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ON THE COUPLED CONTINUUM PIPE FLOW MODEL (CCPF) FOR FLOWS IN KARST AQUIFER

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Abstract. We show that the coupled continuum pipe flow model (CCPF) for flows in karst aquifers is ill-posed in the sense that no reasonable solution exists. We also demonstrate that Hua's modified CCPF model is ill-posed in 3D although it is well-posed in two spatial dimensions. A new modification of the original CCPF model that is consistent with basic physics is proposed and its well-posedness is proved here. We believe that this is the first physically relevant well-posed CCPF type model in 3D.

1. **Introduction.** Karst is a type of landscape that is formed by the dissolution of a layer or layers of soluble bedrocks, including carbonate rocks, limestone and dolomite. Karst regions contain aquifers that are capable of providing large supplies of water. A karst aquifer, in addition to a porous limestone matrix, typically has large cavernous conduits that are known to have great impact on groundwater flow and contaminant transport within the aquifer [26].

Karst aquifers supply a significant portion of the drinking water in the United States (about 40%) and are particularly crucial in states like Florida for which karst aquifers provide more than 90% of the fresh water used [24]. Therefore the study of flows in karst aquifers is of great importance to us, especially since the aquifers are now being seriously threatened by over withdrawals and increasing contamination [34, 26].

The mathematical study of flows in karst aquifers is a great challenge due to the coupling of the flows in the conduits and the flows in the surrounding matrix, the complex geometry of the network of conduits (pipes), the vastly disparate spatial and temporal scales, the strong heterogeneity, and the huge associated uncertainty with the data. Even for a small lab size conceptual model with only one conduit (pipe) imbedded in a homogenous porous media (matrix), significant mathematically rigorous progress has been only achieved recently via the so-called coupled (Navier) Stokes-Darcy model with the classical Beavers-Joseph interface boundary condition [7] or various simplified interface conditions [10, 14, 15, 28].

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On the other hand, geologists have proposed various ad-hoc simplified models in order to make progress. One of the most popular models is the so-called **coupled continuum pipe flow (CCPF)** model where the conduits are simplified into a network of one dimensional pipes [25, 3, 4, 8, 31]. Classical pipe flow formulas (such as the Poiseuille formula in the laminar case) are used in each segment of the network, a Kirchhoff type condition is imposed at each node where different segments of the conduit network meet, and the pipe system is coupled with the continuum system (matrix) through exchange of fluids at the discrete nodes of the conduit system via a Barenblatt type relation. Indeed, this CCPF model has been recently incorporated into the US Geological Survey's popular groundwater flow software system **MODFLOW** [33] where the flow in the conduit network is termed **conduit flow process (CFP)**. However, the validity of this model has not been studied from the mathematical point of view. The purpose of the current short note is to address the following **critical issues**:

- Is the original CCPF model well-posed, i.e., does solution exist and behave nicely?
- What would be a physically relevant and mathematical sound alternative if the original CCPF model is deficient?

We found out that the original CCPF model, both the steady state and the time-dependent cases, are in fact ill-posed in the sense that no solution exists with finite head in the generic case of having fluid exchange between the matrix and the conduit system. The alternative form proposed by Hua [22] is also ill-posed in 3D although it is well-posed in 2D. We then propose a new modification of the original CCPF model that is physically appealing and mathematically well-posedness. We believe this is the first physically relevant and mathematically sound 3D CCPF type model.

The rest of the manuscript is organized as follows. In section 2 we recall the original CCPF model and demonstrate that the model is ill-posed. In section 3 we first recall Hua's CCPF model and its well-posedness in 2D. We then provide a heuristic derivation of the model followed by a demonstration that Hua's model is ill-posed in 3D. In section 4, we propose our new physically relevant CCPF model and provide the proof of its well-posedness. Physically less appealing alternatives will also be presented. We offer concluding remarks in section 5.

2. The original CCPF model.

In this section we first recall the original CCPF model. We then show that this model is ill-posed in the steady state case and time-dependent case. We treat the steady state case separately due to its importance in applications. Indeed, it is expected that for any reasonable flow model for karst aquifer, the flows will converge to a steady state provided that the boundary conditions are time-independent (this is the case for the coupled Stokes-Darcy model for flows in karst aquifers with the simplified Beavers-Joseph-Saffman-Jones interface condition, or the original Beavers-Joseph interface condition with physically small values of the Beaver-Joseph coefficient α_{BJ} for instance [22, 10]).

2.1. The original CCPF model.

2.1.1. *Continuum model for the matrix.* It is well accepted that flows in fluid saturated porous media (matrix) is governed by the following **Darcy equation** [5] which was derived by Darcy on a phenomenological level but can be justified mathematically via homogenization under appropriate assumptions:

$$\mathbf{v} = -\mathbb{K}\nabla h_m. \tag{1}$$

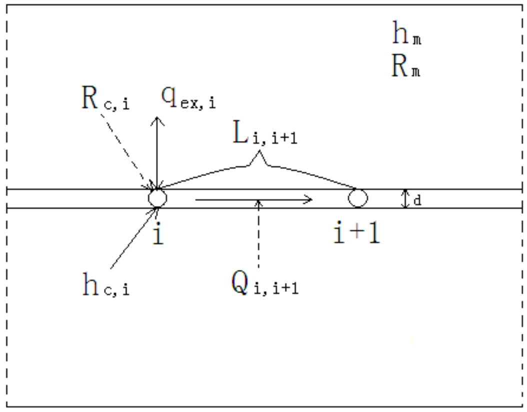

FIGURE 1. Schematic sketch of a domain for CCPF model

Here $\mathbf{v}$ represents the seepage velocity of fluid flows in the matrix, $\mathbb{K}$ represents the hydraulic conductivity (a tensor in general although it will be treated as a scalar below), and h_m represents the hydraulic head in the matrix (continuum).

The **conservation of mass** in matrix (continuum) can be written as [5, 31]

$$\nabla \cdot (\mathbb{K}\nabla h_m) - \Gamma_{ex} + R_m = S\frac{\partial h_m}{\partial t}, \tag{2}$$

where S is the storativity of the water, R_m represents the recharge rate to the continuum matrix, and Γ_{ex} represents the total **exchange rate** of the continuum with other embedded systems (the pipe system in our case). In the CCPF model, the total exchange rate is modeled through the following formula [31, 3, 4]

$$\Gamma_{ex} = \sum_i \delta(\mathbf{x} - \mathbf{x}_i)\, q_{ex,i}\, V^{-1}, \tag{3}$$

where $\mathbf{x}_i$ represents the nodes of the (one dimensional) pipe system embedded in the matrix, δ represents the Dirac delta function, V stands for the unit volume of the continuum, and $q_{ex,i}$ is the fluid exchange rate at the node $\mathbf{x}_i$ which is modeled via a Barenblatt type relation [2] and is of the unit $[L^3T^{-1}]$

$$q_{ex,i} = \alpha_{ex,i}(h_{m,i} - h_{c,i}). \tag{4}$$

Here $h_{m,i}$ and $h_{c,i}$ represent the hydraulic head of the matrix and conduit at the i^{th} node $\mathbf{x}_i$ respectively, and the **exchange coefficient** $\alpha_{ex,i}$ is model through [31]

$$\alpha_{ex}: \ = \ \alpha A K, \ [L^2T^{-1}] \tag{5}$$

$$A: \ = \ \text{exchange surface area}, \ [L^2] \tag{6}$$

$$K: \ = \ \text{hydraulic conductivity}, \ [LT^{-1}] \tag{7}$$

$$\alpha: \ = \ \text{fudging parameter, determined via local geometry, interpreted as inverse fissure spacing}, \ [L^{-1}]$$

2.1.2. *Pipe flow model for the conduit.* The flow in the network of conduits is modeled in the following way. Pipe flow models are utilized in each segment of the conduit system and the exchange of fluids with the surrounding continuum happens at the nodes only. In particular, if the flow is laminar, the following classical

Poiseuille flow formula is applied for a conduit segment from node $\mathbf{x}_i$ to node $\mathbf{x}_j$

$$\begin{aligned} Q_{ij} &= -\frac{\pi d_{ij}^4 g}{128\nu}\frac{h_{c,i}-h_{c,j}}{L_{ij}}, \qquad (8)\\ L_{ij} : &= \text{distance between } i^{th} \text{ and } j^{th} \text{ nodes},\\ d_{ij} : &= \text{diameter of the segment of the pipe},\\ \nu : &= \text{kinematic viscosity}, [L^2T^{-1}],\\ g : &= \text{gravitational acceleration}, [LT^{-2}]. \end{aligned}$$

Here Q_{ij} represents the flow rate ($[L^3T^{-1}]$) from node j to node i.[1]

At each node of the pipe/conduit network the following **Kirchhoff's law** is utilized to couple different segments of the conduit system as well as the surrounding matrix.

$$\sum_j Q_{ij} + q_{ex,i} + R_{c,i} = 0. \qquad (9)$$

Here $R_{c,i}$ represents the recharge rate to the pipe flow system at the i^{th} node.

2.2. **Ill-posedness of the CCPF model.** The coupled continuum pipe flow model (2, 3, 4, 9, 8) is quite appealing due to its intuitive nature and simplicity. Unfortunately the system is not well-posed in the sense that no reasonable solution exists.

Theorem 2.1. *The original CCPF model (2, 3, 4, 9, 8) is ill-posed in the sense that there is no solution to the system that has finite head at the nodes when there is nontrivial fluid exchange between the matrix and the conduits. The ill-posedness occur for both the steady state and time-dependent cases.*

Proof: Indeed, if a reasonably regular solution (h_m, h_c) exists, the head in the matrix and the conduit must be finite at each node so that the exchange term (4) can be defined. We could then treat the exchange term $q_{ex,i}$ at each node in the mass conservation as a simple known source term. In the steady state case, the solution can be represented (assuming given head b on the boundary, and constant hydraulic conductivity for simplicity)

$$\begin{aligned} h_m(\mathbf{x}) = -K^{-1}V^{-1}\sum_i G(\mathbf{x},\mathbf{x}_i)q_{ex,i} + K^{-1}\int_{\Omega_m} G(\mathbf{x},\mathbf{y})R_m(\mathbf{y})\,d\mathbf{y} \\ + \int_{\partial\Omega_m} b(\mathbf{y})\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y})\,dS(\mathbf{y}) \end{aligned} \qquad (10)$$

where $G(\mathbf{x},\mathbf{y})$ is the Green's function of the steady state equation (Poisson equation) with the associated boundary conditions (Dirichlet) and diffusive coefficient set to 1, Ω_m is the region occupied by the matrix with its boundary denoted by $\partial\Omega_m$, $\frac{\partial G}{\partial n}$ is the normal derivative of the Green's function [16]. Although the second and third terms in the formula are harmless and produce smooth functions with given regular boundary value (head) b and recharge rate function R_m, the first term induced by the fluid exchange at the discrete nodes is problematic. Indeed,

$$G(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \phi^{\mathbf{x}}(\mathbf{y})$$

where $\Phi(\mathbf{x})$ is the fundamental solution to the Poisson equation which possesses a singularity at the origin ($-\frac{1}{2\pi}\ln|\mathbf{x}|$ in 2D, and $\frac{1}{4\pi|\mathbf{x}|}$ in 3D), and $\phi^{\mathbf{x}}(\mathbf{y})$ is a

[1] In 2D the formula is given by $Q = \frac{d^3 g}{12\nu}$.

nice harmonic function. This implies that $|h_m(\mathbf{x}_i)| = \infty$ if $q_{ex,i} \neq 0$ which is contradictory to the assumption and (necessity) that the head at node $\mathbf{x}_i$, $h_m(\mathbf{x}_i)$, is finite.

For a given set of boundary data and recharge functions, the continuum matrix and the conduit system do not exchange fluid if and only if the systems decouple and the heads agree at each node $\mathbf{x}_i$. Of course the agreement of heads for the decoupled system at each node happens only for special data satisfying particular compatibility conditions. Therefore the steady state CCPF model is ill-posed in generic situation.

As for the time-dependent problem, we follow a similar approach and utilize the Green's function for the heat equation [23, 27]. Suppose that there exists a solution with finite head as required by basic physics. We can treat the total exchange term (3) in the mass conservation (2) as a given function. Utilizing the Green's function representation and noticing that the leading order term in the Green function is given by the fundamental solution to the heat equation [27](Chap. IV, section 16), we deduce that the leading order contribution to the head in the matrix due to the singular fluid exchange is given by

$$h_m(\mathbf{x}) \approx -\sum_i \frac{1}{SV} \int_0^t \frac{1}{(4\pi(t-s)K/S)^{\frac{n}{2}}} \exp(-\frac{|\mathbf{x}-\mathbf{x}_i|^2}{4(t-s)K/S}) q_{ex,i}(s)\, ds \tag{11}$$

which blows-up as $\mathbf{x} \to \mathbf{x}_i$ if $q_{ex,i} \neq 0$ for spatial dimension 2 or 3 ($n = 2, 3$).

This ends the proof of the theorem. □

Remark: No blow-up of heads have been reported although there have been observations that the heads are over-estimated near the conduit for relatively small exchange coefficent α_{ex} as is suggested in (5). We speculate that no-blow-up is due to the simple fact that the Dirac delta functions must be mollified in the finite difference approximation utilized in MODFLOW [33] and conduit genesis studies [31, 3, 4] and the speculation that the grid size is not small in these cited references. The over estimate of the head close to the conduit is consistent with the analysis above which suggests blow-up. We speculate that we will observe blow-up phenomena under successive grid refinement. The ill-posedness is also consistent with the reported high sensitivity of the CCPF model with respect to α_{ex} [31, 3, 4]. This and other related works will be reported elsewhere.

3. **Hua's modified CCPF model.**

3.1. **Hua's modified CCPF model.**

In the case of one conduit and the system in a laminar steady state, Hua proposed the following version of the coupled continuum pipe flow model in his PhD thesis [22]

$$\begin{cases} -\nabla(\mathbb{K}\nabla h_m) &= -\tilde{\alpha}_{ex}(h_m - h_c)\delta_{\Omega_c} + R_m \quad in\ \Omega_m \\ -\frac{\partial}{\partial\tau}\Big(D\frac{\partial h_c}{\partial\tau}\Big) &= \tilde{\alpha}_{ex}(h_m\big|_{\Omega_c} - h_c) + R_c \quad in\ \Omega_c, \end{cases} \tag{12}$$

where Ω_m, Ω_c are the regions occupied by the matrix and the conduit (conceptualized to a one dimensional curve) respectively, $\frac{\partial}{\partial\tau}$ represents tangential derivative along the conduit, and the laminar Poisieulle constant D is given by

$$D = \begin{cases} \dfrac{\pi d^4 g}{128\nu} & \text{for 3D,} \\ \dfrac{d^3 g}{12\nu} & \text{for 2D.} \end{cases}$$

The well-posedness of Hua's modified CCPF model in two space dimension (2D) as well as its accurate numerical approximation have been addressed by Hua and collaborators [22, 11]. However, no derivation of this model was provided.

3.2. **Heuristic derivation of Hua's model.** Since a derivation of Hua's model is not available in existing literature, we provide a heuristic derivation based on the original CCPF model. The derivation is not only appealing intellectually, but also provides insight into the construction of our new CCPF model which is valid in 3D, and important information on the choice of the new continuum exchange coefficient $\tilde{\alpha}_{ex}$.

We will focus on the case of one straight conduit along the x-axis with uniform diameter d. More general one conduit case can be considered in a similar fashion.

We criticize the original CCPF model in restricting fluid exchange to the discrete nodes only which is unphysical. We believe it probably makes more sense to have fluid exchange everywhere along the conduit. Following this idea, we partition our original conduit of length L into $J, J \gg 1$ segments with equal length L/J. Heuristically the continuum fluid exchange should be the limit of the discrete fluid exchange as J approaches infinity.

For a fixed node i, Kirchhoff's law (9) leads to

$$D\frac{h_{c,i-1} + h_{c,i+1} - 2h_{c,i}}{L/J} + \alpha\pi d\frac{L}{J}K(h_{m,i} - h_{c,i}) + R_{c,i} = 0. \tag{13}$$

Dividing the equation by $\frac{L}{J}$ and taking the limit as $J \to \infty$ we arrive at the second equation in Hua's model (12) under the assumption that $\frac{R_{c,i}}{L/J} \to R_c$. The last assumption is natural since for a continuum (1D) recharge function $R_c(\mathbf{x})$, a first order discrete approximation at a node $\mathbf{x}_i$ with grid size $\frac{L}{J}$ is given by $R_{c,i} \approx R_c(\mathbf{x}_i)\frac{L}{J}$.

Another very useful and important information that we obtained from this simple exercise is that the continuum exchange coefficient $\tilde{\alpha}_{ex}$ (dimension $[LT^{-1}]$) must be different from the discrete exchange coefficient $\alpha_{ex,i}$ (dimension $[L^2T^{-1}]$) proposed in the original CCPF model. The calculation above suggests that the new continuum exchange coefficient is given by

$$\tilde{\alpha}_{ex} = \alpha\pi dK. \tag{14}$$

More general case with variable conduit diameter can be derived as well.

The convergence of the conservation of mass can be formally derived in a similar fashion.

We remark that this derivation cannot be justified rigorously since the original CCPF model is ill-posed.

3.3. **Ill-posedness of Hua's model in 3D.** Although Hua's modified CCPF model is well-posed in two spatial dimension [22, 11], the model is not well-posed in three spatial dimension in general. Indeed, consider the case of x independent data and solution, and steady state, Hua's model reduces to a 2D Poisson equation with atomic source at the origin which implies that the head must be infinite at the origin (the conduit) unless there is no fluid exchange. The detailed proof would be the same as the case for the original CCPF model.

4. **A new CCPF model in 3D.**

4.1. **The new CCPF model in 3D.** We observe that the mathematical problem with the original CCPF and Hua's CCPF is that the fluid exchanges occur on a very singular space: point singularity in the original CCPF case and line singularity in Hua's model. On the physical side, we see that these simplified models do not reflect the physical reality of fluid exchange happening over the interface between the conduit and matrix. We may therefore view the mathematical difficulty as a reflection of the physical deficiency of the simplified models. Taking into consideration the physical observation of exchange of fluids over the whole interface Γ and retaining the Barenblatt type fluid exchange relation, we arrive at the 1st equation in our new model below (15). On the other hand, we do not want to resolve the conduit flow completely, and hence it makes sense to use the averaged head of the matrix on a proper cross section of the interface to calculate the Barenblatt exchange term in the conduit so that the conduit equation remains one dimensional. This leads to the second equation below. To summarize, we propose the following new CCPF model assuming the simple case of an one dimensional conduit centered at the x-axis and laminar flow:

$$\begin{cases} S\frac{\partial h_m}{\partial t} - \nabla(\mathbb{K}\nabla h_m) &= -\tilde{\alpha}_{ex}(h_m\delta_\Gamma - h_c\delta_\Gamma)/|\Gamma_x| + R_m \quad in\ \Omega_m \\ -\dfrac{\partial}{\partial x}\Big(D\dfrac{\partial h_c}{\partial x}\Big) &= \tilde{\alpha}_{ex}(\frac{1}{|\Gamma_x|}\int_{\Gamma_x} h_m\, dl_x - h_c) + R_c \quad in\ \Omega_c; \end{cases} \tag{15}$$

where Γ is the boundary of the circular horizontal conduit centered at the x-axis, Γ_x is the cross section of Γ at x (a circle), dl_x represents the infinitesimal increment of arc length on Γ_x (equivalent to $r(x)\,d\theta$ in the cylindrical coordinates with $r(x)$ being the radius and θ being the angle), and $|\Gamma_x|$ is the length of Γ_x which is $\pi d(x) = 2\pi r(x)$.

4.2. **Well-posedness of the new CCPF model.** We demonstrate that the new CCPF model is well-posed in three spatial dimension in this section. We assume the following simple geometry for the sake of simplicity: $\Omega_m := (0,1)\times B_{y,z} \subset \mathbb{R}^3$, $\Omega_c := (0,1)\times\{0,0\}$, where $B_{y,z}$ is a bounded domain in the y,z plane that contains the origin. The physical pipe is a cylinder with constant radius r centered at Ω_c (hence $|\Gamma_x| = 2\pi r$) but is represented as a line Ω_c in the mathematical domain. We will assume that we have Dirichlet boundary condition, i.e. imposed head.

Without loss of generality we can assume that we have homogeneous Dirichlet boundary condition since the boundary condition can be easily homogenized after a translation of the head and a new re-defined recharge term. Therefore the space that we shall work with for the steady state case is the classical Sobolev space H_0^1 consisting of functions that the function itself and all its first derivatives are square integrable, and the function vanishes on the boundary [1].

We define the following **bilinear form** $a(\cdot,\cdot)$ on $\mathbf{H}\times\mathbf{H}$, where $\mathbf{H} := H_0^1(\Omega_m)\times H_0^1(\Omega_c)$,

$$\begin{aligned} a(\mathbf{h},\mathbf{v}) := \int_\Omega \mathbb{K}\nabla h_m(\mathbf{x})\cdot\nabla v_m(\mathbf{x})d\mathbf{x} + \int_0^1 D h_c'(x) v_c'(x)dx \\ +\frac{\tilde{\alpha}_{ex}}{2\pi}\int_0^1\int_0^{2\pi}(h_m(x,r,\theta)-h_c(x))v_m(x,r,\theta)\,d\theta\,dx \\ -\tilde{\alpha}_{ex}\int_0^1(\frac{1}{2\pi}\int_0^{2\pi}h_m(x,r,\theta)\,d\theta - h_c(x))v_c(x)dx, \end{aligned} \tag{16}$$

where $\mathbf{h}=(h_m,h_c)$, $\mathbf{v}=(v_m,v_c)\in\mathbf{H}$, where cylindrical coordinates are used in the last two integrals.

Then the **weak formulation** of the new CCPF model (15) for the steady state case after homogenizing the boundary conditions is

$$a(\mathbf{h},\mathbf{v}) = (R_m, v_m)_{L^2(\Omega_m)} + (R_c, v_c)_{L^2(\Omega_c)} \quad \forall\ \mathbf{v}\in\mathbf{H}. \tag{17}$$

The well-posedness of the weak problem (17) can be easily established.

Theorem 4.1. *The new model (15) is well-posed in the steady state case, i.e., the weak formulation (17) is well-posed. The time-dependent version of the new model is also well-posed.*

Proof: The proof is a straightforward application of the Lax-Milgram theorem [29] for the steady state case. Indeed, it is easy to see that the bilinear form a is continuous on $\mathbf{H}\times\mathbf{H}$ since H^1 functions have well-defined square integrable traces on the interface $\Gamma=\{(x,r,\theta)|0<x<1, 0<\theta<2\pi\}$ [1]. We only need to check the coercivity. For this purpose we notice that

$$\begin{aligned} a(\mathbf{h},\mathbf{h}) &= \int_\Omega \mathbb{K}\nabla h_m(\mathbf{x})\cdot\nabla h_m(\mathbf{x})d\mathbf{x} + \int_0^1 D|h_c'(x)|^2dx \\ &\quad +\frac{\tilde{\alpha}_{ex}}{2\pi}\int_0^1\int_0^{2\pi}(h_m(x,r,\theta)-h_c(x))h_m(x,r,\theta)\,d\theta\,dx \\ &\quad -\tilde{\alpha}_{ex}\int_0^1(\frac{1}{2\pi}\int_0^{2\pi}h_m(x,r,\theta)\,d\theta-h_c(x))h_c(x)dx \\ &= \int_\Omega \mathbb{K}\nabla h_m(\mathbf{x})\cdot\nabla h_m(\mathbf{x})d\mathbf{x} + \int_0^1 D|h_c'(x)|^2dx \\ &\quad +\frac{\tilde{\alpha}_{ex}}{2\pi}\int_0^1\int_0^{2\pi}(h_m(x,r,\theta)-h_c(x))^2\,d\theta\,dx \\ &\geq \int_\Omega \mathbb{K}\nabla h_m(\mathbf{x})\cdot\nabla h_m(\mathbf{x})d\mathbf{x} + \int_0^1 D|h_c'(x)|^2dx \end{aligned}$$

which implies the coercivity due to the positivity of the hydraulic conductivity $\mathbb{K}$ and the Poiseuille constant D. This ends the proof of the well-posedness of the steady state problem.

An alternative way to show the well-posedness is that we could view h_c as a function of h_m through the second equation in the new CCPF model (15). Indeed, the weak formulation of $(15)_2$ for given $h_m\in H^1(\Omega_m)$ and $R_c\in H^{-1}(\Omega_c)$ is given by

$$a_2(h_c, v_c) = \int_0^1 R_c(x)v_c(x)\,dx + \frac{\tilde{\alpha}_{ex}}{2\pi}\int_0^1 \left(\int_0^{2\pi} h_m(x,r,\theta)\,d\theta\right) v_c(x)\,dx, \quad (18)$$

$$a_2(h_c, v_c) : = \int_0^1 \left(Dh_c'(x)v_c'(x) + \tilde{\alpha}_{ex}h_c(x)v_c(x)\right)\,dx. \quad (19)$$

Here and below, some of the integrations must be interpreted in terms of duality between appropriate dual spaces (such as H_0^1 and H^{-1}) with the duality induces by the L^2 inner product. The well-posedness of this problem is straightforward via a Lax-Milgram argument. We denote the linear solution operator as Ψ, i.e. $a_2(h,v) = (R,v)_{L^2(\Omega_c)}, \forall v \in H_0^1(\Omega_c)$ if and only if $h = \Psi(R)$, and hence

$$h_c = \Psi(R_c) + \frac{\tilde{\alpha}_{ex}}{2\pi}\Psi\left(\int_0^{2\pi} h_m(x,r,\theta)\,d\theta\right) \in H_0^1(\Omega_c). \quad (20)$$

The solution operator can be represented explicitly in terms of Fourier sine series

$$\Psi(f)(x) = \sum_{k=1}^{\infty} \frac{\hat{f}_k}{\pi^2 k^2 D + \tilde{\alpha}_{ex}}\sin(\pi k x), \text{ for } f(x) = \sum_{k=1}^{\infty}\hat{f}_k \sin(\pi k x). \quad (21)$$

This solution operator and the first equation in (15) can be combined to form a single (nonlocal) linear equation for the head h_m in the matrix whose weak formulation takes the form

$$a_1(h_m, v_m) = \int_{\Omega_m} R_m(\mathbf{x})v_m(\mathbf{x})\,d\mathbf{x} + \frac{\tilde{\alpha}_{ex}}{2\pi}\int_0^1 \left(\Psi(R_c)(x)\int_0^{2\pi} v_m(x,r,\theta)\,d\theta\right)dx, \quad (22)$$

$$a_1(h_m, v_m) := \int_{\Omega_m} \mathbb{K}\nabla h_m(\mathbf{x})\cdot\nabla v_m(\mathbf{x})d\mathbf{x} + \frac{\tilde{\alpha}_{ex}}{2\pi}\int_0^1\int_0^{2\pi} h_m(x,r,\theta)v_m(x,r,\theta)\,d\theta\,dx$$
$$-\frac{\tilde{\alpha}_{ex}^2}{4\pi^2}\int_0^1\left(\Psi\left(\int_0^{2\pi} h_m(x,r,\theta)\,d\theta\right)\int_0^{2\pi} v_m(x,r,\theta)\,d\theta\right)dx. \quad (23)$$

The continuity of a_1 on $H_0^1(\Omega_m)\times H_0^1(\Omega_m)$ follows from the standard trace theorem [1] as well as the well-posedness of the second equation through a_2 and the regularity of Ψ. As for the coercivity, we have

$$
\begin{aligned}
a_1(h_m,h_m) &= \int_{\Omega_m} \mathbb{K}\nabla h_m(\mathbf{x})\cdot\nabla h_m(\mathbf{x})d\mathbf{x} + \frac{\tilde{\alpha}_{ex}}{2\pi}\int_0^1\int_0^{2\pi} h_m^2(x,r,\theta)\,d\theta\,dx \\
&\quad -\frac{\tilde{\alpha}_{ex}^2}{4\pi^2}\int_0^1\left(\Psi(\int_0^{2\pi} h_m(x,r,\theta)\,d\theta)\int_0^{2\pi} h_m(x,r,\theta)\,d\theta\right)dx \\
&= \int_{\Omega_m} \mathbb{K}\nabla h_m(\mathbf{x})\cdot\nabla h_m(\mathbf{x})d\mathbf{x} + \frac{\tilde{\alpha}_{ex}}{2\pi}\int_0^1\int_0^{2\pi} h_m^2(x,r,\theta)\,d\theta\,dx \\
&\quad +\frac{\tilde{\alpha}_{ex}}{4\pi^2}\int_0^1\left(D(\Psi(\int_0^{2\pi} h_m(x,r,\theta)\,d\theta))'(\int_0^{2\pi} h_m(x,r,\theta)\,d\theta)'\right)dx \\
&\quad -\frac{\tilde{\alpha}_{ex}}{4\pi^2}\int_0^1\left(\int_0^{2\pi} h_m(x,r,\theta)\,d\theta\int_0^{2\pi} h_m(x,r,\theta)\,d\theta\right)dx \\
&\geq \int_{\Omega_m} \mathbb{K}\nabla h_m(\mathbf{x})\cdot\nabla h_m(\mathbf{x})d\mathbf{x} \\
&\quad +\frac{\tilde{\alpha}_{ex}}{4\pi^2}\int_0^1\left(D(\Psi(\int_0^{2\pi} h_m(x,r,\theta)\,d\theta))'(\int_0^{2\pi} h_m(x,r,\theta)\,d\theta)'\right)dx \\
&\geq \int_{\Omega_m} \mathbb{K}\nabla h_m(\mathbf{x})\cdot\nabla h_m(\mathbf{x})d\mathbf{x},
\end{aligned}
$$

where $'$ stands for differentiation with respect to x and we have utilized the weak formulation for the 2nd equation as well as the explicit formula (21) for the solution operator Ψ. This proves the coercivity.

This alternative approach of viewing h_m as the single unknown is useful, especially for the time-dependent case since the dynamics is in terms of h_m only with the h_c slaved (through Ψ) by h_m. Since we already have the continuity and coercivity of the bilinear form associated with the steady state problem, the time-dependent case can be established via classical semi-group theory, or utilising backward Euler time discretization and taking the limit [35, 10]. We leave the detail to the interested reader.

This ends the proof of the theorem. □

Remark: The new CCPF model (15) can be integrated into the MODFLOW system just as the the original CCPF model. Indeed, we may view the second equation in (15) as the new conduit flow process (CFP) and the new fluid exchange term defined as

$$\Gamma_{ex,new} = -\tilde{\alpha}_{ex}(h_m\delta_\Gamma - h_c\delta_\Gamma)/|\Gamma_x|.$$

The same iterative process that was used in incorporating the original CFP model into MODFLOW can be utilized here as well.

4.3. **Other possible modified CCPF models.** Of course there are many other possible ways of modifying the original CCPF model or Hua's model without properly taking care of the physics. For instance, we could replace the conservation of mass in matrix (2) by the following mollified version while maintain a discrete conduit network with (fixed) finitely many nodes

$$
\begin{aligned}
\nabla\cdot(\mathbb{K}\nabla h_m) - \sum_i \alpha_{ex,i}\left(\frac{1}{|\partial B(\mathbf{x}_i,r_i)|}\int_{\partial B(\mathbf{x}_i,r_i)} h_m(\mathbf{x})\,dS(\mathbf{x}) - h_{c,i}\right)V^{-1} \\
+R_m = S\frac{\partial h_m}{\partial t},
\end{aligned}
\tag{24}
$$

where $\partial B(\mathbf{x}_i, r_i)$ denotes the sphere centered at $\mathbf{x}_i$ with radius r_i. It can be shown that this together with the Kirchhoff type relation (9) leads to mathematical well-posedness. We do not believe that this is a good model however since it does not reflect the basic physics. A slight modification of the model above is to replace the average over the sphere by the average over the ball. We believe that it suffers from similar deficiency although it is mathematically well-posed and may be closer to the numerically mollified version of the original CCPF model.

A good competitor of (15) is to replace the average over the interface in (15) by the average over the whole (solid) cylinder/conduit/pipe. The mathematical well-posedness is straightforward and the numerical treatment of this model might be slightly easier since the forcing is now regular (versus a singular forcing on the interface in (15)). The disadvantage is that the fluid exchange is now assumed to take place on the whole solid conduit instead of the interface only which is physically less appealing.

The validation and invalidation of these models using available lab/field data or the well established Stokes-Darcy model will be reported elsewhere.

5. Conclusion and remarks.

We have shown that the original CCPF model is ill-posed in the sense that no reasonable solution exists if there exists fluid exchange between the conduit system and the matrix system. We have also shown that Hua's modification of the original CCPF model suffers from similar deficiency as well in three spatial dimension although it is well-posed in 2D. We have proposed a new modified CCPF model in 3D that mimics the true physics while maintain the one-dimensionality of the conduit dynamics. The new model is shown to be mathematically sound in both the steady state case and the time-dependent case.

There are many questions to be answered:

- MODFLOW behavior. Our analysis indicates that solution to the original CCPF may blow-up near the discrete nodes of the conduit system. It would be very interesting to perform numerical experiments with MODFLOW to see the behavior of the discrete solution under successive grid refinement. It is also of interest to figure out the exact numerical mollification of the Dirac delta function used in MODFLOW.
- Numerical approximation of the new model. Although we believe developing a convergent numerical scheme to the new model is not difficult, developing fast and accurate numerical solver may be non-trivial due to the existence of singular forcing concentrated on the surface, and the disparate spatial scales. Another important practical issue is the efficient integration with the MODFLOW system (a crude coupling is proposed in section 4).
- Validation of the new model. Although the new CCPF model (15) is mathematically sound, it should be further validated utilizing either lab/field data or existing well established models such as the Stokes-Darcy model with Beavers-Joseph interface condition [10, 17] or simplified interface conditions [14, 15]. In particular, the Stokes-Darcy system can be used as a benchmark to calibrate the continuum exchange parameter $\tilde{\alpha}_{ex}$. Moreover, since we have filled the physical conduit by artificial matrix, the hydraulic conductivity in this fictitious domain needs to be quantified as well. Information such as the parameter regime where the new CCPF model (15) provides reasonable approximation

and parameter regime where CCPF cannot be used and the Navier-Stokes-Darcy model must be used would be also very useful. Validation of other related models discussed above should also be investigated.
- Hua's model works fine in 2D but fails in 3D. His model is apparently more efficient that the new model (15). Is there a regime where the simple 2D model can be used in 3D reality?
- The case with turbulent pipe flow. We have dealt with the laminar flow case only. In many applications the flows in the conduits are turbulent and therefore there is a need to develop parallel theory for the turbulent case.
- Complex geometry. We have only dealt with the simple case of one straight conduit. Real application calls for network of conduits. In this case Kirchhoff's law is still needed to couple different segments of the conduit system. More complex formula involving local geometry may be required to deal with curved conduits.
- Uncertainty and sensitivity. Apparently there are lots of uncertainties in the karst system beyond the usual uncertainty associated with the matrix (hydraulic conductivity for instance). It would be very interesting to determine the sensitivity of the solution on various parameters of the system, such as the geometry of a single pipe and the geometry of the network of conduits.
- Determination of the conduits. It would be very interesting and perhaps a great challenge to try to utilise the model and certain easily measurable physically data (not the conduit geometry though) to locate the conduit and determine its geometry from the data.
- Scaling-up. It would be very interesting to study if there is a good effective large scale equation which avoids resolving the small conduits. Barenblatt's dual porosity is an example under the assumption that the pipes are extremely small and no preferred direction exists (locally homogeneous and isotropic).

There are many potential applications of the new model: it could be used to study conduit genesis just as the original CCPF model; Monte-Carlo simulations could be performed to obtain statistics of conduit distributions within a given physical matrix and external applied recharge; the statistics of conduits could in term be used to reduce uncertainty in prediction as well as numerical scaling-up; possible application in petroleum industry for reservoir in karst region, carbon dioxide sequestration in karst region etc.

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