Links in S^3 of S^1-category 2

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Abstract

A non-splittable link of S^1-category 2 is a Burde-Murasugi link. [1] [2]

1 Introduction

A subset W in an n-manifold M is S^1*-contractible* if there are maps $f : W \to S^1$ and $\alpha : S^1 \to M$ such that the inclusion map $i : W \to M$ is homotopic to $\alpha \cdot f$. The S^1*-category* $cat_{S^1} M$ of M is the smallest number of sets, open and S^1-contractible needed to cover M. Note that if M is closed, $2 \leq cat_{S^1} M \leq n + 1$.

For dimension 3 it was shown in [3] that a closed 3-manifold M^3 has $cat_{S^1} M^3 = 2$ if and only if $\pi_1(M^3)$ is cyclic. By results of Olum [7] and Perelman [6] this implies that $cat_{S^1} M^3 = 2$ if and only if M^3 is a lens space or M^3 is the non-orientable S^2-bundle over S^1.

In this paper we consider the question of S^1-category for knot spaces and more generally for compact irreducible 3-manifolds with boundary. Note that if $\partial M \neq \emptyset$ and $cat_{S^1} M^3 = 1$ then $\pi_1(M)$ is trivial or cyclic and it follows from [6] that M is a ball, a solid torus or a solid Kleinbottle. Our main result is that an orientable and irreducible 3-manifold M with $cat_{S^1} M^3 = 2$ is a Seifert fiber space with handles and at most 2 exceptional fibers. In particular M can then be obtained from two solid tori by glueing their boundaries along incompressible annuli and disks. As a corollary we obtain that the space of a non-splittable

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link L in S^3 has S^1-category 2 if and only if L is a Burde-Murasugi link different from the trivial knot. (For the definition of Burde-Murasugi link see section 3 and [1]).

2 Irreducible and incompressible S^1-contractible submanifolds.

When $cat_{S^1} M = 2$, there are two open subsets W_0, W_1 of M such that $M = W_0 \cup W_1$ and for $i = 0, 1$, there are maps f_i and α_i such that the inclusion $W_i \hookrightarrow M$ is homotopic to $\alpha_i \cdot f_i$. Note that a compact 3-submanifold of an S^1-contractible subset is S^1-contractible.

In [3] (Corollary 1) it was shown that the open sets W_i can be replaced by compact submanifolds meeting only along their boundaries (the hypothesis that M be closed is not used in the proof):

Proposition 1. *Let M be an n-manifold with* $\mathrm{cat}_{S^1} M = 2$*. Then M can be expressed as a union of two compact S^1-contractible n-submanifolds W_0, W_1 such that $W_0 \cap W_1 = \partial W_0 \cap \partial W_1$ is a properly embedded $(n-1)$-submanifold F.*

From now on we assume that $M = W_0 \cup W_1$ is a compact 3-manifold, where W_0 and W_1 are S^1-contractible 3-submanifolds as in Proposition 1; so there are maps f_i and α_i such that the diagram $(*)$ is homotopy commutative.

$$(*) \qquad \begin{array}{ccc} W_i & \longrightarrow & M \\ & {\scriptstyle f_i} \searrow \quad \nearrow {\scriptstyle \alpha_i} & \\ & S^1 & \end{array}$$

Proposition 2. *For $i = 0, 1$, we can take α_i so that $\alpha_i(S^1) \cap F = \emptyset$.*

Proof. This is proposition 1 of [3]. (The hypothesis that M be closed is not used in the proof).

□

Lemma 1. *Suppose that M is a compact, orientable, irreducible and $\partial M \neq \emptyset$. Then there is a decomposition $M = W_0^* \cup W_1^*$ where W_0^* and W_1^* are S^1-contractible 3-submanifolds as in Proposition 1 such that every component F^* of $W_0^* \cap W_1^*$ is incompressible in W_0^* and W_1^* or a 2-sphere.*

Proof. Suppose there is a compressing disk $D \subset W_0$, $\partial D = D \cap \partial W_0$ not $\simeq 0$ on ∂W_0. For a regular neighborhood $U(D)$ in W_0 let $W_0' = \overline{W_0 - N(D)}$, $W_1' = W_1 \cup N(D)$. Then $W_0' \subset W_0$ is S^1-contractible and we claim that W_1' is S^1-contractible as well.

Consider the diagram $(*)$ for W_1 with a homotopy $H : W_1 \times I \to M$ from $\alpha_1 f_1$ to the inclusion. Extend H to $D \times 1$ by inclusion. For a loop t representing a generator of $\pi_1(S^1)$ we have $f_1(\partial D) \simeq t^m$ for some integer m.

If $f_1(\partial D) \simeq 0$ then f_1 extends to $f_1' : W_1' \to S^1$ and we define H on $D \times 0$ to be $\alpha_1 f_1'$. Now H is defined on $W_1 \times I \cup (D \times \partial I) \cup (\partial D \times I)$ and since $\pi_2(M) = 0$, H can be extended to $H' : W_1' \times I \to M$, which is a homotopy from $\alpha_1 f_1'$ to the inclusion.

If $f_1(\partial D)$ is not homotopic to 0 then since $\alpha_1 f_1(\partial D) \simeq 0$ it follows that $\alpha_1(t^m) \simeq 0$. Since M is aspherical, $\pi_1(M)$ is torsion free. Hence $\alpha_1(t) \simeq 0$ and we can replace α_1 and hence f_1 in diagram $(*)$ by a constant map. Then extending f_1 and H by the constant map on $D \times 0$ we can again extend H to H' since $\pi_2(M) = 0$.

For the new decomposition $M = W_0' \cup W_1'$ into S^1-contractible subspaces the Euler characteristic $-\kappa(W_0' \cap W_1') < -\kappa(W_0 \cap W_1)$. Hence the Lemma follows. □

Lemma 2. *Suppose that M is compact, orientable, irreducible and $\partial M \neq \emptyset$. Then there is a decomposition $M = W_0 \cup W_1$ as in Lemma 1 and such that each component of W_0 and W_1 is irreducible.*

Proof. Let $M = W_0 \cup W_1$ be as in Lemma 1 such that the sum $c(W_0, W_1)$ of the number of components of W_0 and W_1 is minimal.

Suppose there is a 2-sphere Σ in a component W_0^i of W_0 that does not bound a ball in W_0^i. Let B be the ball in M bounded by Σ, let $W' = \overline{W_0^i - W_0^i \cap B}$, and let $W'' = W' \cup B$. Then W' is S^1-contractible and we have maps f', α' as in $(*)$ with a homotopy $H : W' \times I \to M$ from $\alpha' f'$ to the inclusion. f' can be extended to $f'' : W'' \to S^1$ and we get a homotopy $H : W' \times I \cup (B \times \partial I) \cup (\partial B \times I) \to M$. Since $\pi_3(M) = 0$, H can be extended to $H : W'' \times I \to M$. Now let W_0' be obtained from W_0 by replacing the component W_0^i with W'' and let W_1' be obtained from W_1 by deleting all components that lie in B. Note that $M = W_0' \cup W_1'$ is as in Lemma 1 and $c(W_0', W_1') < c(W_), W_1)$, contradicting minimality. □

Corollary 1. *Suppose that M is compact, orientable, irreducible and $\partial M \neq \emptyset$. If $cat_{S^1}(M) = 2$ then there is a decomposition $M = W_0 \cup W_1$ such that $W_0 \cap W_1 = \partial W_0 \cap \partial W_1$ and every component of W_0 and W_1 is a ball or a solid torus. Furthermore each component of $W_0^i \cap W_1^j$ is incompressible in both W_0^i and W_1^j.*

Proof. By lemmas 1 and 2 we may assume that every component of W_0 and W_1 is irreducible and every component of $W_0 \cap W_1$ is incompressible in W_0 and W_1. It follows that that for each component W_l^k of W_k the inclusion induces an injection $\pi_1(W_l^k) \to \pi_1(M)$. Since each W_l^k is S^1-contractible, $im(\pi_1(W_l^k) \to \pi_1(M))$ is cyclic. Hence W_l^k is irreducible with trivial or infinite cyclic fundamental group and the Lemma follows. □

3 Main Theorem

By a *Seifert fiber space with handles* we mean a 3-manifold that is a Seifert fiber space or is obtained from a Seifert fiber space by attaching 1-handles along the boundary.

Theorem 1. *(a) A compact, orientable, irreducible 3-manifold with S^1-category 2 is a Seifert fiber space with handles and with at most two exceptional fibers.*

(b) A Seifert fiber space with handles and with at most two exceptional fibers has S^1-category ≤ 2.

Proof. (a) If M is closed then by [3] $\pi_1(M)$ is cyclic and by Perelman ([6]) M is a lens space. Thus assume $\partial M \neq \emptyset$. By Corollary 1, $M = W_0 \cup W_1$ such that $W_0 \cap W_1 = \partial W_0 \cap \partial W_1$ and every component of W_0 and W_1 is a ball or solid torus and each component of $W_0^i \cap W_1^j$ is a disk or an annulus, incompressible in both W_0^i and W_1^j. Let N be a regular neighborhood of the disk components of $W_0 \cap W_1$ and let B be the union of the ball components of W_0 and W_1. We choose a Seifert fibration of the solid torus components of W_0 and W_1 such that $M' = \overline{M - (N \cup B)}$ is a Seifert fiber space (not necessarily connected) and M is obtained from M' by attaching 1-handles. Hence M is a "Seifert web" (see [4]) of the form $M = S_1 \cup \cdots \cup S_n \cup H$, where the S_i's are disjoint Seifert fiber spaces obtained from solid torus components of W_0 and W_1 by identifying along essential annuli, H is a collection of 1-handles, and $H \cap (S_1 \cup \cdots \cup S_n)$ is a collection of disks.

We first show that M is a Seifert fiber space with handles.

If every S_i is a solid torus then M is a handlebody and we can think of M as being a Seifert fiber space S_1 with handles. Thus assume at least one S_i is not a solid torus. Then we may assume that no S_i is a solid torus, otherwise we put it together with H. Hence each S_i contains solid torus components of both W_0 and W_1. We show that $n = 1$.

Assume $n > 1$ and let $\beta = \alpha_0(S^1) \subset W_i^j$ ($i = 0$ or 1) and let $W_i^j \subset \overline{M - S_1}$, say. Let $W_0^k \subset S_1$ be a solid torus component of W_0 and let γ represent a generator of $\pi_1(W_0^k)$. Since $S_1 \cap \overline{M - S_1}$ consists of disks, $H_1(S_1 \cap \overline{M - S_1}) \to H_1(S_1)$ is not onto and so γ is not 0 in $H_1(S_1, S_1 \cap \overline{M - S_1})$. Now $\gamma \simeq \alpha_0 f_0 \gamma \simeq \beta^m$ for some $m \in \mathbb{Z}$ and so γ is 0 in $H_1(M, \overline{M - S_1})$. By excision, inclusion induces an isomorphism $H_1(S_1, S_1 \cap \overline{M - S_1}) \to H_1(M, \overline{M - S_1})$, a contradiction.

We complete the proof by showing that at most one solid torus component of W_0 and at most one solid torus component of W_1 is exceptionally fibered.

First assume $\beta = \alpha_0(S^1) \subset W_0^j \subset W_0$.
Suppose some W_0^i for $i \neq j$ is exceptionally fibered with exceptional fiber γ of multiplicity $q > 1$. It follows that $H_1(W_0^i \cap \overline{M - W_0^i}) \to H_1(W_0^i)$ is

not onto and so γ is not 0 in $H_1(W_0^i, W_0^i \cap \overline{M - W_0^i})$. As before $\gamma \simeq \beta^m$ is 0 in $H_1(M, \overline{M - W_0^i})$ and inclusion induces an isomorphism $H_1(W_0^i, W_0^i \cap \overline{M - W_0^i}) \to H_1(M, \overline{M - W_0^i})$, a contradiction.

Hence W_0^j is the only component of W_0 that could be exceptionally fibered.

Now assume $\beta = \alpha_0(S^1) \subset W_1$.
The above argument shows that in this case no component of W_0 is exceptionally fibered.

The same proof applies to the components of W_1.

(b) Let $M = S \cup H$ be a Seifert fiber space with handles and with at most two exceptional fibers. Decompose the orbit surface of S into two disks D_0, D_1, each with at most one exceptional point and $D_0 \cap D_1 = \partial D_0 \cap \partial D_1$ (see e.g. [5]). Then $S = V_0 \cap V_1$, where V_i is the solid torus corresponding to D_i. For each handle $H_k \approx D^2 \times [-1, 1]$ whose ends $D^2 \times \{-1\}$ and $D^2 \times \{1\}$ are attached to V_i and V_{1-i} resp., replace V_i by the solid torus $V_i \cup D^2 \times [-1, 0]$ and V_{1-i} by $V_{1-i} \cup D^2 \times [0, 1]$. Then let $W_i = V_i \cup$ all handles for which both ends are attached to V_{1-i}. Now $M = W_0 \cup W_1$ and each W_i is S^1-contractible.

□

Corollary 2. *Let M be a compact, orientable, irreducible 3-manifold with $H_1(M) = \mathbf{Z}$. If $cat_{S^1}(M) = 2$ then M is a Seifert fiber space with orbit surface a disk and at most 2 exceptional fibers.*

Proof. M is not closed since otherwise M would be a lens space. Since $H_1(M) = \mathbf{Z}$ the boundary of M is a torus. By Theorem 1, M is a Seifert fiber space with at most two exceptional fibers. The projection of M to the orbit surface $\bar{M}$ induces an epimorphism $H_1(M) \to H_1(\bar{M})$ and it follows that $\bar{M}$ is a disk.

□

We say that a link L in S^3 that is not the unlink of more than one component is a *Burde-Murasugi link* if the components of L can be chosen to be fibers of some Seifert fibration of S^3, including the singular fibration and its unknotted circle (see [1]). Such a link is non-splittable and the components are components of a torus link on the boundary of an unknotted solid torus V in S^3, possibly together with the core curve of V and a curve isotopic to a meridian curve of V.

We say that a link $L \subset S^3$ is of S^1-category m if its space $\overline{S^3 - N(L)}$ has S^1-category m.

Corollary 3. *(a) A non-splittable link L of at least two components in S^3 has S^1-category 2 if and only if L is a Burde-Murasugi link.*
(b) A knot in S^3 has S^1-category 2 if and only if it is a non-trivial torus knot.

Proof. The trivial knot has S^1-category 1. In any other case, since L is non-splittable, $M = \overline{S^3 - N(L)}$ is irreducible. If L is of S^1-category 2 then by Theorem 1, M a Seifert fiber space and the result follows from [1] and [2]. Conversely the complements of these links are Seifert fiber-spaces with at most two exceptional fibers.

□

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