p-Dirac Operators

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Abstract

We introduce non-linear Dirac operators in $\mathbb{R}^n$ associated to the p-harmonic equation and we extend to other contexts including spin manifolds and the sphere.

1 Introduction

Associated to each type of Laplacian one usually sees a first order linearization, to a Dirac operator. For instance associated to the Laplacian in $\mathbb{R}^n$ is the euclidean Dirac operator arising in Clifford analysis. For the Laplace-Beltrami operator associated to a Riemannian manifold there is the Hodge-Dirac operator $d + d^\star$, where d is the exterior derivative and $d^\star$ is the Hodge codifferential which is the formal adjoint to d. Further, in reverse order, to the Atiyah-Singer-Dirac operator on a spin manifold there is the spinorial Laplacian. Also on S^n one has a conformal Dirac operator and the conformal Laplacian. See for instance [3, 5, 9, 11] for details.

Besides the Laplacian in $\mathbb{R}^n$ there are also the non-linear operators referred to as p-Laplacians. See for instance [7, 8, 10]. Despite being non-linear these second order operators posses properties very similar to the usual Laplacian in euclidean space. Further when $p = 2$ this operator corresponds to the usual Laplacian in euclidean space.

Here we shall introduce a first order nonlinear differential operator which in the case $p = 2$ coincides with the euclidean Dirac operator. The conformal covariance of these operators are established. The n-harmonic equation arising here is Clifford algebra valued and the invariance of weak solutions to this equation under conformal transformations is only an invariance up to a factor of the pin group, the double covering of the orthogonal group. This is in contrast to weak solutions to the usual n-harmonic equation. We illustrate that we have a prpoer covariance not involving the pin group when we restrict to the scalar part of our clifford valued equations.

Further a non-linear Cauchy-Riemann equation is introduced and its covariance under composition with non-constant holomorphic functions is described.

Also a p-Dirac and a p-harmonic equation are set up on spin manifolds. We describe the behaviour of weak solutions to the n-Dirac equation under conformal rescaling of the metric on a spin manifold.

We conclude by introducing p-Dirac and p-harmonic equations on the sphere S^n and introducing solutions to these equations.

Dedication This paper is dedicated to the memory of J. Bures.

2 Preliminaries

The p-Laplace equation is the non-linear differential equation $div\|\nabla f\|^{p-2}\nabla f = 0$, where f is a a sufficently smooth, scalar valued function defined on a domain in $\mathbb{R}^n$. Further the operator ∇ is one of the simplest examples of a Dirac operator. One role here is to see how introducing a Dirac operator to the setting of p-Laplace equations might deepen ones perspective of such an equation. In order to introduce Dirac operators we need to first look at some basics of Clifford algebras.

Following [12] and elsewhere one can consider $\mathbb{R}^n$ as embedded in the real Clifford algebra Cl_n. For each $x \in \mathbb{R}^n$ we have within Cl_n the multiplication formula $x^2 = -\|x\|^2$. If $e_1, \ldots, e_n$ is an orthonormal basis for $\mathbb{R}^n$ this relationship defines an anti-commuting relationship $e_ie_j + e_je_i = -2\delta_{ij}$. If this relationship is the only relationship assumed on Cl_n then $1, e_1, \ldots, e_n, \ldots, e_{j_1} \ldots e_{j_r}, \ldots, e_1 \ldots e_n$ is a basis for Cl_n. Here $1 \leq r \leq n$ and $j_1 < \ldots j_r$. It follows that the dimension of Cl_n is 2^n. Further this algebra is associative.

We shall need the following antiautomorphism

$$\sim: Cl_n \to Cl_n :\sim (e_{j_1} \ldots e_{j_r}) = e_{j_r} \ldots e_{j_1}.$$

For each $A \in Cl_n$ we shall write $\tilde{A}$ for $\sim (A)$.

Note for $x = x_1e_1 + x_2e_2 + \ldots + x_ne_n$ that $e_1(x)e_1 = -x_1e_1 + x_2e_2 + \ldots + x_ne_n$. So we have a reflection in the e_1 direction. Similarly for $y \in S^{n-1}$, the unit sphere in $\mathbb{R}^n$, one has that yxy is a reflection in the y direction. Consequently for $y_1, \ldots y_J \in S^{n-1}$ we have that $y_1 \ldots y_J x y_J \ldots y_1$ is an orthogonal transformation acting on the vector x. In fact we have the group $Pin(n) := \{a \in Cl_n : a = y_1 \ldots y_J : y_1, \ldots y_J \in S^{n-1}$ and $J = 1, 2, 3, \ldots\}$. In [12] and elsewhere it is shown that $Pin(n)$ is a double covering of the orthogonal group $O(n)$. When we restrict J to be even we obtain a subgroup known as the spin group and denoted by $Spin(n)$. Further $Spin(n)$ is a double covering of the special orthogonal group, $SO(n)$. We shall also need the Lipschitz group $L(n) = \{a = x_1 \ldots x_J : x_1, \ldots, x_J \in \mathbb{R}^n \backslash \{0\}$ and $J \in \mathbb{N}\}$.

For $A = a_0 + a_1e_1 + \ldots + a_{1\ldots n}e_1 \ldots e_n \in Cl_n$ we define the norm, $\|A\|$, of A to be $(a_0^2 + \ldots + a_{1\ldots n}^2)^{\frac{1}{2}}$. Conjugation on the Clifford algebra is defined to be the anti-automorphism $- : Cl_n \to Cl_n : -(e_{j_1} \ldots e_{j_r}) = (-1)^r e_{j_r} \ldots e_{j_1}$. For $A \in Cl_n$ we write $\overline{A}$ for $-(A)$. Note that the real part, $Sc(A\overline{A})$, of $A\overline{A}$ is $\|A\|^2$. Further for A and $B \in Cl_n$ the product $\overline{A}B$ defines a Clifford algebra valued inner product on Cl_n for which $Sc(\overline{A}B)$ is the standard dot product on $\mathbb{R}^{2^n}$.

It is well known, see [12], that as a vector space Cl_n is canonically isomorphic to the alternating al gebra $\Lambda(\mathbb{R}^n)$.

To the vector $x \in \mathbb{R}^n$ we can associate the differential operator $D := \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$. This is the Dirac operator in euclidean space. Note that if f is a C^1 real valued function defined on a domain U in $\mathbb{R}^n$ then $Df = \nabla f$. Further $D^2 = -\triangle_n$ where $\triangle_n$ is the Laplacian in $\mathbb{R}^n$.

In [1] it is shown that given a Möbius transformation $M(x)$ over the one point compactification of $\mathbb{R}^n$ one can write this transformation as $(ax + b)(cx+d)^{-1}$ where a, b, c and $d \in Cl_n$ and they satisfy the following conditions
(i) a, b, c and d are all products of vectors
(ii) $\tilde{a}c$, $\tilde{c}d$, $\tilde{d}b$ and $\tilde{b}a \in \mathbb{R}^n$
(iii) $\tilde{a}d - \tilde{b}c = 1$.

If $y = M(x)$ then, [13], we have $cx + d \in L(n)$. Consequently $cx + d$ has a multiplicative inverse in Cl_n. It is shown in [4] that $J_{-1}(M, x)^{-1} D_x J_1(M, x) = D_y$ where D_x is the Dirac operator with respect to x and D_y is the Dirac operator with respect to y. Further $J_{-1}(M, x) = \frac{\widetilde{cx+d}}{\|cx+d\|^{n+2}}$ and $J_1(M, x) = \frac{\widetilde{cx+d}}{\|cx+d\|^n}$. Moreover $DJ_1(M, x) = 0$. See [13]. Consequently we have:

Lemma 1 *Suppose ψ is a C^1 function with compact support and $y = M(x)$.*

Then $D_y\psi(y) = (\widetilde{cx+d})^{-1}D_x(\widetilde{cx+d})\psi(M(x))$.

Proof We know that $D_y\psi(y) = J_{-1}(M,x)^{-1}D_xJ_1(M,x)\psi(M(x))$. But $DJ_1(M,x) = 0$. The result now follows from Leibniz rule. $\square$

It may be seen that $(cx+d)^{-1}D_x(\widetilde{cx+d})$ is a dilation and orthogonal transformation acting on D_x.

3 n-Dirac and n-Laplace Equations and Conformal Symmetry

If $v(x)$ is a C^1 vector field then the real or scalar part of Dv is $divv(x)$. Keeping this in mind we formally define the n-Dirac equation for a C^1 function $f: U \to Cl_n$, with U a domain in $\mathbb{R}^n$, to be $D\|f\|^{n-2}f = 0$. This is a non-linear first order differential equation for $n > 2$. When $f = Dg$ for some Cl_n valued function g then the n-Dirac equation becomes $D\|Dg\|^{n-2}Dg = 0$. Further when g is a real valued function the scalar part of this equation becomes $div(\|\nabla g\|^{n-2}\nabla g) = 0$ which is the n-Laplace equation described earlier. In the Clifford algebra context the n-Laplace equation extends to the equation $D\|Du\|^{n-2}Du = 0$. We shall refer to this equation as the n-Cl_n Laplace equation. The function $ln\|x\|$ is a solution to this equation on $\mathbb{R}^n\backslash\{0\}$. When u is scalar valued on identifying the Clifford algebra Cl_n with the alternating algebra $\Lambda(\mathbb{R}^n)$, the nonscalar part of the equation $D\|Du\|^{n-2}Du = 0$ becomes $d\|du\|^{n-2}du = 0$ where d is the exterior derivative. When $n = 3$ using the Hodge star map this equation becomes in vector calculus terminology $\nabla \times \|\nabla u\|^{n-2}\nabla u = 0$.

Noting that $D\frac{x}{\|x\|^n} = 0$ one may see that $\frac{x}{\|x\|^2}$ is a solution to the n-Dirac equation on $\mathbb{R}^n\backslash\{0\}$. We will assume that all Cl_n valued test functions have components in $C_0^\infty(U)$.

Definition 1 *Suppose $f: U \to Cl_n$ is in $L^n_{loc}(U)$, so each component of f is in $L^n_{loc}(U)$. Then f is said to be a weak solution to the n-Dirac equation if for each Cl_n valued test function η defined on U*

$$\int_U (\overline{\|f\|^{n-2}f}D\eta)dx^n = 0.$$

Note that for $g \in W^{1,n}_{loc}(U)$ then Dg is a weak solution to the n-Dirac equation.

We now proceed to establish a conformal covariance for the n-Dirac equation.

Theorem 1 *Suppose $f : U \to Cl_n$ is a weak solution to the n-Dirac equation. Suppose also $y = M(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation such that $cx+d$ is non-zero on the closure of $M^{-1}(U)$. Then $(cx+d)^{-1}f(M(x))$ is a weak solution to the n-Dirac equation on $M^{-1}(U)$.*

Proof: Consider $\int_U(\overline{\|f(y)\|^{n-2}f(y)}D_y\eta(y))dy^n$. As the Jacobian of M is $\frac{1}{\|cx+d\|^{2n}}$ and $D_y = J_{-1}(M,x)^{-1}D_xJ_1(M,x)$ this integral transforms to

$$\int_{M^{-1}(U)}(\overline{\|f(M(x))\|^{n-2}f(M(x))}J_1(M,x)D_xJ_1(M,x)\eta(M(x))dx^n.$$

Redistributing terms in $J_1(M,x)$ this integral becomes

$$\int_{M^{-1}(U)}(\overline{\|(cx+d)^{-1}f(M(x))\|^{n-2}(cx+d)^{-1}f(M(x))}D_xJ_1(M,x)\eta(M(x)))dx^n.$$

As $cx+d$ is bounded on $M^{-1}(U)$ then $J_1(M,x)\eta(M(x))$ is a test function on $M^{-1}(U)$. Further as $cx+d$ is bounded and C^∞ on $M^{-1}(U)$ then $(cx+d)^{-1}$ is a bounded C^∞ function on $M^{-1}(U)$. Consequently $(cx+d)^{-1}f(M(x)) \in L^n_{loc}(M^{-1}(U))$. The result follows. □

Definition 2 *Suppose $f : U \to Cl_n$ belongs to $W^{1,n}_{loc}(U)$ and*

$$\int_U(\overline{\|Df\|^{n-2}Df}D\eta)dx^n = 0$$

for each Cl_n valued test function defined on U. Then f is called a weak solution to the n-Cl_n Laplace equation.

We shall now examine the conformal symmetry of weak solutions to the n-Cl_n Laplace equation. Our arguments follow the lines for A-harmonic morphisms given in [7].

Theorem 2 *Suppose $f : U \to Cl_n$ is a weak solution to the n-Cl_n Laplace equation. Suppose further that $y = M(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation and $cx+d$ is non-zero on the closure of $M^{-1}(U)$. Then $f(M(x))$ is a weak solution to the equation $D_M\|Df(M(x))\|^{n-2}D_Mf(M(x)) = 0$ on $M^{-1}(U)$, where $D_M := \Sigma_{j=1}^n \frac{(cx+d)}{\|cx+d\|}e_j\frac{\widetilde{cx+d}}{\|cx+d\|}\frac{\partial}{\partial x_j}$.*

Proof As $DJ_1(M,x)=0$ then on changing variables and applying Lemma 1 the integral $\int_U(\overline{\|Df\|^{n-2}Df}D\eta)dy^n$ becomes

$$\int_{M^{-1}(U)}\|cx+d\|^{2n}(\|D_Mf(M(x))\|^{n-2}\overline{D_Mf(M(x))}D_M\eta(M(x)))\frac{dx^n}{\|cx+d\|^{2n}}$$

$$=\int_{M^{-1}(U)}\|D_Mf(M(x))\|^{n-2}\overline{D_Mf(M(x))}D_M\eta(M(x))dx^n.$$

In [2] it is shown that $\|(cx+d)A\|=\|cx+d\|\|A\|$ for any $A\in Cl_n$. Consequently $\|D_Mf(M(x))\|=\|Df(M(x))\|$. The result follows. $\square$

Note that $\frac{\widetilde{cx+d}}{\|cx+d\|}$ belongs to the pin group $Pin(n)$. So the covariance we have described here for weak solutions to the n-harmonic equation is not the same as for the classical n-harmonic equation descibed in [10] and elsewhere. We shall return to this point in the next section. First though let us note that it follows from [2] that $Sc(D_M\|Df(M(x))\|^{n-2}D_Mf(M(x)))=Sc(D\|Df(M(x))\|^{n-2}Df(M(x)))$ and

$$Sc(\|Df(M(x)\|^{n-2}\overline{D_Mf(M(x))}D_M\eta(M(x)))=Sc(\|Df(M(x))\|^{n-2}\overline{Df(M(x))}D\eta(M(x))).$$

When f is scalar valued this establishes the conformal invariance of the n-Laplace equation.

4 p-Dirac and p-Cl_n Laplace Equations and Möbius Transformations

We now turn to the more general case. For any real positive number p a differentiable function $f:U\rightarrow Cl_n$ is said to be a solution to the p-Dirac equation if $D\|f\|^{p-2}f=0$. For $1<p<n$ the function $\frac{x}{\|x\|^{\frac{n+p-2}{p-1}}}$ is a solution to this equation on $\mathbb{R}^n\backslash\{0\}$. We obtain this solution by again noting that $D\frac{x}{\|x\|^n}=0$ and solving the equation $\|f\|^{p-2}f=\frac{x}{\|x\|^n}$.

Definition 3 *Suppose that $f:U\rightarrow Cl_n$ belongs to $L^p_{loc}(U)$. Then f is a weak solution to the p-Dirac equation if for each Cl_n valued test function η defined on U we have $\int_U(\overline{\|f\|^{p-2}f}D\eta)dx^n=0$.*

Besides the p-Dirac equation we also need the following equation

$$D\|g\|^{p-2}A(x)g(x)=0$$

where $g : U \to Cl_n$ is a differentiable function and $A(x)$ is a real valued, smooth function. We shall call this equation the A, p-Dirac equation. The A, p-Dirac equation is a natural generalization of the A-harmonic functions defined in [7] and elsewhere.

Definition 4 *Suppose that $g : U \to Cl_n$ is in $L^p_{loc}(U)$ and $A : U \to \mathbb{R}^+$ is a smooth bounded function. Then g is a weak solution to the A, p-Dirac equation if for each Cl_n valued test function η defined on U we have*

$$\int_U (\overline{A(x)\|g(x)\|^{p-2}g(x)}D\eta(x))dx^n = 0.$$

By similar arguments to those used to prove Theorem 1 we now have:

Theorem 3 *Suppose $g : U \to Cl_n$ is a weak solution of the p-Dirac equation and $y = M(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation with $cx+d$ non-zero on the closure of $M^{-1}(U)$. Then $(cx+d)^{-1}g(M(x))$ is a weak solution to the A, p-Dirac equation on $M^{-1}(U)$, with $A(x) = \|cx+d\|^{p-n}$.*

Definition 5 *Suppose $h : U \to Cl_n$ is a solution to the equation*

$$D\|Dh(x)\|^{p-2}Dh(x) = 0$$

then h is called a p-harmonic function.

For $1 < p < n$ the function $\|x\|^{\frac{p-n}{p-1}}$ is a solution to this equation. Again when u is scalar valued one may identify Cl_n with $\Lambda(\mathbb{R}^n)$. In this case the non-scalar part of this p-harmonic equation becomes $d\|du\|^{p-2}du = 0$. Also when $n = 3$ the Hodge star map may be used to see that this equation becomes $\nabla \times \|\nabla u\|^{p-2}\nabla u = 0$.

Note that when h is real valued then the real part of the equation appearing in Definition 5 is the usual p-harmonic equation described in [7].

Definition 6 *For a function $h : U \to Cl_n$ in $W^{1,p}_{loc}(U)$, then h is called a weak solution to the p-harmonic equation if for each test function $\eta : U \to Cl_n$*

$$\int_U (\overline{\|Dh\|^{p-2}Dh}D\eta)dx^n = 0.$$

Definition 7 *For $h : U \to Cl_n$ a differentiable function and A as in Definition 4, then h is called an A, p-harmonic function if*

$$DA(x)\|Dh(x)\|^{p-2}Dh(x) = 0.$$

Further if $M(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation then h is called an A, p, M-harmonic function if

$$D_M A(x)\|Dh(M(x))\|^{p-2}D_M h(M(x)) = 0.$$

Definition 8 *Suppose $h : U \to Cl_n$ belongs to $W^{1,p}_{loc}(U)$ and A is as in Definition 4. Then h is called a weak solution to the A, p-Laplace equation, or A, p-harmonic equation if for each test function $\eta : U \to Cl_n$*

$$\int_U (\overline{A(x)\|Dh(x)\|^{p-2}Dh(x)}\eta(x))dx^n = 0.$$

Further it is a weak solution to the A, p, M-harmonic equation if

$$\int_{M^{-1}(U)} \overline{A(x)\|Dh(M(x))\|^{p-2}D_M h(M(x))}D_M\eta(M(x))dx^n = 0.$$

Further by similar arguments to those used to prove Theorem 2 we have:

Theorem 4 *Suppose that $h : U \to Cl_n$ is a weak solution to the p-harmonic equation and $M(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation with $cx+d$ non-zero on the closure of $M^{-1}(U)$. Then $h(M(x))$ is a weak solution to the A, p, M-harmonic equation on $M^{-1}(U)$ where $A(x) = \|cx+d\|^{2(p+2-n)}$.*

Again $Sc(D_M A(x)\|Dh(M(x))\|^{p-2}D_M h(M(X))) = Sc(DA(x)\|Dh(M(x))\|^{p-2}Dh(M(x)))$ and

$$Sc(A(x)\|Dh(M(x))\|^{p-2}\overline{D_M h(M(x))}D_M\eta(M(x))) = Sc(A(x)\|Dh(M(x))\|^{p-2}\overline{Dh(M(x))}D\eta(M($$

So when h is scalar valued this again re-establishes the A, p covariance of the p-harmonic equation.

5 The p-Cauchy-Riemann Equation

So far we have considered p-Dirac equations in dimensions $n \geq 3$. We now turn to look at the case $n = 2$. In this setting the Dirac operator is $e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y}$. This can be written as $e_1(\frac{\partial}{\partial x} + e_1^{-1} e_2 \frac{\partial}{\partial y})$ and $\frac{\partial}{\partial x} + e_1^{-1} e_2 \frac{\partial}{\partial y} = \frac{\partial}{\partial x} - e_1 e_2 \frac{\partial}{\partial y} = \frac{\partial}{\partial x} + e_2 e_1 \frac{\partial}{\partial y}$. Now $(e_2 e_1)^2 = -1$. Consequently we can identify $e_2 e_1$ with i, the square root of minus one. Then the operator $\frac{\partial}{\partial x} + e_2 e_1 \frac{\partial}{\partial y}$ can be identified with the Cauchy-Riemann operator $\frac{\partial}{\partial \overline{z}}$. If we restrict attention to functions taking values in the even subalgebra of Cl_2 spanned by 1 and $e_1 e_2$ and identify this algebra with $\mathbb{C}$ in the usual way then such a solution to the Dirac equation becomes a holomorphic function and vice versa.

A differentiable function $g : U \to \mathbb{C}$ is said to be a solution to the p-Cauchy-Riemann equation if it satisfies $\frac{\partial}{\partial \overline{z}} \|g(z)\|^{p-2} g(z) = 0$. A function $g : U \to \mathbb{C}$ belonging to $L^p_{loc}(U)$ is said to be a weak solution to the p-Cauchy-Riemann equation if for each test function η defined on U

$$\int_U \|g(z)\|^{p-2} \overline{g(z)} \frac{\partial}{\partial \overline{z}} \eta(z) dx dy = 0.$$

Note that if $h : U \to \mathbb{C}$ is a p-harmonic function then $g(z) := \frac{\partial}{\partial z} h(z)$ is a solution to the p-Cauchy-Riemann equation.

Let us now suppose that U is a bounded domain in the complex plane and $f : U \to \mathbb{C}$ is a non-constant holomorphic function with $f'(z) \neq 0$ on U. Using the identities $\eta(\zeta) = \frac{1}{\pi} \int_U \frac{\partial \eta(z)}{\partial \overline{z}} \frac{1}{z - \zeta} dx dy$ and $\eta(\zeta) = \frac{1}{\pi} \frac{\partial}{\partial \overline{z}} \int_U \frac{\eta(z)}{z - \zeta} dx dy$ for any test function $\eta : U \to \mathbb{C}$, and placing $z = f(u)$, then one may determine that

$$\frac{\partial}{\partial \overline{w}} f'(\zeta)^{-1} \eta(w) = \overline{f'}(\zeta)^{-1} \frac{\partial}{\partial \overline{\zeta}} \eta(f(\zeta))$$

where $w = f(\zeta)$.

Theorem 5 *Suppose $g : U \to \mathbb{C}$ is a weak solution to the p-Cauchy-Riemann equation and $f(z)$ is a holomorphic function defined on U with $f'(z) \neq 0$. Then $f'(\zeta) \|g(f(\zeta))\|^{p-2} g(f(\zeta))$ is a weak solution to the equation*

$$\frac{\partial}{\partial \overline{\zeta}} f'(\zeta) \|g(f(\zeta))\|^{p-2} g(f(\zeta)) = 0.$$

The proof follows the same lines as the proof of Theorem 1.

Note that if $f'(\zeta)\|g(f(\zeta))\|^{p-2} g(f(\zeta))$ is differentiable, then as f is holomorphic, $g(f(\zeta))$ is a solution to the p-Cauchy-Riemann equation.

6 p-Dirac and p-Harmonic Sections on Spin Manifolds

The material presented here depends heavily on the dot product in $\mathbb{R}^n$. In fact one can readily extend many of the basic concepts given here to more general inner product spaces. We shall turn to the context of spin manifolds. Amongst other sources basic facts on spin manifolds can be found in [9].

Suppose that M is a connected, orientable, Riemannian manifold. Associated to such a manifold is a principle bundle with each fiber isomorphic to the group $SO(n)$. If this bundle has a lifting to a further principle bundle with each fiber isomorphic to the group $Spin(n)$, then M is said to have a spin structure and M is called a spin manifold. Associated to a spin manifold is a vector bundle $Cl(M)$ with each fiber isomorphic to Cl_n.

The Levi-Civita connection ∇ on M lifts to a connection ∇' on the spin structure. Associated to that connection is the Atiyah-Singer-Dirac operator D'. If $e_1(x), \ldots, e_n(x)$ is a local orthonormal basis on M then locally $D' = \Sigma_{j=1}^n e_j(x) \nabla_{e_j(x)}$. Further [6] the inner product associated to the Riemannian structure of M lifts to a Clifford algebra valued inner product on $Cl(M)$. We denote this inner product by $< \, , \, >$.

Suppose now that U is a domain in M and $f : U \to Cl(M)$ is a differentiable section. Then f is said to be a solution to the p-Atiyah-Singer-Dirac equation if $D'\|f(x)\|^{p-2}f(x) = 0$. Further a section $f : U \to Cl(M)$ belonging to $L^p_{loc}(U)$ is said to be a weak solution to the p-Atiyah-Singer-Dirac equation if for each test section $\eta : U \to Cl(M)$ we have $\int_U < \|f\|^{p-2}f, D'\eta > dU = 0$ where dU is the volume element induced by the metric on M.

Besides the p-Atiyah-Singer-Dirac equation we may also introduce p-spinorial harmonic functions. A twice differentiable section $h : U \to Cl(M)$ is said to be p-spinorial harmonic if $D'\|D'h\|^{p-2}D'h = 0$. Further if we assumed that $h \in W^{1,p}_{loc}(U)$, then h is a weak solution of the p-spinorial harmonic equation if for each test section $\eta : U \to Cl(M)$ we have

$$\int_U < \|D'h\|^{p-2}D'h, D'\eta > dU = 0.$$

As Cl_n contains an identity there is a projection operator $Sc : Cl(M) \to Cl_{\mathbb{R}}(M)$ where $Cl_{\mathbb{R}}(M)$ is the line bundle of $Cl(M)$ with each fiber the real part of the fiber of $Cl(M)$. It makes sense to now talk of the equation

$$\int_U Sc < \|D'h\|^{p-2}D'h, D'\eta > dU = 0. \tag{1}$$

This last integral arises from the vanishing of the first variation associated to the Dirichlet integral $\int_U \|D'h\|^p dU$. When $p = 2$ this integral gives rise to the spinorial Laplace equation $D'^2 h = 0$.

We can go a little further if our manifold is both a spin manifold and a conformally flat manifold. A manifold is said to be conformally flat if it has an atlas whose transition functions are Möbius transformations.

Suppose M is a conformally flat spin manifold. For a Möbius transition function $M : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ with $M(x) = (ax+b)(cx+d)^{-1} = (-ax-b)(-cx-d)^{-1}$ we can make an identification $(x, X) \leftrightarrow (M(x), \pm(cx+d)^{-1}X)$ where $x \in U$ and $X \in Cl_n$. As M is a spin manifold signs can be chosen so that these identifications are globally compatible over the manifold M. Consequently we have a vector bundle on M. Given the conformal covariance of the n-Dirac equation described in Theorem 1 it now follows from Theorem 1 that one can set up weak solutions to the n-Dirac equation over domains in M, and taking values in this vector bundle.

Similarly one can now use Theorem 2 and the remarks following it to see the conformal invariance of Equation 1.

Two metrics g_{ij} and g'_{ij} on a Riemannian manifold are said to be conformally equivalent if there is a function $k : M \to \mathbb{R}^+$ such that $g'_{ij}(x) = k(x)g_{ij}(x)$ for each $x \in M$. We now investigate how weak solutions to the n-Dirac equation transform under such conformal changes of metric on a spin manifold. We shall denote the inner product on the spinor bundle of M associated to the metric g_{ij} by $< \, , >_1$ and the inner product associated to g'_{ij} by $< \, , >_2$. Further we denote the respective norms by $\| \, \|_1$ and $\| \, \|_2$. We denote the Dirac operator associated to $<, >_1$ by D_1 and the Dirac operator associated to $< \, , >_2$ by D_2. Consequently the integral

$$\int_U < \|f\|_2^{n-2} f, D_2\eta >_2 dU$$

becomes

$$\int_U << f, f >_2^{\frac{n-2}{2}} f, D_2\eta >_2 dU$$

$$= \int_U << f(x), f(x) >_2^{\frac{n-2}{2}} f(x), D_2\eta(x) >_1 k(x)^{2n} dU.$$

However, $D_1 k^{n-1}(x) = k^{n+1}(x) D_2$. See for instance [6].

Consequently the previous integral becomes

$$\int_U << f(x), f(x) >_1^{\frac{n-2}{2}} k^{n-2}(x) f(x), k(x)^{-n-1}(x) D_1 k^{n-1}(x)\eta(x) >_1 k^{2n}(x) dU.$$

This is equal to

$$\int_U < \|k(x)f(x)\|_1^{n-2}k(x)f(x), k(x)^{n-2}D_1k^{n-1}(x)\eta(x) >_1 dU.$$

This calculation describes the change in the n-Dirac equation under conformal rescaling of the metric on a spin manifold. Similar transformations are possible for weak solutions to the p-Dirac equation under conformal changes in metric.

7 The p-Dirac and p-Harmonic Equation on S^n

Here we shall consider the unit sphere S^n in $\mathbb{R}^{n+1} = span\{e_1, \ldots, e_{n+1}\}$, and we shall consider functions defined on domains on S^n and taking values in the Clifford algebra Cl_{n+1}. The stereographic projection from $S^n \setminus \{e_{n+1}\}$ to $\mathbb{R}^n$ corresponds to the Cayley transformation. Consequently one might expect that p-Dirac and p-harmonic equations can be set up on S^n. This indeed is the case. In [11] and elsewhere it is shown that the Dirac operator on $\mathbb{R}^n$ conformally transforms to the conformal Dirac operator $D_S := x(\Gamma + \frac{n}{2})$ on S^n where $\Gamma = \Sigma_{1\leq j<k\leq n} e_i e_j (x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i})$ and $x \in S^n$.

Definition 9 *Suppose U is a domain on S^n and $f : U \to Cl_{n+1}$ is a differentiable function. Then f is called a solution to the p-spherical Dirac equation if $D_S\|f\|^{p-2}f = 0$.*

Note that for $y \in S^n$ the function $\frac{x-y}{\|x-y\|^2}$ is a solution to the n-spherical Dirac equation. This follows as under the Cayley transformation the Clifford-Cauchy kernel $\frac{u-v}{\|u-v\|^n}$ in $\mathbb{R}^n$ conformally transforms to $\frac{x-y}{\|x-y\|^n}$ on S^n and $D_S \frac{x-y}{\|x-y\|^n} = 0$. For the same reason for $1 < p \leq n$ the function $\frac{x-y}{\|x-y\|^{\frac{n+p-2}{p-1}}}$ is a solution to the p-spherical Dirac equation.

Definition 10 *Suppose U is a domain on S^n and $f : U \to Cl_{n+1}$ belongs to $L^p(U)$. Then f is a weak solution to the p-spherical Dirac equation if for each test function $\eta : U \to Cl_{n+1}$*

$$\int_U (\overline{\|f\|^{p-2}f} D_S\eta) dU = 0$$

where dU is a volume element arising from the Lebesgue measure on S^n.

One needs to be a bit careful in setting up a p-harmonic equation on the sphere. This is because the differential operator on S^n that is conformally equivalent to the Laplacian in $\mathbb{R}^n$ is not D_S^2 but is the conformal Laplacian or Yamabe operator Y_S described in [3] and elsewhere. In [2] it is shown that $Y_S = D_S(D_S - x)$.

In [11] it is shown that $(D_S + \frac{p}{2}x)\|x-y\|^{-n+p} = \frac{-n+p}{2}\frac{x-y}{\|x-y\|^{n-p}}$. Bearing this in mind and that the fundamental solution to D_S is $\frac{x-y}{\|x-y\|^n}$ we define the p-spherical harmonic equation as follows.

Definition 11 *Suppose U is a domain on S^n and $f : U \to Cl_n$ belongs to $W^{1,p}_{loc}(U)$. Then f is a weak solution to the p-spherical harmonic equation if weakly $D_S\|(D_S + \frac{p}{2}x)f(x)\|^{p-2}(D_S + \frac{p}{2}x)f(x) = 0$.*

Solutions to the p-spherical harmonic equation include $\|x-y\|^{\frac{p-n}{p-1}}$.

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