Mapping classes associated to mixed-sign Coxeter systems

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Abstract

Given an ordered fat graph with labeled vertices we define a generalized Coxeter system, and Coxeter element. We then construct an associated mapping class with the property that its image under the symplectic homomorphism is conjugate to the negative of the Coxeter element. Applying a result of Lanneau and Thiffeault, we show that the minimum dilatations for orientable pseudo-Anosov mapping classes of surfaces with small genus can be realized by mapping classes of this form.

1 Introduction

In this paper we study a natural correspondence between Coxeter graphs and mapping classes of surfaces. For braids, this correspondence is well known. The Artin braid group, which is defined in terms of the Coxeter graph A_n, generalizes naturally to Artin groups $\mathcal{A}(\Gamma)$ associated to arbitrary Coxeter graphs Γ. For a surface S, and a distinguished collection of simple closed curves $\mathcal{C} = \{\gamma_1, \dots, \gamma_n\}$ whose geometric intersection graph equals Γ, there is a natural homomorphism

$$\begin{array}{rcl} \rho : \mathcal{A}(\Gamma) & \to & \mathrm{Mod}(S) \\ s_i & \mapsto & \delta_{\gamma_i} \end{array}$$

defined by taking the ith generator of $\mathcal{A}(\Gamma)$ to the Dehn twist around γ_i. Several authors have studied group presentations of and relations in $\mathrm{Mod}(S)$ using this correspondence (see, for example, [PV] [Lab] [LP] [Mat]).

The goal in this paper is to describe small dilatation pseudo-Anosov mapping classes using graphs. Heuristically, compositions of Dehn twists along curves with high self- and mutual intersections can be expected to have large dilatation. We canonically define a surface and mapping class $\phi_{\Gamma,\mathfrak{s}}$ associated to each ordered fat graph $\Gamma, \mathfrak{s}$ where

$$\mathfrak{s} : \mathcal{V} \to \{+, -\}$$

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associates a sign to each element of the vertex set $\mathcal{V}$ of Γ. The mapping class $\phi_{\Gamma,\mathfrak{s}}$ is a composition of Dehn twists along simple closed curves that intersect pairwise at most once. As an example, we combine our results with those of E. Lanneau and J.L. Thiffeault [LT] to observe the following (see Section 4.4).

Theorem 1 *The minimum dilatations for orientable pseudo-Anosov mapping classes on closed oriented surfaces of genus $g = 2, 3, 4, 5$ are realizable by mapping classes of the form $\phi_{\Gamma,\mathfrak{s}}$.*

As tools for studying the $\phi_{\Gamma,\mathfrak{s}}$, we define two combinatorial objects associated to the pair $\Gamma, \mathfrak{s}$. One is a generalized Coxeter system $(\mathcal{W}, \mathcal{S})$ and its associated Coxeter element. The other is a surface with a distinguished ordered system of oriented simple closed curves $(S, \mathcal{C})$ and an associated mapping class ϕ. If all signs are positive, then $(\mathcal{W}, \mathcal{S})$ is the classical simply-laced Coxeter system associated to Γ as defined in, for example, [Bour]. If Γ is bipartite and positively signed, then the pair $(S, \mathcal{C})$ and ϕ belongs to a class of examples studied by W. Thurston in [Thu], [Lei]. If Γ is bipartite and signs are compatible with the bipartite partition, the associated mapping classes can be studied using the techniques of Penner in [Pen]. The genus 3 minimum dilatation orientable pseudo-Anosov mapping class cannot be constructed from a positive Thurston, or a mixed-sign Penner example, but can be realized by our construction using more general signed graphs (see Section 4).

Geometric realizations of the Artin group of Γ are usually defined as follows. Let $\mathcal{C}$ be an ordered collection of simple closed curves $\mathcal{C} = \{\gamma_1, \dots, \gamma_n\}$ on S that meet in general position with no pair intersecting more than once, and whose incidence matrix agrees with the adjacency matrix for a graph Γ. We will make the added assumption that the curves in $\mathcal{C}$ are oriented so that, for $i < j$, the algebraic and geometric intersection numbers of γ_i with γ_j agree. Hence, for example, $\iota_{\text{alg}}(\gamma_i, \gamma_j) \geq 0$ when $i < j$. If Γ is the (ordered) incidence graph for $\mathcal{C}$, we say $(S, \mathcal{C})$ is an *(oriented) geometric realization* of the graph Γ. The extra conditions are necessary for ρ to behave naturally with respect to the geometric representation of the Artin group in $\text{Mod}(S)$, and the symplectic representation of $\text{Mod}(S)$ (see Section 2).

Let Γ be a graph, and $\mathfrak{s} : \mathcal{V} \to \{\pm\}$ a labeling of the vertices $\mathcal{V}$ of Γ. To a signed graph $(\Gamma, \mathfrak{s})$ we associate a generalized Coxeter reflection group $\mathcal{W}$ with generators $\mathcal{S} = \{s_1, \dots, s_n\}$, which act as reflections on $\mathbb{R}^n$ and preserve a certain symmetric bilinear form (see Section 2). As in the classical case, the Coxeter element of $\mathcal{W}$ is the product $\omega_{\Gamma,\mathfrak{s}} = s_1 \cdots s_n$. The *Coxeter polynomial* associated to $(\Gamma, \mathfrak{s})$ is the characteristic polynomial of $\omega_{\Gamma,\mathfrak{s}}$, and is denoted $C_{\Gamma,\mathfrak{s}}(x)$.

Theorem 2 *Let $(S, \mathcal{C})$ be a geometric realization of an ordered signed graph $(\Gamma, \mathfrak{s})$. Let*

$$\phi_{\Gamma,\mathfrak{s}} = \delta_{\gamma_1}^{\mathfrak{s}(1)} \cdots \delta_{\gamma_n}^{\mathfrak{s}(n)}$$

be the product of positive Dehn twists δ_{γ_i} and their inverses, as prescribed by the ordering of $\mathcal{C}$ and the sign function $\mathfrak{s}$. The embedding of $\mathcal{C}$ in S defines a natural map $\eta : \mathbb{R}^n \to H_1(S; \mathbb{R})$ such that

$$(\phi_{\Gamma,\mathfrak{s}})_* \circ \eta = -\eta \circ \omega_\Gamma.$$

Given $\phi \in \text{Mod}(S)$ the norm of the largest eigenvalue of $\phi_* : H_1(S; \mathbb{R}) \to H_1(S, \mathbb{R})$, denoted $\lambda_{\text{hom}}(\phi)$, is called the *homological dilatation* of ϕ. For example, if the mapping torus of ϕ is a fibered knot K, then the characteristic polynomial Δ_ϕ of ϕ_* is the Alexander polynomial of K.

Given a polynomial $p(x)$ let $|p|$ be the maximum complex norm for roots of $p(x)$. This is known as the *house* of $p(x)$. For a linear automorphism A let $|A|$ be the maximum norm of an eigenvalue of A, also known as the *spectral radius* of A.

Corollary 3 *Up to cyclotomic factors, the Coxeter polynomial and Alexander polynomial are related as follows:*

$$\Delta_{\phi_{\Gamma,\mathfrak{s}}}(x) = C_{\Gamma,\mathfrak{s}}(-x).$$

Thus, the homological dilatation of $\phi_{\Gamma,\mathfrak{s}}$ is given by

$$\lambda_{hom}(\phi_{\Gamma,\mathfrak{s}}) = |\omega_{\Gamma,\mathfrak{s}}|.$$

For any pseudo-Anosov mapping class ϕ, we have

$$\lambda_{\mathrm{hom}}(\phi) \leq \lambda(\phi)$$

(see, for example, [Ryk]). This implies the following.

Corollary 4 *For any signed graph $(\Gamma, \mathfrak{s})$ such that $\phi_{\Gamma,\mathfrak{s}}$ is pseudo-Anosov, we have*

$$\lambda(\phi_{\Gamma,\mathfrak{s}}) \geq |\omega_{\Gamma,\mathfrak{s}}|.$$

In Section 2, we describe the natural linear representation of the Artin group of a graph and its relation with the Coxeter representation. Section 3 describes a canonical geometric realization of an ordered fat graph. This will complete the proof of Theorem 2. Section 4 contains examples and applications.

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2 Linear representation of the Artin group of a graph

In this section, we define a representation of the Artin group associated to a graph Γ with n-vertices into $\mathrm{SL}(\mathbb{R}^n)$. This representation will be compatible with the symplectic representation of $\mathrm{Mod}(S)$ for any oriented geometric realization $(S, \mathcal{C})$ of Γ. We also define a generalized Coxeter system, called a *mixed-sign Coxeter system*, associated to a pair $(\Gamma, \mathfrak{s})$, and prove a relation between an element of the Artin group determined by $(\Gamma, \mathfrak{s})$ and the Coxeter element of the mixed-sign Coxeter system.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be an ordered graph. The *Artin group* of an unsigned ordered graph Γ is the group

$$\mathcal{A}_\Gamma = \langle \sigma_1, \ldots, \sigma_n \ : \ [\sigma_i \sigma_j]_{m_{i,j}} = [\sigma_j \sigma_i]_{m_{i,j}} \rangle$$

where

$$m_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } v_i \text{ and } v_j \text{ are not connected by an edge} \\ 3 & \text{if } v_i \text{ and } v_j \text{ are connected by an edge} \end{cases}$$

and $[\sigma_i\sigma_j]_m$ is the alternating product

$$[\sigma_i\sigma_j]_m = \sigma_i\sigma_j\sigma_i \dots$$

of length m. If Γ is the classical Coxeter graph A_n, then $\mathcal{A}_\Gamma$ is the the braid group on the disk with $n+1$ punctures (see, e.g., [Bir]).

Let V be the dual vector space of $\mathbb{R}^{\mathcal{V}}$. We can think of $\mathbb{R}^{\mathcal{V}}$ as the space of ways to label the vertices of Γ by real numbers. The ordered vertices $\mathcal{V}$ determine a canonical ordered basis $v_1, \dots, v_n$ of V, thus identifying V with $\mathbb{R}^n$. Let $A = [a_{i,j}]$ be the adjacency matrix for Γ. Let F be the skew-symmetric bilinear form on V defined with respect to $v_1, \dots, v_n$ by

$$F = A^+ - A^-,$$

where A^+ is the upper triangular part of A, and A^- is the transpose of A.

There is a natural representation

$$\eta : \mathcal{A}_\Gamma \to \mathrm{SL}(V)$$

preserving F defined by

$$\eta(\sigma_i)(v_j) = \begin{cases} v_j & \text{if } i = j, \\ v_j + a_{i,j}v_i & \text{if } i < j, \\ v_j - a_{i,j}v_i & \text{if } i > j. \end{cases}$$

We now define a mixed-sign Coxeter system. Let

$$\mathfrak{s} : \mathcal{V} \to \{1, -1\}$$

be a choice of sign. Let $I_\mathfrak{s}$ be the $n \times n$ matrix with 0 on the off-diagonal, and diagonal entries: $\mathfrak{s}(v_i)$. Let $U = I_\mathfrak{s} - A^+$, where A is the adjacency matrix of Γ, and A^+ is the upper triangular part of A and I is the identity matrix. Then U determines a symmetric form

$$B = U + U^T$$

on V.

The Coxeter system $(\mathcal{W}, \mathcal{S})$ associated to Γ is the subgroup of $SL(V)$ generated by $\mathcal{S} = \{s_1, \dots, s_n\}$, where s_i are defined by

$$\begin{aligned} \rho(s_i)(v_j) &= v_j - \mathfrak{s}(v_i)B(v_i, v_j)v_i; \\ &= \begin{cases} -v_j & \text{if } i = j, \\ v_j + \mathfrak{s}(v_i)a_{i,j}v_i & \text{if } i \neq j. \end{cases} \end{aligned}$$

The group $\mathcal{W}$ preserves the symmetric form B. If $\mathfrak{s} \equiv 1$, then $(\mathcal{W}, \mathcal{S})$ is the classical Coxeter system.

Theorem 5 *Given* Γ *and* $\mathfrak{s}$*, let*

$$\omega_{\Gamma,\mathfrak{s}} = s_1 \cdots s_n \in \mathcal{W}_{\Gamma,\mathfrak{s}}$$

and let

$$\phi = \sigma_1^{\mathfrak{s}(v_1)} \cdots \sigma_n^{\mathfrak{s}(v_n)} \in \mathcal{A}_\Gamma.$$

Then

$$\rho(\omega) = -\eta(\phi),$$

where we consider the images of ρ *and* η *as subsets of* $SL(V)$*.*

Let $B_c = U + cU^T$. Then we have $B = B_1$ and $F = B_{-1}$. Now define elements $f_1^{(c)}, \ldots, f_n^{(c)} \in \mathrm{SL}(V)$ by

$$\begin{aligned} f_i^{(c)}(v_j) &= v_j - \mathfrak{s}(v_1) B_c(i,j) v_i \\ &= \begin{cases} -cv_i & \text{if } i = j \\ v_j + \mathfrak{s}(v_i) a_{i,j} v_i & \text{if } i < j \\ v_j + \mathfrak{s}(v_i) c a_{i,j} v_i & \text{if } i > j \end{cases} \end{aligned}$$

Then $f_i^{(1)} = s_i$ and $f_i^{(-1)} = \rho(\sigma_i^{\mathfrak{s}(v_n)})$.

Lemma 6

$$f_1^c \cdots f_n^c = -cU^{-1}U^T.$$

Howlett proved this Lemma in the case when $c = 1$ [How]. We present the generalized proof here.

Proof. First we notice that

$$\begin{aligned} Uf_1^{(c)} &= \begin{bmatrix} \mathfrak{s}(v_1) & -a_{1,2} & -a_{1,3} & \ldots & -a_{1,n} \\ 0 & \mathfrak{s}(v_1) & -a_{2,3} & \ldots & -a_{2,n} \\ \ldots & & & & \\ 0 & & & \ldots & \mathfrak{s}(v_n) \end{bmatrix} \begin{bmatrix} -c & \mathfrak{s}(v_1)a_{1,2} & & \ldots & \mathfrak{s}(v_1)a_{1,n} \\ 0 & 1 & 0 & \ldots & 0 \\ \ldots & & & & \\ 0 & & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} -c\mathfrak{s}(v_1) & 0 & & \ldots & 0 \\ 0 & \mathfrak{s}(v_1) & -a_{2,3} & \ldots & -a_{2,n} \\ \ldots & & & & \\ 0 & & & \ldots & \mathfrak{s}(v_n) \end{bmatrix} \end{aligned}$$

Assume that

$$\begin{aligned} Uf_1^{(c)} \cdots f_k^{(c)} &= \begin{bmatrix} -c\mathfrak{s}(v_1) & 0 & & & & \ldots & & & 0 \\ -ca_{1,2} & -c\mathfrak{s}(v_1) & 0 & & & & & \ldots & 0 \\ & \ldots & & & & & & & \\ -ca_{k,1} & \ldots & -ca_{k,k-1} & -c\mathfrak{s}(v_k) & 0 & & & \ldots & 0 \\ 0 & & \ldots & 0 & \mathfrak{s}(v_{k+1}) & -a_{k+1,k+2} & & \ldots & -a_{k+1,n} \\ & \ldots & & & & & & & \\ 0 & & & \ldots & & & 0 & \mathfrak{s}(v_{n-1}) & -a_{n-1,n} \\ 0 & & & \ldots & & & & 0 & \mathfrak{s}(v_n) \end{bmatrix} \\ &= \begin{bmatrix} L_k & 0 \\ 0 & U_k \end{bmatrix}. \end{aligned}$$

Multiplying on the right by

$$f_{k+1}^{(c)} = \left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline -\mathfrak{s}(v_{k+1})ca_{k+1,1} \ldots - \mathfrak{s}(v_{k+1})ca_{k+1,k} & -c & \mathfrak{s}(v_{k+1})a_{k+1,k+2} \ldots \mathfrak{s}(v_{k+1})a_{k+1,n} \\ \hline 0 & 0 & I \end{array}\right]$$

amounts to replacing the $k+1$st row of $Uf_1^{(c)} \cdots f_k^{(c)}$ by

$$[a_{k+1,1}, \ldots, a_{k+1,k}, -\mathfrak{s}(v_{k+1}), 0, \ldots, 0].$$

Thus,

$$Uf_1^{(c)} \cdots f_{k+1}^{(c)} = \begin{bmatrix} L_{k+1} & 0 \\ 0 & U_{k+1} \end{bmatrix}$$

and $L_n = -cU^T$ as desired. □

Corollary 7 *The transformation* $f = f_1^{(c)} \cdots f_n^{(c)}$ *satisfies*

$$f^T B_c f = B_c.$$

We also have

Corollary 8

$$\rho(\omega) = -U^{-1}U^T,$$

and

$$\eta(\phi) = U^{-1}U^T.$$

This completes the proof of Theorem 5.

3 Geometric realizations of Coxeter elements

Let Γ be a graph with no double- or self-edges, and ordered vertices $\{v_1, \ldots, v_n\}$. Let $\mathfrak{s} : \mathcal{V} \to \{\pm 1\}$ be a choice of signs attached to each vertex. In this section, we show how to construct a geometric realization of Γ in a way that is compatible with the linear representation of the Artin group described in Section 2.

Theorem 9 *Given an ordered graph* Γ*, there is a compact oriented surface with boundary* S_Γ*, and a collection of oriented simple closed curves* $\mathcal{C} = \{\gamma_1, \ldots, \gamma_n\}$ *so that the following holds.*

(i) The oriented loops in $\mathcal{C}$ *form an oriented dual system to* Γ*.*

(ii) The loops in $\mathcal{C}$ generate an n-dimensional subspace of $H_1(S_\Gamma, \mathbb{R})$, and hence determine a homomorphism (extending by the identity)

$$SL(\mathbb{R}^n) \to SL(H_1(S_\Gamma, \mathbb{R})).$$

(iii) the diagram

$$\begin{array}{ccccc} \mathcal{A}(\Gamma) & \xrightarrow{h} & Mod(S_\Gamma) & \longrightarrow & SL(H_1(S_\Gamma;\mathbb{R})) \\ & \searrow^{\eta} & & \nearrow & \\ & & SL(\mathbb{R}^n) & & \end{array}$$

commutes.

Property (i) implies that the intersection form on S_Γ restricted to the ordered oriented curves in $\mathcal{C}$ is given by the matrix $F = A^+ - A^-$, where A is the adjacency matrix of Γ, using the notation of Section 2. Let $\delta_1, \dots, \delta_n$ be the positive Dehn twists along $\gamma_1, \dots, \gamma_n$, and define

$$\phi_{\Gamma,\mathfrak{s}} = \delta_1^{\mathfrak{s}(v_1)} \cdots \delta_n^{\mathfrak{s}(v_n)}.$$

Since the extension map $SL(\mathbb{R}^n) \to SL(H_1(S_\Gamma;\mathbb{R}))$ preserves spectral radius, Theorem 5 implies the following.

Corollary 10 *The mapping class*

$$\phi_{\Gamma,\mathfrak{s}} : S_\Gamma \to S_\Gamma,$$

satisfies

$$\rho(\omega_{\Gamma,\mathfrak{s}}) = -(\phi_{\Gamma,\mathfrak{s}})_*.$$

Thus, Theorem 9 and Corollary 10 imply Theorem 2.

The proof of Theorem 9 will consist of a construction of S_Γ from an ordered graph Γ. As we will show, the surface described in Theorem 9 is unique only if we give Γ an additional fat graph structure, that is, an ordering on the edges emanating from each vertex.

To fix notation, we define an *ordering* on a graph Γ with vertices $\mathcal{V}$ to be a bijection

$$o : \mathcal{V} \to \{1, \dots, n\}.$$

A *fat graph structure* on a graph Γ is a bijection for each vertex $v \in \mathcal{V}$,

$$o_v : \mathcal{V}_v \to \{1, \dots, k\},$$

where $\mathcal{V}_v \subset \mathcal{V}$ is the set of vertices connected to v by an edge and k is the order of $\mathcal{V}_v$.

Given an ordered graph Γ, there is an induced fat graph structure on Γ by ordering the $\mathcal{V}_v$ as they are ordered in the larger set $\mathcal{V}$, but, in general, we allow the two kinds of orderings to be independent from each other.

Construct a system of oriented arcs ℓ_v (say in $\mathbb{R}^3$) corresponding to the vertices v of Γ so that

1. ℓ_{v_1} and ℓ_{v_2} intersect transversally,
2. if $o(v_1) < o(v_2)$, then ℓ_{v_1} meets ℓ_{v_2} with positive intersection, and
3. If v is a vertex, and $v_1, \ldots, v_k$ is the set of vertices adjacent to v, with

$$o_v(v_1) < o_v(v_2) < \cdots < o_v(v_k),$$

then the intersections of $\ell_1, \ldots, \ell_k$ with ℓ_v are arranged in order respecting the orientation of ℓ_v.

Figure 1 shows the arrangement of paths where

(a) $o_v(v_1) < o_v(v_2) < o_v(v_3)$,

(b) $o(v) < o(v_2), o(v_3)$, and

(c) $o(v) > o(v_1)$.

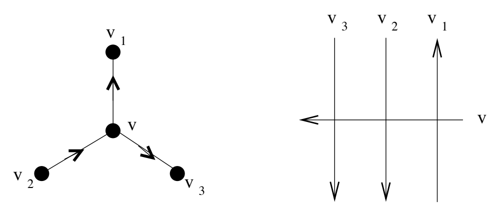

Figure 1: Dual Configuration of line segments

Thicken the line segments in the dual system to produce rectangular strips intersecting in rectangular patches, as in Figure 2. Let Σ_Γ be the surface obtained by gluing the rectangular strips together along the two opposite ends through which the arcs pass. The surface S_Γ has a collection of simple closed loops $\{\gamma_1, \ldots, \gamma_n\}$, which, together with its intersections, are dual to Γ. This completes the

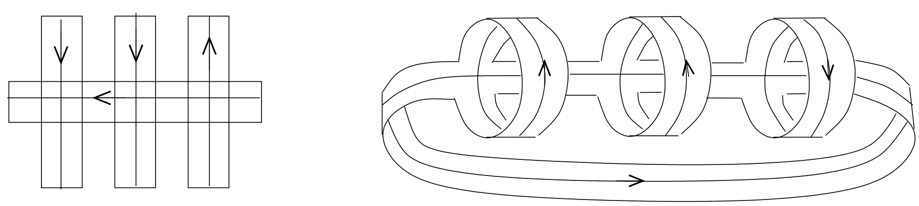

Figure 2: Surface created by thickening the line segments and identifying opposite edges

proof of Theorem 9.

Remark: If we think of the initial system of thickened arcs as lying in S^3, the one point compactification of $\mathbb{R}^3$, then it is natural to glue opposite ends of the rectangles after a full twist in the negative or positive direction, depending on the sign defined by $\mathfrak{s}$. This determines a Seifert surface $S_{\Gamma,\mathfrak{s}}$ for its boundary, which is a link. Since $S_{\Gamma,\mathfrak{s}}$ is obtained by Hopf plumbings on a disk, it is the fiber of the fibration of S^3. The monodromy of the fibration is $\phi_{\Gamma,\mathfrak{s}}$ (cf. [Hir]).

If a graph Γ is bipartite, and is given the bipartite ordering, then there is a fat graph structure on Γ so that the rectangles can be placed in vertical or horizontal directions on a plane as in Figure 3.

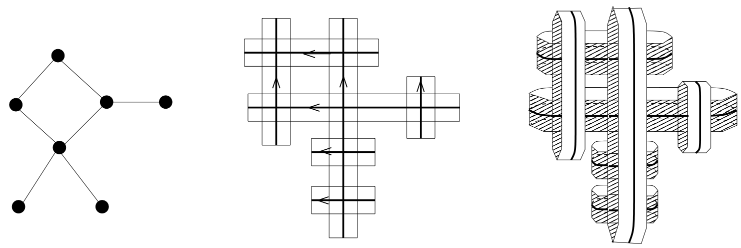

Figure 3: Surface associated to a bipartite graph

Consider the hexagonal graph shown Γ_6 in Figure 4. Since each vertex has order two, there is only one fat graph structure on Γ_6. For the middle diagram, there is no way to orient the core curves on the rectangles in a way that is orientation compatible with any ordering on Γ_6. The orientation on one curve, determines the orientations on its adjacent ones. Thus, a $2n$-gon has an orientation compatible diagram of this form if and only if n is even. The right diagram is compatible with the bipartite ordering. In this example, the middle surface has genus 2 and 4 boundary components, while the right hand surface has genus 3 and 2 boundary components.

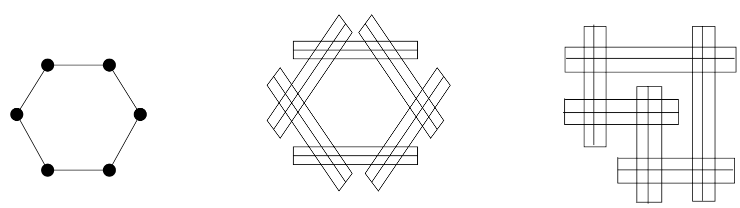

Figure 4: Two surfaces obtained from the hexagonal graph by identifying opposite short edges of the rectangles.

It also may not be possible to make the surface from a globally planar configuration of straight paths, as one can in the bipartite case. Consider for example the cyclic graph with 4 vertices. The bipartite ordering gives rise to a planar configuration (see Figure 5), while for the cyclic ordering, one can verify that there is no configuration of straight line paths that realize the graph (see Figure 6). This example also illustrates that while the graph determines the Euler characteristic of the surface (with boundary), but ordering of the vertices can affect the genus. In the bipartite case, the surface has $(g, n) = (1, 4)$, while in the cyclic case the surface has $(g, n) = (2, 2)$.

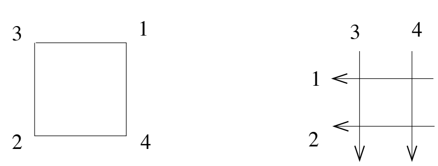

Figure 5: Dual configuration associated to a 4 cycle with bipartite ordering

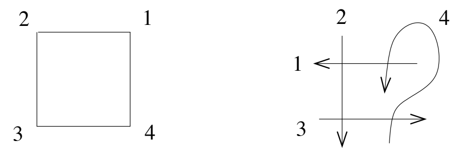

Figure 6: Dual configuration associated to a 4 cycle with cyclic ordering

Remark: We could extend the definition of $\phi_{\Gamma,\mathfrak{s}}$, replacing $\mathfrak{s}$ with

$$\epsilon : \mathcal{V} \to \mathbb{Z}^*,$$

where $\mathbb{Z}^*$ is the set of nonzero integers, and

$$\phi_{\Gamma,\epsilon} = \delta_1^{\epsilon_1} \circ \cdots \circ \delta_n^{\epsilon_n}.$$

Each $\phi_{\Gamma,\epsilon}$ is also associated to a signed graph. The signed graph $(\Gamma', \mathfrak{s})$ is obtained from (Γ, ϵ) by successively replacing each vertex v_i in Γ with $m_i = |\epsilon_n|$ copies $v_i^{(j)}$. Each of the new vertices $v_i^{(j)}$ of Γ' will have edges connecting it to all the vertices to which v_i was connected. The new labels on $v_i^{(j)}$ would equal the sign of ϵ.

4 Examples

Although the mapping classes (S_Γ, ϕ_Γ) constructed in the previous sections have been surfaces with boundary, we can also consider the mapping classes defined on the S_Γ, $(\overline{S_\Gamma}, \phi_\Gamma)$. This section addresses the following question.

Question 11 *For each closed surface is the minimum dilatation mapping class realizable as* $(\overline{S_\Gamma}, \phi_\Gamma)$, *where* Γ *is a mixed-sign Coxeter graph* Γ*?*

We begin in Section 4.1 by studying the classical Coxeter graphs

We will show in Section 4.4 and Section 4.5, that for the known minimum dilatation orientable examples up to genus 5 and the conjectural minimum dilatation braid monodromies for odd number of strands, and hence for hyperelliptic mapping classes, the answer is true.

4.1 Classical Coxeter systems and their associated mapping classes.

By a *classical* Coxeter graph, we mean one with positively signed vertices. Classical Coxeter graphs have been classified according to whether the signature of the associated bilinear form is positive definite, positive semi-definite, or definite with mixed signature (p, q). These graphs are respectively called spherical, affine, and higher rank. In the positive definite case, the Coxeter group is a finite group, fixing the unit sphere in Euclidean space defined by the bilinear form. In the affine case the Coxeter group acts on an infinite cylinder. The simply-laced spherical Coxeter systems have been completely classified (see Figure 7). There are two infinite families A_n and D_n, and the exceptional diagrams E_6, E_7, and E_8. The simply-laced affine Coxeter graphs are obtained by adding an extra node to the spherical diagrams in the way shown in the figure.

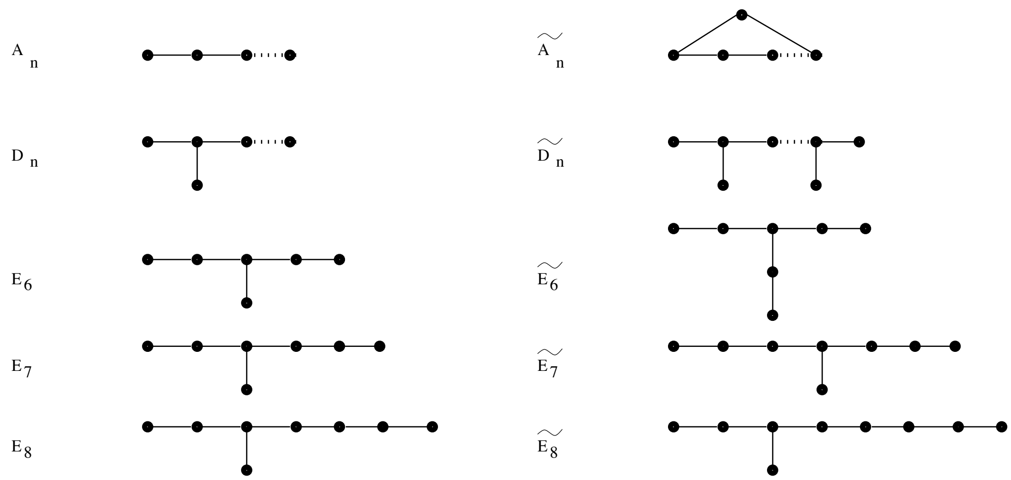

Figure 7: Classification of simply laced Coxeter graphs that are spherical (left) and affine (right).

A'Campo showed that spherical and affine Coxeter systems can be detected through the properties of the Coxeter element ω_Γ of Γ.

Theorem 12 (A'Campo [A'C]) *A Coxeter graph Γ is spherical or affine if and only if $|\omega_\gamma| = 1$.*

Theorem 2 and Theorem 12 imply the following.

Theorem 13 *If Γ is a connected positively signed non-spherical and non- affine Coxeter graph, then ϕ_Γ is pseudo-Anosov and*

$$\lambda(\phi_\Gamma) \geq |\omega_\Gamma|.$$

Proof. By the Nielsen-Thurston classification, any mapping class is either periodic, reducible or pseudo-Anosov [Thu]. Since Γ is connected, ω_Γ is irreducible, and hence so is the homological action of ϕ_Γ. This implies that ϕ_Γ is not reducible. Since Γ is non-spherical and non-affine, Theorem 12 implies that $|\omega_\Gamma| > 1$. The rest follows from Corollary 4. □

The following is a theorem is a consequence of a monotonicity result in graph theory.

Proposition 14 *If Γ and Γ' are connected Coxeter graphs and Γ is a proper subgraph of Γ', then if Γ is non-spherical and non-affine, then so is Γ'. Furthermore, we have the inequality*

$$\mu(\Gamma) \leq \mu(\Gamma'),$$

with equality if and only if Γ *and* Γ' *are spherical or affine. Thus, equality holds if and only if* $\mu(\Gamma) \leq 2$.

McMullen used this analysis in [Mc] to show that the smallest Coxeter eigenvalue for non-spherical and non-affine Coxeter graphs is attained by E_{10} (see Figure 11) and equals Lehmer's number, approximately 1.17628.

Thus, Theorem 2 implies the following result.

Theorem 15 (Leininger [Lei]) *Lehmer's number is the smallest dilatation of a pseudo-Anoosv mapping class associated to a positively signed graph.*

This result was proved by Leininger in the bipartite case [Lei].

4.2 Bipartite graphs and Thurston's examples.

Let Γ be a positively signed bipartite graph with bipartite ordering and (S_Γ, ϕ_Γ) the mapping class associated to a fat graph structure of Γ. Let (S, ϕ) be obtained from (S_Γ, ϕ_Γ) by filling the boundary components with disks. Thurston in [Thu] defines a flat structure on S_Γ so that ϕ has constant derivative outside singularities given by

$$D\phi = \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\mu & 1 \end{bmatrix} = \begin{bmatrix} 1-\mu^2 & \mu \\ -\mu & 1 \end{bmatrix}$$

where $\mu = \mu(\Gamma)$ is the spectral radius of the adjacency matrix of Γ. The heights and widths of the annuli corresponding to the vertices of Γ are computed from the bipartite adjacency matrix, and the characteristic polynomial of $D\phi$ is

$$P_\mu(x) = x^2 - (2-\mu^2)x + 1. \tag{1}$$

When $\mu > 2$, ϕ_Γ and ϕ are pseudo-Anosov, with

$$\lambda(\phi) = \lambda(\phi_\Gamma) = |P_\mu|.$$

For Thurston's examples, the fat graph structure chosen for Γ need not be the one induced by the bipartite structure. In the bipartite case, the following nice property holds (cf. [Thu]).

Theorem 16 *If* S_Γ *is constructed from the fat graph structure of* Γ *associated to a bipartite ordering of the vertices, and if* Γ *does not define a spherical or affine Coxeter system, then* ϕ_Γ *is pseudo-Anosov and orientable.*

Proof. If Γ is not spherical or affine, then by Theorem 2 and Theorem 12,

$$\lambda_{\text{hom}}(\phi_\Gamma) = |\omega_\Gamma| > 1.$$

Furthermore, the action of ω_Γ is irreducible. It follows that ϕ_Γ is pseudo-Anosov.

The core curves of the rectangles define an orientation compatible realization of Γ, and by Theorem 2, the homological dilatation $\lambda_{hom}(\phi_\Gamma)$ is the spectral radius of the bipartite Coxeter element of Γ. The spectral radius of the bipartite Coxeter element can in turn be computed directly from the incidence matrix of Γ and satisfies Equation (1) ([Mc] Prop. 5.3). Hence in the pseudo-Anosov case we have

$$\lambda_{hom}(\phi_\Gamma) = \lambda(\phi_\Gamma).$$

The equality between the homological and geometric dilatations implies that ϕ_Γ is an orientable pseudo-Anosov mapping class. □

4.3 Bipartite graphs and Penner's examples.

In [Pen], Penner shows how to construct train tracks for mapping classes defined by multi-twists of opposite sign. In the language developed in this paper, a subclass of these examples are defined as follows. Let Γ be bipartite, and let $\mathcal{V}_+$ and $\mathcal{V}_-$ be the associated partition of the vertices of Γ. Let

$$\mathfrak{s} : \mathcal{V}_+ \cup \mathcal{V}_- \to \{\pm 1\}$$

take the elements of $\mathcal{V}_+$ to 1 and $\mathcal{V}_-$ to -1. We say in this case that the signs on Γ are compatible with the bipartite ordering.

Theorem 17 (Penner [Pen]) *If $\mathcal{V}_+$ and $\mathcal{V}_-$ are non-empty, $(S_{\Gamma,\mathfrak{s}}, \phi_{\Gamma,\mathfrak{s}})$ is pseudo-Anosov.*

Unlike in Theorem 16, the hypotheses of Theorem 17 do not require that the ordering of Γ be bipartite. Thus, Penner's Theorem is useful for constructing many examples with computable dilatation. It is also possible to get both orientable and non-orientable examples.

Question 18 *Is the minimum dilatation for mapping class $\phi_{\Gamma,\mathfrak{s}}$, where $(\Gamma, \mathfrak{s})$ is of Penner type, bounded from below by a number $\delta > 1$?*

4.4 Minimum dilatation orientable examples.

As we will show in this section, mixed-sign Coxeter graphs can be used to describe mapping classes with small dilatation. Let S_g be a closed genus g surface. Define

$$d_g = \min\{\lambda(\phi) \ : \ \phi : S_g \to S_g \text{ is a pseudo-Anosov mapping class}\},$$

and

$$d_g^+ = \min\{\lambda(\phi) \ : \ \phi : S_g \to S_g \text{ is an orientable pseudo-Anosov mapping class}\}.$$

The known values for d_g and d_g^+ are $d_2 = d_2^+$ ([CH] [Zhi]) and d_g^+ for $g = 2, 3, 4, 5, 7, 8$ ([LT]).

The minimal orientable examples for genus $g = 2, 4, 5$ can be realized by Thurston type examples (see Figure 8, Figure 10, and Figure 11). In these figures, a filled dot denotes a positively signed vertex, while an unfilled dot denotes a negatively signed vertex. For genus $g = 2$, the mapping class that realizes the minimum dilataiton is known to be unique up to obvious equivalences, and is orientable. For $g = 4$ and 5, Lanneau and Thiffeault show that d_g^+ is realized by the Thurston type mapping class associated to the minimal hyperbolic extensions of Coxeter systems E_7 and E_8. In the genus 5 case, $hE_8 = E_{10}$, and the minimal dilatation for orientable mapping classes equals Lehmer's number λ_L.

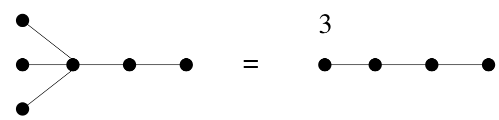

Figure 8: Genus 2 minimal graph.

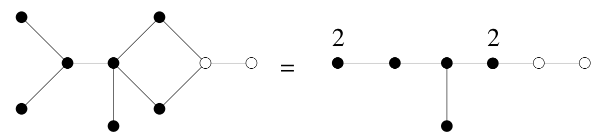

Figure 9: Genus 3 minimal graph.

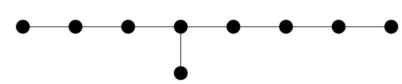

Figure 10: Genus 4 minimal graph.

Lanneau and Thiffeault [LT] also found a mapping class $\phi_3 \in \mathrm{Mod}(S_3)$ realizing d_3^+. A mixed-sign Coxeter graph giving rise to the minimal genus 3 orientable example is shown in Figure 9.

Remark: Numerically d_3^+ (see Figure 9) equals the house of the hyperbolic extension of E_6 (see [Mc], Table 5). Let (S_{hE_6}, ϕ_{hE_6}) be the mapping class obtained from hE_6 using the bipartite ordering. Then S_{hE_6} has genus 4 and 2 boundary components. The genus 3 example obtained from the graph Γ_3 in Figure 9 has 4 boundary components. Thus, the surfaces S_{hE_6} and S_{Γ_3} have the same topological Euler characteristic. This suggests that the topological Euler characteristic are a more natural index by which to study minimum dilatation than genus.

4.5 Small dilatation braids and hyperelliptic mapping classes.

Unlike in the positive case, mixed sign Coxeter graphs give rise to pseudo-Anosov mapping classes with dilatations that are arbitrarily close to 1. In [HK], we investigate hyperelliptic mapping classes associated to the signed graphs $\Gamma_{m,n}$ shown in Figure 13. Here m is the number of filled (+1) nodes of $\Gamma_{m,n}$ and n is the number of unfilled (-1) nodes.

The mapping classes $(S_{\Gamma_{m,n}}, \phi_{\Gamma_{m,n}})$ are pseudo-Anosov with dilatations satisfying thepolynomial

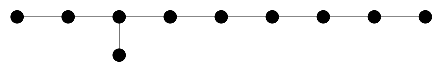

Figure 11: Genus 5 minimal graph.

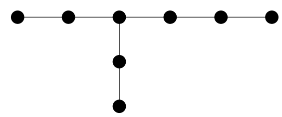

Figure 12: A positive Coxeter graph related to genus 3 example.

sequences

$$P_{m,n}(x) = x^{n+1} R_m(x) + (R_m)^*(x),$$

where $R_m(x) = x^m(x-1) - 2$. (These examples were also studied by P. Brinkmann [Bri], and correspond to certain 2-bridge knots and links.)It follows that $\lambda(\phi_{\Gamma_{m,n}})$ converges to 1 as m and n go to ∞ along positive rays in the (m, n) plane.' The smallest dilatation mapping classes for fixed $m + n$ come from $(S_{\Gamma_{m,m}, \phi_{\Gamma_{m,m}}})$.

Still smaller dilatations are obtained when the boundary components of $S_{\Gamma_{m,n}}$ are filled. When $m = n$ or $|m - n| = 1$, the induced mapping class on the closure of $S_{\Gamma_{m,n}}$, $\overline{\phi}_{\Gamma_{m,n}}$ is periodic or reducible. Otherwise $\phi_{\Gamma_{m,n}}$ is pseudo-Anosov and their dilatations satisfy polynomials

$$Q_{m,n}(x) = x^{n+1} R_m(x) - (R_m)^*(x).$$

The smallest dilatation mapping class for fixed $m + n$ in this family is attained by

$$(\overline{\phi}_{\Gamma_{m-1,m+1}}, \overline{S}_{\Gamma_{m-1,m+1}}).$$

For odd g, these realize the smallest known dilatations amongst pseudo-Anosov hyperelliptic mapping classes [KT].

4.6 Minimum dilatation and Euler characteristic

In this paper, we have been interested in mapping classes on closed surfaces. If we instead keep track of the surface with boundary S_Γ, which retains topological information about the graph, we can ask the question in a different way.

Question 19 *For a fixed Euler characteristic, is the minimum dilatation mapping class always realizable as* (S_Γ, ϕ_Γ) *for some mixed-sign Coxeter graph* Γ*?*

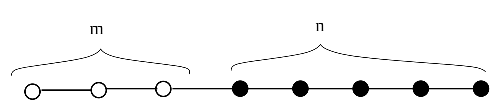

Figure 13: Mixed-sign Coxeter graph $\Gamma_{m,n}$

This topic will be explored further in a future paper.

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