

# Cyclotomic factors of Coxeter polynomials

Benedict H. Gross, Eriko Hironaka and Curtis T. McMullen\*

September 8, 2008

## Contents

1	Introduction . . . . .	1
2	Roots of unity . . . . .	4
3	Joins . . . . .	7
4	Decorating $A_n$ . . . . .	9

## 1 Introduction

In this paper we determine which roots of unity are zeros of the  $E_n$  Coxeter polynomial. We show these roots come exclusively from splittings of  $E_n$  into spherical subdiagrams; in particular they always have order 2, 3, 5, 8, 12, 18, or 30, and they only depend on the value of  $n \pmod{360}$  (provided we exclude the special case  $n = 9$ ).

The proof uses Mann’s theorem on linear relations between roots of unity, and generalizes to other sequences of Coxeter diagrams where nodes are added to a separating edge.

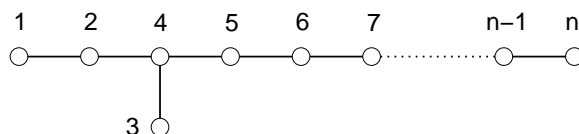


Figure 1. The  $E_n$  diagram.

**The  $E_n$  diagram.** Coxeter systems are a useful source of Salem numbers, Pisot numbers and other interesting algebraic integers. For example, the smallest known Salem number arises from the Coxeter system  $E_{10}$ .

The  $E_n$  Coxeter diagram, defined for  $n \geq 3$ , is shown in Figure 1. Note that  $E_3 \cong A_2 \oplus A_1$ . The  $E_n$  diagram determines a quadratic form  $B_n$  on  $\mathbb{Z}^n$ , and a reflection group  $W_n \subset O(\mathbb{Z}^n, B_n)$  (see §3). The product of the generating

---

\*This research was supported in part by the NSF.

reflections is a *Coxeter element*  $w_n \in W_n$ ; it is well-defined up to conjugacy, since  $E_n$  is a tree [Hum, §8.4].

The *Coxeter number*  $h_n$  is the order of the Coxeter element  $w_n \in W_n$ , and its characteristic polynomial

$$E_n(x) = \det(xI - w_n)$$

is the *Coxeter polynomial*. Explicitly, for  $n \geq 3$  we have:

$$E_n(x) = \frac{x^{n-2}Q(x) + R(x)}{(x-1)},$$

where  $Q(x) = x^3 - x - 1$  and  $R(x) = x^3 + x^2 - 1$ . (See e.g. [MRS, Lemma 5], [Hir2, §4.2] or Corollary 4.3 below.)

We can write  $E_n(x)$  uniquely as a product of monic integral polynomials

$$E_n(x) = C_n(x)S_n(x),$$

where the zeros of the *cyclotomic factor*  $C_n(x)$  are roots of unity, and those of the *Salem factor*  $S_n(x)$  are not. Table 2 lists  $E_n(x)$  for  $n \leq 10$ , along with its factorization into irreducibles and the Coxeter number  $h_n$ . Here  $\Phi_k(x)$  is the cyclotomic polynomial for the primitive  $k$ th roots of unity.

**The spherical and affine cases.** Since  $E_i$  is a spherical diagram ( $B_i$  is positive definite) when  $3 \leq i \leq 8$ , we have  $E_i(x) = C_i(x)$  (and  $S_i(x) = 1$ ) in this range.

The diagram  $E_9$  is the affine version of  $E_8$ ; its Coxeter element has infinite order, but still  $E_9(x) = C_9(x)$ . This is the only case where  $E_n(x)$  has a multiple root (see Lemma 2.4 below).

**The hyperbolic case.** For  $n \geq 10$ , the diagram  $E_n$  is hyperbolic; that is, the signature of  $B_n$  is  $(n-1, 1)$ . By [A'C] this implies that the factor  $S_n(x)$  is a *Salem polynomial*: it is an irreducible, reciprocal polynomial, with a unique root  $\lambda > 1$  outside the unit disk. For  $n = 10$ ,  $E_n(x)$  coincides with *Lehmer's polynomial*, and its root  $\lambda \approx 1.1762808 > 1$  is the smallest known Salem number.

We can now state our main result on the Coxeter polynomials  $E_n(x)$ .

**Theorem 1.1** *For all  $n \neq 9$ :*

1. *The cyclotomic factor  $C_n(x)$  is the least common multiple of the polynomials  $\Phi_2(x), \Phi_3(x)$  and  $E_i(x)$ ,  $3 \leq i \leq 8$ , that divide  $E_n(x)$ ;*
2.  *$E_n(x)$  is divisible by  $E_i(x)$ ,  $3 \leq i \leq 8$ , iff  $n \equiv i \pmod{h_i}$ ; and*
3.  *$E_n(x)$  is divisible by  $\Phi_2(x)$  iff  $n \equiv 1 \pmod{2}$ , and by  $\Phi_3(x)$  iff  $n \equiv 0 \pmod{3}$ .*

**Corollary 1.2** *The cyclotomic factor  $C_n(x)$  only depends on  $n \pmod{360}$ .*

**Corollary 1.3** *The Salem factor  $S_n(x)$  satisfies  $n - 15 \leq \deg(S_n) \leq n$ .*

The value  $n - 15$  is first attained when  $n = 349$ .

$n$	$h_n$	Coxeter polynomial $E_n$	Factorization
3	6	$1 + 2x + 2x^2 + x^3$	$\Phi_2(x)\Phi_3(x)$
4	5	$1 + x + x^2 + x^3 + x^4$	$\Phi_5(x)$
5	8	$1 + x + x^4 + x^5$	$\Phi_2(x)\Phi_8(x)$
6	12	$1 + x - x^3 - x^4 + x^6 + x^7$	$\Phi_3(x)\Phi_{12}(x)$
7	18	$1 + x - x^3 - x^4 + x^6 + x^7$	$\Phi_2(x)\Phi_{18}(x)$
8	30	$1 + x - x^3 - x^4 - x^5 + x^7 + x^8$	$\Phi_{30}(x)$
9	$\infty$	$1 + x - x^3 - x^4 - x^5 - x^6 + x^8 + x^9$	$\Phi_1(x)\Phi_2(x)^2\Phi_3(x)\Phi_5(x)$
10	$\infty$	$1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10}$	$S_{10}(x)$

Table 2. Coxeter polynomials for small  $n$ .

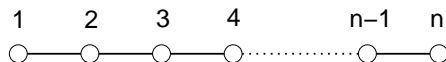


Figure 3. The  $A_n$  diagram.

**Corollary 1.4** For  $n \geq 10$ , the polynomial  $E_n(x)$  is irreducible (and hence  $E_n(x) = S_n(x)$ ) iff  $n \equiv 2, 10, 16, 20, 22, 26$  or  $28 \pmod{30}$ .

**Joins of diagrams and periodicity.** This behavior of  $E_n$  can be understood as a consequence of two general phenomena.

For the first, recall that the  $A_n$  diagram (Figure 3) has Coxeter polynomial

$$A_n(x) = \frac{x^{n+1} - 1}{x - 1} = 1 + x + \cdots + x^n.$$

In §3 we will show:

**Theorem 1.5** Let  $F$  be the Coxeter diagram obtained by joining together diagrams  $F_1, \dots, F_n$  at a single new vertex  $t$ . Then any zero of two or more of the Coxeter polynomials  $F_i(x)$  is also a zero of  $F(x)$ .

Noting that  $E_n$  is a join of  $E_i$  and  $A_{n-i-1}$ , we obtain:

**Corollary 1.6**  $E_n(x)$  is divisible by  $\gcd(E_i(x), A_{n-i-1}(x))$  for  $3 \leq i < n - 1$ .

This result explains why the spherical Coxeter polynomials  $E_i(x)$ ,  $3 \leq i \leq 8$ , occur as factors of  $E_n(x)$ . For example,  $E_{38}$  is the join of  $E_8$  and  $A_{29}$ . The zeros of  $A_{29}(x)$  are the 30th roots of unity (save  $\zeta = 1$ ); thus they include the zeros of  $E_8(x)$ , and consequently  $E_8(x)$  divides  $E_{38}(x)$ . It also explains the occurrence of the cyclotomic factors  $\Phi_2$ ,  $\Phi_3$  and their product; these can occur as  $\gcd(E_3, A_{n-4})$ , depending on the value of  $n \pmod{6}$ .

The second phenomenon underlying the behavior of  $E_n$  is the following periodicity result, proved in §4.

**Theorem 1.7** *Let  $F_n$  be a sequence of Coxeter diagrams obtained by adjoining two fixed diagrams to the ends of  $A_n$ . Assume  $F_n(x) \in \mathbb{Z}[x]$ . Then either*

- (i) *The cyclotomic factor of  $F_n(x)$  is periodic for all  $n \gg 0$ , or*
- (ii) *The diagram  $F_n$  is spherical or affine for all  $n$ .*

In case (ii),  $F_n$  (if connected) must be a re-indexing of one of the well-known spherical or affine series  $A_n, B_n, D_n, \widetilde{B}_n, \widetilde{C}_n$  or  $\widetilde{D}_n$ .

This result, made effective, reduces Theorem 1.1 to a finite computation.

It would be interesting to find a general condition to insure that the cyclotomic factors of  $F_n(x)$  come exclusively from its spherical subdiagrams, as is the case for  $E_n(x)$ .

**Notes and references.** For background on Coxeter systems, see e.g. [Bou] and [Hum]. More on the relationship between Coxeter systems, Salem numbers and Pisot numbers can be found in [Mc], [MRS], [Hir1] and [MS]. A version of Theorem 1.1 was proved independently, and by different arguments, by Bedford and Kim [BK, Thm. 2.4].

## 2 Roots of unity

Let  $\zeta_k$  denote the primitive  $k$ th root of unity  $\exp(2\pi i/k)$ . In this section we formulate Mann's theorem, and use it to prove:

**Theorem 2.1** *Let  $Q, R \in \mathbb{Z}[x]$  be polynomials, not both zero, such that*

$$\zeta_k^n Q(\zeta_k) + R(\zeta_k) = 0$$

*for some  $k \geq 1$  and  $n \in \mathbb{Z}$ . Then either  $Q(x) = \pm x^i R(x)$  for some  $i \in \mathbb{Z}$ , or we have*

$$k \leq 2s \max(\deg Q, \deg R),$$

*where  $s$  is the product of the primes  $p \leq \ell(Q) + \ell(R)$ .*

Here  $\ell(P)$  denotes the number of terms in the polynomial  $P$  (see below).

We then deduce Theorem 1.1 on the cyclotomic factor of  $E_n(x)$ .

**Polar rational polygons.** Let  $\text{Div}(\mathbb{C})$  denote the group of finite divisors on the complex plane. Any  $D \in \text{Div}(\mathbb{C})$  can be expressed as  $D = \sum_I a_i \cdot z_i$  where each coefficient  $a_i \in \mathbb{Z}$  is nonzero and  $\text{supp } D = \{z_i : i \in I\}$  is a set of distinct points forming the *support* of  $D$ . There is a natural evaluation map  $\text{Div}(\mathbb{C}) \rightarrow \mathbb{C}$  defined by

$$D \mapsto \sigma(D) = \sum a_i z_i.$$

We say  $D$  is *effective* if its coefficients are positive.

A *polar rational polygon* (prp) is an effective divisor  $D = \sum a_i \cdot z_i$  such that each  $z_i$  is a root of unity and  $\sigma(D) = 0$ . For each ordering of  $I$ ,  $D$  determines an (immersed) polygon in the plane with vertices  $v_i = \sum_{j < i} a_j z_j$ ; its angles are rational multiples of  $\pi$ , and its sides are of integral length.

The *length* of a prp is given by  $\ell(D) = |\text{supp } D|$ . Its *order* is the cardinality  $o(D)$  of the subgroup of  $\mathbb{C}^*$  generated by the roots of unity  $\{z_i/z_j : i, j \in I\}$ .

A prp is *primitive* if it cannot be expressed as a sum  $D = D' + D''$  of two other nonzero prp's. Every prp is a sum of primitive prp's.

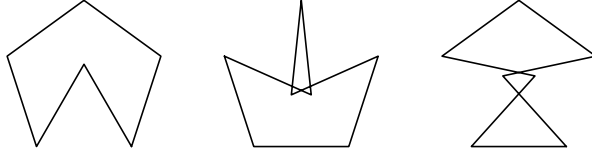


Figure 4. Three primitive polar rational polygons.

We can now state the main result of [Man]:

**Theorem 2.2 (Mann)** *Let  $D$  be a primitive prp. Then the order  $o(D)$  divides the product of the primes  $p$  less than or equal to the length  $\ell(D)$ .*

**Examples.** The regular  $p$ -gons are primitive prp's whenever  $p$  is prime. The smallest primitive prp other than these has length 6 and order 15; it is given by

$$D = \zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4 + \zeta_6 + \zeta_6^{-1}.$$

The corresponding hexagon (for a suitable ordering of the terms in the prp), with sides of length one, is shown at the left in Figure 4. Two other primitive prp's, of length 7 and order 30, are shown in the center and at the right. Together with the regular  $p$ -gons for  $p = 3, 5, 7$ , these are (up to rotation) all the primitive prp's of length  $< 8$  [Man].

**Polynomials.** Any polynomial  $P(x) \in \mathbb{Z}[x]$  can be uniquely expressed in the form

$$P(x) = \sum_{i \in I} \epsilon_i a_i x^i,$$

where  $a_i > 0$  and  $\epsilon_i = \pm 1$ . The *length*  $\ell(P) = |I|$  is the number of terms in  $P$ .

Given  $\zeta \in \mathbb{C}$ , let  $DP(\zeta)$  denote the effective divisor

$$DP(\zeta) = \sum_{i \in I} a_i \cdot (\epsilon_i \zeta^i).$$

If  $\zeta$  is a root of unity and  $P(\zeta) = 0$ , then  $DP(\zeta)$  is a prp.

**Proof of Theorem 2.1.** Let  $P(x) = x^n Q(x) + R(x)$ . Then there are finite sums  $Q(x) = \sum Q_j(x)$  and  $R(x) = \sum R_j(x)$  such that

$$DP(\zeta_k) = \sum_j DP_j(\zeta_k) = \sum_j \zeta_k^n DQ_j(\zeta_k) + DR_j(\zeta_k)$$

gives a decomposition of  $DP(\zeta_k)$  into primitive prps.

If  $\ell(Q_j) > 1$  for some  $j$ , then we have  $o(DP_j(\zeta_k)) \geq k/(2 \deg(Q))$ , since the ratio of any two roots of unity occurring in  $DQ_j(\zeta_k)$  has the form  $\pm \zeta_k^e$  with  $1 \leq e \leq \deg(Q)$ . By Mann's theorem,  $o(DP_j(\zeta_k))$  is bounded above by the product of the primes less than or equal to  $\ell(P_j) \leq \ell(Q) + \ell(R)$ , and so the desired upper bound for  $k$  follows. The same argument applies if  $\ell(R_j) > 1$  for some  $j$ .

Now assume  $\ell(Q_j) = \ell(R_j) = 1$  for all  $j$ , but the desired bound on  $k$  fails. Then  $k > 4m$ , where  $m = \max(\deg(Q), \deg(R))$ . Writing  $Q_j(x) = a_j x^{e_j}$  and  $R_j(x) = b_j x^{f_j}$ , we have

$$\zeta_k^n Q_j(\zeta_k) + R_j(\zeta_k) = a_j \zeta_k^{n+e_j} + b_j \zeta_k^{f_j} = 0$$

for all  $j$ . Consequently  $\zeta_k^{f_j - e_j} = \pm \zeta_k^n$  for all  $j$ . This implies  $f_j - e_j$  is constant mod  $k$  or mod  $(k/2)$  (depending on the parity of  $k$ ). But  $k > 4m$  and  $(f_j - e_j) \in [-m, m]$ , so the difference of exponents  $i = f_j - e_j$  is also constant in  $\mathbb{Z}$ . We then have

$$a_j \zeta_k^{n-i+f_j} + b_j \zeta_k^{f_j} = 0$$

for all  $j$ ; thus  $\epsilon = \zeta_k^{n-i} = \pm 1$  and  $\epsilon a_j + b_j = 0$ , which gives  $\epsilon x^i Q_j(x) + R_j(x) = 0$  and hence  $Q(x) = \pm x^{-i} R(x)$ . ■

**Application to  $E_n$ .** Now recall that for  $n \geq 3$  we have

$$E_n(x)(x-1) = x^{n-2}(x^3 - x - 1) + (x^3 + x^2 - 1) = x^{n-2}Q(x) + R(x).$$

Since  $\deg(Q) = \deg(R) = 3$  and  $\ell(Q) + \ell(R) = 6$ , the Theorem above implies:

**Corollary 2.3** *If  $E_n(\zeta_k) = 0$ , then  $k \leq 180$ .*

**Lemma 2.4** *The polynomial  $E_n(x)$  is separable for all  $n \neq 9$ .*

**Proof.** The only possible multiple roots of  $E_n(x)$  are in its cyclotomic factor  $C_n(x)$ . But for  $|x| = 1$  we have

$$|(E_n(x)(x-1))'| > (n-2)|Q(x)| - |Q'(x)| - |R'(x)| > 0.3(n-2) - 9,$$

so  $E_n(x)$  is separable once  $n \geq 32$ . The remaining cases are easily checked individually. ■

**Proof of Theorem 1.1.** It is straightforward to verify that the Theorem is correct for  $3 \leq n \leq 182$ . Thus  $E_n(\zeta_k) = 0$  for some  $n$  in this range,  $n \neq 9$ , iff  $k \in \{2, 3, 5, 8, 12, 18, 30\} = K$ .

By separability, the cyclotomic factor only depends on the roots of unity where  $E_n(\zeta_k) = 0$ . But the vanishing of  $E_n(\zeta_k)$  only depends on the value of  $n \bmod k$ , so by Corollary 2.3 no new roots of unity can occur as zeros of  $E_n(x)$  for  $n > 182$ . So once the Theorem is checked for all  $n \leq 182$  it also holds for all larger values of  $n$ . ■

### 3 Joins

In this section we define the *join* of a collection of Coxeter systems, and establish the following more precise version of Theorem 1.5.

**Theorem 3.1** *Let  $(W, S)$  be the join of Coxeter systems  $(W_i, S_i)_{i=1}^m$ , with bicolored Coxeter elements  $w_i$ . Suppose  $\lambda$  is an eigenvalue of  $w_i$  with multiplicity  $m_i \geq 0$ . Then  $\lambda$  occurs as an eigenvalue of the bicolored Coxeter element  $w \in W$  with multiplicity at least  $(\sum m_i) - 1$ .*

**Coxeter systems.** Recall that a *Coxeter system*  $(W, S)$  is an abstract group  $W$  with a distinguished set of generators  $S$ , such that the product  $st \in W$  of two generators has finite order  $m_{st} \geq 2$ , the generators themselves have order 2, and these relations give a presentation for  $W$ .

The pair  $(W, S)$  determines a quadratic form  $B$  on  $\mathbb{R}^S$  with matrix  $B_{st} = -2 \cos(\pi/m_{st})$ , and a geometric representation  $W \hookrightarrow O(\mathbb{R}^S, B)$  where the generators act by the reflections

$$s \cdot v = v - B(e_s, v)e_s. \quad (3.1)$$

The *Coxeter diagram*  $F$  of  $(W, S)$  is the (undirected) graph with vertex set  $S$  and an edge of weight  $m_{st} - 2$  joining  $s$  to  $t$  whenever  $m_{st} > 2$ . By convention an unlabeled edge has weight one, and  $i$  parallel unlabeled edges indicate a single edge of weight  $i$ .

The product of the generators  $w = s_1 \cdots s_n$  of  $W$ , taken in any order, is a *Coxeter element* of  $(W, S)$ . If the diagram  $F$  is a tree, then the conjugacy class of  $w$  is independent of the choice of ordering. If  $F$  is bipartite (meaning we can write  $S = S_0 \sqcup S_1$  with all edges connecting  $S_0$  to  $S_1$ ), then the *bicolored* Coxeter element

$$w = \prod S_0 \prod S_1$$

is well-defined up to conjugacy (cf. [Mc, §5]). Thus in Theorem 3.1 we implicitly assume the Coxeter systems  $(W_i, s_i)$  are bipartite.

The *Coxeter polynomial* of a bipartite Coxeter system  $(W, S)$  is the characteristic polynomial

$$F(x) = \det(xI - w)$$

of its bicolored Coxeter elements. We generally denote it using the same symbol as the diagram. Note that if the diagram  $F$  has no multiple edges, then  $W$  preserves the lattice  $\mathbb{Z}^S$  and thus  $F(x) \in \mathbb{Z}[x]$ .

**Pointed Coxeter systems.** A *pointed* Coxeter system is a triple  $(W, S, s)$  with  $s \in S$ . It is determined up to isomorphism by a pointed diagram  $(F, s)$ . By deleting  $s$ , we obtain a Coxeter subsystem  $(\widehat{W}, \widehat{S})$  with Coxeter polynomial  $\widehat{F}(x)$ .

We let  $(A_n, i)$  and  $(E_n, i)$  denote the  $A_n$  and  $E_n$  diagrams with the  $i$ th vertex distinguished, using the numbering in Figures 1 and 3.

**Joins.** The *join*  $(W, S)$  of pointed Coxeter systems  $(W_i, S_i, s_i)_{i=1}^m$  is defined by taking an independent generator  $t$ , setting  $S = \{t\} \cup S_i$ , and setting

$$W = (W_1 * \cdots * W_m * \langle t \rangle) / \langle t^2 = (s_1 t)^3 = \cdots = (s_m t)^3 = \text{id} \rangle.$$

The corresponding diagram  $F$  is obtained from  $\sqcup F_i$  by adding a new vertex  $t$  and connecting it to each  $s_i$  with a single edge (see Figure 5). If all the diagrams  $F_i$  are bipartite, so is  $F$ .

In Theorem 3.1, basepoints  $s_i \in S_i$  must be chosen to make the join  $(W, S)$  well-defined, but the conclusion holds independent of the choice of basepoints.

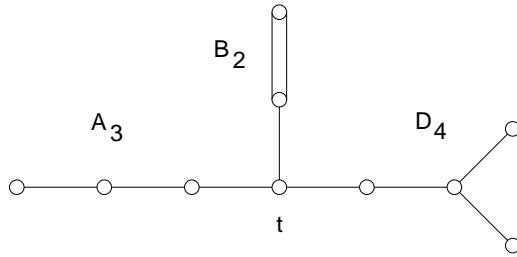


Figure 5. The join of  $A_3$ ,  $B_2$  and  $D_4$ .

**Proof of Theorem 3.1.** Let  $(W, S)$  be the join of  $(W_i, S_i)_1^m$ . By equation (3.1), a given reflection  $s(v)$  only changes the coordinate  $v_s$  of a vector  $v \in \mathbb{R}^S$ . Thus we have natural inclusions  $W_i \subset W$  compatible with the inclusions  $\mathbb{R}^{S_i} \subset \mathbb{R}^S$ .

Since  $s, t \in S$  commute whenever they are not joined by an edge in the Coxeter diagram, we can write the bicolored Coxeter element  $w \in W$  in the form

$$w = tw_1 \cdots w_m.$$

Let  $E_i \subset \mathbb{C}^{S_i} \subset \mathbb{C}^S$  be the  $\lambda$ -eigenspaces for  $w_i$ , extended by zero in the remaining coordinates. By (3.1) we have  $w_i|_{E_j} = \text{id}$  for  $i \neq j$ . Thus  $\oplus E_i$  is a  $\lambda$ -eigenspace for  $w_1 \cdots w_m$ . Since  $t(v)$  only changes  $v_t$ , there is a codimension-one subspace  $E \subset \oplus E_i$  such that  $t|_E = \text{id}$ . Consequently the multiplicity of  $\lambda$  as an eigenvalue for  $w$  is bounded below by

$$\dim(E) = \left( \sum \dim(E_i) \right) - 1 = \left( \sum m_i \right) - 1.$$

■

**The Coxeter polynomial of a join.** Here is an alternative approach to the result above. When  $F$  is the join of  $(F_i, s_i)_1^m$ , a straightforward matrix computation yields the following useful formula for its Coxeter polynomial:

$$F(x) = F_1(x) \cdots F_m(x) \left( (x+1) - x \sum_1^m \frac{\widehat{F}_i(x)}{F_i(x)} \right). \quad (3.2)$$



Cf. [CDS, Prob 9, p.78], [MRS, Cor. 4].

By writing the Coxeter element of  $(W_i, S_i)$  with  $s_i$  at the end, one can verify that the order of vanishing of its Coxeter polynomial satisfies  $\text{ord}(P_i, \lambda) - 1 \leq \text{ord}(\widehat{P}_i, \lambda)$ . Thus equation (3.2) implies

$$\text{ord}(F, \lambda) \geq -1 + \sum \text{ord}(F_i, \lambda).$$

This inequality is equivalent to Theorem 3.1 when the quadratic form  $B$  of  $(W, S)$  is non-degenerate, as it is for  $E_n$ ,  $n \neq 9$ .

## 4 Decorating $A_n$

In this section we generalize our results on  $E_n$  to more general diagrams  $F_n$  of the form shown in Figure 6. Our main result is:

**Theorem 4.1** *Let  $F_n$  be the sequence of Coxeter diagrams obtained by attaching pointed diagrams  $(B, s)$  and  $(C, t)$  to the ends of  $A_n$ . Assume  $F_n(x) \in \mathbb{Z}[x]$  for all  $n$ . Then either*

1. *The diagram  $F_n$  is spherical or affine for all  $n$ , or*
2. *The cyclotomic factor of  $F_n(x)$  is periodic for  $n \gg 0$ .*

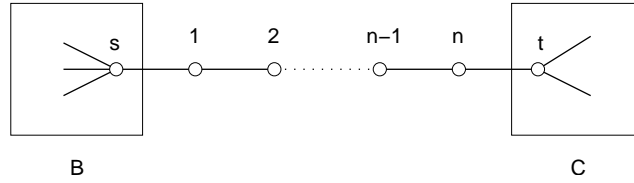


Figure 6. The diagram  $F_n$  obtained by attaching  $(B, s)$  and  $(C, t)$  to the ends of  $A_n$ .

**Coxeter polynomials.** We begin by determining the Coxeter polynomial  $F_n(x)$ . First, by repeatedly applying equation (3.2) with  $m = 1$ , we obtain:

**Proposition 4.2** *The Coxeter polynomial of the diagram  $B_n$  obtained by attaching  $(B, s)$  to one end of  $A_n$  satisfies:*

$$B_n(x)(x - 1) = x^{n+1}(B(x) - \widehat{B}(x)) + (x\widehat{B}(x) - B(x)).$$

Here is an example:

**Corollary 4.3** *For  $n \geq 4$ , we have*

$$E_n(x)(x - 1) = x^{n-2}(x^3 - x - 1) + (x^3 + x^2 - 1).$$

**Proof.** Take  $(B, s) = (A_4, 2)$ ; then  $B(x) = A_4(x)$ ,  $\widehat{B}(x) = A_1(x)A_2(x)$ , and  $B_n(x) = E_{n+4}(x)$ . Thus  $B(x) - \widehat{B}(x) = x(x^3 - x - 1)$  and  $x\widehat{B}(x) - B(x) = x^3 + x^2 - 1$ , which gives

$$E_{n+4}(x)(x - 1) = x^{n+2}(x^3 - x - 1) + (x^3 + x^2 - 1).$$

■

Since  $F_n$  is the join of  $B_{n-1}$  and  $C$ , by applying equation (3.2) once more we find:

**Proposition 4.4** *The Coxeter polynomials of  $F_n$ ,  $(B, s)$  and  $(C, t)$  are related by  $F_n(x)(x - 1) = x^{n+1}Q(x) - R(x)$ , where*

$$\begin{aligned} Q(x) &= (B(x) - \widehat{B}(x))(C(x) - \widehat{C}(x)) \quad \text{and} \\ R(x) &= (x\widehat{B}(x) - B(x))(x\widehat{C}(x) - C(x)). \end{aligned}$$

We will also need the following result. Let  $\beta(F_n) \geq 1$  denote the largest real zero of  $F_n(x)$ ; equivalently, the spectral radius of the bicolored Coxeter element for  $F_n$ .

**Proposition 4.5 (Hoffman–Smith)** *If  $\beta(F_n) > 1$ , then  $\beta(F_n) \neq \beta(F_{n+1})$ .*

**Proof.** Let  $A_{st} = 2I - B_{st}$  denote the symmetric ‘adjacency matrix’ for the  $F_n$  diagram, and  $\alpha(F_n)$  its spectral radius. Then since  $\beta(F_n) > 1$ , we have

$$\alpha(F_n) = (2 + \beta(F_n) + \beta(F_n)^{-1})^{1/2} > 2,$$

(see e.g. [Mc, Thm. 5.1]).

By [HS, Lem. 2.3 and Prop. 2.4], the condition  $\alpha(F_n) > 2$  implies that  $\alpha(F_{n+1}) < \alpha(F_n)$  if we are adding nodes to an internal path, and that  $\alpha(F_{n+1}) > \alpha(F_n)$  if we are adding nodes to an external path (i.e. if  $(B, s)$  or  $(C, t)$  is equal to  $(A_i, 1)$ .) (The proof in [HS] is given for graphs, but it applies without change to Coxeter diagrams, using the following key fact: if  $s$  is an endpoint of a maximal  $A_k$  embedded in  $F_n$ , either  $s$  is an endpoint of  $F_n$ , or  $\sum_{t \neq s} A_{st} \geq 1 + \sqrt{2}$ .)

In particular, we have  $\alpha(F_{n+1}) \neq \alpha(F_n)$ , and hence  $\beta(F_n) \neq \beta(F_{n+1})$ . ■

**Proof of Theorem 4.1.** Since  $\deg B > \deg \widehat{B}$  and  $\deg C > \deg \widehat{C}$ , we have  $Q(x) \neq 0$ . By Theorem 2.1, either:

- (i)  $F_n(x)(x - 1) = (x^n \pm x^i)Q(x)$ , or
- (ii) Only finitely many  $k$  satisfy  $F_n(\zeta_k) = 0$  for some  $n$ .

In case (i), the zeros of  $F_n(x)$  outside the unit circle must be constant as  $n$  varies. This implies the spectral radius  $\beta(F_n)$  of the bicolored Coxeter element is constant; hence  $\beta(F_n) = 1$  by Proposition 4.5, which means  $F_n$  is spherical or affine by [A’C].

For case (ii), fix  $k$  such that  $F_n(\zeta_k) = 0$ . Clearly the values of  $F_n(\zeta_k)$  are periodic in  $n$ . To complete the proof, we must show the order of vanishing of  $F_n$  at  $\zeta_k$  is also periodic. For this we may assume  $Q(x)$  and  $R(x)$  are relatively prime. Then  $Q(\zeta_k) \neq 0$ , and hence for all  $n \gg 0$ ,  $F'_n(\zeta_k) \neq 0$ , since the dominant term in the derivative is  $(n+1)\zeta_k^n Q(\zeta_k)$ . Consequently the cyclotomic zeros of  $F_n$  are simple for all  $n \gg 0$ , and the proof is complete. ■

**Notes.** For a survey of results on the largest eigenvalues of graphs, including the inequality of Hoffman and Smith used above, see [CR].

## References

- [A’C] N. A’Campo. Sur les valeurs propres de la transformation de Coxeter. *Invent. math.* **33**(1976), 61–67.
- [BK] E. Bedford and K. Kim. Dynamics of rational surface automorphisms: Linear fractional recurrences. *Preprint, 2006*.
- [Bou] N. Bourbaki. *Groupes et algèbres de Lie, Ch. IV–VI*. Hermann, 1968; Masson, 1981.
- [CDS] D. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs*. Academic Press, 1980.
- [CR] D. Cvetković and P. Rowlinson. The largest eigenvalue of a graph: a survey. *Linear and Multilinear Algebra* **28**(1990), 3–33.
- [Hir1] E. Hironaka. Salem-Boyd sequences and Hopf plumbing. *Osaka J. Math.* **43**(2006), 497–516.
- [Hir2] E. Hironaka. Hyperbolic perturbations of algebraic links and small Mahler measure. In *Singularities in Geometry and Topology 2004*, pages 77–94. Math. Soc. of Japan, 2007.
- [HS] A. J. Hoffman and J. H. Smith. On the spectral radii of topologically equivalent graphs. In *Recent Advances in Graph Theory (Proc. Second Czechoslovak Sympos., Prague, 1974)*, pages 273–281. Academia, 1975.
- [Hum] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, 1990.
- [Man] H. B. Mann. On linear relations between roots of unity. *Mathematika* **12**(1965), 107–117.
- [MRS] J. F. McKee, P. Rowlinson, and C. J. Smyth. Pisot numbers from stars. In *Number Theory in Progress, Vol. I*, pages 309–319. de Gruyter, 1999.
- [MS] J. F. McKee and C. J. Smyth. Pisot numbers, Mahler measure, and graphs. *Experiment. Math.* **14**(2005), 211–229.
- [Mc] C. McMullen. Coxeter groups, Salem numbers and the Hilbert metric. *Publ. Math. IHES* **95**(2002), 151–183.

MATHEMATICS DEPARTMENT  
HARVARD UNIVERSITY  
CAMBRIDGE, MA 02138-2901

MATHEMATICS DEPARTMENT  
FLORIDA STATE UNIVERSITY  
TALLAHASSEE, FL 32306-4510