Nonlinear A-Dirac Equations

Craig A. Nolder
Department of Mathematics
Florida State University
Tallahassee, FL 32306-4510, USA
nolder@math.fsu.edu

February 5, 2010

Abstract

This paper is a study of solutions to nonlinear Dirac equations, in domains in Euclidean space, which are generalizations of the Clifford Laplacian as well as elliptic equations in divergence form. A Caccioppoli estimate is used to prove a global integrability theorem for the image of a solution under the Euclidean Dirac operator. Oscillation spaces for Clifford valued functions are used which generalize the usual spaces of bounded mean oscillation, local Lipschitz continuity or local order of growth of real-valued functions.

1 Introduction

We develop tools for the study of solutions to nonlinear Dirac equations of the form $DA(x, Du) = 0$. Here u is a function valued in the universal Clifford algebra over a domain in Euclidean space, D is the usual Euclidean Dirac operator and A satisfies boundedness and ellipticity conditions. We refer to these as A-Dirac equations. A study of the conformal invariance of p-Dirac equations, a special case of A-Dirac equations, appears in [17]. These equations are nonlinear generalizations of the Dirac Laplace equation as well as generalizations of elliptic equations of A-harmonic type $\text{div}A(x, \nabla u) = 0$. The study of these equations is partially motivated by the fact that some arise as the Euler-Lagrange equations to variational integrals. There is an extensive literature concerning A-harmonic equations. See [9], [11] and [7]. For other recent work on nonlinear Dirac equations see [4],[5],[6],[21] and [22].

This paper is organized as follows. Section 2 presents preliminaries. In Section 3, A-Dirac equations are defined and a Caccioppoli estimate is given for solutions. In Section 4 we define the space of Clifford valued functions of p, k-oscillation. These spaces generalize bounded mean oscillation, local Lipschitz spaces and local order of growth spaces for real-valued functions. These spaces are used in the hypotheses for a global integrability theorem for Du when u

is a solution to an A-Dirac equation. Theorem 4.4 generalizes previous results for bounded harmonic functions in a ball [12] as well as results for A-harmonic functions which satisfy a Lipschitz condition in more general domains [3]. See Section 4 for details. In the last section, Section 5, we present a Poincaré inequality. We obtain relationships, for so-called uncoupled solutions, between the p, k-oscillation condition and the local Lipschitz and order of growth conditions. It is also shown that the p, k-oscilation condition is equivalent to a certain order of growth of $|Du|$. **Dedication** : This paper is dedicated to the memory of Juha Heinonen.

2 Preliminaries

We write $\mathcal{U}_n$ for the real universal Clifford algebra over $\mathbb{R}^n$. The Clifford algebra is generated over $\mathbb{R}$ by the basis of reduced products

$$\{e_1, e_2, ..., e_n, e_1e_2, ..., e_1 \cdots e_n\}$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ with the relation $e_ie_j+e_je_i = -2\delta_{ij}$. We write e_0 for the identity. The dimension of $\mathcal{U}_n$ is $\mathbb{R}^{2^n}$. We have an increasing tower $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathcal{U}_3 \subset \cdots$. Here $\mathbb{H}$ is the quaternions. The Clifford algebra $\mathcal{U}_n$ is a graded algebra as $\mathcal{U}_n = \oplus_l \mathcal{U}_n^l$ where $\mathcal{U}_n^l$ are those elements whose reduced Clifford products have length l. We use the conjugation $\overline{(e_{j_1}...e_{j_l})} = (-1)^l e_{j_l}...e_{j_1}$. The product $\overline{A}B$ defines an inner product on $\mathcal{U}_n$. We have $\overline{AB} = \bar{B}\bar{A}$ and $\overline{\overline{A}} = A$. For $A \in \mathcal{U}_n$, $\mathrm{Sc}(A)$ denotes the scalar part of A, that is the coefficient of the element e_0. The scalar part of a Clifford inner product $\mathrm{Sc}(\overline{A}B)$ is the usual inner product in $\mathbb{R}^{2^n}$ when A and B are identified as vectors. Throughout, $\Omega \subset \mathbb{R}^n$ is a connected and open set with boundary $\partial\Omega$. A Clifford valued function $u : \Omega \to \mathcal{U}_n$ can be written as $u = \Sigma_\alpha u_\alpha e_\alpha$ where each u_α is real valued and the e_α are reduced products. The norm used here is given by $|\Sigma_\alpha u_\alpha e_\alpha| = (\Sigma_\alpha u_\alpha^2)^{1/2}$. This norm is submultiplicative, $|AB| \le C|A||B|$.

The Dirac operator used here is as follows :

$$D = \Sigma_{j=1}^n e_j \frac{\partial}{\partial x_j}. \tag{1}$$

When u is a scalar function, Du can be identified with ∇u. Also $D^2 = -\Delta$. Here Δ is the Laplace operator with operates only on coefficients. A function is monogenic when $Du = 0$. It follows that if u is monogenic, then its coefficients are harmonic. When $u = u_1e_1 + ... + u_ne_n$ and $Du = 0$, the coefficients are the Stein-Weiss conjugate harmonic system. When $n = 2$ these are conjugate harmonic functions. See [20].

We use the isometric isomorphism between the Grassman algebra and the Clifford algebra. The Grassman algebra is denoted by $\Lambda^*(\Omega)$ with grading $\oplus_l \Lambda_l^*(\Omega)$. We write d for the exterior derivative and d^* for its formal adjoint. The linear extension of the map $\lambda : \Lambda^*(\Omega) \to \mathcal{U}_n(\Omega)$ defined on reduced multivectors by

$$\lambda : e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_l} \rightarrow e_{\alpha_1} \cdots e_{\alpha_l}$$

is an isometric isomorphism between vector spaces independent of the choice of basis. See [8].

For a Clifford valued function u we write $u^{\#}$ for $\lambda^{-1}(u)$. Via λ, the operator $d - d^*$ is mapped to D. By this we mean, in terms of reduced products,

$$(Du)^{\#} = du^{\#} - d^* u^{\#}.$$

Notice if $du^{\#} = d^* u^{\#} = 0$, that is $u^{\#}$ is a harmonic field, then $Du = 0$ so that u is monogenic. Conversely, if $Du = 0$, then $du^{\#} = d^* u^{\#}$ and so $d^* du^{\#} = dd^* u^{\#} = 0$. Moreover $(-\Delta u)^{\#} = (D^2 u)^{\#} = (d - d^*)^2 u^{\#} = (-dd^* - d^* d) u^{\#} = -\Delta u^{\#}$. Throughout Q is a cube in Ω with volume $|Q|$. We write σQ for the cube with the same center as Q and with sidelength σ times that of Q. For $q > 0$ we write $L^q(\Omega, \mathcal{U}_n)$ for the space of Clifford valued functions in Ω whose coefficients belong to the usual $L^q(\Omega)$ space. Also, $W^{1,q}(\Omega, \mathcal{U}_n)$ is the space of Clifford valued functions in Ω whose coefficients as well as their first distributional derivatives are in $L^q(\Omega)$. We also write $L^q_{loc}(\Omega, \mathcal{U}_n)$ for $\cap L^q(\Omega', \mathcal{U}_n)$, where the intersection is over all Ω' compactly contained in Ω. We similarly write $W^{1,q}_{loc}(\Omega, \mathcal{U}_n)$. For a form ω, we write $\omega \in L^p(\Omega, \Lambda^*)$ when $\lambda(\omega) \in L^p(\Omega, \mathcal{U}_n)$. We similarly denote Sobolev and local spaces for forms.

3 A-Dirac Equations and a Caccioppoli Estimate

We consider nonlinear generalizations of the Clifford-Laplace equation $\Delta u = 0$. We are partially motivated by the fact that these equations may arise as the Euler-Lagrange equations of variational integrals. To this purpose, we define the operator :

$$A(x, \xi) : \Omega \times \mathcal{U}_n \rightarrow \mathcal{U}_n.$$

Here $x \rightarrow A(x, \xi)$ is measurable for all ξ and $\xi \rightarrow A(x, \xi)$ is continuous for a.e. $x \in \Omega$. We assume the structure conditions with $p > 1$:

$$Sc(\overline{A(x, \xi)}\xi) \geq |\xi|^p, \tag{2}$$

and

$$|A(x, \xi)| \leq a|\xi|^{p-1}, \tag{3}$$

for some $a > 0$.

The exponent p will represent this exponent throughout the rest of this paper.

We consider weak solutions to

$$DA(x, Du) = 0 \tag{4}$$

for $u \in W^{1,p}_{loc}(\Omega, \mathcal{U}_n)$.

Using integration by parts we express this as follows :

$$\int_\Omega \overline{A(x, Du)} D\phi = 0 \tag{5}$$

for all $\phi \in W^{1,p}(\Omega, \mathcal{U}_n)$ with compact support.

These equations generalize the important case of the p-Dirac equation :

$$D(|Du|^{p-2} Du) = 0. \tag{6}$$

Here $A(x, \xi) = |\xi|^{p-2}\xi$.

These equations were introduced and their conformal invariance was studied in [17].

In the case of the p-Dirac equation, $|x|^{(p-n)/(p-1)}$, when $p \neq n$, and $\log|x|$, $p \neq n$, are solutions to (6). See [1] and [2] for a study of the scalar part of these equations in the plane.

It is important to notice that when u is a function and A maps $\mathcal{U}^1_n$ to $\mathcal{U}^1_n$, the scalar part of equation (4) is $\mathrm{div}A(x, \nabla u) = 0$. These equations have been extensively studied with many applications, including variational calculus and the theory of quasiregular mappings. See [9].

Notice if u is a solution to (5), then so is $u + u_0$ for any monogenic function u_0.

Next is a Caccioppoli estimate for solutions to (5).

Theorem 3.1 *Let u be a solution to (5) and $\eta \in C^\infty_0(\Omega), \eta > 0$. Then*

$$\left(\int_\Omega |Du|^p \eta^p\right)^{1/p} \leq pa\left(\int_\Omega |u|^p |\nabla \eta|^p\right)^{1/p}. \tag{7}$$

Proof :

Choose $\phi = -u\eta^p$. Then $D\phi = -p\eta^{p-1}(D\eta)u - \eta^p Du$. Hence using (5) and (2),

$$0 = \int_\Omega Sc(\overline{A(x, Du)} D\phi) = \int_\Omega Sc(\overline{A(x, Du)}(-p\eta^{p-1}(D\eta)u - \eta^p Du))$$

$$\leq -\int_\Omega |Du|^p \eta^p + p\int_\Omega |\overline{A(x, Du)}||u||D\eta||\eta|^{p-1}.$$

Using Hölder's inequality and (3) we have

$$\int_\Omega |Du|^p \eta^p \leq pa\int_\Omega |u||\nabla\eta||Du|^{p-1}|\eta|^{p-1}$$

$$\leq C\left(\int_\Omega |u|^p |\nabla\eta|^p\right)^{1/p}\left(\int_\Omega |Du|^p \eta^p\right)^{(p-1)/p}.$$

Corollary 3.2 *Suppose that u is a solution to (5). Let Q be a cube with $\sigma Q \subset \Omega$ where $\sigma > 1$. Then there is a constant C, independent of u, such that*

$$(\int_Q |Du|^p)^{1/p} \leq paC|Q|^{-1/n}(\int_{\sigma Q} |u|^p)^{1/p}. \tag{8}$$

Proof :

Choose $\eta \in C_0^\infty(\sigma Q), \eta > 0, \eta = 1$ in Q and $|\nabla \eta| \leq C|Q|^{-1/n}$.

Notice that u can be replaced by $u + w$ where w is any monogenic function.

4 Oscillation and Integrability

As an application of the Caccioppoli estimate we prove a global integrability theorem for Du when u is a solution to (5).

Notice that when $u \in W^{1,q}(Q, \mathcal{U}_n)$, it follows that $u^\# \in W^{1,q}(Q, \Lambda^*)$. In this case, when $q > 1$, there exists a constant C, depending only q and n, and a harmonic field $u_Q^\#$ such that

$$(\int_Q |u^\# - u_Q^\#|^q)^{1/q} \leq C|Q|^{1/n}((\int_Q |du^\#|^q)^{1/q} + (\int_Q |d^* u^\#|^q)^{1/q}). \tag{9}$$

This result appears in [10]. The constant can be taken as the constant for the unit cube. A scaling argument gives the factor $|Q|^{1/n}$. We write u_Q for $\lambda(u_Q^\#)$ throughout the rest of this paper. Notice that $u_Q = \lambda(u_Q^\#)$ is a monogenic function.

Definition 4.1 *Assume that $u \in L^1_{loc}(\Omega, \mathcal{U}_n)$, $q > 0$ and that $-\infty < k \leq 1$.*

We say that u is of q, k-oscillation in Ω when

$$\sup_{2Q \subset \Omega} |Q|^{-(qk+n)/qn}(\int_Q |u - u_Q|^q)^{1/q} < \infty. \tag{10}$$

When u is a function, u_Q is the average value of u over Q. When u a function, $q = 1$ and $k = 0$, (10) is the usual definition of the bounded mean oscillation. When $0 < k \leq 1$, (10) is equivalent to the usual local Lipschitz condition [14]. See Section 5 for definitions and equivalences about these spaces.

Notice that monogenic functions satisfy (10) just as the space of constants is a subspace of the bounded mean oscillation and Lipschitz spaces of real valued functions.

We remark it follows from Hölder's inequality that if $s \leq q$ and if u is of q, k-oscillation, then u is of s, k-oscillation.

The following lemma shows that Definition 4.1 is independent of the expansion factor of the cube.

Lemma 4.2 *Suppose that* $F \in L^1_{loc}(\Omega, \mathbb{R})$*,* $F > 0$ *a.e.,* $\gamma \in \mathbb{R}$ *and* $\sigma_1, \sigma_2 > 1$*. If*

$$\sup_{\sigma_1 Q \subset \Omega} |Q|^\gamma \int_Q F < \infty,$$

then

$$\sup_{\sigma_2 Q \subset \Omega} |Q|^\gamma \int_Q F < \infty.$$

Proof : If $\sigma_1 \leq \sigma_2$, then the implication is immediate. Assume $\sigma_1 > \sigma_2$. Let Q be a cube with $\sigma_2 Q \subset \Omega$. Dyadically subdivide Q into a finite number of subcubes $\{Q_i\}$ with $l(Q_i) \leq \frac{(\sigma_2 - 1)}{\sigma_1} l(Q)$. Then $\sigma_1 Q_i \subset \Omega$ for all i. Moreover

$$|Q|^\gamma \int_Q F \leq |Q|^\gamma \Sigma_i \int_{Q_i} F$$

$$= C(\sigma_1, \sigma_2, \gamma, n) \Sigma_i |Q_i|^\gamma \int_{Q_i} F.$$

We use a Whitney decomposition $\mathcal{W} = \{Q\}$ of Ω. The decomposition consists of closed dyadic cubes with disjoint interiors which satisfy

(*a*) $\Omega = \cup_{Q \in \mathcal{W}} Q$,

(*b*) $|Q|^{1/n} \leq d(Q, \partial\Omega) \leq 4|Q|^{1/n}$,

(*c*) $\frac{1}{4}|Q_1|^{1/n} \leq |Q_2|^{1/n} \leq 4|Q_1|^{1/n}$ when $Q_1 \cap Q_2$ is not empty.

Here $d(Q, \partial\Omega)$ is the Euclidean distance between Q and the boundary of Ω. See [20].

Definition 4.3 *We write* $N(j)$ *for the number of Whitney cubes with side length* 2^{-j} *where* j *is an integer. We say that* Ω *satisfies a Whitney cube number condition with exponent* $0 < \lambda < n$ *if there is a constant* $M < \infty$ *such that*

$$N(j) \leq M 2^{\lambda j}.$$

Notice that there exists j_0 such that $N(j) = 0$ for $j \leq j_0$. Balls and cubes satisfy a Whitney cube number condition with $\lambda = n - 1$. The geometry of $\partial\Omega$ determines the Whitney cube number condition. See [15]. Notice that the volume of a domain satisfying a Whitney cube number condition is finite.

The following result appears in [18]: There exists a Blaschke product B on the unit disk $\mathbb{D}$ of the complex plane such that

$$\int_{\mathbb{D}} |B'(z)| = \infty.$$

This shows that there exist bounded harmonic functions in the disk whose gradients fail to be L^1-integrable. The proof shows that

$$\int_{\mathbb{D}} |B^{'}(z)|(\log(2+|B^{'}(z)|))^{-1} = \infty.$$

Notice that $B = u_1 + iu_2$ can be embedded in $\mathcal{U}_2$ as $F = -u_1e_1 + u_2e_2$ with $|B^{'}(z)| = |DF(z)|$.

In [12] it is shown that if u is a bounded harmonic function in a ball $B \subset \mathbb{R}^n$, then

$$\int_B |\nabla u|(\log(2+|\nabla u|))^{-1-\epsilon} < \infty$$

for all $\epsilon > 0$.

Furthermore the following result follows from calculations in [3]: Suppose that Ω satisfies a Whitney cube number condition with exponent λ. If u is a solution to an A-harmonic equation, $\mathrm{div}A(x, \nabla u) = 0$, and satisfies a local Lipschitz condition with exponent k, $0 < k < 1$, then

$$\int_\Omega |\nabla u|^q < \infty$$

for $q < (n-\lambda)/(1-k)$.

The following theorem unites and generalizes these results to Clifford valued functions. Notice with the logarithmic term in place we attain the endpoint $(n-\lambda)/(1-k)$. A similar result for differential forms appears in [16]. It is notable here that we include a larger range of regularity than in the previous results, namely $-\infty < k \leq 1$ and the inclusion of order of growth spaces. The proof is similar to the case of the A-harmonic equation. The main step is to replace the Caccioppoli estimate for ∇u when u is A-harmonic with a similar Caccioppoli estimate for Du when u satisfies an A-Dirac equation.

Theorem 4.4 *Suppose that $u \in W^{1,p}_{loc}(\Omega, \mathcal{U}_n)$ and Ω is a domain which satisfies a Whitney cube number condition with exponent λ. If u is of p, k -oscillation, $-\infty < k < 1$ and $(n-\lambda)/(1-k) \leq p$, then*

$$\int_\Omega |Du|^{(n-\lambda)/(1-k)}(\log(2+|Du|))^{-1-\epsilon} < \infty$$

for all $\epsilon > 0$.

Proof : Let Q be a cube with $2Q \subset \Omega$. Using the Caccioppoli estimate (7), the p, k-oscillation condition (10) and Lemma 4.2 we have

$$|Q|^{(p-pk-n)/pn}(\int_Q |Du|^p)^{1/p} \leq C|Q|^{-(pk+n)/pn}(\int_{\sqrt{2}Q} |u-u_Q|^p)^{1/p} \leq C, \quad (11)$$

where C is a constant independent of Q.

Using Hölder's inequality and (11),

$$\int_Q |Du|^{(n-\lambda)/(1-k)} \leq C|Q|^{\lambda/n}. \tag{12}$$

Next let $Q_j \subset \Omega$ be a Whitney cube with side length 2^{-j}. Define, for $-\infty \leq k < 1$,

$$U_j = \{x \in Q_j : |Du(x)| \geq 2^{(1-k)j/2}\}.$$

We have using (12),

$$\int_{Q_j} |Du|^{(n-\lambda)/(1-k)}(\log(2+|Du|))^{-1-\epsilon}$$

$$\leq |Q_j|2^{(n-\lambda)j/2} + \int_{U_j} |Du|^{(n-\lambda)/(1-k)}(\log(2+|Du|))^{-1-\epsilon}$$

$$\leq 2^{-nj}2^{(n-\lambda)j/2} + Cj^{-1-\epsilon}\int_{Q_j} |Du|^{(n-\lambda)/(1-k)}$$

$$\leq 2^{-(n+\lambda)j/2} + Cj^{-1-\epsilon}|Q_j|^{\lambda/n}$$

$$= 2^{-(n+\lambda)j/2} + Cj^{-1-\epsilon}2^{-\lambda j}.$$

Let R_j denote the set of Whitney cubes in Ω with side length 2^{-j}. Using the Whitney cube number condition it follows that

$$\int_{R_j} |Du|^{(n-\lambda)/(1-k)}(\log(2+|Du|))^{-1-\epsilon} \leq N(j)\{2^{-(n+\lambda)j/2} + Cj^{-1-\epsilon}2^{-\lambda j}\}$$

$$\leq C\{2^{-(n-\lambda)j/2} + j^{-1-\epsilon}\}.$$

Altogether

$$\int_\Omega |Du|^{(n-\lambda)/(1-k)}(\log(2+|Du|))^{-1-\epsilon}$$

$$\leq \Sigma_{-\infty}^{\infty}\int_{R_j} |Du|^{(n-\lambda)/(1-k)}(\log(2+|Du|))^{-1-\epsilon} < \infty.$$

Notice that by Hölder's inequality Theorem 4.4 holds with $(n-\lambda)/(1-k)$ replaced with q when $q < (n-\lambda)/(1-k)$.

5 Poincaré Inequality

We continue to use the notation given in Section 2. We introduce a subclass of functions valued in $\mathcal{U}_n$.

Definition 5.1 *A Clifford valued function u is uncoupled in Ω if there is a constant C such that*

$$\max(|du^{\#}|, |d^*u^{\#}|) \leq C|Du|$$

a.e. in Ω.

Recall that we have

$$(Du)^{\#} = du^{\#} - d^* u^{\#}.$$

The difference on the right hand side can only have cancelation when $du^{\#}$ and $d^* u^{\#}$ have terms for the same basis element (and hence the same length) with appropriate coefficients. If this is not the case, and there is no cancelation, it follows that u is uncoupled. Notice since $d : \Lambda^l \to \Lambda^{l+1}$ and $d^* : \Lambda^l \to \Lambda^{l-1}$ cancelation is only possible if u has two reduced products such that the difference of their lengths is 2. For example, if $u \in \mathcal{U}_n^l$ for some l, then u is uncoupled. Notice that it is possible for u to be uncoupled even if some cancelation occurs in the above difference. We give some examples. Throughout these examples we consider functions valued in the algebra $\mathcal{U}_2$ and defined in domains in $\mathbb{R}^2$. They can be embedded in higher spaces if the coefficients depend only on two variables.

Example 5.2 *Let $u = u_1 e_1 + u_2 e_2$. Since u is a 1-form, it is uncoupled.*

Here

$$Du = (\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2})e_1 e_2 - (\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}).$$

Hence if $U + iV$ is analytic and we set $u_1 = U$ and $u_2 = -V$, then u is a monogenic function.

Notice that if this is the case, then $du^{\#} = 0$ and $d^* u^{\#} = 0$ so that $u^{\#}$ is a harmonic field.

Example 5.3 *Let $u = u_0 + u_{12} e_1 e_2$. In this case*

$$Du = (\frac{\partial u_0}{\partial x_1} + \frac{\partial u_{12}}{\partial x_2})e_1 + (\frac{\partial u_0}{\partial x_2} - \frac{\partial u_{12}}{\partial x_1})e_2.$$

Now set $u_0 = x_2$ and $u_{12} = x_1$. It follows that $Du = 0$ while $du^{\#} = d^* u^{\#} = dx_2$. Hence $|Du| = 0$, $|du^{\#}| = |d^* u^{\#}| = 1$ and u is not uncoupled.

Example 5.4 *For u in Example 5.3, let $u_0 = 2x_2$ and $u_2 = x_1$. Now $Du = e_2$, $du^{\#} = 2dx_2$ and $d^* u^{\#} = dx_2$.*

Hence u is uncoupled with constant $C = 2$.

We now give a version of the Poincaré inequality for Clifford valued functions.

Theorem 5.5 *Let $1 < q < \infty$ and suppose that $u \in W^{1,q}_{loc}(\Omega, \mathcal{U}_n)$ is uncoupled. Then for each cube $Q \subset \Omega$, there is a monogenic function u_Q and a constant C, independent of u, such that*

$$(\int_Q |u - u_Q|^q)^{1/q} \leq C|Q|^{1/n} (\int_Q |Du|^q)^{1/q}. \qquad (13)$$

Proof : The result follows from (9).

Notice $u_Q = \lambda(u_Q^{\#})$, where $u_Q^{\#}$ appears in (9), is a monogenic function. Moreover $|u - u_Q| = |u^{\#} - u_Q^{\#}|$ and since u is uncoupled,

$$\max(|du^{\#}|, |d^* u^{\#}|) \leq C|Du|.$$

Notice if u is monogenic, then $u = u_Q$.

Definition 5.6 *Suppose that $u : \Omega \to \mathcal{U}_n$. We write $u \in locLip_k(\Omega)$, $0 < k \leq 1$, when*

$$\sup\{\frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^k} : x_1, x_2 \in \Omega, |x_1 - x_2| < \frac{1}{2\sqrt{n}} d(x_1, \partial\Omega)\} < \infty. \tag{14}$$

Also we write $u \in locOrd_k(\Omega), k \leq 0$, when

$$\sup\{\frac{|u(x_1) - u(x_2)|}{d(x_1, \partial\Omega)^k} : x_1, x_2 \in \Omega, |x_1 - x_2| < \frac{1}{2\sqrt{n}} d(x_1, \partial\Omega)\} < \infty. \tag{15}$$

The factor $\frac{1}{2\sqrt{n}}$ is used for convenience and any factor between zero and one could be used.

Notice that $u = \Sigma_\alpha u_\alpha e_\alpha$ is in one of these classes if and only if each coefficient u_α is also. Local order of growth spaces are studied in [13].

Theorem 5.7 *a) Suppose that u is an uncoupled solution to (5). If $u \in locLip_k(\Omega)$, $0 < k \leq 1$, then u is of p, k-oscillation. If $u \in locOrd_k(\Omega)$, $k \leq 0$, then u is of p, k-oscillation.*

b) Suppose that u is a scalar function and a solution to (5). If u is of p, k-oscillation, then $u \in locLip_k(\Omega)$ when $0 < k \leq 1$ and $u \in locOrd_k(\Omega)$ when $k \leq 0$.

Proof : We use Lemma 4.2 in the proof.

a) Let Q be a cube with $4\sqrt{n}Q \subset \Omega$ and suppose that $x_1 \in Q$. Notice that $\sqrt{n}|Q|^{1/n} < \frac{1}{2\sqrt{n}} d(x_1, \partial\Omega)$. Using the Poincaré and the Caccioppoli inequalities we have

$$\begin{aligned} &(\int_Q |u - u_Q|^p)^{1/p} \\ &\leq C|Q|^{1/n}(\int_Q |Du|^p)^{1/p} \\ &\leq C(\int_{2Q} |u - u(x_1)|^p)^{1/p}. \end{aligned} \tag{16}$$

If $u \in locLip_k(\Omega)$ and $0 < k \leq 1$, then (16) is

$$\leq C \sup\{|x_2 - x_1|^k : x_2 \in 2Q\}|Q|^{1/p}$$

$$= C|Q|^{(pk+n)/pn}.$$

If $u \in locOrd_k(\Omega)$ and $k \leq 0$, then (16) is

$$\leq Cd(x_1, \partial\Omega)^k |Q|^{1/p}$$

$$\leq C|Q|^{(pk+n)/pn}.$$

Hence u is of p, k-oscillation in Ω.

b) If u is a function and a solution to (5), then u is a solution to the scalar part of this equation which is an A-harmonic equation. In this case u can be redefined on a set of measure zero as a continuous solution. Also it follows from Theorem 3.34 in [9] that if u is such a solution in $\sigma Q \subset \Omega$ and $x_1, x_2 \in Q$, then

$$|u(x_1) - u(x_2)| \leq C|Q|^{1/p} (\int_{\sigma Q} |u - u_{\sigma B}|^p)^{1/p}. \tag{17}$$

for $\sigma > 1$. If an expansion of σQ is contained in Ω then the p, k-oscillation condition is used with (17) to obtain

$$|u(x_1) - u(x_2)| \leq C|Q|^{k/n}. \tag{18}$$

Next let $|x_1 - x_2| \leq \frac{1}{2\sqrt{n}} d(x_1, \partial\Omega)$. Choose a cube Q with center x_1 and side length equal to $\frac{1}{2\sqrt{n}} d(x_1, \partial\Omega)$. It follows that $2\sqrt{n}|Q|^{1/n} = d(x_1, \partial\Omega)$. In this case (18) with $\sigma = \sqrt{2}$ shows that $u \in locOrd_k(\Omega)$ for $k \leq 0$.
Next suppose that $\frac{1}{8\sqrt{n}} d(x_1, \partial\Omega) < |x_1 - x_2| \leq \frac{1}{2\sqrt{n}} d(x_1, \partial\Omega)$. Then using the same cube as in the $locOrd$ case above gives

$$|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^k.$$

Otherwise, in the case that $|x_1 - x_2| \leq \frac{1}{8\sqrt{n}} d(x_1, \partial\Omega)$, choose a cube Q centered at x_1 with $|x_1 - x_2| = \frac{\sqrt{n}}{2}|Q|^{1/n}$. It follows from (18) that $u \in locLip_k(\Omega)$.

Notice using the Poincaré inequality we have a converse to (11) when u is uncoupled :

$$|Q|^{-(pk+n)/pn} (\int_Q |u - u_Q|^p)^{1/p} \leq C|Q|^{(p-pk-n)/pn} (\int_Q |Du|^p)^{1/p}. \tag{19}$$

Hence the p, k-oscillation condition is equivalent to this local order of growth of $|Du|$.

In conclusion, we have presented basic tools for the study of the A-Dirac equation. Following up on the work in [17] it is natural to consider the invariance of the solutions under compositions with a quasiregular mapping from $\Omega \to \mathbb{R}^n$. We conjecture that, just as in the case of the A-harmonic equation, the A-Dirac morphisms, for $p = n$, are exactly the quasiregular mappings.

References

[1] G. Aronsson, Construction of singular solutions to the p-harmonic equation and its limit equation for p = ∞, *Manuscripta Math.* **56**(1986), 135–158.

[2] G. Aronsson, On certain p-harmonic functions in the plane, *Manuscripta Math.* **61**(1988), 79–101.

[3] K. Astala and P. Koskela, Quasiconformal mappings and global integrability of the derivative, *J. Anal. Math.* **57**(1991), 203–220.

[4] Q. Chen, J. Jost, J.Y. Li and G.F. Wang, Dirac-harmonic maps, *Math Z.* **254**(2006), 409–432.

[5] Q. Chen, J. Jost, J.Y. Li and G.F. Wang, Regularity theorems and energy identities for Dirac-harmonic maps, *Math Z.* **251**(2005), 61–84.

[6] Q. Chen, J. Jost and G.F. Wang, Nonlinear Dirac equtions on Riemann surfaces, *Ann. Global Anal. Geom.* **33**, no. 3(2008), 253–270.

[7] L. D'Onofrio and T. Iwaniec, Notes on the p-harmonic equation, Contemporary Mathematics 370, The p-Harmonic Equation and Recent Advances in Analysis, 3rd Prarie Anaysis Seminar, Kansas State University, Manhatten, Kansas, Oct. 17-18, 2003, 25–50.

[8] J. Gilbert and M.A.M. Murray, Clifford algebras and Dirac operators in harmonic analysis, Cambridge University Press, 1991.

[9] J. Heinonen, T. Kilpelainen and O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford University Press, 1993.

[10] T. Iwaniec, Nonlinear Differential Forms, Lectures in Jyväskylä, International Summer School, August 1998.

[11] T. Iwaniec and G. Martin, Geometric Function Theory and Non-linear Analysis, Oxford University Press, 2001.

[12] T. Kilpelainen and P. Koskela, Global integrability of the gradients of solutions to partial diffierential equations, *Nonlinear Anal.* **23**(1994), 899–909.

[13] N. Langmeyer, The quasihyperbolic metric, growth, and John domains, *Ann. Acad. Sci. Fenn. Math.* **23**(1998), 205–224.

[14] N.G. Meyers, Mean oscillation over cubes and Hölder continuity, *Proc. Amer. Math. Soc.* **15**(1964),717–721.

[15] O. Martio and M. Vuorinen, Whitney cubes, p-capacity and Minkowski content, *Exposition Math.* **5**(1987), 17–40.

[16] C.A. Nolder, Global integrability theorems for A-harmonic tensors, *Jour. Math. Anal. and Appl.* **247**(2000), 236–245.

[17] C.A. Nolder and J. Ryan, p-Dirac operators, to appear *Advances in Applied Clifford Algebras*.

[18] W. Rudin, The radial variation of analytic functions, *Duke Math. J.***22**(1955), 235-242.

[19] W. Sprössig, On the treatment of classes of nonlinear boundary value and extension problems by Clifford analytic methods, Dirac operators in analysis, J. Ryan and D. Struppa (eds.), Pitman Research Notes in Mathematics Series **394**, Addison Wesley, 1998.

[20] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.

[21] C. Wang, A remark on nonlinear Dirac equations, preprint.

[22] C. Wang and D. Xu, Regularity of Dirac-Harmonic maps, preprint.