t) + \tau'(z) \frac{\partial}{\partial z} \phi(\mathbf{x}) \psi(t) \} d\mathbf{x} dt \\ & = \theta_0(\mathbf{x}) \phi(\mathbf{x}) \psi(0). \end{aligned} \quad (41)$$

Utilizing the strong $L^q(L^2)$ convergence (39) and the weak $L^2(H_{0,per}^1)$ convergence (40) together with elliptic regularity, we can pass to the limit as $k \rightarrow 0$ and we arrive at

$$\int_0^{T^*} \int_{\Omega} \{-\theta\phi\psi' + RaA^{-1}(\mathbf{k}\theta) \cdot \nabla\theta\phi\psi + RaA^{-1}(\mathbf{k}\theta)_3\tau'\phi\psi + \nabla\theta \cdot \nabla\phi\psi + \tau' \frac{\partial}{\partial z}\phi\psi\} dxdt = \theta_0\phi\psi(0) \quad (42)$$

which is exactly the weak form of the infinite Prandtl number model. Since the infinite Prandtl number model possesses a unique solution, θ must be the unique solution and hence the whole sequence of θ_k and $\tilde{\theta}_k$ converges to θ as any subsequence has a sub-subsequence that converges to the same limit θ .

We summarize the result here as

Lemma 2 (Consistency and convergence) *For any given $T^* > 0$ and $\theta_0 \in L^2$, the solution to the numerical scheme (25) converges to the solution of the infinite Prandtl number model, i.e.,*

$$\theta_k(\cdot), \theta_k(\cdot + k), \tilde{\theta}_k(\cdot), \tilde{\theta}_k(\cdot + k) \rightarrow \theta(\cdot), \text{ in } L^q(0, T^*; L^2), \forall q \in [1, \infty), \quad (43)$$

$$\theta_k(\cdot), \theta_k(\cdot + k), \tilde{\theta}_k(\cdot), \tilde{\theta}_k(\cdot + k) \rightharpoonup \theta(\cdot), \text{ weakly in } L^2(0, T^*; H_{0,per}^1) \quad (44)$$

where θ is the unique solution to the infinite Prandtl number model.

Moreover, if $\theta_0 \in H_{0,per}^1 \cap H^2$, then there exists a generic constant c_{15} independent of k, n such that

$$\|\theta^{n+1} - \theta^n\| \leq c_{15}k. \quad (45)$$

For a given (fixed) finite interval, the uniform in time convergence ($L^\infty(L^2)$) of the scheme can be established easily if we assume all the compatibility condition needed (so that the exact solution is smooth enough up to the initial time $t = 0$, see [36] for the case of Navier-Stokes equations). If no high order compatibility condition is assumed, one can show the uniform in time convergence on any finite interval modulus an initial layer (see [18] for the case of Navier-Stokes equations). Uniform convergence without enough compatibility conditions is needed for the proof of convergence of the global attractors (see the next section).

2.6 Convergence of the stationary statistical properties

As we mentioned earlier, for complex systems with chaotic/turbulent behavior, statistical properties are much more important than individual trajectories. In fact it is essentially hopeless to try to find approximation schemes that possess the property that the approximate trajectory remain close to the "true" trajectory for all time due to abundant sensitive dependence on data and positive Lyapunov exponents.² Therefore, the natural question to ask is if stationary statistical properties are well approximated. These stationary statistical properties are characterized by stationary statistical solutions or invariant measures of the

²There is a notable exception when the system possess explicit hyperbolic structure for which numerical shadowing may be possible [33]

system. Hence the question that we ask here is if the invariant measures of the discrete time approximation approximate the invariant measures of the continuous in time infinite Prandtl number model.

We first observe that the numerical scheme (6) can be viewed as a *discrete time dynamical system* on the phase space L^2 with the notation

$$\theta^{n+1} = F_k(\theta^n). \quad (46)$$

Thanks to the wellposedness result we see that F_k in fact maps L^2 into $H_{0,per}^1$, and F_k^2 maps L^2 into $H_{0,per}^1 \cap H^2$ by elliptic regularity. Moreover, the discrete dynamical system is uniformly (in k) dissipative thanks to the uniform H^1 estimate (20). Therefore, these dynamical system possess a compact global attractor in H^1 which attracts all bounded sets in L^2 . This leads to the existence of invariant measures via a classical Krylov-Bogliubov argument [40, 13] for the numerical scheme (the discrete dynamical system).

We recall the definition of invariant measures.

Definition 1 (Invariant measures) *A Borel probability measure μ_k on L^2 is called an invariant measure for F_k if*

$$\int_{L^2} \Phi(F_k(\theta)) d\mu_k = \int_{L^2} \Phi(\theta) d\mu_k \quad (47)$$

for all bounded continuous test functional Φ .

The set of all invariant measures for F_k is denoted \mathcal{IM}_k .

We also recall that a Borel probability measure μ on L^2 is an invariant measure, or stationary statistical solution for the infinite Prandtl number model for convection, if

1.
$$\int_{L^2} \|\nabla\theta\|^2 d\mu(\theta) < \infty, \quad (48)$$

- 2.

$$\int_{L^2} \langle -Ra A^{-1}(\mathbf{k}\theta) \cdot \nabla\theta - Ra A^{-1}(\mathbf{k}\theta)_3 \tau'(z) + \Delta\theta + \tau''(z), \Phi'(\theta) \rangle d\mu(\theta) = 0 \quad (49)$$

for any cylindrical test functional $\Phi(\theta) = \phi((\theta, w_1), \dots, (\theta, w_m))$ where ϕ is a C^1 function on R^m , $\{w_j, j \geq 1\}$ are the eigenfunctions of Δ which form an ortho-normal basis for L^2 and $w_j \in H_{0,per}^1 \cap C^2, \forall j$, and \langle, \rangle denotes the $H^{-1}, H_{0,per}^1$ duality.

- 3.

$$\int_{L^2} \int_{\Omega} \{|\nabla\theta|^2 + Ra(A^{-1}(\mathbf{k}\theta))_3 \theta \tau' - \tau'' \theta\} d\mathbf{x} d\mu(\theta) \leq 0. \quad (50)$$

The set of all stationary statistical solutions for the infinite Prandtl number model is denoted \mathcal{IM} .

Roughly speaking, the first condition says that the invariant measures are supported on the smaller and finer space of H^1 , the second condition is the differential form of the weak formulation of the invariance of the measure under the flow, and the third condition is a statistical version of the energy inequality.

Now let $\mu_k \in \mathcal{IM}_k$ be a sequence of invariant measures. Thanks to the uniform estimate in H^1 (20), we see that the support of μ_k is contained in a bounded ball in H^1 independent of k . Therefore, thanks to the Prokhorov compactness theorem and Rellich compactness theorem [3, 28], the sequence μ_k is weakly pre-compact in the set of all Borel probability measures on L^2 and hence it must contain a weakly convergent subsequence (still denoted $\{\mu_k\}$) which converges to a Borel probability measure μ . Our goal is to show that μ must be an invariant measure of the infinite Prandtl number model.

The first condition in the definition is easily verified since the global attractors for the discrete dynamical systems are uniformly bounded in H^1 independent of the time step k , and the invariant measures are supported on the global attractor.

In order to check the second condition, i.e., the differential form of the weak formulation of invariance, we let $\Phi(\theta) = \phi((\theta, w_1), \dots, (\theta, w_m)) = \phi(y_1, \dots, y_m)$ be a cylindrical test functional. Notice that

$$\Phi'(\theta) = \sum_{j=1}^m \frac{\partial}{\partial y_j} \phi((\theta, w_1), \dots, (\theta, w_m)) w_j, \quad (51)$$

hence, denoting \langle, \rangle the duality between H^{-1} and $H_{0,per}^1$, we have

$$\begin{aligned} & \int_{L^2} \langle -Ra A^{-1}(\mathbf{k}\theta) \cdot \nabla \theta - Ra A^{-1}(\mathbf{k}\theta)_3 \tau'(z) + \Delta \theta + \tau''(z), \Phi'(\theta) \rangle d\mu(\theta) \\ &= \int_{L^2} \langle -Ra A^{-1}(\mathbf{k}\theta) \cdot \nabla \theta - Ra A^{-1}(\mathbf{k}\theta)_3 \tau'(z) + \Delta \theta + \tau''(z), \sum_{j=1}^m \frac{\partial \phi}{\partial y_j}((\theta, w_1), \dots, (\theta, w_m)) w_j \rangle d\mu(\theta) \\ &= \int_{L^2} \sum_{j=1}^m \frac{\partial \phi}{\partial y_j} \int_{\Omega} (Ra A^{-1}(\mathbf{k}\theta) \cdot \nabla w_j \theta - Ra A^{-1}(\mathbf{k}\theta)_3 \tau'(z) w_j + \Delta w_j \theta + \tau''(z) w_j) d\mathbf{x} d\mu(\theta) \\ &= \lim_{k \rightarrow 0} \int_{L^2} \sum_{j=1}^m \frac{\partial \phi}{\partial y_j} \int_{\Omega} (Ra A^{-1}(\mathbf{k}\theta) \cdot \nabla w_j \theta - Ra A^{-1}(\mathbf{k}\theta)_3 \tau'(z) w_j + \Delta w_j \theta + \tau''(z) w_j) d\mathbf{x} d\mu_k(\theta) \\ &= \lim_{k \rightarrow 0} \int_{L^2} \sum_{j=1}^m \frac{\partial \phi}{\partial y_j} \int_{\Omega} (Ra A^{-1}(\mathbf{k}\theta) \cdot \nabla w_j F_k(\theta) - Ra A^{-1}(\mathbf{k}F_k(\theta))_3 \tau'(z) w_j + \Delta w_j F_k(\theta) + \tau''(z) w_j) d\mathbf{x} d\mu_k(\theta) \\ &= \lim_{k \rightarrow 0} \int_{L^2} \langle -Ra A^{-1}(\mathbf{k}\theta) \cdot \nabla F_k(\theta) - Ra A^{-1}(\mathbf{k}F_k(\theta))_3 \tau'(z) + \Delta F_k(\theta) + \tau''(z), \Phi'(\theta) \rangle d\mu_k(\theta) \\ &= \lim_{k \rightarrow 0} \int_{L^2} \langle \frac{F_k(\theta) - \theta}{k}, \Phi'(\theta) \rangle d\mu_k(\theta) \\ &= \lim_{k \rightarrow 0} \int_{L^2} \frac{1}{k} (\Phi(F_k(\theta)) - \Phi(\theta)) d\mu_k(\theta) \\ &= 0 \end{aligned}$$

where we have used the boundedness of $\frac{\partial \phi}{\partial y_j}$ on the union of the support of μ_k , the consistency estimate (24), the invariance of μ_k under F_k , and the following straightforward estimates valid

on the union of the support of the μ_k s (uniformly bounded in $H_{0,per}^1 \cap H^2$)

$$\begin{aligned}
\left| \int_{\Omega} A^{-1}(\mathbf{k}\theta) \cdot \nabla w_j(F_k(\theta) - \theta) d\mathbf{x} \right| &\leq c \|F_k(\theta) - \theta\| = \mathcal{O}(k) \rightarrow 0, \\
\left| \int_{\Omega} (A^{-1}(\mathbf{k}(F_k(\theta) - \theta)))_3 \tau'(z) w_j d\mathbf{x} \right| &\leq c \|F_k(\theta) - \theta\| = \mathcal{O}(k) \rightarrow 0, \\
\left| \int_{\Omega} \Delta w_j(F_k(\theta) - \theta) d\mathbf{x} \right| &\leq c \|F_k(\theta) - \theta\| = \mathcal{O}(k) \rightarrow 0, \\
\langle F_k(\theta) - \theta, \Phi'(\theta) \rangle &= \Phi(F_k(\theta)) - \Phi(\theta) + \mathbf{o}(\|F_k(\theta) - \theta\|) = \Phi(F_k(\theta)) - \Phi(\theta) + \mathbf{o}(k),
\end{aligned}$$

the weak convergence of μ_k to μ , the scheme, and the invariance of μ_k .

This proves the differential form of the weak invariance of μ under the infinite Prandtl number dynamics, i.e., no. 2.

The energy type inequality no.3 can be verified easily as well. For this purpose, we first show that any invariant measure μ_k of the numerical scheme (6) must satisfy the same energy type estimate. The desired continuous one will be the limit as the time step approaches zero.

We first show that the invariant measures for F_k also satisfy the energy inequality. For this purpose we multiply the scheme (6) by θ^{n+1} and integrate over the domain, and we have

$$\frac{1}{2k} (\|\theta^{n+1}\|^2 - \|\theta^n\|^2 + \|\theta^{n+1} - \theta^n\|^2) + \|\nabla \theta^{n+1}\|^2 + \int_{\Omega} (-\tau'' \theta^{n+1} + Ra \tau' A^{-1}(\mathbf{k}\theta^{n+1}))_3 \theta^{n+1} = 0.$$

This can be re-written using the discrete dynamical system notation F_k as

$$\frac{1}{2k} (\|F_k(\theta)\|^2 - \|\theta\|^2 + \|F_k(\theta) - \theta\|^2) + \|\nabla F_k(\theta)\|^2 + \int_{\Omega} (-\tau'' F_k(\theta) + Ra \tau' A^{-1}(\mathbf{k}F_k(\theta)))_3 F_k(\theta) = 0. \tag{52}$$

Integrating this identity with respect to the invariant measure μ_k and utilizing the invariance we have

$$\int_{L^2} \int_{\Omega} (|\nabla F_k(\theta)|^2 - \tau'' F_k(\theta) + Ra \tau' A^{-1}(\mathbf{k}F_k(\theta)))_3 F_k(\theta) d\mathbf{x} d\mu_k(\theta) = -\frac{1}{2k} \int_{L^2} \|F_k(\theta) - \theta\|^2 d\mu_k(\theta) \leq 0. \tag{53}$$

Utilizing the invariance of μ_k under F_k again in the lower order terms we have

$$\int_{L^2} (\|\nabla F_k(\theta)\|^2 + \int_{\Omega} (-\tau'' \theta + Ra \tau' (A^{-1}(\mathbf{k}\theta)))_3 d\mathbf{x}) d\mu_k(\theta) \leq 0. \tag{54}$$

Now we recall that the support of μ_k is uniformly bounded in H^1 and hence, since the w_j s are the eigenfunctions of Δ which form an orthonormal basis in L^2 , w_j also form a complete orthogonal system in $H_{0,per}^1$ (with the inner product between f and g given by $\int_{\Omega} \nabla f \cdot \nabla g$),

$$\|\nabla F_k(\theta)\|^2 = \lim_{m \rightarrow \infty} \sum_{j=1}^m \frac{(\nabla F_k(\theta), \nabla w_j)^2}{\|\nabla w_j\|^2} = \lim_{m \rightarrow \infty} \sum_{j=1}^m \frac{(F_k(\theta), \Delta w_j)^2}{\|\nabla w_j\|^2}$$

Therefore

$$\begin{aligned}
\int_{L^2} \|\nabla F_k(\theta)\|^2 d\mu_k(\theta) &= \int_{L^2} \lim_{m \rightarrow \infty} \sum_{j=1}^m \frac{(F_k(\theta), \Delta w_j)^2}{\|\nabla w_j\|^2} d\mu_k(\theta) \\
&= \lim_{m \rightarrow \infty} \int_{L^2} \sum_{j=1}^m \frac{(F_k(\theta), \Delta w_j)^2}{\|\nabla w_j\|^2} d\mu_k(\theta) \\
&= \lim_{m \rightarrow \infty} \int_{L^2} \sum_{j=1}^m \frac{(\theta, \Delta w_j)^2}{\|\nabla w_j\|^2} d\mu_k(\theta) \\
&= \int_{L^2} \|\nabla \theta\|^2 d\mu_k(\theta)
\end{aligned} \tag{55}$$

where we have used the uniform boundedness of $\|\nabla \theta\|$ and $\|\nabla F_k(\theta)\|$ on the support of μ_k , the Lebesgue Dominated Convergence theorem and the invariance of μ_k under F_k .

Hence we see that μ_k also satisfies the energy inequality, i.e.,

$$\int_{L^2} \int_{\Omega} (|\nabla \theta|^2 - \tau'' \theta + Ra \tau'(A^{-1}(\mathbf{k}\theta))_3 \theta) d\mu_k(\theta) \leq 0. \tag{56}$$

Next, we take the limit as k approaches zero. The last two terms in the discrete energy inequality above converge to the right limit by the very definition of weak convergence of μ_k to μ . As for the leading order quadratic term, we have

$$\begin{aligned}
\int \|\nabla \theta\|^2 d\mu(\theta) &= \lim_{m \rightarrow \infty} \sum_{j=1}^m \int \frac{(\theta, \Delta w_j)^2}{\|\nabla w_j\|^2} d\mu(\theta) \\
&= \lim_{m \rightarrow \infty} \lim_{k \rightarrow 0} \sum_{j=1}^m \int \frac{(\theta, \Delta w_j)^2}{\|\nabla w_j\|^2} d\mu_k(\theta) \\
&\leq \liminf_{k \rightarrow 0} \sum_{j=1}^{\infty} \int \frac{(\theta, \Delta w_j)^2}{\|\nabla w_j\|^2} d\mu_k(\theta) \\
&= \liminf_{k \rightarrow 0} \int \|\nabla \theta\|^2 d\mu_k(\theta).
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{L^2} \int_{\Omega} (|\nabla \theta|^2 - \tau'' \theta + Ra \tau'(A^{-1}(\mathbf{k}\theta))_3 \theta) d\mu(\theta) \\
&\leq \liminf_{k \rightarrow 0} \int \|\nabla \theta\|^2 d\mu_k(\theta) + \lim_{k \rightarrow 0} \int_{L^2} \int_{\Omega} (-\tau'' \theta + Ra \tau'(A^{-1}(\mathbf{k}\theta))_3 \theta) d\mu_k(\theta) \\
&\leq \liminf_{k \rightarrow 0} \int_{L^2} \int_{\Omega} (|\nabla \theta|^2 - \tau'' \theta + Ra \tau'(A^{-1}(\mathbf{k}\theta))_3 \theta) d\mu_k(\theta) \\
&\leq 0.
\end{aligned} \tag{57}$$

This completes the proof of the energy type inequality (no.3 in the definition) for the limit probability measure μ . Therefore we conclude that the limit μ must be an invariant measure of the infinite Prandtl number model.

Sometimes we impose a stronger version of the statistical energy inequality in the definition of stationary statistical solutions: we require that the statistical version of the energy inequality be true on any energy shells $e_1 \leq \|\theta\| \leq e_2$ instead of just one infinite shell from zero to infinity. Such kind of energy inequalities are useful in some applications in the Navier-Stokes case (see for instance [13]). They can be verified with a little bit extra work which involves approximating the finite difference by differentiation and utilizing the uniform in H^2 estimates (invariant measures are supported in a bounded ball in H^2). We shall supply details elsewhere. Likewise it is sometimes useful to have a stronger version of the invariance of μ under the continuous in time dynamics either as a straightforward pull-back invariance or differential form of the weak invariance with a broader class of test functionals that are bounded on bounded sets of L^2 , Fréchet differentiable for $\theta \in H_{0,per}^1$ with $\Phi'(\theta) \in H_{0,per}^1$ and the derivative is continuous and bounded as a function from $H_{0,per}^1$ to $H_{0,per}^1$. It can be shown that these variations yield the same definition just as in the case of two dimensional Navier-Stokes equations [13].

Next, we consider the limit of heat transport in the vertical direction, i.e., the Nusselt number, as the step size approaches zero. We first recall the definition of the *Nusselt number*.

Definition 2 (Nusselt number) *For the infinite Prandtl number model, the non-dimensional averaged heat transport in the vertical direction is defined as*

$$\begin{aligned} Nu &= \sup_{\theta_0 \in L^2} \limsup_{t \rightarrow \infty} \frac{1}{tL_xL_y} \int_0^t \int_{\Omega} |\nabla T(\mathbf{x}, s)|^2 d\mathbf{x}ds \\ &= 1 + Ra \sup_{\theta_0 \in L^2} \limsup_{t \rightarrow \infty} \frac{1}{tL_xL_y} \int_0^t \int_{\Omega} A^{-1}(\mathbf{k}T(\mathbf{x}, s))_3 T(\mathbf{x}, s) d\mathbf{x}ds \\ &= 1 + Ra \sup_{\theta_0 \in L^2} \limsup_{t \rightarrow \infty} \frac{1}{tL_xL_y} \int_0^t \int_{\Omega} A^{-1}(\mathbf{k}\theta(\mathbf{x}, s))_3 \theta(\mathbf{x}, s) d\mathbf{x}ds. \end{aligned} \quad (58)$$

Likewise, the non-dimensional averaged heat transport in the vertical direction for the discrete in time scheme (6) is defined as

$$Nu_k = 1 + Ra \sup_{\theta_0 \in L^2} \limsup_{N \rightarrow \infty} \frac{1}{NL_xL_y} \sum_{n=1}^N \int_{\Omega} A^{-1}(\mathbf{k}\theta^n(\mathbf{x}))_3 \theta^n(\mathbf{x}) d\mathbf{x} \quad (59)$$

It is well-known that long time averages defined through Banach (generalize) limits are spatial averages with respect to appropriate invariant measures of the underlying dynamical system [2, 13, 40, 41, 45]. Moreover, for a given continuous test functional φ_0 (in the application here $\varphi_0(\theta) = 1 + \frac{Ra}{L_xL_y} \int_{\Omega} A^{-1}(\mathbf{k}\theta(\mathbf{x}))_3 \theta(\mathbf{x}) d\mathbf{x}$), and a particular trajectory (initial data), there exists a particular Banach limit, LIM_0 , so that $LIM_0 \frac{1}{t} \int_0^t \varphi_0(\theta(s)) ds = \limsup \frac{1}{t} \int_0^t \varphi_0(\theta(s)) ds$ [44, 45]. Therefore, when combined with the Prokhorov's compactness theorem, we deduce the existence of an invariant measures $\mu_k \in \mathcal{IM}_k$ such that

$$Nu_k = 1 + \frac{Ra}{L_xL_y} \int_{L^2} \int_{\Omega} A^{-1}(\mathbf{k}\theta(\mathbf{x}))_3 \theta(\mathbf{x}) d\mathbf{x}d\mu_k. \quad (60)$$

Hence by the weak convergence result that we just proved, we see that for any sequence of Nu_k (and hence μ_k) there exists a subsequence (still denoted Nu_k and μ_k), and $\mu \in \mathcal{IM}$ such that

$$\begin{aligned}
\lim_{k \rightarrow 0} Nu_k &= 1 + \frac{Ra}{L_x L_y} \lim_{k \rightarrow 0} \int_{L^2} \int_{\Omega} A^{-1}(\mathbf{k}\theta(\mathbf{x}))_3 \theta(\mathbf{x}) \, d\mathbf{x} d\mu_k \\
&= 1 + \frac{Ra}{L_x L_y} \int_{L^2} \int_{\Omega} A^{-1}(\mathbf{k}\theta(\mathbf{x}))_3 \theta(\mathbf{x}) \, d\mathbf{x} d\mu \\
&\leq 1 + \frac{Ra}{L_x L_y} \sup_{\nu \in \mathcal{IM}} \int_{L^2} \int_{\Omega} A^{-1}(\mathbf{k}\theta(\mathbf{x}))_3 \theta(\mathbf{x}) \, d\mathbf{x} d\nu \\
&= 1 + \frac{Ra}{L_x L_y} \sup_{\theta_0 \in L^2} \lim_{t \rightarrow \infty} \frac{1}{t L_x L_y} \int_0^t \int_{\Omega} A^{-1}(\mathbf{k}\theta(\mathbf{x}, s))_3 \theta(\mathbf{x}, s) \, d\mathbf{x} ds \quad (61)
\end{aligned}$$

where we have used the weak convergence of μ_k to μ , the compactness of the set of all invariant measures due to Prokhorov's theorem and the a priori estimates, the fact that extremal points of the set of invariant measures are ergodic (in the sense that phase space spatial average and time average are the same) [2, 41, 44].

To summarize, we have proved the following main result.

Theorem 1 (Convergence of stationary statistical properties) *Let μ_k be an arbitrary invariant measure of the numerical scheme (6) with time step k , i.e. $\mu_k \in \mathcal{IM}_k$, and let Nu_k be the Nusselt number characterizing the heat transport in the vertical direction for the scheme with time step k defined in (59). Then each subsequence of μ_k must contain a sub-subsequence (still denoted $\{\mu_k\}$) and an invariant measure μ of the infinite Prandtl number model so that μ_k weakly converges to μ , i.e.,*

$$\mu_k \rightharpoonup \mu, k \rightarrow 0. \quad (62)$$

Moreover, the Nusselt number converges in an upper semi-continuous fashion in the sense that

$$\limsup_{k \rightarrow 0} Nu_k \leq Nu. \quad (63)$$

In particular, this implies that the convergent numerical schemes will not overestimate the Nusselt number asymptotically.

3 Conclusions and Remarks

Our main result clearly demonstrated the usefulness of uniformly dissipative schemes in terms of approximating stationary statistical properties of (possibly) complex/chaotic dissipative dynamical systems since the stationary statistical properties of the scheme converge to those of the continuous time model. To the best of our knowledge, this is the first rigorous result proving convergence of stationary statistical properties of numerical schemes to those of the continuous in time dynamical system that they are approximating. Our result may be

viewed as a partial generalization of Lax's equivalence theorem in the sense that consistency and long time stability (uniform dissipativity) implies convergence of stationary statistical properties. We would like to emphasize that the methodology here can be applied to much more general dissipative systems (with chaotic behavior for relevance) although we have treated the infinite Prandtl number model only. We hope that our work will stimulate further study on numerical schemes for approximating statistical properties of dissipative dynamical systems.

The convergence of the stationary statistical properties rely on the uniform bound in a space which is compactly embedded in the phase space (it is $H_{0,per}^1$ in the infinite Prandtl number case which is compactly imbedded in the phase space L^2 by Rellich's theorem). Simply having a uniform bound in the phase space may not imply the convergence of the statistical properties (we could construct schemes that possess absorbing ball in L^2 but not in H^1). Therefore, we would rather use the *uniformly dissipative* terminology instead of the *global in time stability* used by many other authors which could mean uniform bound in the phase space only.

The convergence that we derived here is actually semi-convergence since different subsequences may converge to different invariant measures of the continuous in time dynamical system. There is no convergence rate either. This is perhaps the generic picture in the sense that the result here is nearly optimal without additional assumption on the continuous dynamical system. One very useful physical assumption is mixing of the continuous system. Indeed, if we assume that the continuous system is exponentially mixing with a rate of r [41], i.e., there is a physically relevant invariant measure μ so that

$$\left| \frac{1}{t} \int_0^t \Phi(\theta(\mathbf{x}, s)) ds - \int \Phi(\theta(\mathbf{x})) d\mu \right| \leq c \exp(-rt) \quad (64)$$

for all appropriate test functionals $\Phi(\theta)$ and almost all trajectories, then approximating a specific statistical quantity $\int \Phi(\theta(\mathbf{x})) d\mu$ becomes a finite time integration problem. Indeed, suppose the given tolerance level is 2ϵ , we first fix a time t so that $c \exp(-rt) \leq \epsilon$ (the time t is usually large for small mixing rate r). We then adjust our mesh size (small time step or mesh size) so that

$$\left| \frac{1}{t} \int_0^t \Phi(\theta(\mathbf{x}, s)) ds - \frac{1}{N} \sum_{n=1}^N \Phi(\theta^n) \right| \leq \epsilon$$

where $Nk = t$ with k being the time step. Hence the infinite time approximation of stationary statistical property becomes the problem of approximation on finite time interval $[0, t]$ for appropriate numerical schemes (say uniformly dissipative). This motives us to work on higher order schemes so that the integration on $[0, t]$ can be calculated quickly.

Of course we do not have exponential mixing for general dissipative complex/chaotic dynamical systems. One way to circumvent this difficulty is by considering noisy systems since our environment is intrinsically noisy. Exponential mixing can be verified for many dissipative systems with appropriate additive white noise [10, 12, 30, 27]. Hence there is a strong incentive to generalize the notion of uniformly dissipative schemes to approximations of continuous in time stochastic dynamical systems (both SDE and SPDE, see [30] for the

case of SDE and fully implicit approach). We will report results in this direction somewhere else.

Of course the scheme that we presented here is not the only uniformly dissipative scheme. The fully implicit backward Euler scheme is a uniformly dissipative scheme as one can readily verify. However, the backward Euler is nonlinear in the unknown and hence the computational cost at each time step is expected to be higher. There are other linear implicit uniformly dissipative schemes. For instance one may check that the following family of schemes are uniformly dissipative for $\lambda \in [0, 1]$,

$$\frac{\theta^{n+1} - \theta^n}{k} + Ra A^{-1}(\mathbf{k}\theta^n) \cdot \nabla \theta^{n+1} + Ra (A^{-1}(\mathbf{k}(\lambda\theta^{n+1} + (1-\lambda)\theta^n)))_3 \tau'(z) = \Delta \theta^{n+1} + \tau''(z). \quad (65)$$

However, it seems that this family of scheme is uniformly dissipative under small time step restriction $k \leq \frac{1}{Ra^2}$. On the one hand this kind of restriction is expected since the linearly unstable modes grows as the Rayleigh number grows and hence the time step should reflect this through CFL condition. On the other hand, we observe that for the case of $\lambda = 0$, the scheme is the same as

$$\frac{T^{n+1} - T^n}{k} + Ra A^{-1}(\mathbf{k}T^n) \cdot \nabla T^{n+1} = \Delta T^{n+1}$$

whose inviscid part is stable (satisfies maximum principle). Hence if the viscous scheme is unstable, the viscous term plays a destabilizing role.

We also remark that at the time discretization only stage we should anticipate implicit scheme due to the CFL condition.

An issue that we have not addressed here is spatial discretization. Since we have utilized the background temperature profile τ in our uniform dissipativity argument, it is expected that we need to resolve small scales within the background profile. Similar issue for the Navier-Stokes was investigated earlier [9]. We will report the details somewhere else.

Another issue that we have not addressed here is the behavior of the global attractors. We fully anticipate an upper semi-continuity result. The proof is a modification of the classical one [37, 32, 17] since we will not have uniform in time convergence of trajectories on finite time interval. In fact, we will have uniform in time convergence after a transitional period of time due to the fact that the points on the global attractors of the scheme may not satisfy high order compatibility condition for the infinite Prandtl number model and hence the solution may have an initial transitional layer (see [18] for the case of two dimensional Navier-Stokes equations). We will report this elsewhere as well.

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