Coverings of 3-manifolds by open balls and two open solid tori

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1 Introduction

The Lusternik-Schnirelmann category $cat(M)$ of a closed n-manifold M is defined to be the smallest number of sets, open and contractibe in M needed to cover M. A generalization to an $\mathcal{A}$-category of M was introduced in [CP]. Let A be a closed connected k-manifold, $0 \leq k \leq n-1$. A subset B in the n-manifold M is A-*contractible* if there are maps $\varphi : B \longrightarrow A$ and $\alpha : A \longrightarrow M$ such that the inclusion map $i : B \longrightarrow M$ is homotopic to $\alpha \cdot \varphi$. The $\mathcal{A}$-category $cat_A(M)$ of M is the smallest number of sets, open and A-contractible needed to cover M. Note that when A is a point P, $cat_P(M) = cat(M)$. For3-dimensional manifolds the invariant $cat(M)$ was studied in [GG]. In the case $A = S^1$ it was shown in [GGH2] that the fundamental group of a closed n-manifold M with $cat_{S^1}(M) = 2$ is cyclic if $n = 3$, and is cyclic or a free product of two cyclic groups with nontrivial amalgamation if $n > 3$. We now know that if $n > 3$, then in fact $\pi(M)$ is trivial or infinite cyclic.

For $n = 3$ it is now natural to ask about minimal covers of M by open sets, each homotopy equivalent to A. In particular when A is a point or S^1 one may consider covers of M by open 3-balls or open solid tori. It is well known that if M is covered by two open balls then $M = S^3$ and the existence of a Heegaard-splitting shows that every M can be covered by four open balls. Hempel and McMillan [HM] proved that if M is covered by three open balls, then M is a connected sum of S^3 and finitely many S^2-bundles over S^1. By the Poincarè Conjecture the same is true when $cat(M) = 3$ ([GG]).

A new proof of the slightly more generalized Hempel-McMillan result was

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given in [GGH1]. These proofs did not use the Poincarè Conjecture; a much shorter proof can be given by using it, since then it suffices to compute the fundamental group of M. In this paper we follow this approach, accepting Perelman's proof of the Poincarè Conjecture and the 3-dimensional spherical space-form conjecture, to obtain a classification of all closed 3-manifolds that can be covered by two open balls and one open solid torus or by one open ball and two open solid tori.

2 One ball and one solid torus

From the prime decomposition of closed 3-manifolds M it follows that if $\pi(M)$ is free then M is a connected sum of S^2-bundles over S^1 and a homotopy sphere. By Perelman's proof of the Poincaré Conjecture, M is in fact a connected sum of S^2-bundles over S^1. Moreover it follows from Perelman's work that a closed 3-manifold M with cyclic fundamental group is a lens space (here we include S^3 and $S^2 \times S^1$) [MT]. By Kneser's conjecture ([H], Chapter 7), a decomposition of $\pi(M)$ as a free product of cyclic groups can be realized as a connected sum decomposition of M into prime factors. Thus we have the following Theorem:

Theorem 1 (Perelman) *If the fundamental group of a closed* 3*-manifold M is a free product of cyclic groups then M is a connected sum of manifolds each of (possibly nonorientable) Heegard genus* ≤ 1.

To construct a 3-manifold from three open submanifolds it is useful to be able to work in the pl-category. The following lemma, proved in ([GGH1], Corollary 1(a)), allows us to do this.

Lemma 2 *Suppose M is a closed* 3*-manifold covered by three open sets U_1, U_2, U_3, such that U_i is homeomorphic to the interior of a compact connected* 3*-manifold M_i (i=1,2,3). Then M admits a covering $M = M_1 \cup M_2 \cup M_3$ such that ∂M_1 is transverse to ∂M_2 , $\partial M_3 \subset int(M_1 \cup M_2)$, and M_1, M_2, M_3 are pl embedded.*

We first compute the fundamental groups of compact manifolds that are a union of two balls or of a ball and a solid torus.

Lemma 3 *(a) If a compact* 3*-manifold $N = B_1 \cup B_2$ is a union of two balls then $\pi(N)$ is free.*
(b) If a compact 3*-manifold $N = B \cup V$ is a union of a ball and a solid torus then $\pi(N)$ is a free product of a free group and a cyclic group.*

Proof.

Proof. Consider the graph of groups G (see e.g. [SW]) of $(B_1 \cup B_2, B_1 \cap \partial B_2)$ in case (a) and $(B \cup V, B \cap \partial V)$ in case (b).

G has one vertex associated to B_2 (resp. V) labeled by the trivial group $im(\pi(B_2) \to \pi(B_1 \cup B_2))$ (resp. the cyclic group $C = im(\pi(V) \to \pi(B \cup V))$) and one vertex associated to each component K of $B_1 - B_2$ (resp. of $B - V$) labeled by the trivial group $im(\pi(K) \to \pi(B_1 \cup B_2))$ (resp. $im(\pi(K) \to \pi(B \cup V))$).

The edges of G are in one-to-one correspondence with the components K' of $B_1 \cap \partial B_2$ (resp. $B \cap \partial V$) and are labeled by the trivial group $im(\pi(K') \to \pi(B_1 \cup B_2))$ (resp. $im(\pi(K') \to \pi(B \cup V))$).

Then $\pi(B_1 \cup B_2)$ (resp. $\pi(B \cup V)$) is the fundamental group of G (see e.g. [GGH2], section 4) which is F in case(a) and $C * F$ in case (b), where F is free.

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3 One ball and two solid tori

Theorem 4

(a) If M is a closed 3-manifold that is a union of three open balls then M is a connected sum of S^3 and S^2-bundles over S^1.
(b) If M is a closed 3-manifold that is a union of two open balls and an open solid torus then M is a connected sum of S^2-bundles over S^1 and a lens space.
(c) If M is a closed 3-manifold that is a union of an open ball and two open solid tori then M is a connected sum of S^2-bundles over S^1 and two lens spaces.

Proof.

By Lemma 2 we get a decomposition of M as a union of balls B_1 and solid tori V_j as follows:

(a) $M = B_1 \cup B_2 \cup B_3$ where ∂B_1 is transverse to ∂B_2 and $\partial B_3 \subset int(B_1 \cup B_2)$.
(b) $M = B_1 \cup B_2 \cup V_3$ where ∂B_1 is transverse to ∂B_2 and $\partial V_3 \subset int(B_1 \cup B_2)$.
(c) $M = B_1 \cup V_2 \cup V_3$ where ∂B_1 is transverse to ∂V_2 and $\partial V_3 \subset int(B_1 \cup V_2)$.

Case(a). Let $N = B_1 \cup B_2$. Then $M = N \cup B_3$. The sphere $S = \partial B_3$ separates N into submanifolds W_1 and W_2 and since M is closed one of them W_1, say, does not meet ∂N, and W_2 contains ∂N. Then M is the union of W_1 and B_3 along S, and $\overline{M - B_3} \subset int(N)$. Since $\partial \overline{M - B_3}$ is a 2-sphere, $\pi(\overline{M - B_3})$ is a subgroup $\pi(N)$ and from Lemma 3 it follows that $\pi(\overline{M - B_3})$ is free and hence $\pi(M)$ is free. Now the result follows from Theorem 1.

Cases(b) and (c). Let $N = B_1 \cup B_2$ in case(b) and $N = B_1 \cup V_2$ in case (c). Then $M = N \cup V_3$ and by Lemma 3 $\pi(N)$ is free in case(b) and $\pi(N)$ is a free product of a cyclic group and a free group in case(c). If N is closed then $M = N$ and M is the union of the two balls B_1 and $B_2 - int(B_1)$ in case (b), or M is the union of B_1 and $V_2 - int(B_1)$ in case (c), and so M is a 3-sphere. Thus we now assume $\partial N \neq \emptyset$.

Since $Z \times Z$ is not a subgroup of a free product of cyclic groups, $T = \partial V_3$ is compressible in N and hence separates N into submanifolds with closures W_1 and W_2 such that T is compressible in W_1 or W_2. Again, since M is closed, W_1, say, does not meet ∂N, and W_2 contains ∂N.

If T is compressible in W_1 with compressing disk D, a regular neighborhood (in W_1) of $T \cup D$ is a once punctured solid torus V and $W_1 = V \cup M_0$ where M_0 is a submanifold of W_1 such that $V \cap M_0 = \partial V \cap \partial M_0 = \partial M_0$ is a 2-sphere. Then $\pi(N) = \pi(M_0) * \pi(\overline{N - M_0})$ and by uniqueness of the free product decomposition it follows that in case(b) $\pi(M_0)$ is free, and in case(c) that $\pi(M_0)$ is a free product of a cyclic group and a free group. Since M is closed, $M = V_3 \cup W_1 = (V_3 \cup V) \cup M_0 = L \cup M_0$, where L is a once punctured lens space. Then in case(b) $\pi(M) = \pi(L) * \pi(M_0)$ is a free product of a cyclic group and a free group, and in case (b) $\pi(M)$ is a free product of two cyclic groups and a free group. Now the result follows from Theorem 1.

If T is compressible in W_2 with compressing disk D, a regular neighborhood (in W_2) of $T \cup D$ is a once punctured solid torus V. Note that the compressing disk D is also a meridian disk of V_3, so we must have $V_3 = V \cup B$, with B a ball such that $V \cap B = \partial V \cap \partial B$ is a 2-sphere. Since $V \subset N$ we have $M = N \cup B$.

Case(b) In this case $N = B_1 \cup B_2$ and $M = N \cup B$ is a union of three balls. So the result follows from case(a).

Case(c) In this case $N = B_1 \cup V_2$ and $M = N \cup B$ is a union of two balls and one solid torus. So the result follows from case(b).

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Remark: The converse of Theorem 4 is also true. For example, if M is a connected sum of S^2-bundles over S^1 and two lens spaces, then we can even decompose M as a union of a ball and two solid tori such that their interiors are pairwise disjoint, see ([GHN], Example 1.6).

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