Manifolds with S^1-category 2 have cyclic fundamental groups

J. C. Gómez-Larrañaga* F. González-Acuña† Wolfgang Heil‡

Dedicated to José Maria Montesinos on the occasion of his 65th birthday

Abstract

A closed topological n-manifold M^n is of S^1-category 2 if it can be covered by two open subsets W_1,W_2 such that the inclusions $W_i \to M^n$ factor homotopically through maps $W_i \to S^1$. We show that for $n > 3$ the fundamental group of such an n-manifold is either trivial or infinite cyclic. [1] [2]

1 Introduction

The concept of the *A-category* of a manifold was introduced by Clapp and Puppe [1]. For a closed, connected n-manifold M it is defined as follows: Let A be a closed connected k-manifold, $0 \leq k \leq n-1$. A subset B in the 3-manifold M is *A-contractible* if there are maps $\varphi : B \longrightarrow A$ and $\alpha : A \longrightarrow M$ such that the inclusion map $i : B \longrightarrow M$ is homotopic to $\alpha \cdot \varphi$. The *A-category* $cat_A M$ of M is the smallest number of sets, open and A-contractible needed to cover M. Note that $2 \leq cat_A M \leq n+1$. For A a point P we obtain the classical Lusternik-Schnirelman category $cat\, M = cat_P M$. We are interested here in the case $A = S^1$.

In dimension 3, $cat\, M^3 = 2$ if and only if $\pi_1(M^3) = 1$, hence by the Poincaré conjecture $cat\, M^3 = 2$ if and only if $M^3 = S^3$. In [5] it was shown that

*Centro de Investigación en Matemáticas, A.P. 402, Guanajuato 36000, Gto. México. jcarlos@cimat.mx

†Instituto de Matemáticas, UNAM, Ciudad Universitaria, 04510 México, D.F. México and Centro de Investigación en Matemáticas, A.P. 402, Guanajuato 36000, Gto. México. fico@math.unam.mx

‡Department of Mathematics, Florida State University, Tallahasee, FL 32306, USA. heil@math.fsu.edu

[1]**AMS classification numbers: 57N10, 57N13, 57N15, 57M30**

[2]**Key words and phrases: Lusternik-Schnirelmann category, coverings of n-manifolds with open S^1-contractible subsets**

$cat\, M^3 = 3$ if and only if $\pi_1(M)$ is a non-trivial free group, hence it follows from the Poincaré conjecture $cat\, M^3 = 3$ if and only if M^3 is a connected sum of S^2-bundles over S^1. It was conjectured that also for dimensions $n > 3$, $cat\, M^n = 3$ implies that $\pi_1(M)$ is a non-trivial free group. This was proven to be true in [3].

In [6] it was shown that for a closed 3-manifold M^3 we have $cat_{S^1} M^3 = 2$ if and only if $\pi_1(M^3)$ is cyclic. By results of Olum [10] and Perelman [9] this implies that $cat_{S^1} M^3 = 2$ if and only if M^3 is a lens space or M^3 is the non-orientable S^2-bundle over S^1. For the case $n > 3$ we showed [6] that $cat_{S^1} M^n = 2$ implies that $\pi_1(M^n)$ is cyclic or a nontrivial product with amalgamation $A *_C B$ of cyclic groups. In this paper we show that in this case $\pi_1(M^n)$ is in fact cyclic (Corollaries 1 and 2):

Theorem: If M is a closed n-manifold for $n > 3$ and with $\mathrm{cat}_{S^1} M = 2$ then $\pi(M)$ is trivial or infinite cyclic.

The paper is organized as follows: In section 2 we relate $cat\, M$ to $cat_K M$ for a 1-dimensional CW-complex K to show that for $n > 3$, $\pi_1(M)$ is either trivial or infinite. In section 3 we describe the main technique used in [6], the graph of groups associated to a decomposition of M into two S^1-contractible submanifolds, and review the propositions of [6] needed for our proof. Homology considerations are then used to obtain the Theorem in the orientable case. Finally in section 4 we prove the Theorem in the non-orientable case.

2 $\pi_1(M)$ is trivial or infinite

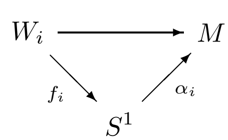
For a cell-complex K, a subspace W of the manifold M^n is K-contractible (in M^n) if there exist maps $f : W \to K$ and $\alpha : K \to M^n$ such that the inclusion $\iota : W \to M^n$ is homotopic to $\alpha \cdot f$.

Notice that a subset of a K-contractible set is also K-contractible.

$cat_K M$ is the smallest m such that there exist m open K-contractible subsets of M whose union is M.

In particular, when $cat_{S^1} M^n = 2$, there are two open subsets W_0, W_1 of M such that $M = W_0 \cup W_1$ and for $i = 0, 1$, there are maps f_i and α_i as in the diagram below, such that the inclusion $W_i \hookrightarrow M$ is homotopic to $\alpha_i \cdot f_i$.

(∗)



Lemma 1. *Let K be a cell-complex of dimension $< n$. Then*

$$cat\, M \leq cat_K M \cdot cat\, K$$

.

Proof. Suppose W is an open K-contractible subset of M with inclusion factoring homotopically through $W \xrightarrow{f} K \xrightarrow{\alpha} M$. If $cat\, K = k$, there is a cover of K by open subsets $K_1, \ldots, K_k$, each contractible in K. It is now easy to see that $\{f^{-1}(K_1), \cdots, f^{-1}(K_k)\}$ is an open cover of W with each $f^{-1}(K_i)$ contractible in M. Hence if $cat_K M = m$, then M can be covered by $m \cdot k$ open sets, each contractible in M.

□

In the next lemma we assume that $\alpha : K \to M$ is an inclusion map. We can do this by the following

Remark 1. Every map of a finite 1-complex into an n-manifold, $n \geq 3$, is homotopic to an embedding.

This follows since such a map can be approximated by an embedding ([2], Corollary 26.3A).

Lemma 2. *Let K be a cell-complex of dimension $< n$ embedded in M and let $p : \tilde{M} \to M$ be a covering map. If $W \subset M$ is K-contractible then $\tilde{W} := p^{-1}(W)$ is $p^{-1}(K)$-contractible in $\tilde{M}$.*
In particular, if $dim(K) \leq 1$ and $\tilde{M}$ is simply connected, then $\tilde{W}$ is contractible in $\tilde{M}$.

Proof. There is a map $f : W \to K \subset M$ and a homotopy $h_t : W \to M$ such that h_0 is the inclusion and $h_1 = f$. Define $\tilde{h}_0 : \tilde{W} \to \tilde{M}$ to be the inclusion. By the homotopy lifting theorem $\tilde{h}_0$ extends to a homotopy $\tilde{h}_t : \tilde{W} \to \tilde{M}$ such that $\tilde{h}_1(\tilde{W}) \subset p^{-1}(f(W)) \subset p^{-1}(K)$.
If $dim(K) \leq 1$ and $\tilde{M}$ is simply connected then $p^{-1}(K)$ is contractible in $\tilde{M}$ and therefore $\tilde{h}_0$ is homotopic to the constant map.

□

We now consider $K = S^1$.

Proposition 1. *Suppose M is a closed n-manifold for $n > 3$ with $cat_{S^1} M = 2$. Then $\pi_1(M)$ is trivial or infinite.*

Proof. Suppose $\pi_1(M)$ is finite and non-trivial. For the universal cover $\tilde{M}$ of M, a cover $\{W_1, W_2\}$ of M by S^1-contractible subsets lifts to a cover $\{\tilde{W}_1, \tilde{W}_2\}$ of $\tilde{M}$ by (in $\tilde{M}$) contractible subsets (Lemma 2). Hence $cat\, \tilde{M} = 2$ and $\tilde{M}$ is a (homotopy) n-sphere ([4]). By a theorem of Krasnosel′skii′ ([8]) this implies that $cat\, M = n+1$. By Lemma 1, $cat_{S^1} M \geq \frac{cat\, M}{cat\, S^1} = \frac{n+1}{2} > 2$, a contradiction.
□

3 The orientable case

The next proposition asserts that instead of open sets we can choose compact submanifolds intersecting only along their boundary.

Proposition 2. *Let M be an n-manifold with $\operatorname{cat}_{S^1} M = 2$. Then M can be expressed as a union of two compact S^1-contractible n-submanifolds W_0, W_1 such that $W_0 \cap W_1 = \partial W_0 \cap \partial W_1$ is a properly embedded $(n-1)$-submanifold F. Hence for $i = 0, 1$, there are maps f_i and α_i such that the diagram $(*)$ is homotopy commutative. Furthermore for $n > 2$, we may assume that α_i is an embedding and $\alpha_i(S^1)$ does not intersect $W_0 \cap W_1$.*

This follows from Corollary 1 and Proposition 1 of [6]; the hypothesis that M be closed is not used in the proofs. (The statement that we may assume that α_i is an embedding was used without justification in [6], but follows from Remark 1).

For a decomposition as in proposition 2, we consider the graph G of (M, F). The vertices correspond to the components W_i^j of W_i, $i = 0, 1$; for a component F_{jk} of $W_0 \cap W_1$ there is an edge joining the corresponging vertices. $\pi_1(M)$ is the fundamental group of $\mathcal{G}$, the graph of cyclic groups of (M, F). The vertex groups are the cyclic groups $G(W_i^j) := \operatorname{im}(\pi_1 W_i^j \longrightarrow \pi_1 M)$, the edge groups are the cyclic groups $G(F_{jk}) := \operatorname{im}(\pi_1 F_{jk} \longrightarrow \pi_1 M)$. Note that the edge homomorphisms $G(F_{jk}) \longrightarrow G(W_i^j)$ are injective. If every component of F is separating then the graph G is a tree. Since W_i can be deformed into a circle contained in $M - F$ $(i = 0, 1)$ we can show ([6] Lemma 9) that there is a sub-graph G_Q of G homeomorphic to a point or a segment such that the fundamental group of the restriction of $\mathcal{G}$ to G_Q is all of $\pi_1(M)$. Furthermore at most two of the edge monomorphisms corresponding to edges of G_Q are not epimorphisms. From this we obtain the following Proposition below (Theorem 2 of [6]). Again the hypothesis that M be closed is not used in the proofs.

For a path-connected subspace Y of M we let $G(Y) := \operatorname{im}(\pi_1 Y \to \pi_1 M)$

Proposition 3. *Suppose $M = W_0 \cap W_1$ is as in proposition 2. Assume that $n > 2$ and every component of F is separating.*
(a) If $\alpha_0(S^1)$ is contained in W_1 or $\alpha_1(S^1)$ is contained in W_0 then $\pi_1(M)$ is cyclic.
(b) If $\alpha_i(S^1)$ is contained in W_i (i=0,1) and F' is a component of F separating $\alpha_0(S^1)$ from $\alpha_1(S^1)$ let X_i be the component of $M - F'$ containing $\alpha_i(S^1)$. Then $G(X_i) = G(\alpha_i(S^1))$ and $\pi_1(M) = G(X_0) *_{G(F')} G(X_1)$.

By abelianizing $\pi_1(M) = G(X_0) *_{G(F')} G(X_1)$ (and noting that $G(X_i)$ and $G(F')$ are cyclic) we obtain the "abelianized free product with amalgamation" $H_1(M) = G(X_0) \oplus_{G(F')} G(X_1)$.

This implies that $G(X_0) = \operatorname{im}(H_1(W_0) \to H_1(M))$, $G(X_1) = \operatorname{im}(H_1(W_1) \to H_1(M))$ and $G(F') = \operatorname{im}(H_1(F) \to H_1(M))$.

Lemma 3. *Suppose M is a closed orientable n-manifold, $n > 2$, and $M = W_0 \cup W_1$ where W_0 and W_1 are compact K-contractible n-submanifolds with K a cell-complex of dimension $< n-1$. If every component of $F = W_0 \cap W_1 = \partial W_0 \cap \partial W_1$*

separates then
(1) $im(H_1(W_0) \to H_1(M)) = im(H_1(F) \to H_1(M)) = im(H_1(W_1) \to H_1(M))$
and
(2) $H_1(F) \to H_1(M))$, $H_1(F) \to H_1(W_i)$ *is onto (for* $i = 0, 1$*).*

Proof. In the exact sequence

$$H^n(W_i) \leftarrow H^n(M) \leftarrow H^n(M, W_i) \overset{\delta}{\leftarrow} H^{n-1}(W_i) \overset{i^*_{n-1}}{\leftarrow} H^{n-1}(M)$$

we have $H^n(W_i) = 0$ since W_i is not closed and $H_n(M) = \mathbb{Z}$ since M is closed and orientable. By excision and Poincaré duality, $H^n(M, W_i) \cong H^n(W_{1-i}, \partial W_{1-i}) = H_0(W_{1-i}) = \mathbb{Z}^{\omega_{i-1}}$, where ω_i is the number of components of W_i. Now $i^*_{n-1} = 0$ because it factors through $H^{n-1}(K) = 0$. Hence δ is surjective and $H^{n-1}(W_i) = \mathbb{Z}^{\omega_{i-1}-1}$.

Since every component of F separates, the graph of (M, F) is a tree and therefore $\omega_0 + \omega_1 - \gamma = 1$, where γ is the number of components of F.

Hence by Poincaré duality $H_1(W_i, F) \cong H^{n-1}(W_i) = \mathbb{Z}^{\gamma - \omega_i}$.

From the homology sequence of (W_i, F) we have $\mathbb{Z}^{\gamma-\omega_i} = ker(H_0(F) \to H_0(W_i)) = im(j_* : H_1(W_i, F) \to H_0(F))$. Hence j_* is injective and $H_1(F) \to H_1(W_i)$ is surjective. From this (1) follows.

In the Mayer Vietoris sequence
$H_1(W_0) \oplus H_1(W_1) \overset{\kappa_*}{\to} H_1(M) \overset{\delta_*}{\to} H_0(F) \overset{\beta}{\to} H_0(W_0) \oplus H_0(W_1) \twoheadrightarrow H_0(M) = \mathbb{Z}$,
β must be injective since $\omega_0 + \omega_1 = \gamma + 1$. Hence $\delta_* = 0$ and κ_* is onto, hence (1) implies that $H_1(W_i) \to H_1(M)$ is onto (for $i = 0, 1$) and (2) follows.

□

Corollary 1. *If M is a closed orientable n-manifold for $n > 3$ and with* $\mathrm{cat}_{S^1} M = 2$ *then $\pi(M)$ is trivial or infinite cyclic.*

Proof. Write M as a union of two S^1-contractible submanifolds as in Proposition 2. Since a component F' of $W_0 \cap W_1$ is S^1-contractible, the inclusion induced homomorphism factors as $H_{n-1}(F'; \mathbb{Z}_2) \to H_{n-1}(S^1; \mathbb{Z}_2) \to H_{n-1}(M^n; \mathbb{Z}_2)$, and since $H_{n-1}(S^1; \mathbb{Z}_2) = 0$, F' is separating. From Proposition 3, $\pi_1(M)$ is cyclic or $\pi_1(M) = G(X_0) *_{G(F')} G(X_1)$ where the images of $G(X_0)$ and $G(X_1)$ in $H_1(M)$ are the images of $H_1(W_0)$ and $H_1(W_1)$ in $H_1(M)$. By lemma 3 they coincide; hence $G(X_0) = G(X_1) = G(F')$ and $\pi_1(M)$ is again cyclic. Now the result follows from Proposition 1.

□

4 The non-orientable case

Lemma 2. Suppose M is a closed non-orientable n-manifold, $n > 2$, and $M = W_0 \cup W_1$ where W_0 and W_1 are compact K-contractible n-submanifolds with K a cell-complex of dimension $< n-1$. If every component of $F = W_0 \cap W_1 = \partial W_0 \cap \partial W_1$ separates, then $coker(H_1(F) \to H_1(M))$ and $coker(H_1(W_i) \to H_1(M)$ are groups of odd order (for $i = 0, 1$).

Proof. The proof is similar to that of Lemma 3, taking $\mathbb{Z}_2$ coefficients. □

Corollary 2. *If M is a closed non-orientable n-manifold for $n > 3$ and with* $\text{cat}_{S^1} M = 2$ *then $\pi(M)$ is infinite cyclic.*

Proof. As in the proof of Corollary 1 we obtain M as a union of two S^1-contractible submanifolds as in Propositions 2 and 3. Then $\pi_1(M)$ is cyclic or $\pi_1(M) = G(X_0) *_{G(F')} G(X_1)$ where the images of $G(X_0)$ and $G(X_1)$ in $H_1(M)$ are the images of $H_1(W_0)$ and $H_1(W_1)$ in $H_1(M)$. By lemma 3 the index of $G(X_i)$ in $H_1(M) = G(X_0) \oplus_{G(F')} G(X_1)$ is odd. This index is the same as the index of $G(F')$ in $G(X_i)$.

Now let $p : \tilde{M} \to M$ be the orientable two-fold covering. Since $im(H_1(\alpha_i(S^1)) \to H_1(M)) = im(H_1(W_i) \to H_1(M))$ has odd order, α_i is an orientation reversing loop and $\tilde{S} := p^{-1}(\alpha(S^1))$ is homeomoprhpic to S^1. By Lemma 2, $\tilde{M} = \tilde{W_0} \cup \tilde{W_1}$, where $\tilde{W_i} = p^{-1}W_i$ is $\tilde{S}$-contractible. Hence $cat_{S^1}(\tilde{M}) = 2$ and it follows from Corollary 1 that $\pi_1(\tilde{M})$ is 1 or $\mathbb{Z}$. If $\pi_1(\tilde{M}) = 1$ then $\pi_1(M) = \mathbb{Z}_2$ which is impossible by Proposition 1.

Hence $\pi_1(\tilde{M}) = \mathbb{Z}$, a subgroup of index 2 of $\pi_1(M) = G(X_0) *_{G(F')} G(X_1)$, with $[G(X_0) : G(F')]$ and $[G(X_1) : G(F)]$ odd. The only noncyclic extensions of $\mathbb{Z}$ by $\mathbb{Z}_2$ are $\mathbb{Z} \times \mathbb{Z}_2$ and the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$ (see e.g. Lemma 1.2 of [7]) and it is easy to see that these two groups are not a free product with amalgamation $A *_C B$ of cyclic groups with $[A : C]$ and $[B : C]$ odd. Hence $\pi_1(M) = \mathbb{Z}$.

□

References

[1] M. Clapp and D.Puppe, Invariants of the Lusternik-Schnirelmann type and the topology of critical sets, Trans. Amer. Math. Soc. 298 (1986), 603–620.

[2] R.J. Daverman, Decompostions of Manifolds, Academic Press (1986).

[3] Dranishnikov, A; Katz, M.; Rudyak, Y.: Small values of the Lusternik-Schnirelmann category for manifolds. Geometry and Topology 12 (2008), issue 3, 1711-1727.

[4] Fox, Ralph H. On the Lusternik-Schnirelmann category, Ann. of Math. (2) 42, (1941), 333–370.

[5] J. C. Gómez-Larrañaga and F. González-Acuña, Lusternik–Schnirelmann category of 3–manifolds, Topology 31 (1992), 791–800.

[6] J. C. Gómez-Larrañaga, F. González-Acuña and Wolfgang Heil, Fundamental groups of manifolds with S^1-category 2, Math. Z. 259 (2008), 419–432.

[7] B. Jahren, S. Kwasik, Free involutions on $S^1 \times S^n$, preprint.

[8] Krasnosel′skii′ M. A. On special coverings of a finite-dimensional sphere. (Russian) Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 961–964. MR0081454 (18,406a)

[9] John Morgan and Gang Tian, *Ricci Flow and the Poincarè Conjecture*, AMS Clay Math. Monographs, vol 3 (2007).

[10] Paul Olum, Mappings of manifolds and the notion of degree, Ann. of Math. (2) 58, (1953). 458–480