A generalized birth-death stochastic model for high-frequency order book dynamics

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Abstract

We use a generalized birth-death stochastic process to model the high-frequency dynamics of the limit order book, and illustrate it using parameters estimated from Level II data for a stock on the London Stock Exchange. A new feature of this model is that limit orders are allowed to arrive in multiple sizes, an important empirical feature of the order book. We can compute various quantities of interest without resorting to simulation, conditional on the state of the order book, such as the probability that the next move of the mid-price will be upward, or the probability, as a function of order size, that a limit ask order will be executed before a downward move in the mid-price. This generalizes a successful model of Cont *et al.* (2010) by means of a new technical approach to computing the distribution of first passage times.

1 Introduction

High-frequency trading lately accounts for a significant portion of the trading volume on equity, futures and options exchanges worldwide. Trading time

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measured in milliseconds is possible due to the advent of electronic "order-driven" trading systems instituted by the most important exchanges, including NYSE, Nasdaq, and the Tokyo and London Stock Exchanges. These automated systems operate by aggregating all outstanding limit orders in a *limit order book* that is visible to traders, and incoming market orders are executed against the best available prices.

The high-frequency behavior of the limit order book is a natural candidate for stochastic modeling, both from the perspective of the trader who wishes to forecast, and from the perspective of the Exchange which would like to control the stability of the system. Thanks to the availability of Level II data (quotes and number of shares at different prices), dynamical models can be formulated, estimated, and tested.

In recent literature, there have been equilibrium models of a limit order market like those of Parlour (1998), Foucault (1999), and Goettler *et al.* (2005). Another category is a dynamic expected utility maximization model like those of Avellaneda and Stoikov (2008), and Rous (2009). All these models contain unobservable parameters of utility functions for different investors that are hard to calibrate. Recently, Cont *et al.* (2010) have successfully proposed a continuous-time Markov model in which all the parameters can be estimated from the order book "Level II" data, the model is analytically tractable, and it reproduces various typical empirical features of observed order books.

Cont *et al.* (2010) view the limit order book as a queuing system where incoming orders and cancellations of existing orders arrive in unit size according to independent Poisson processes, and this allows them to compute various conditional probabilities, such as the probability that the next mid-price move will be upward. This kind of queuing system can be described by what is called a birth-death process, where the states represent the number of shares at a given order price, and transitions take place by birth (the entry of a new limit order), or death (removal from the limit order book by cancellation or matching with a new market order). Birth-death processes have been well-studied in the queuing literature, which makes them attractive for use as statistical models.

However, one drawback of this approach is that in traditional queuing theory births and death occur (with probability one) one at a time. For example, in a telephone exchange handling a large number of calls, the probability that two calls arrive simultaneously can safely be taken to be zero. Therefore a pure birth-death model of the order book restricts orders to be of unit size (which can be set to the average order size, for example).

In practice, the limit order book does not behave this way. Empirical studies on statistical properties of limit orders by Knez and Ready (1996), and Bouchaud *et al.* (2002) show that one of the most important conditioning variables for price impact is the size of an order relative to the volume at the best price, so a model that can deal with multiple order sizes is of particular interest. The difficulty is that the theory of more general birth-death processes which allow multiple births or deaths is less well developed.

In this paper, we solve a generalized version the model of Cont *et al.* (2010) in which the technical requirement that limit orders be of unit size is no longer needed. We employ a generalized birth-death process that allows multiple births to model the arrival of limit orders of various sizes. We use Laplace transforms to compute the distribution of first passage times, which allows for the calculation of quantities such as the probability that the mid-price increases/decreases at its next move and the probability that an order is executed before the mid-price moves. We illustrate this model for parameters estimated from Level II data for a stock on the London Stock Exchange. This model, to the best of our knowledge, is the first one to deal with multiple order sizes quantitatively.

The technical problem of evaluating the Laplace transform of the relevant first passage time for multiple order sizes is overcome by a new technique analyzing the limit of truncated state spaces, each of which is tractable via recursion. The continued fraction method employed by Cont *et al.* (2010) apparently does not carry over.

The structure of this paper is as follows. Section 2 describes the dynamics of a limit order book and introduces a general birth-death process with multiple births to capture the characteristics of this system. In section 3, we estimate the parameters required for the model using high-frequency data from the London Stock Exchange. Section 4 derives the probability density function of the first-passage time in a general birth-death process, which plays a key role in calculating the quantities of interest in a limit order book. In section 5, we use the result of section 4 to forecast illustrative quantities from the order book: the probability that the mid-price increases/decreases at its next move, and the probability that an ask order is executed before the mid-price declines. Section 6 concludes.

2 Order book dynamics and the model

Market participants trading an asset in an order-driven market can take three different kinds of actions. They may (1) place a *limit order* to buy or sell a specified number of shares of the asset at a particular price specified at the time of the order, (2) place a *market order* to buy or sell a specified number of shares of the asset at the best currently available price, which is executed immediately, or (3) cancel a previously placed limit order that has not yet been executed.

The outstanding limit orders are summarized in a *limit order book*, which lists the total number of shares of buy and sell limit orders at each price. This limit order book is constantly changing as new orders arrive, and we are interested in modeling statistical properties of the limit order book state.

The limit buy orders are called *bids*, and the limit sell orders are called *asks*. The *size of the order* is the number of shares specified by the order. The lowest price for which there is an outstanding limit sell order is called the *best ask* and the highest price for which there is an outstanding limit buy order is called the *best bid*. The gap between the best ask and the best bid is called the *bid-ask spread*. The average of the best ask and best bid is called the *mid-price*. We consider the mid-price instead of transaction prices to avoid the tendency of transaction prices to bounce back and forth between the best ask and the best bid. Prices are restricted to be integer multiples of a minimum price difference called a *tick value* or *tick*.

Market orders. A trade occurs as a market order arrives and it is matched against limit orders of the opposite side. If the size of the market order is bigger than the the quantity in the current best price, the remaining will be matched to the second best price. So a market order bigger than the opposite best price widens the bid-ask spread by increasing the best ask (and hence the mid-price) if it is a buy order, or decreasing the best bid (and hence the mid-price) if it is a sell order.

Limit orders. When the bid-ask spread is more than one tick, a limit order that falls between the best ask and bid narrows the bid-ask spread by increasing the best bid (and hence the the mid-price)if it is a buy order, or decreasing the best ask (and the hence the mid-price) if it is a sell order.

Cancellations. Market participants are allowed to cancel any limit orders they previously posted. The cancellation of all the limit orders at the best price widens the bid-ask spread by increasing the best ask if the cancellation takes place at the best ask, or decreasing the best bid if it takes place at the best bid.

To capture the above dynamics of a limit order book, Cont *et al.* (2010) proposed a stochastic model where the events above are modeled using independent Poisson processes. Their model successfully captures the characteristics of the speed of the order arrivals by modeling it with the intensity rate of a Poisson process. However, the model analysis requires them to assume that all orders are of unit size (where the unit is the average observed size of limit orders). Empirical studies (Chakraborti *et al.* (2009); Toke (2010)) show that new limit order sizes are in fact randomly distributed according to an exponential law. To deal with the size of limit orders and also consider the tractability of the model, we develop the following model.

Model Framework and Notation

Limit orders are allowed to be placed on a finite price grid $\Pi = \{1, 2, \dots, n\}$ representing prices measured in multiples of a tick, and where the upper boundary n is chosen large enough to exceed all reasonable order price levels that might occur during the time frame of the analysis. Let $\mathbf{Z}^+ = \{0, 1, 2, 3, \dots\}$ denote the non-negative integers, which represents the possible sizes of limit orders in multiples of S_m shares, where S_m is the average size of a market order (see below).

We describe the state of the order book with two $\mathbf{Z}^+$-valued continuous-time Markov processes, the ask process

$$A(t) = (A_1(t), \dots, A_n(t)),$$

and the bid process

$$B(t) = (B_1(t), \dots, B_n(t)),$$

where $t \geq 0$, $A_k(t)$ represents the number of asks (in multiples of S_m shares) outstanding at price k and time t, and similarly for $B_k(t)$. Since there cannot be asks and bids at the same price (they would be traded against each other), we must have $A_k(t) \wedge B_k(t) = 0$ for all k and t, where $\wedge$ denotes the minimum operator.

The *best ask price* at time t is then defined by

$$p_A(t) = \inf\{p \in \Pi : A_p(t) > 0\} \wedge (n+1),$$

the *best bid price* at time t is

$$p_B(t) = \sup\{p \in \Pi : B_p(t) > 0\} \vee 0,$$

and in our market $p_B(t) < p_A(t)$ for all t.

The *mid-price* at time t is denoted

$$p_M(t) = (p_A(t) + P_B(t))/2,$$

and the *bid-ask spread* is

$$p_S = p_A(t) - p_B(t).$$

We impose the following model assumptions governing how the processes A and B evolve.

✓ Market orders and cancellations of existing orders are of constant size S_m, where S_m denotes the average size of a market order over the period of interest;

✓ Market buy and sell orders arrive at independent, exponentially distributed times with rate μ;

✓ Limit orders of size k, $k = 1, 2, \ldots, M$ (in multiples of S_m shares) arrive at a distance of j ticks from the opposite best quote at independent, exponentially distributed times with rates denoted $\lambda_j^{(k)}, j \geq 1$. Here M is the number of order sizes to be handled by the model, which is chosen at the discretion of the modeler;

✓ Cancellations of limit orders at a distance of j ticks from the same-side best quote arrive at independent, exponentially distributed times at a rate proportional to the number of outstanding shares: if the number of outstanding shares at that level is kS_m, then the cancellation rate is $k\theta_j$, $j, k \geq 0$;

✓ All the above events are mutually independent.

The parameters S_m, μ, θ_j, and $\lambda_j^{(k)}$ must all be estimated from market data. The rate parameters $\mu, \lambda_j^{(k)}$ are measured in "orders per second" and

θ_j is can be thought of as measured in "orders per second per existing order" (since $k\theta_j$ is a rate in orders per second) where one order represents S_m shares.

A and B are then continuous time Markov chains with state space $(\mathbf{Z}^+)^n$ and transition rates given by

$$
\begin{array}{llll}
A_i(t) \to A_i(t) + k & \text{at rate} & \lambda^{(k)}_{i-p_B(t)} & \text{for } i > p_B(t), k \geq 0 \\
A_i(t) \to A_i(t) - 1 & \text{at rate} & A_i(t)\theta_{i-p_A(t)} & \text{for } i \geq p_A(t) \\
A_i(t) \to A_i(t) - 1 & \text{at rate} & \mu & \text{for } i = p_A(t) > 0
\end{array}
$$

$$
\begin{array}{llll}
B_i(t) \to B_i(t) + k & \text{at rate} & \lambda^{(k)}_{p_A(t)-i} & \text{for } i < p_A(t), k \geq 0 \\
B_i(t) \to B_i(t) - 1 & \text{at rate} & B_i(t)\theta_{p_B(t)-i} & \text{for } i \leq p_B(t) \\
B_i(t) \to B_i(t) - 1 & \text{at rate} & \mu & \text{for } i = p_B(t) < n + 1
\end{array}
$$

Notice we choose the same-side best quote instead of the opposite side for cancellations since there is nothing to cancel between the best bid and the best ask when the bid-ask spread is greater than 1. The model above may be viewed as a direct generalization of the model of Cont *et al.* (2010) to account for the arrival of limit orders of multiple sizes.

The use of an infinite state space $\mathbf{Z}^+$ at each price level is no more than a convenience idealization, since in reality the number of shares in the order book will be bounded above by a finite bound κ, which could be taken to be any suitably large value that will never be exceeded in practice, such as ten times the total number of outstanding shares of the stock in question. It turns out it is technically convenient for us also to consider the finite model in which the state space is truncated at $\kappa >> 1$; we will show that the probabilities of interest are not sensitive to the particular choice of κ because they converge to a limit as κ tends to infinity.

To define the κ-truncated model, let $\mathbf{Z}_\kappa = \{0, 1, \ldots, \kappa\}$ be a finite state space representing the number of bids or asks at a given price in the order book. We merely need to modify the continuous time Markov chains defined above to have state space $(\mathbf{Z}_\kappa)^n$ with the same transition rates, except that (1) if a transition would cause the state of $A_i < \kappa$ or $B_i < \kappa$ to exceed κ, the state is reset to κ, and (2) at state κ, if j is the number of ticks from κ to the opposite best quote, the rate of incoming limit orders $\lambda^{(k)}_j$ is reset to zero for all k (so the only transitions at state κ are downward).

If κ is set large enough, the behavior of the truncated model will be identical to the behavior of the infinite model for values relevant to the data, so there is no theoretical disadvantage in considering the truncated model when needed.

3 Parameter estimation

Our raw data is Level II order book data for Vodaphone (VOD.L), traded on the London Stock Exchange from September 7th, 2009 to September 11th, 2009 (five trading days, and all day long for each trading day). This data consists of snapshots of time-stamped sequences of transactions (i.e., actual trades) and quotes (prices and quantities of shares) for the ten best price levels on each side of the order book. These snapshots are time-stamped in seconds (starting zero at midnight), with a resolution of one millisecond, and taken whenever there is a change at any of the price levels. Since we study the European market and want to avoid the impact of the US market, we use only four hours of data, from 9:30 am to 1:30 pm.

From the raw data, Table 1 shows an example of quotes from time 39301.481 to 39308.359 and Table 2 shows the transactions that took place during this period. Since we need the price, quantity and time of each order (market, limit, and cancellation) to calibrate the parameters in our model, the following criteria are employed to derive such numbers from the raw data:

✓ if the quantity at a given price has increased, then we count a limit order at that price, with a volume equal to the difference of the quantities observed;

✓ if the quantity at a given price has decreased and there is no transaction at that time, then we count a cancellation order at that price, with a volume equal to the difference of the quantity;

✓ if the quantity at a given price has decreased and there is a transaction at that time, then we count a market order at that price, with a volume equal to the difference of the quantity, and we record the time in the quote instead of the transaction field since there might be some delay of recording market orders in the transaction field in the raw data we have.

So from Table 1 and Table 2 we can conclude that an order of size 10000 was canceled at the 4th level bid at time 39302.891, an order of size 320 was executed at the best bid at time 39305.192, and an order of size 7500 was canceled at the 4th level bid at time 39308.359.

To calibrate the parameters, we need to compute the average size S_m (in shares) of market orders. Let T be the length of time investigated, N_m the number of shares in market orders placed during that time, and $N_c(i)$ the number of shares cancelled at the ith level quote during that time. Here "ith level quote" means the price of a bid (respectively, ask) at a distance $i-1$ ticks from the best bid (respectively, best ask) price.

The arrival rate of market orders (at both best bid and best ask) is then estimated (in orders per second) by

$$\hat{\mu} = \frac{N_m}{S_m T}$$

and the cancellation rate for the ith level quote (both bid and ask) can be estimated by

$$\hat{\theta}(i-1) = \frac{N_c(i)}{S_m T}\left(\frac{L(i)}{S_m}\right)^{-1}$$

where $L(i)$ is the average number of shares in the order book at the ith level quote over time period T, and $\frac{L(i)}{S_m}$ then represents the average number of unit orders outstanding.

For limit orders at a distance j ticks from the opposite best quote, we divide the data by size into M groups, where M is a number of sizes to be handled by the model and is at the discretion of the modeler. An order is in the kth group, $k = 1, \ldots, M$, if its size in shares is closer to kS_m than to $k'S_m$ for any other k', and $N_l^{(k)}$ denotes the total number of shares in the limit orders of the kth group. We then estimate the arrival rate of our model limit orders of size k (in orders) by

$$\hat{\lambda}_j^{(k)} = \frac{N_l^{(k)}(j)}{S_m T}$$

Using our data from 9:30 am to 1:30 pm on September 7th, 2009, and setting $M = 2$, we estimate the parameters for the first level quote as reported in Table 3.

4 First-passage times of general birth-death processes with multiple births

We are interested in forecasting a number of quantities in the order book, such as the probability that the mid-price increases/decreases at its next move, or the probability that an order is executed before the mid-price moves. These calculations boil down to calculating the probability density function of the first passage time of a general birth-death process to reach state 0. As indicated in Section 2, the number of orders at the best quote (ask/bid) follows a general birth-death process with multiple birth rates $\lambda_S^{(1)}, \lambda_S^{(2)}, \ldots, \lambda_S^{(M)}$ of sizes $1, 2, \ldots M$, respectively (in multiples of S_m shares), and death rates $\mu + i\theta(0)$ of size 1 at state $i \geq 1$. We formulate and solve the problem of deriving the probability density function of the first passage time of such a process to reach state 0 in this section.

For simplicity, let's assume $M = 2$, since larger values can be handled similarly. Now consider a general $\mathbf{Z}^+$-valued birth-death process $X(t)$, where we call the value of $X(t)$ the "state" at time t. We assume $X(t)$ has birth rates $\lambda^{(1)}, \lambda^{(2)}$ of sizes 1, 2, respectively, and death rates μ_i of size 1 at state $i \geq 1$, and let τ_b denote the first-passage time of this process to state 0 given it begins in state b. We now consider the probability density function of τ_b.

Notice that, if the current state is $i \geq 2$, in order to reach state $i-2$ this process must first reach state $i-1$. So by the Markovian property (Asmussen (2003)), we can write τ_b as the sum of independent random variables

$$\tau_b = \tau_{b,b-1} + \tau_{b-1,b-2} + \cdots + \tau_{1,0} \tag{1}$$

where $\tau_{i,i-1}$ denotes the first-passage time of the above process from state i to state $i-1$, for $i = 1, 2, \ldots b$.

Let $v_i = \lambda^{(1)} + \lambda^{(2)} + \mu_i$ and let $g_i(t)$ denote the probability density function of $\tau_{i,i-1}$. Then the dwell time at state i has density $v_i e^{-v_i t}$ and with probability $\frac{\lambda^{(1)}}{v_i}$ the subsequent transition is to state $i+1$, with probability $\frac{\lambda^{(2)}}{v_i}$ the subsequent transition is to state $i+2$, and with probability $\frac{\mu_i}{v_i}$ the transition is to state $i-1$. Hence we have the recursive formula

$$g_i(t) = \frac{\mu_i}{v_i} v_i e^{-v_i t} + \frac{\lambda^{(1)}}{v_i} v_i e^{-v_i t} * g_{i+1}(t) * g_i(t) + \frac{\lambda^{(2)}}{v_i} v_i e^{-v_i t} * g_{i+2}(t) * g_{i+1}(t) * g_i(t) \tag{2}$$

where $*$ means convolution. The idea to derive the probability density function of τ_b is to derive the Laplace transform of $\tau_{i,i-1}$ first, and then the Laplace transform of τ_b by multiplying these together, and finally numerically invert the Laplace transform of τ_b (Abate and Whitt (1995); Abate and Whitt (1999)).

Definition 4.1 *The Laplace transform of a function $f(x)$ is*

$$\hat{f}(s) = \int_0^\infty e^{-sx} f(x) dx \tag{3}$$

when the above improper integral converges, where s is a complex variable.

Obviously, if $f(x)$ is bounded and s is a complex variable with positive real part, then the above integral converges.

By the Laplace transform of $\tau_{i,i-1}$ we mean the Laplace transform of its pdf $g_i(t)$, which is

$$\hat{g}_i(s) = \mathbb{E}[e^{-s\tau_{i,i-1}}] = \int_0^\infty e^{-st} g_i(t) dt \tag{4}$$

where s is a complex variable with positive real part.

If we take the Laplace transform on both sides of equation (2) and use

$$\int_0^\infty e^{-st} e^{-v_i t} dt = \frac{1}{v_i + s},$$

we have

$$\hat{g}_i(s) = \frac{\mu_i}{v_i + s} + \frac{\lambda^{(1)}}{v_i + s} \hat{g}_{i+1}(s) \hat{g}_i(s) + \frac{\lambda^{(2)}}{v_i + s} \hat{g}_{i+2}(t) \hat{g}_{i+1}(s) \hat{g}_i(s) \tag{5}$$

i.e.

$$\hat{g}_i(s) = \frac{\mu_i}{v_i + s - \lambda^{(1)} \hat{g}_{i+1}(s) - \lambda^{(2)} \hat{g}_{i+2}(s) \hat{g}_{i+1}(s)} \tag{6}$$

These formulas provide a recursive way to compute the functions $\hat{g}_i$ if we make use of the truncated model described previously. That is, consider a truncated birth-death process $X^{(\kappa)}(t)$ defined on $\mathbf{Z}_\kappa$ with birth rates $\lambda^{(1)}$ of size 1 and $\lambda^{(2)}$ of size 2 at states $i \leq \kappa - 2$, birth rate $\lambda^{(1)} + \lambda^{(2)}$ of size 1 at state $\kappa - 1$, and death rates μ_i of size 1 at states $i \geq 1$.

The processes X and $X^{(\kappa)}$ are identical for trajectories that remain below state $\kappa - 2$, so will be interchangeable in application if κ is chosen large enough. Let $T_i^{(\kappa)}$ denote the first passage time down from state i to state $i-1$ in the truncated process $X^{(\kappa)}$, and let $\tau_b^{(\kappa)}$ be the corresponding first passage time from state b to zero. We show below that $T_i^{(\kappa)}$ tends to a limit in probability as κ tends to infinity, and hence so does $\tau_b^{(\kappa)}$. (In fact the limit is τ_b, but we don't need that here.) We will see in examples that the limit is closely approached already for fairly low values of κ, which makes the recursive problem numerically easy.

Let $\hat{g}_i^{(\kappa)}(s)$ be the Laplace transform of the first-passage time $T_i^{(\kappa)}$ from state i to state $i-1$. Since the first passage time from κ to $\kappa - 1$ is an exponential random variable with rate μ_κ, we have the following similar recursive formulas determining each $\hat{g}_i^{(\kappa)}(s)$:

$$\hat{g}_\kappa^{(\kappa)}(s) = \frac{\mu_\kappa}{s + \mu_\kappa} \tag{7}$$

$$\hat{g}_{\kappa-1}^{(\kappa)}(s) = \frac{\mu_{\kappa-1}}{s + \lambda^{(1)} + \lambda^{(2)} + \mu_{\kappa-1} - (\lambda^{(1)} + \lambda^{(2)})\hat{g}_\kappa^{(\kappa)}(s)} \tag{8}$$

$$\hat{g}_i^{(\kappa)}(s) = \frac{\mu_i}{\lambda^{(1)} + \lambda^{(2)} + \mu_i + s - \lambda^{(1)}\hat{g}_{i+1}^{(\kappa)}(s) - \lambda^{(2)}\hat{g}_{i+2}^{(\kappa)}(s)\hat{g}_{i+1}^{(\kappa)}(s)} \tag{9}$$

for $i = \kappa - 2, \kappa - 3, \ldots, 1$.

Next, we prove $\hat{g}_i^{(\kappa)}(s)$ converges as κ tends to $+\infty$. Let $F_i^{(\kappa)}(t)$ be the cdf of $T_i^{(\kappa)}$.

Lemma 4.2 *We have the following stochastic order relation as a function of* κ*:*

$$T_i^{(\kappa)} \leq T_i^{(\kappa+1)} \quad \textit{for } \kappa \geq i$$

Sketch of Proof. This is equivalent to the statement

$$F_i^{(\kappa)}(t) \geq F_i^{(\kappa+1)}(t) \quad \text{for } \kappa \geq i \text{ and all } t \geq 0.$$

To see this, note that $X^{(\kappa)}$ and $X^{(\kappa+1)}$ have the same probability of an upward move on $\{1, \ldots, \kappa - 1\}$ and the same probability of a downward move on $\{1, \ldots, \kappa\}$. Hence the probability of either process exceeding state $\kappa - 1$ by time $s < t$ is the same. Conditional on exceeding state $\kappa - 1$ by time s, the process $X^{(\kappa)}$ now has a greater probability of arriving at state $i-1$ by time t, because $X^{(\kappa+1)}$ may first detour to state $\kappa+1$, causing a delay. QED

The above inequality means that $F_i^{(\kappa)}(t)$ decreases to a limit $F_i^{(\infty)}(t)$ as $k \to \infty$. By a similar argument as above,

$$F_i^{(\kappa)}(t) \geq F_i(t) \quad \text{for } k \geq i \text{ and all } t \geq 0,$$

where $F_i(t)$ denotes the cdf of $\tau_{i,i-1}$, the first passage time from i to $i-1$ for the unbounded process X. Therefore, in the limit,

$$F_i^{(\infty)}(t) \geq F_i(t)$$

and hence $F^{(\infty)}(t) \to 1$ as $t \to \infty$, so it is the cdf of a random variable on $[0, \infty)$.

Since

$$\hat{g}_i^{(\kappa)}(s) = \int_0^\infty e^{-st} dF_i^{(\kappa)}(t)$$

and letting

$$\hat{g}_i(s) = \int_0^\infty e^{-st} dF_i^{(\infty)}(t)$$

then $\hat{g}_i^{(\kappa)}(s) \to \hat{g}_i(s)$ at any complex number s with positive real part, as $k \to \infty$, by virtue of the following standard lemma.

Lemma 4.3 *If probability distribution functions $F_n(x) \to F(x)$ for all $x \in \mathbb{R}$ as $n \to \infty$, then $\mathbb{E}_n(u) \to \mathbb{E}(u)$ for every bounded piecewise-continuous function u*

$$u(x) : \mathbb{R} \to \mathbb{C}$$

where $\mathbb{E}_n$ and $\mathbb{E}$ are expectations with respect to F_n and F, respectively.

Table 4 illustrates the convergence of the Laplace transforms of the pdf of the first passage times for $X^{(\kappa)}$ from state 10 to 9, from 9 to 8 and from 8 to 7 evaluated at $s = 10 + 5\mathrm{i}$, as κ ranges from 10 to 20. Convergence to a limit is seen to be very fast for these parameter values, and is similar for other values of s.

From now on we will consider first passage times $\tau = \tau^{(\infty)}$ as the limit of the corresponding times $\tau^{(\kappa)}$ as $\kappa \to \infty$ for the bounded processes $X^{(\kappa)}$. These first passage times will be indistinguishable in practice from those of the original process X (they are in fact equal), and we approximate them by recursion using a finite but sufficiently large value of κ.

Let the probability density function of τ_b, the first passage time from b to 0, be $f_{b,0}(t)$, and the cummulative distribution function be $F_{b,0}(t)$. Denote the Laplace transform of $f_{b,0}(t)$ by $\hat{f}_{b,0}(s)$ and the Laplace transform of $F_{b,0}(t)$ be $\hat{F}_{b,0}(s)$. Then by Abate and Whitt (1999)

$$\hat{f}_{b,0}(s) = \prod_{i=1}^{b} \hat{g}_i(s) \tag{10}$$

since $\mathbb{E}[e^{-s(X+Y)}] = \mathbb{E}[e^{-sX}e^{-sY}] = \mathbb{E}[e^{-sX}]\mathbb{E}[e^{-sY}]$, when X and Y are independent variables. Also,

$$\begin{aligned}
\hat{F}_{b,0}(s) &= \int_0^\infty e^{-st} F_{b,0}(t)dt && (11)\\
&= -\frac{1}{s}F_{b,0}(t)e^{-st}|_{t=0}^{\infty} - (-\frac{1}{s})\int_0^\infty e^{-st} f_{0,b}(t)dt && (12)\\
&= \frac{1}{s}\int_0^\infty e^{-st} f_{0,b}(t)dt && (13)\\
&= \frac{1}{s}\hat{f}_{b,0}(s) && (14)
\end{aligned}$$

since s has a positive real part, where $\hat{g}_i(s)$ is defined in (4).

With the Laplace transform $\hat{f}(s)$ in hand, $f(t)$ can be computed by numerically inverting $\hat{f}(s)$, provided $f(t) = 0$ for $t < 0$, as follows.

$$\begin{aligned}
f(t) &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{ts}\hat{f}(s)ds \quad \text{(for a fixed } a > 0\text{)}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(a+iu)t}\hat{f}(a+iu)du\\
&= \frac{e^{at}}{2\pi}\left\{\int_{-\infty}^{+\infty} \Re[\hat{f}(a+iu)]\cos utdu - \int_{-\infty}^{+\infty} \Im[\hat{f}(a+iu)]\sin utdu\right\}
\end{aligned}$$

since $f(t)$ is real. If $t \geq 0$, then $f(-t) = 0$, i.e.

$$\frac{e^{-at}}{2\pi}\left\{\int_{-\infty}^{+\infty} \Re[\hat{f}(a+iu)]\cos(-ut)du - \int_{-\infty}^{+\infty} \Im[\hat{f}(a+iu)]\sin(-ut)du\right\} = 0,$$

and hence

$$\int_{-\infty}^{+\infty} \Re[\hat{f}(a+iu)] \cos utdu = -\int_{-\infty}^{+\infty} \Im[\hat{f}(a+iu)] \sin utdu.$$

So for $t \geq 0$, we have

$$\begin{aligned} f(t) &= \frac{e^{at}}{\pi} \int_{-\infty}^{+\infty} \Re[\hat{f}(a+iu)] \cos utdu \\ &= \frac{2e^{at}}{\pi} \int_{0}^{+\infty} \Re[\hat{f}(a+iu)] \cos utdu \quad (\text{since } \Re[\hat{f}(a+iu)] = \Re[\hat{f}(a-iu)]). \end{aligned}$$

Using the trapezoidal rule with a step size h, the above approximately equals

$$f_h(t) = \frac{he^{at}}{\pi} \Re[\hat{f}(a)] + \frac{2he^{at}}{\pi} \sum_{k=1}^{\infty} \Re[\hat{f}(a+ikh)] \cos(kht).$$

Let $h = \frac{\pi}{2t}$ and $a = \frac{A}{2t}$, then

$$f_h(t) = \frac{e^{A/2}}{2t} \Re[\hat{f}(\frac{A}{2t})] + \frac{e^{A/2}}{t} \sum_{k=1}^{\infty} (-1)^k \Re[\hat{f}(\frac{A+2k\pi i}{2t})] \tag{15}$$

Abate and Whitt (1995) showed that if $f(t)$ is bounded, say $|f(t)| \leq C$, then the discretization error is bounded by

$$|f(t) - f_h(t)| \leq C \frac{e^{-A}}{1-e^{-A}}$$

In our code, we set $A = 15 \log 10$ in (15) in order to get 15 digits of accuracy.

Abate and Whitt (1995) proved that $\Re[f(\frac{A+2k\pi i}{2t})]$ has a constant sign when k sufficiently large, if the second derivative of $f(t)$ is continuous. (This is true for the cdf of the first-passage time of the κ-truncated process, since it has the form $1 - \sum_{i=1}^{l} \gamma_i e^{-\beta_i t}$, where γ_i and β_i are constants. A convenient reference for this algebraic approach is Keilson (1979)). So (15) is an eventually alternating series, and a standard method to accelerate the computation of (15) is the Euler summation to m terms after an initial n terms (Johnsonbaugh (1979)):

$$E(m,n,t) = \sum_{k=0}^{m} \mathrm{C}_m^k 2^{-m} s_{n+k}(t) \tag{16}$$

where
$$s_n(t) = \frac{e^{A/2}}{2t}\Re(\hat{f}(\frac{A}{2t})) + \frac{e^{A/2}}{t}\sum_{k=1}^{n}(-1)^k\Re(\hat{f}(\frac{A+2k\pi i}{2t})) \tag{17}$$
which is the weighted average of the last $m+1$ partial sums by a binodmial probability distribution with parameters m and $p = 1/2$. (Here C_m^k is the binomial coefficient "m choose k".) It is easy to see when fixing m that $E(m,n,t)$ converges to $f_h(t)$ as n goes to infinity. In this paper, after choosing $m = 25$ and $n = 25$ in (16), we obtained an error associated with the difference of successive terms of
$$\frac{|E(m,n+1,t) - E(m,n,t)|}{E(m,n,t)} < 0.01.$$
Tables 5-7 show the results of this calculation for the approximating process $X^{(\kappa)}$, as the truncation state κ increases. The values of $F_{b,0}^{(\kappa)}(t)$ evaluated at $t = 1, 5, 10$ when $b = 10$ are shown to illustrate the dependence on κ. Figures 1-2 show the graphs of $f_{b,0}^{(\kappa)}(t)$ and $F_{b,0}^{(\kappa)}(t)$ when $b = 10$ and the truncation state is $\kappa = 60$. Larger values of κ make no visible difference.

5 Two Examples

5.1 Direction of the next price move

Now we compute the probability that the mid-price increases at its next move using our model and the parameters estimated in Section 3. We choose $M = 2$ in our calibration. Assume that at time 0, there are a orders at the best ask price $p_A(0)$, and b orders at the best bid price $p_B(0)$, and the bid-ask spread is S ticks. Let τ_a be the first time all the orders at the price $p_A(0)$ disappear, and τ_b be the first time all the orders at the price $p_B(0)$ disappear. Let τ_A^i be the first time a limit sell order arrives i ticks away from the best bid and τ_B^i be the first time a limit buy order arrives i ticks away from the best ask, $i = 1, \ldots, S-1$. Let
$$\Delta_S = \sum_{i=1}^{S-1}\sum_{j=1}^{M}\lambda_i^{(j)}$$
where the $\lambda_i^{(j)}$ are the limit order arrival rates defined in section 2.

The Laplace transforms of τ_a and τ_b, denoted by $\hat{f}_a^S(s)$ and $\hat{f}_b^S(s)$, can be computed recursively as in (7)-(9) and (10). Next let

$$\tau_{aB} = \tau_a \wedge \tau_B^1 \wedge \ldots \wedge \tau_B^{S-1}$$

and

$$\tau_{bA} = \tau_b \wedge \tau_A^1 \wedge \ldots \wedge \tau_A^{S-1}$$

Let $f_{\tau_{aB}}(t)$ and $f_{\tau_{bA}}(t)$ be the probability density functions of τ_{aB} and τ_{bA}, respectively. Then τ_{aB} and τ_{bA} are independent since τ_a, τ_B^1, ..., τ_B^{S-1}, τ_b, τ_A^1, ..., τ_A^{S-1} are all independent.

Lemma 5.1 (Cont *et al.* (2010)) *Let Z be an exponentially distributed random variable with parameter Λ, σ be a random variable with Laplace transform $\hat{f}(s)$, and Z and σ be independent. Then the Laplace transform of the random variable $\sigma \wedge Z$ is given by*

$$\hat{f}(\Lambda + s) + \frac{\Lambda}{\Lambda + s}(1 - \hat{f}(\Lambda + s)).$$

Therefore the Laplace transforms of τ_{aB} and τ_{bA} are

$$\hat{f}_{\tau_{aB}}(s) = \hat{f}_a^S(\Delta_S + s) + \frac{\Delta_S}{\Delta_S + s}(1 - \hat{f}_a^S(\Delta_S + s))$$

and

$$\hat{f}_{\tau_{bA}}(s) = \hat{f}_b^S(\Delta_S + s) + \frac{\Delta_S}{\Delta_S + s}(1 - \hat{f}_b^S(\Delta_S + s)).$$

Obviously, τ_{aB} and τ_{bA} are both nonnegative, so we can apply the numerical inversion (16)-(17) to $\hat{f}_{\tau_{aB}}(s)$ and $\hat{f}_{\tau_{bA}}(s)$ to derive $f_{\tau_{aB}}(t)$ and $f_{\tau_{bA}}(t)$. Then the pdf of $\tau_{aB} - \tau_{bA}$ is

$$f_{\tau_{aB}-\tau_{bA}}(z) = \begin{cases} \int_0^\infty f_{\tau_{aB}}(u) f_{\tau_{bA}}(u - z) du & (z \leq 0) \\ \int_z^\infty f_{\tau_{aB}}(u) f_{\tau_{bA}}(u - z) du & (z > 0) \end{cases}$$

so

$$\mathbb{P}[\tau_{aB} - \tau_{bA} < 0] = \int_{-\infty}^0 \int_0^\infty f_{\tau_{aB}}(u) f_{\tau_{bA}}(u - z) du dz$$

which is the probability that the mid-price increases at its next move when currently, there are a orders at best ask, and b orders at best bid, and the bid-ask spread is S ticks.

Although the double integral is computationally costly, one can use parallel computing to speed the computation. Table 8 shows the result when the bid-ask spread is 1 tick using the parameters calibrated in Table 3, where the truncation bound is 50 plus the larger of the numbers of initial orders at best bid and best ask. The probabilities shown depend, of course, on the model parameters, and are reported here for illustration.

5.2 Best ask order execution

A trader may intend to place a limit order at the current best ask price, and would like to know the probability that the order will be executed, conditional on it not being cancelled, before the mid-price experiences a downward move. A virtue of our model is that we can compute this probability as a function of the size of the limit order, which is not handled by previous models.

Assume again at time 0 there are a orders (that is, aS_m shares) at best ask, and b orders (that is, bS_m shares) at best bid, and the bid-ask spread is S ticks. Let ϵ_a^r be the time when an order of size rS_m shares, placed at the best ask at time 0, is completely executed by incoming market orders (of unit size S_m). The probability we are interested in is

$$\mathbb{P}[\epsilon_a^r - \tau_{bA} < 0]. \tag{18}$$

Because the order book is executed on a first-in first-out basis, the time to execution of our order is not influenced by limit orders that may later arrive at the same price. Also, only the a pre-existing orders are subject to cancellation, since we are conditioning on no cancellation of the order we place. Therefore, according to the assumptions made in Section 2, ϵ_a^r is equivalent to the first-passage time of a pure death process to state 0 given it begins at state $a+r$, where the death rate is $\mu + (i-r)\theta_0$ at state i when $i = r+1, r+2, \ldots, r+a$, and is μ when $i = 1, 2, \ldots, r$. Denote the probability density function of ϵ_a^r by $f_{\epsilon_a^r}(t)$, then the Laplace transform of $f_{\epsilon_a^r}(t)$, denoted by $\hat{f}_{\epsilon_a^r}(s)$, is given by

$$\hat{f}_{\epsilon_a^r}(s) = \left(\frac{\mu}{\mu+s}\right)^r \prod_{i=r+1}^{r+a} \frac{\mu + (i-r)\theta_0}{\mu + (i-r)\theta_0 + s} \tag{19}$$

since the Laplace transform of the first passage time from state i to state

$i - 1$ in this pure death process is

$$\frac{\mu}{\mu + s} \quad \text{when } i = 1, 2, \ldots, r$$

and

$$\frac{\mu + (i-r)\theta_0}{\mu + (i-r)\theta_0 + s} \quad \text{when } i = r+1, r+2, \ldots, r+a.$$

We can take the inverse Laplace transform of $\hat{f}_{\epsilon_a^r}(s)$ to get $f_{\epsilon_a^r}(t)$, since ϵ_a^r is positive. Then the probability density function of $\epsilon_a^r - \tau_{bA}$ is given by

$$f_{\epsilon_a^r - \tau_{bA}}(z) = \begin{cases} \int_0^\infty f_{\epsilon_a^r}(u) f_{\tau_{bA}}(u-z) du & (z \leq 0) \\ \int_z^\infty f_{\epsilon_a^r}(u) f_{\tau_{bA}}(u-z) du & (z > 0) \end{cases}$$

so the target probability (18) is

$$\mathbb{P}[\epsilon_a^r - \tau_{bA} < 0] = \int_{-\infty}^0 \int_0^\infty f_{\epsilon_a^r}(u) f_{\tau_{bA}}(u-z) du dz.$$

Table (9) and (10) show the result when the order size r is $1S_m$ and $2S_m$, respectively, and the bid-ask spread S is 1 tick, using the parameters calibrated in Table 3, where the truncation bound is chosen to be 50 plus the larger of the numbers of initial orders at best bid and best ask. In both tables, the first row is the number of initial shares b (multiples of S_m) at the best bid and the first column is the number of initial shares a (multiples of S_m) at the best ask.

6 Conclusion

In order to deal with different sizes of limit orders in an order book, we have enhanced a recent model of birth-death processes by adding multiple births. We show the convergence of the Laplace transforms of the first passage times for a sequence of truncated processes as the truncation state tends to infinity, which permits us to apply a recursive method to finding a close approximation of the Laplace transform of the first passage time to zero in our model. Numerical results are obtained to illustrate the use of the model by answering two typical questions: conditional on the current state, what is the probability that the next move of the mid-price is upward, and what is the probability, as a function of size, that a limit order placed at best ask is executed before the mid-price moves downward?

The recursive approach used here does not handle the more general case of when limit orders, market orders, and cancellations all may have multiple sizes; this corresponds to analyzing a generalized birth-death process where both births and deaths are allowed to have multiple sizes. We do not know how to handle this situation.

A further open question is how to analyze the model when the arrival rates of limit orders, market orders, and cancellations are not all mutually independent. This case would seem to be important in improving the quality of the model's fit to the real world.

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Table 1: A sample of quotes taken from the raw data. The time-stamp is measured in seconds from midnight, the type B refers to bids, and the level is the number of ticks from best bid (counting from 1 = best bid). Rows appear when an event occurs.

time-stamp	type	level	price	quantity
39301.481	B	4	134.9	203651
39301.722	B	1	135.05	10000
39302.891	B	4	134.9	193651
39302.891	B	2	135	192869
39305.192	B	1	135.05	9680
39308.359	B	4	134.9	186151

Table 2: A sample transaction taken from raw data. A transaction represents an actual trade taking place where a market order arrives to fill an opposite limit order.

time-stamp	price	quantity
39305.192	135.05	320

Table 3: Estimated parameter values, using $M = 2$ sizes of limit orders. Here $\hat{\mu}$ is the arrival rate of market orders in units of S_m shares, $\hat{\theta}(0)$ is the limit order cancellation rate per orders present, $\hat{\lambda}_0^{(1)}$ and $\hat{\lambda}_0^{(2)}$ are the arrival rates of limit orders of size 1 and 2 (in multiples of S_m) at best bid and ask, and $L(1)$ is the average number of shares at the best quote (both bid and ask).

$\hat{\mu}$	$\hat{\theta}(0)$	$\hat{\lambda}_0^{(1)}$	$\hat{\lambda}_0^{(2)}$	S_m	$L(1)$
3.16	0.71	7.46	0.80	8127	59986

Table 4: Laplace Transform, evaluated at $s = 10 + 5\mathrm{i}$, of the pdf of the first passage time $\tau_{i,i-1}$ for $i = 8, 9, 10$ in a truncated birth-death process with birth rates $\lambda^{(1)}, \lambda^{(2)}$ of size $1, 2$, respectively, and death rates $\mu_i = \mu + i\theta$ of size 1 at state $i \geq 1$: $\lambda^{(1)} = 7.46$, $\lambda^{(2)} = 0.80$, $\theta = 0.71$, $\mu = 3.16$. The first column displays different choices for the truncated state $\kappa = 10, \ldots, 20$.

κ	$i = 8$	$i = 9$	$i = 10$
10	0.346738-0.0832812i	0.383182-0.0960662i	0.477343-0.117804i
11	0.345423-0.0812206i	0.366959-0.0865165i	0.402881-0.0986009i
12	0.345356-0.0808634i	0.36551-0.0844016i	0.386282-0.0893613i
13	0.345367-0.08081i	0.365423-0.0840193i	0.384709-0.0872165i
14	0.345372-0.080803i	0.365432-0.0839597i	0.384601-0.0868137i
15	0.345373-0.0808022i	0.365437-0.0839515i	0.384608-0.0867486i
16	0.345373-0.0808022i	0.365438-0.0839506i	0.384613-0.0867392i
17	0.345373-0.0808022i	0.365439-0.0839505i	0.384614-0.0867381i
18	0.345373-0.0808022i	0.365439-0.0839505i	0.384614-0.086738i
19	0.345373-0.0808022i	0.365439-0.0839505i	0.384614-0.086738i
20	0.345373-0.0808022i	0.365439-0.0839505i	0.384614-0.086738i

Table 5: The cdf $F_{b,0}^{(\kappa)}(t)$ of the first passage time from b to zero, for $b = 10$ and $t = 1$. Listed are the values for various choices of the truncation state $\kappa = 10, \ldots, 40$. Parameters are $\lambda^{(1)} = 7.46$, $\lambda^{(2)} = 0.80$, $\theta = 0.71$, $\mu = 3.16$.

κ	$F_{b,0}^{(\kappa)}(1)$
10	0.00489228
11	0.00384194
12	0.00350267
13	0.00341969
14	0.00340075
15	0.00339684
16	0.00339609
17	0.00339596
18	0.00339594
19	0.00339594
20	0.00339594
21	0.00339594
22	0.00339594
23	0.00339594
24	0.00339594
25	0.00339594
26	0.00339594
27	0.00339594
28	0.00339594
29	0.00339594
30	0.00339594
31	0.00339594
32	0.00339594
33	0.00339594
34	0.00339594
35	0.00339594
36	0.00339594
37	0.00339594
38	0.00339594
39	0.00339594
40	0.00339594

Table 6: As above, but evaulated at $t = 5$.

κ	$F_{b,0}^{(\kappa)}(5)$
10	0.140684
11	0.123179
12	0.112249
13	0.105679
14	0.101809
15	0.0995937
16	0.0983638
17	0.0977031
18	0.09736
19	0.0971878
20	0.0971042
21	0.0970649
22	0.0970471
23	0.0970392
24	0.0970358
25	0.0970344
26	0.0970339
27	0.0970337
28	0.0970336
29	0.0970336
30	0.0970335
31	0.0970335
32	0.0970335
33	0.0970335
34	0.0970335
35	0.0970335
36	0.0970335
37	0.0970335
38	0.0970335
39	0.0970335
40	0.0970335

Table 7: As above, but evaluated at $t = 10$

κ	$F_{b,0}^{(\kappa)}(10)$
10	0.29788
11	0.267879
12	0.248017
13	0.235182
14	0.227029
15	0.221965
16	0.218899
17	0.217095
18	0.216065
19	0.215495
20	0.215189
21	0.21503
22	0.214949
23	0.21491
24	0.214891
25	0.214883
26	0.214879
27	0.214877
28	0.214877
29	0.214876
30	0.214876
31	0.214876
32	0.214876
33	0.214876
34	0.214876
35	0.214876
36	0.214876
37	0.214876
38	0.214876
39	0.214876
40	0.214876

Table 8: Probability that the mid-price increases at its next move, with parameters as estimated above. The column labels indicate the number of initial shares (in multiples of S_m) at best bid, and the row labels indicate the number of initial shares at best ask.

	1	2	3	4
1	0.50	0.639607	0.693461	0.71779
2	0.349262	0.50	0.551479	0.590195
3	0.305924	0.435753	0.50	0.533218
4	0.281541	0.403993	0.465104	0.50

Table 9: Probability of executing an ask order before the mid-price moves downward when the order size is $1S_m$. The column labels indicate the number of initial shares (in multiples of S_m) at best bid, and the row labels indicate the number of initial shares at best ask.

	1	2	3	4
1	0.591788	0.825442	0.91758	0.958566
2	0.55933	0.789673	0.892219	0.941529
3	0.5383	0.766822	0.873602	0.92768
4	0.524982	0.751181	0.8597	0.9165

Table 10: Probability of executing an ask order before the price moves downward when the order size is $2S_m$. Rows and columns are as in Table 9.

	1	2	3	4
1	0.550144	0.779358	0.883466	0.934737
2	0.529704	0.756403	0.863987	0.919624
3	0.51695	0.740946	0.849723	0.907672
4	0.508713	0.730555	0.839777	0.899094

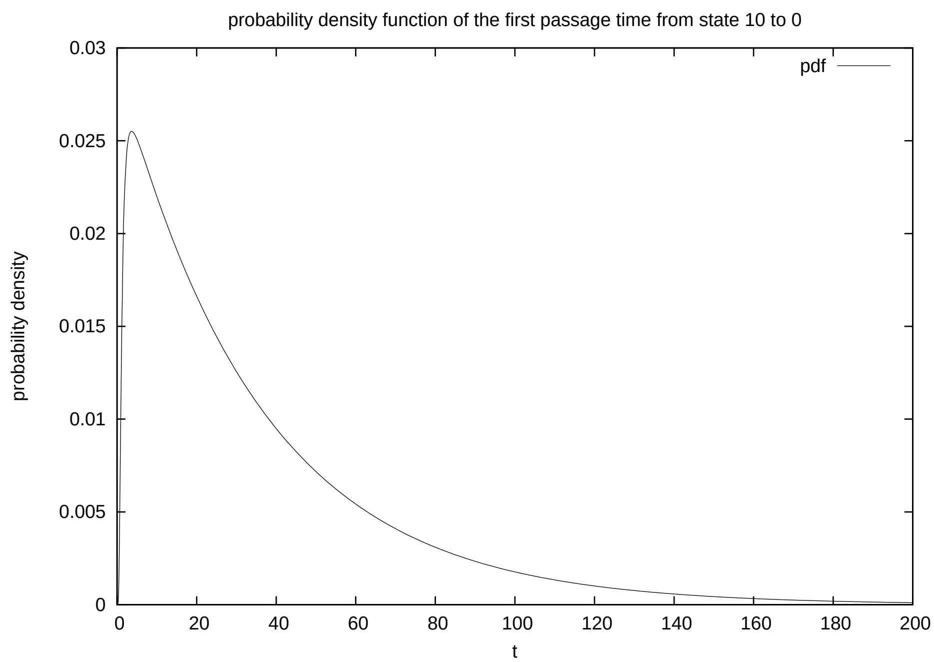

Figure 1: Probability density function of the first passage time from state 10 to state 0, with parameters $\lambda^{(1)} = 7.46$, $\lambda^{(2)} = 0.80$, $\theta = 0.71$, $\mu = 3.16$, truncation state $\kappa = 60$. (Larger values of κ are indistinguishable.)

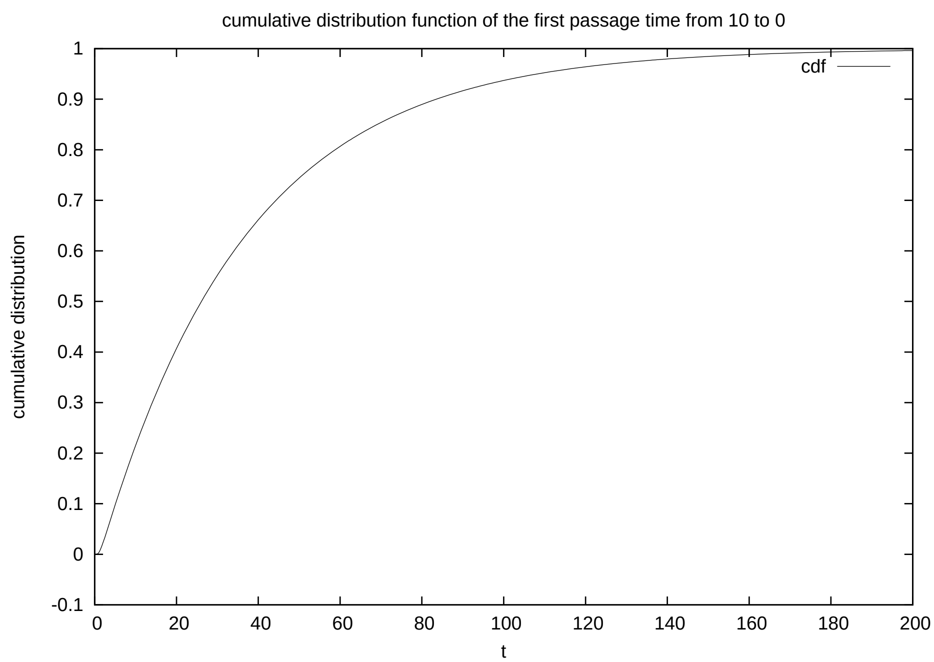

Figure 2: Cumulative distribution function of the first passage time from state 10 to state 0 corresponding to the pdf of Table 1.