A bound on the vertical transport of heat in the ultimate state of slippery convection at large Prandtl numbers

Xiaoming Wang*
Jared P. Whitehead†

June 12, 2012

Abstract

A new upper bound on the rate of vertical heat transport is established in 3 dimensions for stress-free velocity boundary conditions on horizontally periodic plates. For large (but finite) Prandtl numbers this bound is an improvement over the 'ultimate' $Ra^{1/2}$ scaling and in the limit of infinite Pr, agrees with the bound of $Ra^{5/12}$ recently derived in that limit.

keywords: Rayleigh-Bénard convection, Boussinesq equations, Nusselt number, Prandtl number, Rayleigh number, Grashof number, free-slip, background temperature profile

1 Introduction

Since Lord Rayleigh's mathematical description of convection in [27], and computation of the onset of convective instability, scientists have carefully analyzed this idealized model of such a wide-ranging phenomenom. In his seminal paper, Lord Rayleigh illustrated that the stability of the inert, conductive solution was dependent on a single non-dimensional number (since called the Rayleigh number). Further work considered how the flow developed as the Rayleigh number Ra was increased past this critical value Ra_c. Of pertinent interest to the geophysical and astrophysical community is how this system behaves in the limit of very strongly forced convection, i.e. $Ra \gg Ra_c$ [2].

Several recent experimental and numerical investigations have focused on the 'ultimate' regime of strongly forced convection, and in particular on the dependence of the rate of vertical heat transport to the Rayleigh number and other material parameters (the geometry of the system, and the Prandtl number Pr a

*Department of Mathematics, Florida State University, Tallahassee, FL 32306, USA
†Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

material property of the fluid). It is acknowledged that the Nusselt number Nu, a non-dimensional measure of the increase in heat transport due to convection, is dependent on the Rayleigh number through a power law $Nu \sim Ra^{\gamma}$, but the value of γ for $Ra \gg Ra_c$ is a matter of dispute [3, 15, 29, 20]. This paper follows the work of [12] to derive a rigorous bound on Nu for the case of large Prandtl number and stress-free vertical velocities.

Following Lord Rayleigh [27], we consider the rate of vertical heat transport in Rayleigh Bénard convection, as described by the classical (non-dimensional) Boussinesq equations with stress-free vertical boundaries:

$$\frac{1}{Pr}\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \nabla p = \Delta \mathbf{u} + Ra\,\mathbf{k}T, \quad \nabla\cdot\mathbf{u} = 0, \tag{1}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \Delta T, \tag{2}$$

$$\frac{\partial u_j}{\partial z}\Big|_{z=0,1} = u_3|_{z=0,1} = 0, \quad j = 1,2 \tag{3}$$

$$T|_{z=0} = 1, \qquad T|_{z=1} = 0, \tag{4}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \qquad T|_{t=0} = T_0, \tag{5}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the fluid velocity field, p is the kinematic pressure, T is the (scaled) temperature field, $\mathbf{k}$ is the unit upward vector, Ra is the Rayleigh number measuring the ratio of differential heating to dissipation, Pr is the Prandtl number which is the ratio of kinematic viscosity to thermal diffusivity, and the fluid occupies the (non-dimensionalized) region

$$\Omega = [0, L_x] \times [0, L_y] \times [0, 1] \tag{6}$$

where periodicity in the horizontal directions is assumed, implying [1]

$$\int_\Omega u_j \, dxdydz = 0, \quad j = 1,2; \forall t \geq 0, \tag{7}$$

if this is satisfied initially.

At sufficiently large Prandtl number, we can formally consider the infinite Prandtl number limit as:

$$\nabla p^0 = \Delta \mathbf{u}^0 + Ra\,\mathbf{k}T^0, \quad \nabla\cdot\mathbf{u}^0 = 0, \tag{8}$$

$$\frac{\partial T^0}{\partial t} + \mathbf{u}^0\cdot\nabla T^0 = \Delta T^0, \tag{9}$$

$$\frac{\partial u_j^0}{\partial z}\Big|_{z=0,1} = u_3^0|_{z=0,1} = 0, \quad j = 1,2 \tag{10}$$

$$T^0|_{z=0} = 1, \qquad T^0|_{z=1} = 0, \tag{11}$$

which is relevant for fluids such as silicone oil and the earth's mantle as well as many gases under high pressure [6, 16, 5]. This simplification removes the nonlinear term in the momentum equation, greatly simplifying the dynamics.

[1] This is consistent with the horizontal momentum equationss, and guarantees the applicability of the Poincaré type inequality. See for instance [10].

The fact that the velocity field is linearly *slaved* to the temperature field has been exploited in several recent rigorous estimates on the rate of heat transport in the vertical direction in this infinite Prandtl number setting (see [13, 9, 14, 33, 34] and the references therein, as well as the work of [6, 22]). In particular, a bound on the Nusselt number which scales like $Ra^{\frac{1}{3}}$ (modulo logarithmic corrections) in the no-slip case was obtained in [9] (and improved more recently in [26]) with the help of a maximum principle in the temperature field, and in [14] where the maximum principle was not invoked. For horizontal plates with free-slip (stress-free) boundaries at infinite Prandtl number the best known bound on the Nusselt number scales like $Ra^{\frac{5}{12}}$ [34] (a similar scaling is true for arbitrary Prandtl number in 2 dimensions, see [33]). On the other hand, the best known bound on the Nusselt number with arbitrary Prandtl number scales like $Ra^{\frac{1}{2}}$ [8] for the no-slip case (the same argument applies to the free-slip/stress-free case) while the bound calculated via marginal stability in [21] suggests $Ra^{\frac{1}{3}}$. This $\frac{1}{3}$ scaling is in agreement with a heuristic argument set forth in [24], but is in contrast to the proposed 'ultimate' regime first described in [23] and extensively considered in [17, 18, 19].

For large Prandtl number we will consider the full Boussinesq system as a perturbation of the infinite Prandtl number model, implying that the velocity field is only a perturbation from a linear slaving with the temperature field. This near linear relationship was exploited in the no-slip case in [32] in studying statistical properties of the Boussinesq system at large Prandtl number, and in [31] to obtain a bound on the Nusselt number that is in agreement with the infinite Prandtl number bound (for no-slip velocities). Following [34] and [32, 31] we will combine these methods to see that there exists a non-dimensional constant c_0 (30) such that if the Grashof number Gr is sufficiently small, i.e, $\frac{Pr}{Ra} \geq c_0$ then $Nu \leq 0.3546\, Ra^{\frac{5}{12}} + c\, Gr^2 Ra^{\frac{1}{4}}$ where $Gr = \frac{Ra}{Pr}$.

Throughout this manuscript, we assume the physically important case of high Rayleigh number $Ra \gg 1$ so that we may have non-trivial dynamics. We also follow the mathematical tradition of denoting the small parameter as ϵ, i.e.

$$\epsilon = \frac{1}{Pr}. \tag{12}$$

c will denote a generic non-dimensional constant independent of the Rayleigh number and Prandtl number.

The rest of the manuscript is organized as follows. In section 2 we recall a few a priori estimates on the solutions to the Boussinesq system at large Prandtl number. In section 3, we derive the $Ra^{\frac{5}{12}}$ upper bound for the Nusselt numbers at large Prandtl number. In section 4, we offer concluding remarks.

2 A priori estimates

In this section we derive a few a priori estimates on solutions of the Boussinesq system at large Prandtl number. Some of these estimates (or at least closely related estimates) are contained in [30] and [8]. For completeness we review all the estimates needed for the current result. The mathematical inequalities referred to are listed (with references) in Appendix B.

Throughout this manuscript, we will assume that the range of initial temperature T_0 is contained in the unit interval $[0,1]$. Hence we deduce by the maximum principle that the range of T is contained in $[0,1]$ for all time, i.e.,

$$\|T\|_{L^\infty} \leq 1. \tag{13}$$

Following the background method developed in [12, 8] we will also assume that the background temperature profile $\tau(z)$ under consideration is always contained in the unit interval $[0,1]$ (see(54)). Therefore, the fluctuation temperature field $\theta = T - \tau$ satisfies the same estimate

$$\|\theta\|_{L^\infty} \leq 1. \tag{14}$$

Note that the same argument as applied in [8] implies that the following estimates hold

$$\begin{aligned} \langle \|\nabla \boldsymbol{u}\|^2 \rangle &\leq cRa^{\frac{3}{2}}, &(15)\\ \langle \|\nabla T\|^2 \rangle &\leq cRa^{\frac{1}{2}}, &(16) \end{aligned}$$

for all suitable weak solutions of the Boussinesq system with arbitrary Prandtl number where $< \cdot >$ represents long time average, i.e.

$$\langle f(\cdot) \rangle = \limsup_{t\to\infty} \frac{1}{t} \int_0^t f(s)\, ds$$

and $\|\cdot\|$ is the standard L^2 norm.

For the first estimate derived in this section, multiply the velocity equation (1) by $\boldsymbol{u}$ and after integrating over the domain, apply the Cauchy Schwarz and Poincaré inequalities (and using the maximum principle as discussed above) to arrive at:

$$\begin{aligned} \frac{\epsilon}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|_{L^2}^2 + \|\nabla \boldsymbol{u}(t)\|^2 &\leq Ra\|T(t)\|\|u_3(t)\| \\ &\leq c\,Ra\|\nabla \boldsymbol{u}(t)\|. \end{aligned} \tag{17}$$

Applying Young's inequality to the right hand side, yields

$$\frac{\epsilon}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|^2 + \|\nabla \boldsymbol{u}(t)\|^2 \leq c\,Ra^2 + \frac{1}{2}\|\nabla \boldsymbol{u}(t)\|^2. \tag{18}$$

Finally we use Poincaré's inequality in conjunction with the Gronwall inequality to deduce that

$$\limsup_{t\to\infty} \|\boldsymbol{u}(t)\| \leq c\,Ra. \tag{19}$$

For the next estimate, multiply the velocity equation (1) by $A\boldsymbol{u}(t)$ (where A denotes the Stokes operator with viscosity one and the associated boundary conditions, see for instance [10]), and integrate over the domain, again applying Cauchy-Schwarz appropriately

$$\frac{\epsilon}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}(t)\|^2 + \|A\boldsymbol{u}(t)\|^2 \quad \leq \quad Ra\|T(t)\|\|A\boldsymbol{u}(t)\| + \epsilon\|\nabla \boldsymbol{u}(t)\|\|A\boldsymbol{u}(t)\|\|\boldsymbol{u}(t)\|_{L^\infty}.$$

Applying the Agmon inequality and employing the maximum principle on the temperature field $T(t)$ results in

$$\begin{aligned}\frac{\epsilon}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}(t)\|^2 + \|A\boldsymbol{u}(t)\|^2 &\leq Ra|\Omega|^{\frac{1}{2}}\|A\boldsymbol{u}(t)\| + c_A\epsilon\|\nabla \boldsymbol{u}(t)\|^{\frac{3}{2}}\|A\boldsymbol{u}(t)\|^{\frac{3}{2}} \\ &\leq \frac{1}{2}\|A\boldsymbol{u}(t)\|^2 + Ra^2|\Omega| + \frac{27}{4}c_A^4\epsilon^4\|\nabla \boldsymbol{u}(t)\|^6 \qquad (20)\end{aligned}$$

where the generalized Young's inequality was used twice to obtain the last line. The Poincaré inequality on $\nabla \boldsymbol{u}(t)$ with constant c_p is used to arrive at

$$\epsilon\frac{d}{dt}\|\nabla \boldsymbol{u}(t)\|^2 + c_p^2\|\nabla \boldsymbol{u}(t)\|^2 \leq 2|\Omega|Ra^2 + \frac{27}{2}c_A^4\epsilon^4\|\nabla \boldsymbol{u}(t)\|^6. \qquad (21)$$

It follows that the ball of radius $2|\Omega|^{\frac{1}{2}}Ra/c_p$ is invariant for $\|\nabla \boldsymbol{u}(t)\|$ if the following large Prandtl number (small Grashof number) condition holds

$$Gr = \frac{Ra}{Pr} \leq \frac{c_p^{\frac{3}{2}}}{2\cdot 3^{\frac{3}{4}}c_A|\Omega|^{\frac{1}{2}}}. \qquad (22)$$

On the other hand, estimate (15) implies that for Ra sufficiently large, any orbit will enter this ball of radius $2|\Omega|^{\frac{1}{2}}Ra/c_p$. Hence this is an absorbing ball and

$$\limsup_{t\to\infty}\|\nabla \boldsymbol{u}(t)\| \leq 2Ra|\Omega|^{\frac{1}{2}}/c_p. \qquad (23)$$

Inserting this into (20) and taking the long time average (relying on (22)), we have the following estimate

$$\langle\|A\boldsymbol{u}\|^2\rangle \leq cRa^2|\Omega|. \qquad (24)$$

Next, we need an estimate on the time derivative of the velocity. For this purpose we differentiate the velocity equation (1) in time to reach

$$\epsilon\left(\frac{\partial^2\boldsymbol{u}}{\partial t^2} + \left(\frac{\partial \boldsymbol{u}}{\partial t}\cdot\nabla\right)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\frac{\partial \boldsymbol{u}}{\partial t}\right) + \nabla\frac{\partial p}{\partial t} = \Delta\frac{\partial \boldsymbol{u}}{\partial t} + Ra\mathbf{k}\frac{\partial T}{\partial t}. \qquad (25)$$

Multiplying this equation by $\frac{\partial \mathbf{u}}{\partial t}$, integrating over Ω and applying Cauchy-Schwarz and the generalized Hölder inequalities, we deduce that for large t

$$\frac{\epsilon}{2}\frac{d}{dt}\left\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|^2 \quad \leq \quad Ra\left\|\frac{\partial T(t)}{\partial t}\right\|_{H^{-1}}\left\|\nabla\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\| + \epsilon\left\|\boldsymbol{u}(t)\right\|_{L^3}\left\|\nabla\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|\left\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|_{L^6}.$$

Using the Sobolev inequalities (73) we can show that for large t the right hand side of this is less than or equal to

$$
\begin{aligned}
& Ra \left\| \frac{\partial T(t)}{\partial t} \right\|_{H^{-1}} \left\| \nabla \frac{\partial \boldsymbol{u}(t)}{\partial t} \right\| + c_S \epsilon \left\| \nabla \boldsymbol{u}(t) \right\| \left\| \nabla \frac{\partial \boldsymbol{u}(t)}{\partial t} \right\|^2 \\
\leq \; & \frac{1}{4} \left\| \nabla \frac{\partial \boldsymbol{u}(t)}{\partial t} \right\|^2 + Ra^2 \left\| \frac{\partial T(t)}{\partial t} \right\|_{H^{-1}}^2 + \frac{2}{c_p} Ra |\Omega|^{\frac{1}{2}} c_S \epsilon \left\| \nabla \frac{\partial \boldsymbol{u}(t)}{\partial t} \right\|^2 \quad (26)
\end{aligned}
$$

where $c_S = c_{S1} c_{S2}$, and (23) and Young's inequality were used in the last line. This implies that

$$
\left\langle \left\| \nabla \frac{\partial \mathbf{u}}{\partial t} \right\|^2 \right\rangle \leq 2 Ra^2 \left\langle \left\| \frac{\partial T}{\partial t} \right\|_{H^{-1}}^2 \right\rangle \quad (27)
$$

provided the following large Prandtl number (small Grashof number) condition is satisfied

$$
Gr = \frac{Ra}{Pr} \leq \frac{c_p}{8 c_S |\Omega|^{\frac{1}{2}}}. \quad (28)
$$

Setting

$$
c_0 = \frac{c_p}{2|\Omega|^{\frac{1}{2}}} \min \left\{ \frac{c_p^{\frac{1}{2}}}{3^{\frac{3}{4}} \cdot c_A}, \frac{1}{4 c_S} \right\} \quad (29)
$$

we combine the two large Prandtl number conditions (22) and (28) into the following large Prandtl number (small Grasholf number) condition

$$
Gr = \frac{Ra}{Pr} \leq c_0. \quad (30)
$$

Next, we consider the H^{-1} norm applied to the temperature equation (2) to deduce

$$
\begin{aligned}
\left\| \frac{\partial T(t)}{\partial t} \right\|_{H^{-1}} & \leq \| T(t) \boldsymbol{u}(t) \| + \| \nabla T(t) \| \\
& \leq \| \boldsymbol{u}(t) \| + \| \nabla T(t) \| \quad (31)
\end{aligned}
$$

where we have used the maximum principle on the temperature field T. This further implies, thanks to (15) and (16),

$$
\left\langle \left\| \frac{\partial T}{\partial t} \right\|_{H^{-1}}^2 \right\rangle \leq 2 \langle \| \boldsymbol{u} \|^2 + \| \nabla T \|^2 \rangle \leq c Ra^{\frac{3}{2}}. \quad (32)
$$

Inserting this back into (27) we have

$$
\left\langle \left\| \nabla \frac{\partial \mathbf{u}}{\partial t} \right\|^2 \right\rangle \leq c Ra^{\frac{7}{2}}. \quad (33)
$$

For the final estimate, we need to bound $\langle\|\nabla((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\|^2\rangle$ in terms of the non-dimensional parameters of the system. With this in mind, note that the momentum equation can be rewritten as

$$A\boldsymbol{u} = Ra\,P(\mathbf{k}T) - \varepsilon P\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) \tag{34}$$

where P denotes the Leray-Hopf projector from the square integrable space onto the divergence free subspace. Classic elliptic regularity on the Stokes operator A (see for instance [10]), gives

$$\|A^{\frac{3}{2}}\boldsymbol{u}\|^2 \le C\left\{Ra^2\|\nabla T\|^2 + \varepsilon^2\left\|\nabla\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2 + \varepsilon^2\|\nabla((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\|^2\right\}. \tag{35}$$

Sobolev imbedding and elliptic regularity of the Stokes operator imply that

$$\begin{aligned}\|\nabla((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\|^2 &\le C(\|\boldsymbol{u}\|^2_{L^\infty}\|\nabla^2\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{u}\|^4_{L^4})\\ &\le C\|\nabla\boldsymbol{u}\|\|A\boldsymbol{u}\|^3\\ &\le C\|\nabla\boldsymbol{u}\|^{\frac{5}{2}}\|A^{\frac{3}{2}}\boldsymbol{u}\|^{\frac{3}{2}}\end{aligned} \tag{36}$$

where we have utilized the Sobolev inequality (74), elliptic regularity for the Stokes operator, the Agmon inequality, and the interpolation inequality.

Hence

$$\begin{aligned}\langle\|A^{\frac{3}{2}}\boldsymbol{u}\|^2\rangle &\le C\left\langle Ra^2\|\nabla T\|^2 + \varepsilon^2\left\|\nabla\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2 + \varepsilon^8\|\nabla\boldsymbol{u}\|^{10}\right\rangle\\ &\le C\left(Ra^{2+\frac{1}{2}} + \varepsilon^2 Ra^{\frac{7}{2}} + \varepsilon^8 Ra^{\frac{19}{2}}\right).\end{aligned} \tag{37}$$

where we have used the following estimates

$$\langle\|\nabla\boldsymbol{u}\|^{10}\rangle \le \limsup_{t\to\infty}\|\nabla\boldsymbol{u}(t)\|^8\langle\|\nabla\boldsymbol{u}\|^2\rangle \le C\,Ra^{\frac{19}{2}}.$$

This also implies that

$$\begin{aligned}\langle\|\nabla((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\|^2\rangle &\le C\langle\|\nabla\boldsymbol{u}\|^{\frac{5}{2}}\|A^{\frac{3}{2}}\boldsymbol{u}\|^{\frac{3}{2}}\rangle\\ &\le C\langle\|\nabla\boldsymbol{u}\|^{10}\rangle^{\frac{1}{4}}\langle\|A^{\frac{3}{2}}\boldsymbol{u}\|^2\rangle^{\frac{3}{4}}\\ &\le C\left(Ra^{\frac{19}{2}}\right)^{\frac{1}{4}}\left(Ra^{\frac{5}{2}} + \varepsilon^2 Ra^{\frac{7}{2}} + \varepsilon^8 Ra^{\frac{19}{2}}\right)^{\frac{3}{4}}\\ &\le C\left(Ra^{\frac{17}{4}} + \varepsilon^{\frac{3}{2}} Ra^5 + \varepsilon^6 Ra^{\frac{19}{2}}\right)\end{aligned} \tag{38}$$

where we have used Hölder's inequality on the second line.

3 Bound on the Nusselt number

The primary statistical quantity of interest in Rayleigh-Bénard convection is the ratio of the convective heat flux to that provided by a purely conductive temperature profile. This non-dimensional quantity is referred to as the Nusselt number and can be equivalently defined as the following (see for instance [25] for detailed derivations):

$$
\begin{aligned}
Nu &= \sup_{\boldsymbol{u}_0, T_0} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega |\nabla T(\mathbf{x},s)|^2\, d\mathbf{x}ds, \\
&= 1 + \sup_{\boldsymbol{u}_0, T_0} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega u_3(\mathbf{x},s) T(\mathbf{x},s)\, d\mathbf{x}ds, \\
&= 1 + \sup_{\boldsymbol{u}_0, \theta_0} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega u_3(\mathbf{x},s) \theta(\mathbf{x},s)\, d\mathbf{x}ds \qquad (39)
\end{aligned}
$$

where $T(x,y,z,t) = \tau(z) + \theta(x,y,z,t)$ is the temperature field and $\tau(z)$ is the background temperature profile (as proposed in the theory of [8, 12, 9, 14, 33, 34] as a generalization of E. Hopf's original calculation [28]) satisfying the same boundary conditions as T, and $(\boldsymbol{u}, \theta)$ are suitable weak solutions to

$$
\begin{aligned}
\nabla p &= \Delta \mathbf{u} + Ra\, \mathbf{k}\theta + \epsilon \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0, && (40)\\
\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta + u_3\tau'(z) &= \Delta\theta + \tau''(z), && (41)\\
\frac{\partial u_j}{\partial z}|_{z=0,1} = u_3|_{z=0,1} &= 0, \quad j = 1,2 && (42)\\
\theta|_{z=0,1} &= 0, && (43)\\
\mathbf{f} &= -\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) && (44)
\end{aligned}
$$

with appropriate initial conditions $(\boldsymbol{u}_0, \theta_0)$. The Nusselt number is a statistical property of the Boussinesq system in the sense that it is the average of $1 + \frac{1}{|\Omega|}\int_\Omega u_3\theta$ over the whole phase space with respect to some appropriate invariant measure (stationary statistical solution) of the Boussinesq system [32].

Writing the momentum equation as a perturbation of the infinite Prandtl number (8) ($\epsilon = 0$) equation illustrates the methodology used in this paper. Heuristically, the perturbation from the infinite Pr system is small in the limit of $\epsilon \to 0$ or $Pr \to \infty$. Hence we can combine estimates for the infinite Prandtl number case derived in [34] with the a priori estimates obtained in the previous section to deduce an upper bound on the Nusselt number for stress-free velocities that is asymptotically the same as that obtained in [34, 33].

Multiplying the temperature equation (41) by θ and integrating over Ω we have

$$
\frac{1}{2}\frac{d}{dt}\|\theta(t)\|^2 + \|\nabla\theta(t)\|^2 + \int_\Omega \tau' \frac{\partial\theta(t)}{\partial z} + \int_\Omega \tau' u_3(t)\theta(t) = 0. \qquad (45)
$$

From the definition of θ we also have

$$
\|\nabla T(t)\|^2 = \|\nabla\theta(t)\|^2 + 2\int_\Omega \tau' \frac{\partial\theta(t)}{\partial z} + \|\tau'\|^2. \qquad (46)
$$

Following [9, 14] we combine these two estimates to see that

$$\left\langle \|\nabla T\|^2 \right\rangle = \|\tau'\|^2 - \left\langle \int_\Omega (|\nabla\theta|^2 + 2\tau' u_3\theta) \right\rangle . \tag{47}$$

To estimate the indefinite term, we combine the divergence of the momentum equation with the Laplace operator applied to the evolution equation for u_3 to arrive at:

$$\begin{aligned} \Delta^2 u_3 &= -Ra\Delta_H\theta + \epsilon\left(-\Delta_H f_3 + \frac{\partial^2 f_1}{\partial x\partial z} + \frac{\partial^2 f_2}{\partial y \partial z}\right), \qquad (48) \\ u_3|_{z=0,1} &= \left.\frac{\partial^2 u_3}{\partial z^2}\right|_{z=0,1} = 0, \quad \Delta_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \end{aligned}$$

Using the periodic horizontal boundary conditions, we note that this equation can be rewritten in terms of the horizontal Fourier coefficients of each variable as

$$(k^2 - D^2)^2 \hat{u}_{3k} = Ra\, k^2 \hat{\theta}_k + \epsilon(k^2 \hat{f}_{3k} + ik_1 D\hat{f}_{1k} + ik_2 D\hat{f}_{2k}) \tag{49}$$

where $k = |\mathbf{k}|$ is the length of the horizontal wave-number $\mathbf{k}$ and $D = \frac{d}{dz}$ is the derivative operator in the vertical direction, and the $\hat{}$ indicates the Fourier coefficient of the corresponding variable.

Denote the pseudo differential operator $|\nabla_H| = \sqrt{-\nabla_H \cdot \nabla_H} = \sqrt{-\Delta_H}$ and introduce the pseudo-vorticity ω as in [34] as

$$\Delta u_3 = |\nabla_H|\omega, \tag{50}$$

which in terms of the Fourier coefficients, is expressed as

$$(-k^2 + D^2)\hat{u}_{3k} = k\hat{\omega}_k. \tag{51}$$

It follows that we can eliminate $\hat{u}_{3k}$ in (49) to tie the pseudo-vorticity to the temperature fluctuations

$$k(-k^2 + D^2)\hat{\omega}_k = Ra\, k^2\hat{\theta}_k + \epsilon(k^2 \hat{f}_{3k} + ik_1 D\hat{f}_{1k} + ik_2 D\hat{f}_{2k}). \tag{52}$$

Hence, for $k \neq 0$,

$$\begin{aligned} k^2|\hat{\theta}_k|^2 &\geq \frac{1}{Ra^2}\Bigg(|(-k^2+D^2)\hat{\omega}_k|^2 + \epsilon^2 \left| k\hat{f}_{3k} + i\frac{k_1}{k}D\hat{f}_{1k} + i\frac{k_2}{k}D\hat{f}_{2k}\right|^2 \\ &\quad -2\epsilon\left|(-k^2+D^2)\hat{\omega}_k\right| \left| k\hat{f}_{3k} + i\frac{k_1}{k}D\hat{f}_{1k} + i\frac{k_2}{k}D\hat{f}_{2k}\right| \Bigg) \\ &\geq \frac{1}{2Ra^2}\left|(-k^2+D^2)\hat{\omega}_k\right|^2 - \frac{\epsilon^2}{Ra^2}\left| k\hat{f}_{3k} + i\frac{k_1}{k}D\hat{f}_{1k} + i\frac{k_2}{k}D\hat{f}_{2k}\right|^2 \\ &\geq \frac{1}{2Ra^2}\left|(-k^2+D^2)\hat{\omega}_k\right|^2 - \frac{\epsilon^2}{Ra^2}\left(\left|k\hat{f}_{3k}\right| + \left|D\hat{f}_{1k}\right| + \left|D\hat{f}_{2k}\right|\right)^2 \qquad (53) \end{aligned}$$

We now choose a specific background temperature profile following [34]

$$\tau'(z) = \begin{cases} 1 - \left(\frac{1+p}{2\delta} - p\right) z, & 0 \le z \le \delta \\ \frac{1}{2} + p(z - \frac{1}{2}), & \delta \le z \le 1 - z \\ \left(\frac{1+p}{2\delta} - p\right)(1 - z), & 1 - \delta \le z \le 1. \end{cases} \tag{54}$$

Applying the exact same calculations as in [34] in the absence of the balance parameter ($b = 0$ in the notation used in [34]), we can see that

$$\frac{1}{2Ra^2}\|(k^2 - D^2)\hat{\omega}_k\|^2 + \|D\hat{\theta}_k\|^2 + \frac{p}{Ra}\|\hat{\omega}_k\|^2 - \frac{1+p}{\delta}\left\{\int_0^\delta \hat{u}_{3k}\bar{\hat{\theta}}_k + \int_{1-\delta}^1 \hat{u}_{3k}\bar{\hat{\theta}}_k\right\} \ge 0, \quad \forall k \tag{55}$$

is equivalent to

$$\begin{aligned} &\frac{k^4}{Ra^2} + \frac{p}{Ra} - \frac{3^{3/2} \cdot (1+p)^2 \cdot k}{2^2 \cdot 5^2}\delta^3 \ge 0 \\ &\Rightarrow \frac{3^{3/2} \cdot (1+p)^2}{2^2 \cdot 5^2}\delta^3 \le \frac{k^3}{Ra^2} + \frac{p}{kRa} \end{aligned}$$

implying that

$$\delta \le \frac{2^{4/3} \cdot 5^{2/3} \cdot p^{1/4}}{3^{3/4} \cdot (1+p)^{2/3}} \frac{1}{Ra^{5/12}} \tag{56}$$

where

$$k_m = \left(\frac{p}{3}Ra\right)^{1/4} \tag{57}$$

is the horizontal wave-number that saturates the bound on δ. Letting $p = \frac{3}{29}$ (optimally chosen to minimize the prefactor in the final bound (see [34] for details) we choose the optimal size of the boundary layer as

$$\delta = \frac{5^{2/3} \cdot 29^{5/12}}{2^2 \cdot 3^{1/2}} \frac{1}{Ra^{5/12}}. \tag{58}$$

Therefore using (46)

$$\begin{aligned} Nu &= \langle \|\nabla T\|^2 \rangle \\ &= \|\tau'\|^2 - \left\langle \int_\Omega (|\nabla\theta|^2 + 2\tau' u_3\theta) \right\rangle \\ &= \|\tau'\|^2 - \left\langle \|\nabla\theta\|^2 + \frac{2p}{Ra}\int_\Omega \theta u_3 - \frac{1+p}{\delta}\left\{\int_0^\delta u_3\theta + \int_{1-\delta}^1 u_3\theta\right\} \right\rangle. \end{aligned}$$

Using the identity

$$\left\langle \int_\Omega u_3\theta \right\rangle = \frac{1}{Ra}\left\langle \int_\Omega |\nabla \boldsymbol{u}|^2 \right\rangle \tag{59}$$

and (63) derived in the Appendix we can bound the Nusselt number from above as

$$\begin{aligned}
Nu &\leq \|\tau'\|^2 - \sum_k \left\langle \|k\hat{\theta}_k\|^2 + \|D\hat{\theta}_k\|^2 + \frac{p}{Ra}\|\hat{\omega}_k\|^2 - \frac{1+p}{\delta}\left\{\int_0^\delta \hat{u}_{3k}\bar{\hat{\theta}}_k + \int_{1-\delta}^1 \hat{u}_{3k}\bar{\hat{\theta}}_k\right\}\right\rangle \\
&\leq \|\tau'\|^2 + \frac{\epsilon^2}{Ra^2}\left\langle \|\nabla f\|^2 \right\rangle \\
&\quad - \sum_k \left\langle \frac{1}{2Ra^2}\|(k^2 - D^2)\hat{\omega}_k\|^2 + \|D\hat{\theta}_k\|^2 + \frac{p}{Ra}\|\hat{\omega}_k\|^2 - \frac{1+p}{\delta}\left\{\int_0^\delta \hat{u}_{3k}\bar{\hat{\theta}}_k + \int_{1-\delta}^1 \hat{u}_{3k}\bar{\hat{\theta}}_k\right\}\right\rangle \\
&\leq \|\tau'\|^2 + \frac{\epsilon^2}{Ra^2}\left\langle \|\nabla f\|^2 \right\rangle
\end{aligned}$$

so long as the size of the boundary layer is chosen according to (58) so that (55) is satisfied. We can then bound the remainder term using the Poincaré, Hölder and Agmon inequalities and the estimates (15), (24), (33) and (38) obtained in the previous section:

$$\begin{aligned}
\frac{\epsilon^2}{Ra^2}\left\langle \|\nabla f\|^2 \right\rangle &\leq \frac{2\epsilon^2}{Ra^2}\left\langle \left(\left\| \nabla \frac{\partial \boldsymbol{u}}{\partial t} \right\|^2 + \|\nabla((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\|^2 \right) \right\rangle \\
&\leq C\left(\epsilon^2 Ra^{\frac{9}{4}} + \epsilon^{\frac{7}{2}} Ra^3 + \epsilon^8 Ra^{\frac{15}{2}} + \epsilon^2 Ra^{\frac{3}{2}}\right) \qquad (60)
\end{aligned}$$

Inserting this back into the bound on Nu and rewriting the remainder in terms of the Grasholf number Gr we can see that

$$\begin{aligned}
Nu &\leq \frac{2^{11}\cdot 3^{\frac{1}{2}}}{5^{\frac{2}{3}}\cdot 29^{\frac{29}{12}}} Ra^{\frac{5}{12}} + C\left(Gr^2 Ra^{\frac{1}{4}} + \epsilon^{\frac{1}{2}} Gr^3 + \epsilon^{\frac{1}{2}} Gr^{\frac{15}{2}} + \epsilon^{\frac{1}{2}} Gr^{\frac{3}{2}}\right) \\
&\sim 0.3546 Ra^{\frac{5}{12}} + C Gr^2 Ra^{\frac{1}{4}} \qquad (61)
\end{aligned}$$

for $Gr \leq c_0$ for stress-free (slippery) horizontal plates in three dimensions.

4 Concluding Remarks

The bound (61) at small Gr is consistent with the infinite Prandtl ($Gr = 0$) bound obtained in [34] albeit with a less optimal prefactor ($Nu \lesssim 0.28764 Ra^{\frac{5}{12}}$ at $Pr = \infty$, although the current estimate does not consider the impact of the balance parameter which may subtly improve the prefactor). The improvement provided by this result is shown in Fig. 1. Specifically this indicates that for $Pr \gtrsim O\left(10^4\right)$ and within a given range of Rayleigh numbers (dictated by (30)), the current result is an improvement over [12]. The small Grasholf number assumption can be considered a restriction on the strength of the inertia with respect to the forcing placed on the system, i.e. the inertial forces in the flow are restricted to scale sub linearly with Ra. This is not inconsistent with the

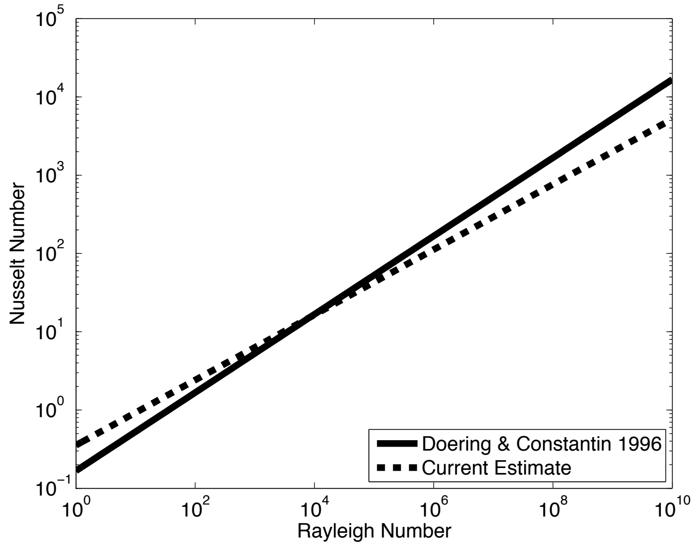

Figure 1: Comparison of the asymptotic bound obtained in this paper with that of [12] for Grasholf number satisfying (30).

theory of the 'ultimate' state of turbulent convection where Nu is thought to scale as $Ra^{\frac{1}{2}}$ because these theories rely on the assumption of large Gr (see [19, 4] for the most recent results, although originally this theory was proposed in [23]). This result in combination with [33, 34] and citeWa2008a, does indicate the need for these theories to establish the necessity of either no-slip boundary conditions or three dimensions at large Gr for the appearance of this 'ultimate' regime of convection.

Heuristically [23] argues that at a sufficiently large Rayleigh number (dependent on Gr or Pr) the large scale circulation (or the fluctuations) in the bulk of the flow will induce a shear velocity forcing the viscous boundary layers to become turbulent. In [19] this theory is expounded to include the case when the thermal and/or viscous boundary layers become turbulent as well, resulting in different scaling laws for the Nusselt number, dependent on which case is considered. The experimental data described in [20, 4] appears to validate this theory for $Pr \leq 1$. However in each case considered in [19] the Nusselt number is proposed to increase with Pr, contrary to the results obtained here and in [34, 26, 31, 32] where the limit $Pr \to \infty$ is considered and strict bounds are set on the heat transport. In concert with these earlier results, the present paper indicates that the large Prandtl number limit should be considered in greater detail to understand the true state of the 'ultimate' regime of convection in such a case.

Not only is the large Prandtl number case not explained in the theory of [23, 19], but the result of [33] is not explainable via the same theory unless one can show that the restriction to stress-free boundaries in two dimensions will force the thermal boundary layer to remain laminar. This highlights a possibility for theoretical improvement, i.e. *if* the appearance of a turbulent thermal boundary layer is indicative of the 'ultimate' state of convection one must question why the thermal boundary layer does not become turbulent for infinite or large Prandtl number or stress-free boundaries in two dimensions. Consideration of the effect of the vortex stretching term (which is absent in 2D and suppressed at large Pr) on the theory of [19] and [23] may yield insight into how these turbulent boundary layers are formed or suppressed and the subsequent impact on the heat transport.

The extension of the current result to $Gr > c_0$ is non-trivial and cannot be accomplished through the current methodology, indicating perhaps that such a result is not possible and that the 'ultimate' state as described in [19] holds for sufficiently small Pr. We note however that to date the $\frac{5}{12}$ scaling has been demonstrated in the analysis for stress-free velocities, but there is no theory that predicts such a scaling and numerical simulations and experiments have not observed such a state so it remains to be seen whether this scaling is a by-product of the analysis or if some physically relevant information can be gathered from the current derivation. Either way the similarity between the current result and that obtained in [34] with the asymptotically motivated numerical results of [22] are indicative that the current result is sharp with respect to the background method.

In addition to the derived unique (to stress-free boundaries) scaling law, we note the appearance of the saturating wave-number (57). Such a dominant horizontal scale distinct from the size of the boundary layer has not been observed in the simulations to date, and it remains to be seen if such a scale is physically important or only a mathematical construct of the variational formulation. This indicates a dominant horizontal scale that can be used to construct asymptotic solutions (both numeric and rigorous) akin to either [7] or [11] that (according to the current analysis) should saturate the bound derived in this paper. In addition, direct numerical simulations in which forcing is added at this scale and/or careful analysis of the energy at these scales is investigated will provide additional insight into the nature of convection between slippery plates.

A An estimate on the pseudo-vorticity

As in the main body, denote the pseudo differential operator $|\nabla_H| = \sqrt{-\nabla_H \cdot \nabla_H} = \sqrt{-\Delta_H}$ and introduce the pseudo-vorticity ω as in [34] as

$$\Delta u_3 = |\nabla_H|\omega, \tag{62}$$

Our key estimate in this appendix is

$$\|\omega\|^2 \le 2\|\nabla \boldsymbol{u}\|^2 \tag{63}$$

We utilize Fourier coefficients in the horizontal directions to deduce this fact. Indeed, the definition of the pseudo-vorticity implies, with $k = (k_x, k_y)^{tr}$

denoting the horizontal wave number, $\hat{u}_{jk}$ representing the horizontal Fourier coefficients of $u_j, j = 1, 2, 3$, $\hat{u}_{Hk} = (\hat{u}_{1k}, \hat{u}_{2k})^{tr}$, $D = \frac{\partial}{\partial z}$,

$$\begin{aligned}
|k|^2|\hat{\omega}_k|^2 &= |D^2\hat{u}_{3k}|^2 + |k|^2|D\hat{u}_{3k}|^2 + |k|^4|\hat{u}_{3k}|^2 \\
&= |k_x D\hat{u}_{1k} + k_y D\hat{u}_{2k}|^2 + |k|^2|D\hat{u}_{3k}|^2 + |k|^4|\hat{u}_{3k}|^2 \quad \text{(since } \boldsymbol{u} \text{ is divergence free)} \\
&\leq 2|k|^2|D\hat{u}_{Hk}|^2 + |k|^2|D\hat{u}_{3k}|^2 + |k|^4|\hat{u}_{3k}|^2 \\
&\leq 2|k|^2(|D\hat{u}_{Hk}|^2 + |D\hat{u}_{3k}|^2 + |k|^2|\hat{u}_{3k}|^2) \qquad (64)
\end{aligned}$$

dividing by $|k|^2$ and summing over k we have the desired bound on the pseudo-vorticity (63).

B List of useful inequalities

In this appendix we provide a list of mathematical inequalities that are used in the body of the text, in particular in Section 2. The same notation used in the body of the text is used here, and the constants are denoted with the same symbol. [1] and [10] are excellent references for the inequalities cited below.

- Cauchy-Schwarz inequality:

$$\int_\Omega |f(\mathbf{x})g(\mathbf{x})|\, d\mathbf{x} \leq \|f(\mathbf{x})\|\|g(\mathbf{x})\|. \qquad (65)$$

- Poincaré Inequality:

$$\|\nabla \boldsymbol{u}\| \geq c_p\|\boldsymbol{u}\| \qquad \text{and} \qquad \|\Delta \boldsymbol{u}\| \geq c_{p'}\|\nabla \boldsymbol{u}\|, \qquad (66)$$

  if $u\big|_{z=0,1} = 0$ or $\int_\Omega u d\mathbf{x} = 0$.

- Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad \text{where} \qquad \frac{1}{p} + \frac{1}{q} = 1. \qquad (67)$$

- Gronwall's inequality:
  Let $f(t)$, $g(t)$ and $h(t)$ be sufficiently smooth on the interval $[a, b]$ where

$$\frac{df(t)}{dt} \leq g(t)f(t) + h(t) \qquad (68)$$

  then

$$f(t) \leq f(a)e^{\int_a^t g(s)ds} + \int_0^t e^{\int_s^t g(\tau)\,d\tau}h(s)\,ds. \qquad (69)$$

- Agmon's inequality:

$$\|\boldsymbol{u}(t)\|_{L^\infty} \leq c_A\|\nabla \boldsymbol{u}(t)\|^{\frac{1}{2}}\|A\boldsymbol{u}(t)\|^{\frac{1}{2}}. \qquad (70)$$

- Generalized Hölder inequality:

  Assume that $r \in (0, \infty)$ and $p_k \in (0, \infty]$ such that

$$\sum_{k=1}^{n} \frac{1}{p_k} = \frac{1}{r} \tag{71}$$

  then for all sufficiently smooth, measurable functions f_k then

$$\left\| \prod_{k=1}^{n} f_k \right\|_r \le \prod_{k=1}^{n} \|f_k\|_{p_k} \tag{72}$$

  where $\|\cdot\|_{p_k}$ is the L^{p_k} norm and $\prod$ is the product operator.

- Sobolev inequalities in three dimension

$$\|\boldsymbol{u}(t)\|_{L^3} \le c_{S1} \|\nabla \boldsymbol{u}(t)\| \quad \text{and} \quad \left\| \frac{\partial \boldsymbol{u}(t)}{\partial t} \right\|_{L^6} \le c_{S2} \left\| \nabla \frac{\partial \boldsymbol{u}(t)}{\partial t} \right\| \tag{73}$$

$$\|\nabla \boldsymbol{u}(t)\|_{L^4} \quad \le \quad C \|\nabla \boldsymbol{u}(t)\|^{\frac{1}{2}} \|\nabla^2 \boldsymbol{u}\|^{\frac{3}{2}}. \tag{74}$$

- Elliptic regularity of the Stokes operator

$$\|\nabla^2 \boldsymbol{u}(t)\| \le C \|A \boldsymbol{u}(t)\|. \tag{75}$$

- Interpolation inequality:

$$\|A \boldsymbol{u}(t)\| \le \|\nabla \boldsymbol{u}(t)\|^{\frac{1}{2}} \|A^{\frac{3}{2}} \boldsymbol{u}(t)\|^{\frac{1}{2}}. \tag{76}$$

Acknowledgments

This work is supported in part by the National Science Foundation through DMS1008852, and the U.S. Department of Energy through the LANL/LDRD Program.

References

[1] R. Adams and J. Fournier. *Sobolev Spaces, 2nd ed.* Elsevier, 2003.

[2] G. Ahlers. Turbulent convection. *Physics*, 2(74):7 pp, 2009.

[3] G. Ahlers, S. Grossmann, and D. Lohse. Heat transfer and large scale dynamics in turbulent Rayleigh-Bénard convection. *Review of Modern Physics*, 81:503–537, 2009.

[4] G. Ahlers, X. He, D. Funfschilling, and E. Bodenschatz. Heat transport by turbulent Rayleigh-Bénard convection for Pr $\simeq 0.8$ and $3 \times 10^{12} \lesssim Ra \lesssim 10^{15}$: Aspect ratio $\gamma = 0.50$. *New Journal of Physics*, submitted, 2012.

[5] E. Bodenschatz, W. Pesch, and G. Ahlers. Recent developments in Rayleigh-Bénard convection. *Annual Review of Fluid Mechanics*, 32:709–778, 2000.

[6] F. H. Busse. *Mantle convection: Plate tectonics and global dynamics. Chapter 2: Fundamentals of thermal convection.* Gordon and Breach, New York, 1989.

[7] G. P. Chini and S. M. Cox. Large Rayleigh number thermal convection: Heat flux predictions and strongly nonlinear solutions. *Physics of Fluids*, 21:083603, 2009.

[8] P. Constantin and C. R. Doering. Heat transfer in convective turbulence. *Nonlinearity*, 9:1049–1060, 1996.

[9] P. Constantin and C. R. Doering. Infinite Prandtl number convection. *Journal of Statistical Physics*, 94:159–172, 1999.

[10] P. Constantin and C. Foias. *Navier-Stokes Equations.* Chicago University Press, Chicago, IL, USA, 1988.

[11] L. T. Corson. Maximizing the heat flux in steady unicellular porous media convection. Technical report, Woods Hole Oceanographic Institution, 2011. WHOI Geophysical Fluid Dynamics Program Fellow Report.

[12] C. R. Doering and P. Constantin. Variational bounds on energy dissipation in incompressible flows. III. Convection. *Physical Review E.*, 53(6):5957–5981, 1996.

[13] C. R. Doering and P. Constantin. On upper bounds for infinite Prandtl number convection with or without rotation. *Journal of Mathematical Physics*, 45(2):784–795, 2001.

[14] C. R. Doering, F. Otto, and M. G. Reznikoff. Bounds on vertical heat transport for infinite-Prandtl-number Rayleigh-Bénard convection. *Journal of Fluid Mechanics*, 560:229–241, 2006.

[15] D. Funfschilling, E. Bodenschatz, and G. Ahlers. Search for the "Ultimate State" in Turbulent Rayleigh-Bénard Convection. *Physical Review Letters*, 103:014503, 2009.

[16] A. V. Getling. Rayleigh-Bénard convection. Structures and dynamics. *Advanced Series in Nonlinear Dynamics*, 11, 1998.

[17] S. Grossmann and D. Lohse. Scaling in thermal convection: a unifying theory. *Journal of Fluid Mechanics*, 407:27–56, 2000.

[18] S. Grossmann and D. Lohse. Thermal convection for large Prandtl number. *Physical Review Letters*, 86:3316–3319, 2001.

[19] S. Grossmann and D. Lohse. Multiple scaling in the ultimate regime of thermal convection. *Physics of Fluids*, 23:045108, 2011.

[20] X. He, D. Funfschilling, H. Nobach, E. Bodenschatz, and G. Ahlers. Transition to the ultimate state of turbulent Rayleigh-Bénard convection. *Physical Review Letters*, 108(2):024502, 2012.

[21] L. N. Howard. Heat transport by turbulent convection. *Journal of Fluid Mechanics*, 17:405–432, 1963.

[22] G. R. Ierley, R. R. Kerswell, and S. C. Plasting. Infinite-Prandtl-number convection. Part 2. A singular limit of upper bound theory. *Journal of Fluid Mechanics*, 560:159–227, 2006.

[23] R. H. Kraichnan. Turbulent thermal convection at arbitrary Prandtl number. *Physics of Fluids*, 5:1374–1389, 1962.

[24] W. V. R. Malkus. The heat transport and spectrum of thermal turbulence. *Proceedings of the Royal Society of London A*, 225:196–212, 1954.

[25] J. Otero. *Bounds for the heat transport in turbulent convection.* PhD thesis, University of Michigan, MI, 2002. Department of Mathematics, 137 pp.

[26] F. Otto and C. Seis. Rayleigh-Bénard convection: Improved bounds on the Nusselt number. *Journal of Mathematical Physics*, 52:083702, 2011.

[27] Lord Rayleigh. On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side. *Philosophical Magazine and Journal of Science*, 32(192):529–546, 1916.

[28] R. Temam. *Navier-Stokes Equations.* AMS Chelsea, Providence, Rhode Island, USA, 2000.

[29] P. Urban, V. Musilová, and L. Skrbek. Efficiency of Heat Transfer in Turbulent Rayleigh-Bénard Convection. *Physical Review Letters*, 107:014302, 2011.

[30] X. Wang. Asymptotic behavior of global attractors to the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number. *Communications in Pure and Applied Mathematics*, 60(9):1293–1318, 2007.

[31] X. Wang. Bound on vertical heat transport at large Prandtl number. *Physica D*, 237:854–858, 2008.

[32] X. Wang. Stationary statistical properties of Rayleigh-Bénard convection at large Prandtl number. *Communications in Pure and Applied Mathematics*, 61:789–815, 2008.

[33] J. P. Whitehead and C. R. Doering. The ultimate regime of two-dimensional Rayleigh-Bénard convection with stress-free boundaries. *Physical Review Letters*, 106:244501, 2011.

[34] J. P. Whitehead and C. R. Doering. Rigid rigorous bounds on heat transport in a slippery container. *(in press) Journal of Fluid Mechanics*, 2012.