Small dilatation pseudo-Anosov mapping classes coming from the simplest hyperbolic braid.

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Abstract

In this paper we study the minimum dilatation pseudo-Anosov mapping classes coming from fibrations over the circle of a single 3-manifold, namely the mapping torus for the "simplest pseudo-Anosov braid". The dilatations that arise include the minimum dilatations for orientable mapping classes for genus $g = 2, 3, 4, 5, 8$ as well as Lanneau and Thiffeault's conjectural minima for orientable mapping classes, when $g = 2, 4 (\text{mod } 6)$. Our examples also show that the minimum dilatation for orientable mapping classes is strictly greater than the minimum dilatation for non-orientable ones when $g = 4, 6, 8$.

1 Introduction

Let S_g be a closed oriented surface of genus $g \geq 1$, and let Mod_g be the *mapping class group*, that is, the group of orientation preserving homeomorphisms of S_g to itself up to isotopy. A mapping class $\phi \in \text{Mod}_g$ is called *pseudo-Anosov* if S_g has a pair of ϕ-invariant, transversally measured, singular foliations on which ϕ acts by stretching along one and contracting along the other by a constant $\lambda(\phi) > 1$. The constant $\lambda(\phi)$ is called the *(geometric) dilatation* of ϕ. A mapping class is pseudo-Anosov if it is neither periodic nor reducible [Thu2] [FLP] [CB].

A pseudo-Anosov mapping class ϕ is defined to be *orientable* if its invariant foliations are orientable. Let $\lambda_{\text{hom}}(\phi)$ be the spectral radius of the action of ϕ on the first homology of S. Then

$$\lambda_{\text{hom}}(\phi) \leq \lambda(\phi),$$

with equality if and only if ϕ is orientable (see, for example, [LT] p. 5).

The dilatations $\lambda(\phi)$ satisfy reciprocal monic integer polynomials of degree bounded from above by $6g - 6$ [Thu2]. If ϕ is orientable the degree is bounded by $2g$. For fixed g, it follows that $\lambda(\phi)$ achieves a minimum $\delta_g > 1$ in Mod_g (cf. [AY] [Iva]). Let δ_g^+ be the minimum dilatation among orientable pseudo-Anosov elements of Mod_g.

In this paper, we address the question:

Question 1.1 *What is the behavior of δ_g and δ_g^+ as functions of g?*

So far, exact values of δ_g have only been found for $g \leq 2$. For $g = 1$, $\text{Mod}_1 = \text{SL}(2; \mathbb{Z})$, and

$$\delta_1 = \frac{3 + \sqrt{5}}{2}.$$

For $g = 2$, Cho and Ham [CH] show that δ_2 is the largest real root of

$$t^4 - t^3 - t^2 - t + 1 = 0$$

or approximately 1.72208.

In the orientable case more is known due to recent results of Lanneau and Thiffeault [LT]. Given $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}$ with $0 < a < b$, let

$$LT_{(a,b)}(t) = t^{2b} - t^b(1 + x^a + x^{-a}) + 1,$$

and let $\lambda_{(a,b)}$ be the largest real root of $LT_{(a,b)}(t)$.

Theorem 1.2 (Lanneau-Thiffeault [LT] Thm. 1.2, Thm. 1.3) *For $g = 2,3,4,6,8$,*

$$\lambda_{(1,g)} \leq \delta_g^+$$

with equality when $g = 2,3,4$.

For $g = 2$, the value of δ_2^+ was first determined by Zhirov [Zhi]. For $g = 5$, Lanneau and Thiffeault show that δ_5^+ equals Lehmer's number. This dilatation is realized as a product of multi-twists in along a curve arrangement dual to the E_{10} Coxeter graph [Lei], and as the monodromy of the (-2,3,7)-pretzel knot [Hir]. Lanneau and Thiffeault also find a lower bound for δ_7^+, but so far no one has found an example with that dilatation.

Based on their calculations, Lanneau and Thiffeault ask: *is $\delta_g^+ = \lambda_g$ for all even g?* We call the affirmative answer to their question the LT-conjecture.

In our first result, we improve on the following previously known best bounds for minimum dilatation of infinite families

$$(\delta_g)^g \leq (\delta_g^+)^g \leq 2 + \sqrt{3}$$

(see [Min] [HK]).

Theorem 1.3 *If $g = 0,1,3,4 (\text{mod } 6)$, $g \geq 3$, then*

$$\delta_g \leq \lambda_{(3,g+1)},$$

and if $g = 2,5 (\text{mod } 6)$ and $g \geq 5$, then

$$\delta_g \leq \lambda_{(1,g+1)}.$$

For the orientable case, our results complement those of Lanneau and Thiffeault for $g = 2,4 (\text{mod } 6)$.

Theorem 1.4 *Let $g \geq 3$. Then*

(i) $\delta_g^+ \leq \lambda_{(3,g+1)}$ *if $g = 1,3 (\text{mod } 6)$,*

(ii) $\delta_g^+ \leq \lambda_{(1,g)}$ *if $g = 2,4 (\text{mod } 6)$, and*

(iii) $\delta_g^+ \leq \lambda_{(1,g+1)}$ *if $g = 5 (\text{mod } 6)$.*

Putting Theorem 1.4 together with Lanneau and Thiffeault's lower bound for $g = 8$ gives

Corollary 1.5 *For $g = 8$, we have*

$$\delta_8^+ = \lambda_{(1,8)}.$$

For large g, it is known that δ_g and δ_g^+ converges to 1. Furthermore, we have

$$\log(\delta_g) \asymp \frac{1}{g} \qquad \text{and} \qquad \log(\delta_g^+) \asymp \frac{1}{g} \tag{1}$$

(see [Pen] [McM1] [Min] [HK]). The LT-conjecture together with (1) leads to the natural question:

Question 1.6 (e.g., [McM1], p.551, [Far], Problem 7.1) *Do the sequences*

$$(\delta_g)^g \qquad \textit{and} \qquad (\delta_g^+)^g$$

converge as g grows? What is the limit?

The examples in this paper show the following.

Proposition 1.7 *If the limit exists, then*

$$\limsup_{g\to\infty}(\delta_g)^g \leq \frac{3+\sqrt{5}}{2}.$$

If the LT-conjecture is true, then δ_{2m}^+ is a monotone strictly decreasing sequence (see Proposition 4.4) that converges to $\frac{3+\sqrt{5}}{2}$. Thus, the LT-conjecture implies equality in Proposition 1.7.

Lanneau and Thiffeault show that $\delta_5^+ \leq \delta_6^+$, and hence δ_g^+ is not strictly monotone decreasing (cf. [Far] Question 7.2). Theorem 1.4 shows the stronger statement.

Proposition 1.8 *If the LT-conjecture is true, then* $\delta_g^+ \leq \delta_{g+1}^+$, *whenever* $g = 5 (\mathrm{mod}\ 6)$.

Another example concerns the question of whether the inequality $\delta_g \leq \delta_g^+$ is strict for any or all g. Table 1 shows the following.

Proposition 1.9 *For* $g = 4, 6, 8$ *we have*

$$\delta_g < \delta_g^+.$$

If the LT conjecture is true, then Theorem 1.3 and Proposition 4.4 imply that the phenomena revealed in Proposition 1.9 repeats itself periodically.

Proposition 1.10 *If the LT-conjecture is true, then for all even* $g \geq 4$ *we have*

$$\delta_g < \delta_g^+.$$

We prove Theorem 1.3 and Theorem 1.4 by exhibiting a family of mapping classes $\phi_{(a,b)}$ that come from a fibered face of a single 3-manifold M. This is interesting in light of the *Universal Finiteness Theorem* due to Farb, Leininger and Margalit [FLM]. For any pseudo-Anosov mapping class $\phi \in \mathrm{Mod}_g$, let $M(\phi)$ be the mapping torus of ϕ after removing tubular neighborhoods of suspensions of the singularities. Let

$$\mathcal{T}_P = \{M(\phi) \ : \ \lambda(\phi) \leq P^g\}.$$

Then $\mathcal{T}_P$ is a finite set for all P ([FLM] Thm. 1.1). The asymptotic equations (1) imply that

$$\mathcal{T} = \{M(\phi) \ : \ \phi \in \mathrm{Mod}_g, \phi \text{ pseudo-Anosov}, \lambda(\phi) = \delta_g\}$$

and

$$\mathcal{T}^+ = \{M(\phi) \ : \ \phi \in \mathrm{Mod}_g, \phi \text{ pseudo-Anosov}, \lambda(\phi) = \delta_g^+\}$$

are finite. For our examples, M is the complement of a two component link L, known as 6_2^2 in Rolfsen's table [Rolf]. (See also [KT] for another example of a single manifold producing small dilatations.)

The following is a table of the minimal dilatations that arise in this paper's examples for genus 1 through 12. All numbers in the table are truncated to 5 decimals. An asterisk $*$ marks the numbers that have been verified to equal δ_g^+ (resp. δ_g). For singularity-type, we use the convention that $(a_1, \ldots, a_k)$ means that the singularities of the invariant foliations have degrees $a_1, \ldots, a_k$ (see Lanneau and Thiffeault's notation[LT], p.3). The singularity-types for our examples are derived from the formula given in Proposition 3.5.

g	orientable	degrees of singularities	unconstrained	degrees of singularities
1	2.61803*	no sing.	2.61803*	no sing.
2	1.72208*	(4)	1.72208*	(4)
3	1.40127*	$(2, 2, 2, 2)$	1.40127	(2,2,2,2)
4	1.28064*	(10,2)	1.26123	(3,3,3,3)
5	1.17628*	(16)	1.17628	(16)
6	-	-	1.1617	(5,5,5,5)
7	1.13694	(6,6,6,6)	1.13694	(6,6,6,6)
8	1.12876*	(22,6)	1.1135	(25,1,1,1)
9	1.1054	(8.8.8.8)	1.1054	(8,8,8,8)
10	1.10149	(28,8)	1.09466	(9,9,9,9)
11	1.08377	(34,2,2,2)	1.08377	(34,2,2,2)
12	-	-	1.07874	(11,11,11,11)

Table 1: Minimal orientable and unconstrained dilatations coming from M

For $g = 1, 2, 3, 4, 5$, our orientable examples agree both in dilatation and in singularity-type with previously found minimizing examples. Thus, for example, we have shown that

$$M \in \mathcal{T} \cap \mathcal{T}^+.$$

For $g = 8$, it agrees with the singularity-type anticipated by Lanneau and Thiffeault (see [LT]). For $g = 6k$, we do not get any orientable examples out of fibrations of M, and for $g = 7$, our minimal example gives a strictly larger dilatation than Lanneau and Thiffeault's lower bound.

Section 2 contains a brief review of Thurston norms, Alexander norms, and the Teichmüller polynomial. These are the basic tools used in this paper. In Section 3 we describe our family of examples, and in Section 4 we prove Theorem 1.3 and Theorem 1.4.

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2 Background and tools

We give a brief review of fibrations of a hyperbolic 3-manifold M and their invariants, emphasizing tools that we will use in the rest of the paper. For more details see, for example, [Thu1] [FLP] [McM1] [McM2].

The theory of fibered faces of the Thurston norm ball and Teichmuller polynomials gives rise to an atlas of all possible fibrations of a given hyperbolic manifold. Assume M is a compact hyperbolic 3-manifold with boundary. Given an embedded surface S on M, let $\chi_-(S)$ be the sum of $|\chi(S_i)|$, where S_i are the irreducible components of S with negative Euler characteristic. The Thurston norm of $\psi \in \mathrm{H}^1(M; \mathbb{Z})$ is defined to be

$$||\psi||_T = \min \chi_-(S),$$

where the minimum is taken over oriented embedded surfaces $(S, \partial S) \subset (M, \partial M)$ such that the class of S in $\mathrm{H}_2(M, \partial M; \mathbb{Z})$ is dual to ψ.

Elements of $\mathrm{H}^1(M;\mathbb{Z})$ are canonically associated with epimorphisms

$$\pi_1(M;\mathbb{Z}) \to \mathbb{Z}$$

which factor through epimorphisms

$$\mathrm{H}_1(M;\mathbb{Z}) \to \mathbb{Z}.$$

Thus, we have a lattice $\Lambda_M \subset \mathbb{R}^{b_1(M)}$ equal to any of the following naturally identified objects.

$$\mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M) \to \mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(M;\mathbb{Z}), \mathbb{Z}).$$

If $\psi \in \Lambda_M$ is induced by a fibration

$$M \to S^1$$

we say that ψ is *fibered.* In this case,

$$||\psi||_T = \chi_-(S),$$

where S is the fiber of ψ. The *monodromy* of ψ is the mapping class $\phi : S \to S$, such that M is the mapping torus of ϕ, and ψ is the natural projection to S^1. Since M is hyperbolic, ϕ is automatically pseudo-Anosov.

Let Σ be the unit sphere in $\mathbb{R}^{b_1(M)}$ with respect to the extended Thurston norm. Then Σ is a polyhedron and Λ_M projects to a dense subset of Σ, called the *rational points* of Σ. The fibered elements of Λ_M project to the (open) faces of Σ. The faces that contain images of fibered elements are called *fibered faces.* Any element that projects to a fibered face is fibered.

Let $\psi \in \Lambda_M$ be a fibered element, and let ψ_0 be the element of Λ_M that lies closest to the origin along the ray containing ψ. Then

$$\psi = r(\psi_0),$$

for some positive integer r, and the fibration associated to ψ is obtained by taking the fibration associated to ψ_0 and composing with the r-cyclic covering of S^1. It follows that ψ_0 has connected fibers, while the fibers of ψ have r-connected components. Such elements ψ_0 are called primitive elements. The dilatation of the monodromy ϕ is given by

$$\lambda(\phi) = \lambda(\phi_0)^{1/r},$$

where ϕ_0 is the monodromy of the associated primitive element.

Theorem 2.1 ([Fri], Theorem E) *There is a continuous function $\mathcal{Y}(\psi)$ defined on the entire fibered cone in $\mathbb{R}^{b_1(M)}$, so that if ψ is fibered with monodromy ϕ, then*

$$\mathcal{Y}(\psi) = \frac{1}{\log(\lambda(\phi))}.$$

The function $\mathcal{Y}$ is homogeneous of degree one, and is a concave function tending to zero along the boundary of the cone.

The Alexander polynomial Δ_M of M is a polynomial in $\mathbb{Z}[G]$, where $G = \mathrm{H}_1(M;\mathbb{Z})$. Each element $\psi \in \Lambda_M$ determines an epimorphism of $\mathbb{Z}$-modules:

$$\rho : \mathbb{Z}[G] \to \mathbb{Z}[t, t^{-1}],$$

where we identify $\mathbb{Z}[t, t^{-1}]$ with the group ring over $\mathbb{Z} = \mathrm{H}_1(S^1; \mathbb{Z})$. This defines a specialization

$$\Delta_{(M,\psi)} = \rho(\Delta_M) \in \mathbb{Z}[t, t^{-1}].$$

The polynomial $\Delta_{(M,\psi)}$ is the characteristic polynomial for the monodromy ϕ of ψ acting on $\mathrm{H}_1(S; \mathbb{Z})$, where S is the fiber of ψ. Thus, the degree of the Alexander polynomial specialized to a particular ψ is the rank of $\mathrm{H}_1(S; \mathbb{Z})$. This is called the Alexander norm of ψ. The *homological dilatation* of ϕ, that is, the spectral radius of the action of ϕ on $\mathrm{H}_1(S; Z)$, is the maximum among norms of roots of $\Delta_{(M,\phi)}$. We denote the homological dilatation by $\lambda_{\mathrm{hom}}(\phi)$.

The *Teichmuller polynomial* Θ associated to a fibered face of Σ_M is analogous to the Alexander polynomial. It is a polynomial in $\mathbb{Z}[G]$ such that for each ψ in the cone over the fibered face, the geometric dilatation $\lambda(\phi)$ of the monodromy is the largest real root of Θ specialized to ψ [McM1].

3 The mapping torus for the simplest pseudo-Anosov braid

We now look at a particular 3-manifold, and study properties of its fibrations. This example has also been studied in ([McM1] §11), and the first part of this section will be a review of what is found there.

Let $M = S^3 \setminus N(L)$, where L is the link drawn in two ways in Figure 1, and $N(L)$ is a tubular neighborhood. As seen from the left diagram in Figure 1, M fibers over the circle with fiber a four

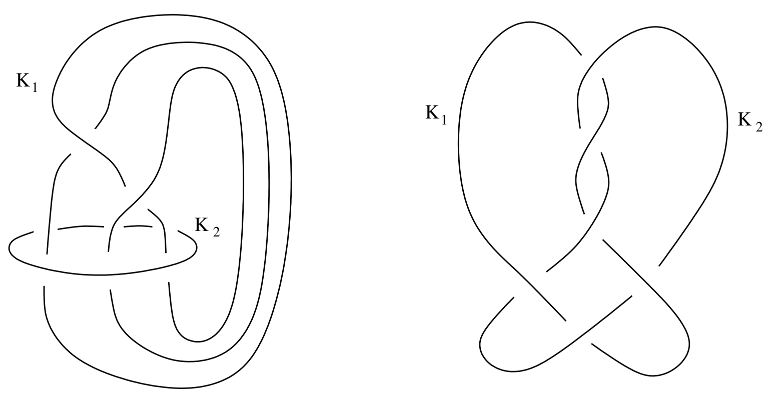

Figure 1: Two diagrams for the link 6^2_2.

times punctured sphere S. Let $\psi \in \Lambda_M$ be the associated element. Let K_1 be the component of L passing through S, and let K_2 be the other component of L.

The monodromy ϕ of ψ is the composition of two Dehn twists determined by 180 degree rotations as drawn in FIgure 2, and has dilatation

$$\lambda(\phi) = \frac{3 + \sqrt{5}}{2}.$$

Its lift to a torus realizes δ_1, and its dilatation is smallest possible for mapping classes defined on S. The associated braid β (which can be written as $\sigma_1 \sigma_2^{-1}$ with respect to the standard basis for the braid group) has been called the "simplest pseudo-Anosov braid"([McM1] §11).

The Thurston norm and the Alexander norm both are given by

$$||(a, b)|| = \max\{2|a|, 2|b|\}, \tag{2}$$

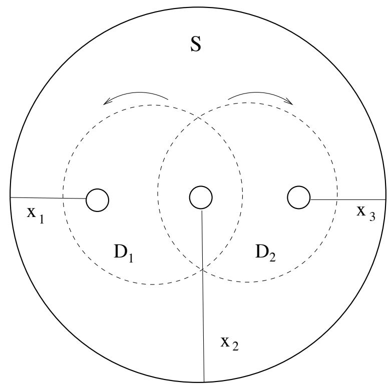

Figure 2: Braid monodromy associated to $\beta = \sigma_1\sigma_2^{-1}$.

where $(a,b) \in \mathrm{H}^1(M;\mathbb{Z})$ denotes the class that evaluates to a on the meridian μ_1 of K_1 and b on the meridian μ_2 of K_2.

The lattice points Λ_M in the fibered cone (of points projecting to the fibered face) defined by $\psi = (0,1)$ is the set

$$\Psi = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \ : \ b > 0, -b < a < b\}$$

as shown in Figure 3. For the rest of this paper, we will only be concerned with the subset $\Psi_0 \subset \Psi$ consisting of elements of Ψ with connected fibers. Thus,

$$\Psi_0 = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \ : \ b > 0, -b < a < b, \gcd(a,b) = 1\}.$$

The elements of Ψ_0 are in one-to-one correspondence with the rational points on the fibered face defined by ψ, which can be thought of as the projectivization of Ψ.

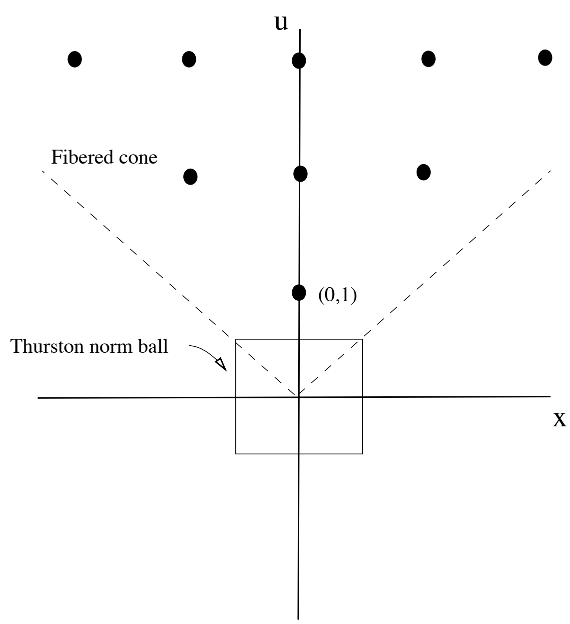

Figure 3: Fibered cone Ψ containing $\psi = (0,1)$.

The Alexander polynomial for L is given by

$$\Delta_L(x,u) = u^2 - u(1 - x - x^{-1}) + 1 \tag{3}$$

(see Rolfsen's table [Rolf]), and the Teichmuller polynomial is given by

$$\Theta_L(x,u) = u^2 - u(1 + x + x^{-1}) + 1. \tag{4}$$

Specialization to the element $(a,b) \in \mathrm{H}^1(M;\mathbb{Z})$ discussed in Section 2 is the same as plugging (t^a, t^b) into the equations for the Alexander and Teichmuller polynomials.

Proposition 3.1 *If $(a,b) \in \Psi_0$, then the associated monodromy $\phi_{(a,b)}$ is pseudo-Anosov and its homological dilatation is the maximum norm among roots of the polynomial*

$$\Delta_L(t^a, t^b) = t^{2b} - t^b(1 - t^a - t^{-a}) + 1,$$

and the geometric dilatation is the largest real root $\lambda_{(a,b)}$ of

$$\Theta_L(t^a, t^b) = t^{2b} - t^b(1 + t^a + t^{-a}) + 1.$$

Corollary 3.2 *If $(a,b) \in \Psi_0$, then the associated monodromy $\phi_{(a,b)}$ is orientable if a is odd and b is even.*

Proof. If a is odd and b is even, then the roots of $\Theta_L(t^a, t^b)$ are the negatives of the roots of $\Delta_L(t^a, t^b)$. This implies that the geometric and homological dilatations of $\phi_{(a,b)}$ are equal, and therefore $\phi_{(a,b)}$ is orientable. □

Later in this section, we will show the converse of Corollary 3.2. First we consider how the monodromy behaves near the boundary of $S_{(a,b)}$.

Proposition 3.3 *Let $\phi_{(a,b)} : S_{(a,b)} \to S_{(a,b)}$ be the monodromy associated to (a,b). The boundary components of $S_{(a,b)}$ consists of $\gcd(3,a)$ components coming from $T(K_1)$ and $\gcd(3,b)$ coming from $T(K_2)$. Thus, the total number of boundary components of $S_{(a,b)}$ is given by*

$$\begin{cases} 2 & \text{if } \gcd(3,ab) = 1 \\ 4 & \text{if } 3 \text{ divides } ab \end{cases}$$

Proof. The number of components in $T(K_i) \cap S_{(a,b)}$ is the index of the image of $\pi_1(T(K_i))$ in $\mathbb{Z}$ under the composition of maps

$$\pi_1(T(K_i)) \to \pi_1(M) \to \mathbb{Z}$$

induced by inclusion and $\psi_{(a,b)}$.

For $i = 1, 2$, let ℓ_i be the longitude of K_i that is contractible in $S^3 \setminus K_i$. Then, for $T(K_1)$ we have

$$\psi_{(a,b)}(\mu_1) = a \quad \text{and} \quad \psi_{(a,b)}(\ell_1) = 3\psi_{(a,b)}(\mu_2) = 3b,$$

so the number of boundary components contributed by $T(K_1)$ is

$$\gcd(a, 3b) = \gcd(3, a),$$

since we are assuming that $\gcd(a,b) = 1$. The contribution of $T(K_2)$ is computed similarly. □

Proposition 3.4 *The genus of $S_{(a,b)}$, for $(a,b) \in \Psi_0$ is given by*

$$
\begin{aligned}
g(S_{(a,b)}) &= |b| + \left(1 - \frac{\gcd(3,a) + \gcd(3,b)}{2}\right) \\
&= \begin{cases} |b| & \text{if } 3 \text{ does not divide } ab \\ |b| - 1 & \text{if } 3|a \text{ or } 3|b. \end{cases}
\end{aligned}
$$

Proof. From (2) we have

$$2|b| = \chi_-(S_{(a,b)}) = 2g - 2 + \gcd(3,a) + \gcd(3,b).$$

□

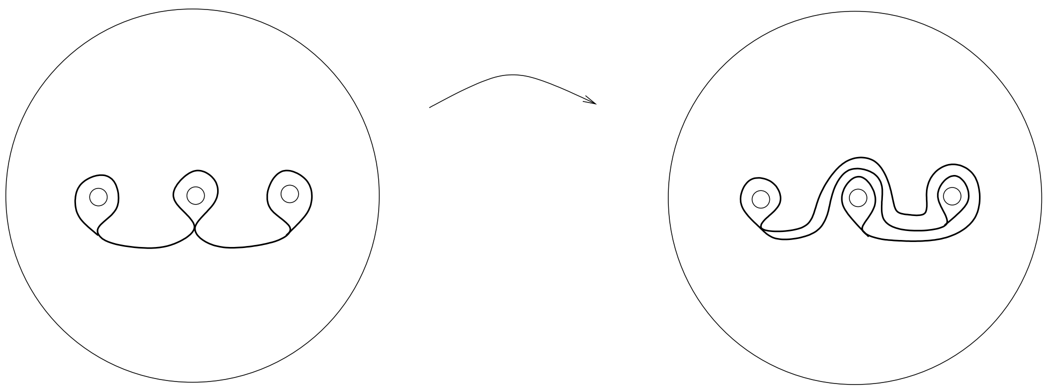

Figure 4: Train track for $\phi : S \to S$.

Proposition 3.5 *Let $(a,b) \in \Psi_0$, and let $\mathcal{F}$ be a $\phi_{(a,b)}$-invariant foliation. Then $\mathcal{F}$*

(i) has no interior singularities,

(ii) has $(3b/\gcd(3,a))$-pronged at the $\gcd(3,a)$ boundary components coming from $T(K_1)$), and

(iii) has $(b/\gcd(3,b))$-pronged at the $\gcd(3,b)$ boundary components coming from $T(K_2)$.

Proof. Let $\mathcal{L}$ be the lamination of M defined by suspending $\mathcal{F}$ over M considered as the mapping torus of ϕ. From the train track for ϕ (Figure 4), one sees that each of the boundary components of S are one-pronged, and that there are no other singularities. It follows that $\mathcal{L}$ has no singularities outside a neighborhood of the K_i, and near each K_i the leaves of $\mathcal{L}$ come together at a simple closed curve $\gamma_i \in \mathrm{H}_1(T(K_i)$. Write

$$\gamma_i = r_i \mu_i + s_i \ell_i$$

for $i = 1, 2$.

For $(a,b) \in \Psi_0$, the number of intersections of γ_i with $S_{(a,b)}$ is the image of γ_i under the epimorphism

$$\psi_{(a,b)} : \pi_1(M) \to \mathbb{Z}$$

defining the fibration. Figure 4 shows that $s_1 = 1$ and $r_2 = 1$. Using the identities

$$
\begin{aligned}
s_1 &= 1 & \lambda_1 &= 3\mu_2, \\
r_2 &= 1 & \lambda_2 &= 3\mu_1,
\end{aligned}
$$

we have

$$\begin{aligned}\psi_{(a,b)}(\gamma_1) &= r_1\psi_n(\mu_1) + 3\psi_n(\mu_2) = r_1 a + 3b \\ \psi_{(a,b)}(\gamma_2) &= \psi_n(\mu_2) + 3s_2\psi_n(\mu_1) = 3s_2 a + b.\end{aligned}$$

This implies that $\phi_{(a,b)}$ is $(r_1 a + 3b)/m_1$-pronged at m_1 boundary components and $(3s_2 a + b)/m_2$-pronged at m_2 boundary components. We find r_1 and s_2 by looking at some particular examples.

In general, if $f : \Sigma \to \Sigma$ is pseudo-Anosov on a compact oriented surface Σ with genus g and and $n_1, \dots, n_k$ are the number of prongs at the singularities and boundary components, then by the Poincaré-Hopf theorem

$$\sum_{i=1}^{k}(n_i - 2) = 4g - 4. \tag{5}$$

For $(a, b) = (1, n)$, n not divisible by 3, we have two singularities with number of prongs given by:

$$\begin{aligned}\psi_n(\gamma_1) &= r_1 + 3n \\ \psi_n(\gamma_2) &= 3s_2 + n.\end{aligned}$$

Plugging into (5) gives

$$r_1 + 3s_2 = 0.$$

Let $s = s_2$. The mapping class $\phi_{(1,2)}$ is the unique genus 2 pseudo-Anosov mapping class with dilatation equal to λ_2 [CH][LT], and has one 6-pronged singularity (see, for example, [HK]). Thus, $s = 0$ and we have

$$\gamma_1 = \ell_1 = 3\mu_2$$

and

$$\gamma_2 = \mu_2.$$

The claim follows. □

Corollary 3.6 *The map $\phi_{(a,b)}$ has singularities with number of prongs (or prong-type) given by*

$$\begin{cases} (3b, b) & \text{if } \gcd(3, ab) = 1 \\ (3b, b/3, b/3, b/3) & \text{if } \gcd(3, b) = 3 \\ (b, b, b, b) & \text{if } \gcd(3, a) = 3 \end{cases}$$

Corollary 3.7 *If b is odd, then $\phi_{(a,b)}$ is not orientable.*

Proof. By Corollary 3.6, the number of prongs at each boundary component is odd if b is odd. Thus, $\phi_{(a,b)}$ is not locally orientable near the boundary components. □

Corollary 3.8 *For $(a, b) \in \Psi_0$, $\phi_{(a,b)}$ is 1-pronged at one or more boundary components of $S_{(a,b)}$ if and only if $(a, b) \in \{(0, 1), (\pm 1, 3), (\pm 2, 3)\}$.*

Corollary 3.9 *If $(a, b) \notin \{(0, 1), (\pm 1, 3), (\pm 2, 3)\}$, then $\phi_{(a,b)}$ extends to the closure of $S_{(a,b)}$ over the boundary components to a mapping class $\overline{\phi}_{(a,b)}$ with the same dilatation as $\phi_{(a,b)}$.*

Proposition 3.10 *Table 2 below describes the pairs $(a, b) \in \Psi_0$ that give rise to an orientable (or non-orientable) genus g pseudo-Anosov mapping class. (Here $g \geq 4$.)*

g (mod 6)	*orientable*	*non-orientable*
0	*no example*	$b = g + 1,\ a = 0 (\text{mod } 3)$
1	$b = g + 1,\ a = 3(\text{mod } 6)$	$b = g,\ a = 1, 2(\text{mod } 3)$
2	$b = g,\ a = 1, 5(\text{mod } 6)$	$b = g + 1,\ a = 1, 2(\text{mod } 3)$
3	$b = g + 1,\ a = 3(\text{mod } 6)$	*no example*
4	$b = g,\ a = 1, 5(\text{mod } 6)$	$b = g + 1, a = 0(\text{mod } 3)$
5	$b = g + 1,\ a = 1, 5(\text{mod } 6)$	$b = g,\ a = 1, 2(\text{mod } 3)$

Table 2: Fibrations of M according to genus.

4 Minimal dilatations for the fibered face.

Let Ψ_0 be the fibered cone discussed in Section 3. Let

$$\begin{aligned} d_g &= \min\{\lambda(\psi) : \psi \in \Psi_0, \text{genus of } \psi \text{ is } g\}, \text{and} \\ d_g^+ &= \min\{\lambda(\psi) : \psi \in \Psi_0, \text{genus of } \psi \text{ is } g, \text{ the monodromy of } \psi \text{ is orientable}\}. \end{aligned}$$

In this section, we finish the proofs of Theorem 1.3 and Theorem 1.4 and their consequences by determining d_g and d_g^+.

Proposition 4.1 *Let $(a, b) \in \Psi_0$. Then*

$$\lambda_{(a,b)} < \lambda_{(a',b')}$$

if either

(1) $|a| < |a'|$ and $|b| = |b'|$; or

(2) $|a| = |a'|$ and $|b| > |b'|$.

Proof. One compares the slopes of rays from the origin to (a, b) and (a', b'). The claim follows from Theorem 2.1. □

Proposition 4.2 *For $b \geq 3$, we have*

$$\lambda_{(1,b)} \geq \lambda_{(3,b+1)},$$

with equality when $b = 3$.

Proof. Let $\lambda = \lambda_{(3,b+1)}$. We will show that $LT_{(1,b)}(\lambda) < 0$. Multiplying by λ^2 and using the fact that $LT_{(3,b+1)}(\lambda) = 0$ gives

$$\begin{aligned} \lambda^2 LT_{(1,b)}(\lambda) &= \lambda^2 LT_{(1,b)}(\lambda) - LT_{(3,b+1)}(\lambda) \\ &= \lambda^{b+4} - \lambda^{b+3} - \lambda b + 2 + \lambda^{b-2} + \lambda^2 - 1 \\ &= (\lambda - 1)(\lambda^{b+3} - \lambda^{b-2}(\lambda^3 + \lambda^2 + \lambda + 1) + \lambda + 1) \\ &= (\lambda - 1)\lambda^{b-2}[\lambda^5 - \lambda^3 - \lambda^2 - \lambda - 1 + \lambda^{2-b}(\lambda + 1)]. \end{aligned}$$

Thus, it is enough to show that

$$C = \lambda^5 - \lambda^3 - \lambda^2 - \lambda - 1 + \lambda^{2-b}(\lambda + 1) < 0.$$

Since $\lambda > 1$ and $b \geq 3$, we have

$$C < \lambda^5 - \lambda^3 - \lambda^2 = \lambda^2(\lambda^3 - \lambda - 1).$$

One can check that the right hand side is negative for $\lambda < 1.3$. By Proposition 4.1, λ decreases as b increases. A check shows that $\lambda_{(3,5)} < 1.3$, and hence $C < 0$ for $b \geq 4$. For $b = 3$, one checks directly that

$$\lambda_{(1,3)} = \lambda_{(3,4)}.$$

□

Remark 4.3 *The mapping class $\phi_{(1,3)}$ is defined on a genus 2 surface with four boundary components, with prong-type (3,1,1,1). The mapping class $\phi_{(3,4)}$ is defined on a genus 3 surface with prong-type (4,4,4,4). By Proposition 4.2 these two examples have the same dilatation.*

Putting together Proposition 4.1 and Proposition 4.2, we have the following.

Corollary 4.4 *The sequences $\lambda_{(1,b)}$ and $\lambda_{(3,b)}$ satisfy:*

$$\lambda_{(1,b)} > \lambda_{(3,b+1)} > \lambda_{(1,b+1)}.$$

Lemma 4.5 *For $n \geq 2$, Then*

$$\lim_{n\to\infty} (\lambda_{(a,n)})^n = \frac{3+\sqrt{5}}{2},$$

for any fixed a.

Proof. The projections of the lattice points $(a, n) \in \Lambda_M$ on the fibered face of ψ converge to $(0, 1/2)$. □

Corollary 4.6 *For the minimal dilatations d_g and d_g^+ that are realized on M, we have*

$$\lim_{g\to\infty} (d_g)^g = \lim_{g\to\infty} (d_g^+)^g = \frac{3+\sqrt{5}}{2}.$$

Proposition 4.7 *The following table describes the pairs $(a, b) \in \Psi_0$ that give rise to the minima d_g and d_g^+ realized on M. Here unconstrained means not constrained to be orientable.*

g mod 6	$\lambda(\phi_{(a,b)}) = d_g^+$, $\phi_{(a,b)}$ *orientable*	$\lambda(\phi_{(a,b)}) = d_g$
0	*no example*	$(3, g+1)$
1	$(3, g+1)$	$(3, g+1)$
2	$(1, g)$	$(1, g+1)$
3	$(3, g+1)$	$(3, g+1)$
4	$(1, g)$	$(3, g+1)$
5	$(1, g+1)$	$(1, g+1)$

Table 3: Pairs (a, b) giving smallest dilatations.

Proposition 4.7 and Corollary 3.9 complete the proofs of Theorem 1.3 and Theorem 1.4. A pictorial view of how the elements of Ψ giving the least dilatations for each genus up to 12 lie on the "atlas" for M is shown in Figure 5.

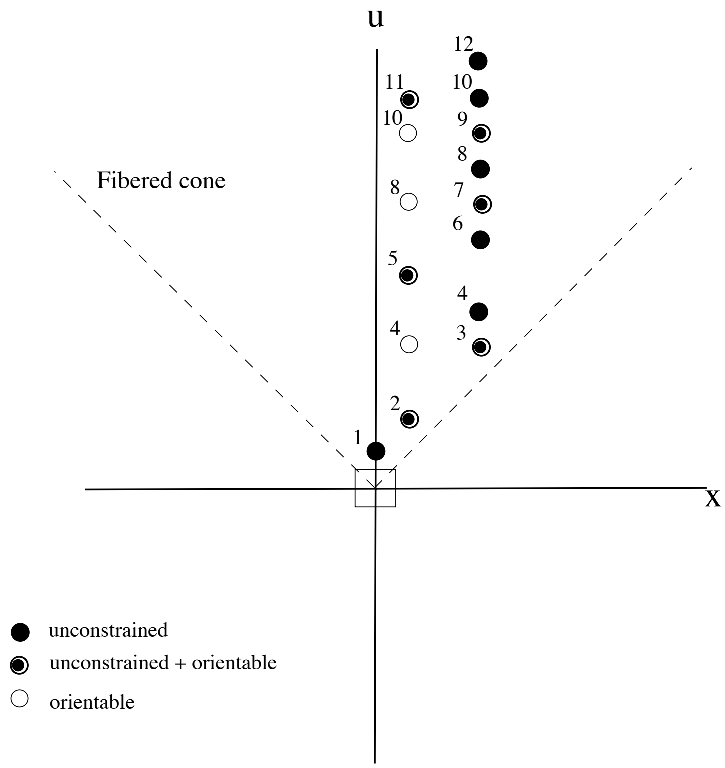

Figure 5: Minima for d and d^+ in genus $g = 1, \ldots, 12$.

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