MONOMIAL PRINCIPALIZATION IN THE SINGULAR SETTING

COREY HARRIS

Abstract. We generalize an algorithm by Goward for principalization of monomial ideals in nonsingular varieties to work on any scheme of finite type over a field. The normal crossings condition considered by Goward is weakened to the condition that components of the generating divisors meet as complete intersections.

1. Introduction

Throughout, X will denote a scheme of finite type over an arbitrary field. By regular sequence, we mean a sequence $x_1, \dots, x_n$ of elements in a ring R such that $(x_1, \dots, x_n) \subset R$ is a proper ideal and, for each i, the image of x_i in $R/(x_1, \dots, x_{i-1})$ is a non-zerodivisor, see [Eis95, p. 243].

Definition (Complete intersection crossings)**.** Let $Y_1, \dots, Y_n \subset X$ be Cartier divisors. We say that $\{Y_1, \dots, Y_n\}$ has *complete intersection crossings*, or *c.i. crossings*, if for every subset $A \subset \{Y_1, \dots, Y_n\}$ and every point $p \in \cap_{Y \in A} Y$, the local equations y_i for the $Y \in A$ form a regular sequence at p.

Note that the definition requires each Y_i to be cut out locally by a non-zerodivisor, making Y_i an effective Cartier divisor in X. Note also that the condition places no restrictions on X. The following definition will be used only in the introduction to compare concepts.

Definition (Simple normal crossings)**.** Let $Y_1, \dots, Y_n \subset X$ be Cartier divisors. We say that $\{Y_1, \dots, Y_n\}$ has *simple normal crossings* if for every subset $A \subset \{Y_1, \dots, Y_n\}$, the intersection $Z = \cap_{Y \in A} Y$ is nonsingular with $\operatorname{codim}_X Z = \#A$. If $D = \sum a_i Y_i$, with $a_i \geq 0$, we say D is a *simple normal crossings divisor* or *s.n.c divisor*.

The s.n.c. condition on singletons requires each Y_i to be nonsingular, and the condition on the empty set means X itself must be nonsingular.

Example 1. Let $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ and consider the curves Y_1 defined by y and Y_2 defined by $y^2 - x^3$. These two curves do not meet with simple normal crossings because $y^2 - x^3$ is not smooth in the intersection. So $Y_1 + Y_2$ is not an s.n.c. divisor, but $\{Y_1, Y_2\}$ does have complete intersection crossings since $y^2 - x^3$ is sent to a non-zerodivisor in the integral domain $k[x, y]/(y)$. Observe (Figure 1a) that these curves do not meet transversally.

Example 2. Now consider the cone $C = \operatorname{Spec} k[x, y, z]/(xy - z^2)$ and the subschemes Y_x, Y_y cut out by the ideals (x) and (y), respectively. Since C is singular,

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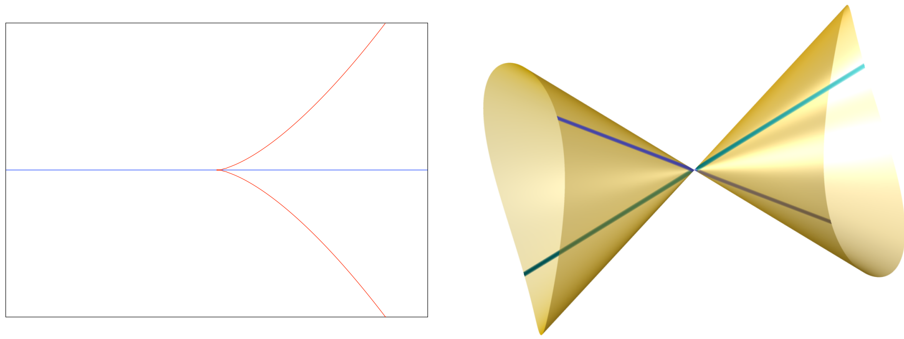

(A) Curves y, $y^2 - x^3$ in the plane

(B) Curves x, y on $xy - z^2$

FIGURE 1. Examples

there are no s.n.c. divisors in sight. However $(k[x,y,z]/(xy-z^2))/(y) \cong k[x,z]/(z^2)$, so y, x forms a regular sequence at every point in C. Thus $\{Y_x, Y_y\}$ has complete intersection crossings in C.

Remark. In both of the above examples, we checked the c.i. crossing condition only affine-locally, even though the definition is in terms of stalks. This is sufficient because if $x_1, \dots, x_n$ is a regular sequence in a Noetherian ring R, then it remains a regular sequence in R_p for any prime ideal $p \subset R$.

Definition. A subscheme $Z \subset X$ is called *monomial* if it is cut out by effective divisors which are supported on a fixed s.n.c. divisor. By analogy, if $\{Y_1, \dots, Y_n\}$ has c.i. crossings, we will say Z is a *c.i. monomial subscheme* (with respect to $\{Y_1, \dots, Y_n\}$) if Z is cut out by divisors of the form $\sum a_i Y_i$ with $a_i \geq 0$. As well, if $\beta : \widetilde{X} \to X$ is the blowup of X at a c.i. monomial subscheme, we will call β (or just $\widetilde{X}$) a *c.i. monomial blowup* (with respect to $\{Y_1, \dots, Y_n\}$).

Theorem 1 (Main Theorem)**.** *Let $\{Y_1, \dots, Y_n\}$ have c.i. crossings. If $D_1, \dots, D_h$ are given by $D_j = \sum a_{ij} Y_i$, where $a_{ij} > 0$, then there exists a sequence of c.i. monomial blowups at codimension 2 centers*

$$\widetilde{X} = X_n \xrightarrow{\beta_n} X_{n-1} \xrightarrow{\beta_{n-1}} \cdots \xrightarrow{\beta_2} X_1 \xrightarrow{\beta_1} X$$

such that $(\mathscr{I}_{D_1} + \cdots + \mathscr{I}_{D_h})\mathcal{O}_{X_i}$ is c.i. monomial for each i and $(\mathscr{I}_{D_1} + \cdots + \mathscr{I}_{D_h})\mathcal{O}_{\widetilde{X}}$ is locally principal.

Goward's theorem [Gow05, Theorem 2] is the analogous statement to Theorem 1 for monomial subschemes. The algorithm here is a direct generalization of Goward's, and our method of proof follows his. We first need to know that c.i. monomial blowups preserve c.i. crossings and so we verify this in section 2. Following this, we give the proof of Theorem 1 in section 3.

2. C.I. MONOMIAL BLOWUPS PRESERVE C.I. CROSSINGS

Assume $\{Y_1, \dots, Y_n\}$ has c.i. crossings and $n \geq 2$. Let $D_1 = \sum a_i Y_i$ and $D_2 = \sum b_i Y_i$ with $a_i, b_j \geq 0$. If $\beta : \widetilde{X} \to X$ is the blowup of X at $Y_1 \cap Y_2$, let E denote the exceptional divisor in $\widetilde{X}$ and let $\widetilde{Y}_i$ denote the proper transform of Y_i in $\widetilde{X}$. We will verify that $\beta^* D_i$, which is supported on $\{E, \widetilde{Y}_1, \dots, \widetilde{Y}_n\}$, has complete intersection crossings.

The proof will be by induction on the number of divisors. The base case is handled in Proposition 2 and the inductive step handled in Proposition 3.

Proposition 2. *Suppose $\{Y_1, Y_2\}$ has c.i. crossings on X. Let $\beta : \widetilde{X} \to X$ denote the blowup of X at $Y_1 \cap Y_2$. Then $\{E, \widetilde{Y}_1, \widetilde{Y}_2\}$ has complete intersection crossings in X.*

Proof. There are eight subsets of $\{\widetilde{Y_1}, \widetilde{Y_2}, E\}$. The intersections corresponding to $\emptyset$, $\{\widetilde{Y_1}, \widetilde{Y_2}\}$, $\{\widetilde{Y_1}, \widetilde{Y_2}, E\}$ are empty, and those corresponding to $\{\widetilde{Y_1}\}$, $\{\widetilde{Y_2}\}$, $\{E\}$ are effective Cartier. This leaves the intersections corresponding to $\{\widetilde{Y_1}, E\}$ and $\{\widetilde{Y_2}, E\}$. We show the result for $\{\widetilde{Y_1}, E\}$.

It suffices to prove the result affine-locally. Since Y_1, Y_2 are effective Cartier divisors on X, there is an affine open cover $\{U_\alpha = \operatorname{Spec} R_\alpha\}$ of X such that $y_{1,\alpha}$ is a local equation for Y_1 on U_α and $y_{2,\alpha}$ is a local equation for Y_2. Now let $U = \operatorname{Spec} R$ be a member of such a cover and let $(y_1), (y_2) \subset R$ be principal ideals defining Y_1 and Y_2, respectively. Then the blowup $\widetilde{Y}$ of Y centered at $Y_1 \cap Y_2$ is $\operatorname{Proj} R[a_1, a_2]/(a_2 y_1 - a_1 y_2)$, see [Ful98, B.6.10]. Consider the open set $D(a_2) := \{[p] \in \operatorname{Spec} R \mid a_2 \notin p\}$ which we can write as

$$D(a_2) = \operatorname{Spec} R[a_1]/(y_1 - a_1 y_2).$$

The pullback $\beta^* Y_1$ is cut out by $a_1 y_2$ in $D(a_2)$ and the exceptional divisor is cut out by y_2, so the proper transform of Y_1 is cut out in $D(a_2)$ by a_1. Since $(R[a_1]/(y_1 - a_1 y_2))/(a_1) \cong R/(y_1)$, and y_2 is not a zerodivisor in $R/(y_1)$ by assumption, we have that a_1, y_2 is a regular sequence in $R[a_1]/(y_1 - a_1 y_2)$ corresponding to the intersection $Y_1 \cap Y_2$.

The proof is completed by noting again that localization preserves regular sequences, so that a_1, y_2 is a regular sequence in $\mathcal{O}_{\widetilde{X},p}$ for each $p \in \widetilde{Y}_1 \cap \widetilde{Y}_2$. $\square$

We note that as a scheme of finite type over a Noetherian ring, X is Noetherian. In particular $\mathcal{O}_{X,p}$ is a Noetherian local ring for all $p \in X$. In the proof of Proposition 3, we will need the following result.

Lemma 1. *If R is a Noetherian local ring and $x_1, \dots, x_r$ is a regular sequence of elements in the maximal ideal of R, then any permutation of $x_1, \dots, x_r$ is again a regular sequence.*

Proof. See [Eis95, Theorem 17.2]. $\square$

We will make repeated use of the lemma, along with the idea that if $x_1, \dots, x_n$ is a regular sequence, then $x_1, \dots, x_k$ is a regular sequence for $1 \leq k \leq n$.

Proposition 3. *Suppose $\{Y_1, \dots, Y_n\}$ has c.i. crossings on X. Let $\beta : \widetilde{X} \to X$ denote the blowup of X at $Y_1 \cap Y_2$. Then $\{E, \widetilde{Y}_1, \dots, \widetilde{Y}_n\}$ has c.i. crossings in $\widetilde{X}$.*

Proof. By induction, assume $\{E, \widetilde{Y}_1, \dots, \widetilde{Y}_{n-1}\}$ has c.i. crossings. We will show that if A contains any (possibly empty) subset of $\{E, \widetilde{Y}_1, \widetilde{Y}_2\}$, then the corresponding intersection is cut out by a regular sequence at each point in an affine chart on the blowup.

Let $A \subset \{E, \widetilde{Y}_1, \dots, \widetilde{Y}_n\}$. We want $\cap_{Z \in A} Z \neq \emptyset$ so $\{\widetilde{Y}_1, \widetilde{Y}_2\}$ cannot be a subset of A. Assume then without loss of generality that $\widetilde{Y}_2 \notin A$. Assume also that

$\widetilde{Y}_n \in A$. Each element of $\{Y_1, \ldots, Y_n\}$ is an effective Cartier divisor so again we can find a cover of X by open affines $\{U_\alpha = \operatorname{Spec} R_\alpha\}$ such that for each i, and each $Y \in \{Y_1, \ldots, Y_n\}$, we have a non-zerodivisor $y \in R_\alpha$ which cuts out Y in U_α. Choose such a $U = \operatorname{Spec} R$ and let $y_i \in R$ be a local equation for Y_i. We can then write the blowup of this chart $\widetilde{U} = \operatorname{Proj} R[a_1, a_2]/(a_2 y_1 - a_1 y_2)$. As before, we work affine-locally in $D(a_2) = \operatorname{Spec} R[a_1]/(y_1 - a_1 y_2)$, and we have that E is cut out by y_2 and $\widetilde{Y}_1$ is cut out by a_1.

Assume that A contains both E and $\widetilde{Y}_1$. By induction we can find a regular sequence for $A \backslash \{\widetilde{Y}_n\}$ in the elements $\{y_2, a_1, s_1, \ldots, s_k\}$ (where $\#(A \backslash \{\widetilde{Y}_n\}) = k + 2$) and so Lemma 1 ensures that $a_1, y_2, s_1, \ldots, s_k$ is a regular sequence at each $p \in \cap_{Y \in A \backslash \{\widetilde{Y}_n\}} Y$. Let r be a local equation for $\widetilde{Y}_n$ on $D(a_2)$ such that we have a regular sequence in the elements $r, y_1, y_2, s_1, \ldots, s_k$ at each point $p \in U$. Then by Lemma 1 these elements form a regular sequence in any order. Thus $y_2, r, s_1, \ldots, s_n$ form a regular sequence in $R/(t_1)$, showing that $a_1, y_2, r, s_1, \ldots, s_n$ is a regular sequence at each point $p \in \cap_{Y \in A} Y$.

If A contains E but does not contain $\widetilde{Y}_1$, then we can use Lemma 1 again to get the regular sequence $s_1, \ldots, s_n, r, y_2, a_1$ which shows that $s_1, \ldots, s_n, r, y_2$ is a regular sequence. Similarly, if A contains $\widetilde{Y}_1$ but not E, we can rearrange to get $s_1, \ldots, s_n, r, a_1, y_2$, which shows $s_1, \ldots, s_n, r, a_1$ is a regular sequence. Truncating this sequence again shows that $s_1, \ldots, s_n, r$ is a regular sequence. This is the case where A contains neither E nor $\widetilde{Y}_1$. □

In this paper, we will only need the result as stated in Proposition 3, but since the fact is true in greater generality, we provide Proposition 4 for completeness. In the following proof we use Proposition 3 as the base case and induct on the number of divisors that cut out the center for the blowup.

Proposition 4. *Suppose $\{Y_1, \ldots, Y_n\}$ has c.i. crossings on X. Let $\beta : \widetilde{X} \to X$ denote the blowup of X at $\cap_{i=1}^r Y_i$, where $r \leq n$. Then $\{E, \widetilde{Y}_1, \ldots, \widetilde{Y}_n\}$ has c.i. crossings in $\widetilde{X}$.*

Proof. Let $A \subset \{E, \widetilde{Y}_1, \ldots, \widetilde{Y}_n\}$ with $\widetilde{Y}_1 \in A$. We want $\cap_{Z \in A} Z \neq \emptyset$ so we require $A \cap \{\widetilde{Y}_2, \ldots, \widetilde{Y}_r\} = \emptyset$. Each element of $\{Y_1, \ldots, Y_n\}$ is an effective Cartier divisor so again we can find a cover of X by open affines $\{U_\alpha = \operatorname{Spec} R_\alpha\}$ such that for each α, and each $Y \in \{Y_1, \ldots, Y_n\}$, we have a non-zerodivisor $y \in R_\alpha$ which cuts out Y in U_α. Choose such a $U = \operatorname{Spec} R$ and let $y_i \in R$ be a local equation for Y_i. We can then write the blowup of this chart

$$\widetilde{U} = \operatorname{Proj} R[a_1, \ldots, a_r]/(\{a_j y_i - a_i y_j \mid i \neq j\})$$

(again see [Ful98, B.6.10]). As before, we work affine-locally in

$$D(a_1) = \operatorname{Spec} R[a_2, \ldots, a_r]/(\{y_i - a_i y_1 \mid i \neq 1\})$$

where we have that E is cut out by y_1, and $\widetilde{Y}_i$ is cut out by a_i for each $i > 2$.

We can use induction on r with the base case Proposition 3. Then assume $\{E, \widetilde{Y}_1, \ldots, \widetilde{Y}_n\}$ has c.i. crossings in the blowup $\beta' : X' \to X$ centered at $\cap_{i=1}^{r-1} Y_i$. If we let U' denote the blowup of the chart U by β' then

$$U' = \operatorname{Proj} R[a_1, \ldots, a_{r-1}]/(\{a_j y_i - a_i y_j \mid i \neq j\})$$

with corresponding open chart

$$D'(a_1) = \operatorname{Spec} R[a_1, \dots, a_{r-1}]/(\{y_i - a_i y_1 \mid i = 2, \dots, r-1\}).$$

Now we observe that $D(a_1)$ is related to $D'(a_1)$ by

$$\begin{aligned} D(a_1) \cap \mathbb{V}(a_r) &= \operatorname{Spec}(R[a_2, \dots, a_r]/(\{y_i - a_i y_1 \mid i \neq 1\}))/(a_r) \\ &= \operatorname{Spec} R[a_2, \dots, a_{r-1}]/(\{y_i - a_i y_1 \mid i = 2, \dots, r-1\} \cup \{y_r\}) \\ &= D'(a_1) \cap \mathbb{V}(y_r). \end{aligned}$$

Recall that $\widetilde{Y}_r \notin A$. The inductive hypothesis says that we can find a regular sequence $s_1, \dots, s_k$ corresponding to A, where $k = \#A$. Then as usual we can localize at a point and rearrange to get that $y_r, s_1, \dots, s_k$ is a regular sequence at each point $p \in D'(a_1)$. Then $s_1, \dots, s_k$ is a regular sequence at each point in $D'(a_1)/(y_r) = D(a_1)/(a_r)$, so $a_r, s_1, \dots, s_k$ is a regular sequence at each point of $D(a_1)$. □

3. Proof of main theorem

In [Gow05], Goward defines an invariant (σ, τ) on the divisors in question. We adopt those definitions for the new context here:

Definition. Let $D_1 = \sum_{i=1}^n a_i Y_i$ and $D_2 = \sum_{i=1}^n b_i Y_i$ where $\{Y_1, \dots, Y_n\}$ has c.i. crossings in X and $a_i, b_j \geq 0$. We define

$$\sigma_{ij}(D_1, D_2) = \begin{cases} \max\{(|a_i - b_i|, |a_j - b_j|), (|a_j - b_j|, |a_i - b_i|)\} & \text{if } a_i - b_i \text{ and } a_j - b_j \text{ have opposite signs,} \\ (-\infty, -\infty) & \text{otherwise,} \end{cases}$$

where the max is taken lexicographically. Now we can define

$$\begin{aligned} \sigma(D_1, D_2) &= \max\{\sigma_{ij}(D_1, D_2) \mid Y_i \cap Y_j \neq 0,\ i \neq j\} \\ \tau(D_1, D_2) &= \#\{(i,j) \mid \sigma_{ij}(D_1, D_2) = \sigma(D_1, D_2),\ i \leq j\} \end{aligned}$$

so $\sigma(D_1, D_2)$ takes the value of the worst intersection in the support, and $\tau(D_1, D_2)$ counts how many intersections share this value.

These invariants are calculated for divisors on X and, after blowing up, for their pullbacks. We show that these calculations go the same way as in the simple normal crossings context and then outline the steps of the proof.

Proposition 5. *Let D_1, D_2 be as defined above. Then $\mathscr{I}_{D_1} + \mathscr{I}_{D_2}$ is principal at $p \in X$ if and only if $\sigma_{ij}(D_1, D_2) = (-\infty, -\infty)$ whenever $p \in Y_i \cap Y_j$.*

Proof. Let $x_1, \dots, x_n$ be a regular sequence at $p \in X$ corresponing to $Y_1, \dots, Y_n$. Then let $f_1 = u_1 x_1^{a_1} \dots x_n^{a_n}$ and $f_2 = u_2 x_1^{b_1} \dots x_n^{b_n}$ be local equations for D_1 and D_2, where $u_i \in \mathcal{O}_{X,p}$ are units. Suppose $\mathscr{I}_{D_1} + \mathscr{I}_{D_2}$ is not principal at p. Then (f_1, f_2) is not principal, so we have some i, j such that $a_i - b_i$ and $a_j - b_j$ have opposite signs. Thus $\sigma_{ij}(D_1, D_2) > 0$. On the other hand, suppose $p \in Y_i \cap Y_j$ and $\sigma_{ij}(D_1, D_2) > 0$. Then $a_i - b_i$ and $a_j - b_j$ have opposite signs, so (f_1, f_2) is not principal and thus $\mathscr{I}_{D_1} + \mathscr{I}_{D_2}$ is not principal at p. □

As a result of this proof, we see that if $\mathscr{I}_{D_1} + \mathscr{I}_{D_2}$ is principal at p, then $(\mathscr{I}_{D_1} + \mathscr{I}_{D_2})_p = (\mathscr{I}_{D_i})_p$ for some $i \in \{1, 2\}$. So by induction, we have that if $\mathscr{I}_{D_1} + \dots + \mathscr{I}_{D_h}$ is principal at p, then $(\mathscr{I}_{D_1} + \dots + \mathscr{I}_{D_h})_p = (\mathscr{I}_{D_i})_p$ for some $i \in \{1, \dots, h\}$.

We have left to show that blowing up at the chosen codimension 2 centers strictly reduces the invariant (σ, τ) and then that such blowups can be taken successively until $(\sigma, \tau) = (-\infty, -\infty)$.

Proposition 6. *Let D_1, D_2 be as defined above. Suppose we have (i, j) such that $\sigma_{ij}(D_1, D_2) = \sigma(D_1, D_2) > (-\infty, -\infty)$ and let $\beta : \widetilde{X} \to X$ be the blowup of X centered at $Y_i \cap Y_j$. Then*

$$(\sigma(D_1, D_2), \tau(D_1, D_2)) > (\sigma(\beta^* D_1, \beta^* D_2), \tau(\beta^* D_1, \beta^* D_2).$$

Sketch of proof. Assume that $(i, j) = (1, 2)$ so that the blowup is centered at $Y_1 \cap Y_2$. The proof relies on calculations of $\sigma_{ij}(\beta^* D_1, \beta^* D_2)$, which depend only on the coefficients a_i, b_i, a_j, b_j. The details can be found in the proof of [Gow05, Thm 1]. Here we verify only that the calculations of $\sigma_{ij}(\beta^* D_1, \beta^* D_2)$ from the s.n.c. case still go through with c.i. crossings.

Let $E \subset \widetilde{X}$ denote the exceptional divisor. Then

$$\beta^* Y_i = \begin{cases} \widetilde{Y}_i + E & \text{if } i = 1, 2 \\ \widetilde{Y}_i & \text{if } i > 2 \end{cases}$$

as seen from the work done in Proposition 2. Thus

$$\beta^* D_1 = (a_1 + a_2)E + \sum_i a_i \widetilde{Y}_i$$

and

$$\beta^* D_2 = (b_1 + b_2)E + \sum_i b_i \widetilde{Y}_i. \qquad \square$$

Theorem 1 (Main Theorem)**.** Let $\{Y_1, \ldots, Y_n\}$ have c.i. crossings. If $D_1, \ldots, D_h$ are given by $D_j = \sum a_{ij} Y_i$, where $a_{ij} > 0$, then there exists a sequence of c.i. monomial blowups at codimension 2 centers

$$\widetilde{X} = X_n \xrightarrow{\beta_n} X_{n-1} \xrightarrow{\beta_{n-1}} \cdots \xrightarrow{\beta_2} X_1 \xrightarrow{\beta_1} X$$

such that $(\mathscr{I}_{D_1} + \cdots + \mathscr{I}_{D_h})\mathcal{O}_{X_i}$ is c.i. monomial for each i and $(\mathscr{I}_{D_1} + \cdots + \mathscr{I}_{D_h})\mathcal{O}_{\widetilde{X}}$ is locally principal.

Proof. By the previous remarks, we can assume $h = 2$. Let $Y_1, \ldots, Y_n$ be the divisors in the support of $D_1 + D_2$. If $\sigma(D_1, D_2) = (-\infty, -\infty)$ we are done, so assume not. Then let

$$(i, j) = \max\{(k, l) \mid \sigma_{kl}(D_1, D_2) = \sigma(D_1, D_2)\}$$

where the max is taken lexicographically. If we take the blowup $\beta_1 : X_1 \to X$ centered at $Y_i \cap Y_j$, then Proposition 6 gives that

$$(\sigma(D_1, D_2), \tau(D_1, D_2)) > (\sigma(\beta^* D_1, \beta^* D_2), \tau(\beta^* D_1, \beta^* D_2).$$

Now we note that $\beta_1^* D_1$, $\beta_1^* D_2$ are divisors supported on $\{E, \widetilde{Y}_1, \ldots, \widetilde{Y}_n\}$ and have c.i. crossings by Proposition 3. Thus, $(\mathscr{I}_{D_1} + \mathscr{I}_{D_2})\mathcal{O}_{\widetilde{X}}$ defines a c.i. monomial subscheme of $\widetilde{X}$.

We can repeat this process, with (σ, τ) decreasing at each iteration. Since (σ, τ) takes values in $\mathbb{N}^2 \times \mathbb{N}$, we must get $(\sigma, \tau) = (-\infty, -\infty)$ after finitely many steps, yielding the desired sequence of blowups. $\square$

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Department of Mathematics, Florida State University, Tallahassee FL, 32306, USA
E-mail address: charris@math.fsu.edu
URL: http://www.math.fsu.edu/~charris