

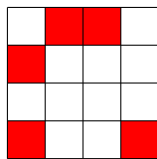
DEGREES OF PROJECTIONS OF RANK LOCI

PAOLO ALUFFI

ABSTRACT. We provide formulas for the degrees of the projections of the locus of square matrices with given rank from linear spaces spanned by a choice of matrix entries. The motivation for these computations stem from applications to ‘matrix rigidity’; we also view them as an excellent source of examples to test methods in intersection theory, particularly computations of Segre classes. Our results are generally expressed in terms of intersection numbers in Grassmannians, which can be computed explicitly in many cases. We observe that, surprisingly (to us), these degrees appear to match the numbers of *Kekulé structures* of certain *benzenoid hydrocarbons*, and arise in many other contexts with no apparent direct connection to the enumerative geometry of rank conditions.

1. INTRODUCTION

Consider the space \mathbb{P}^{n^2-1} of $n \times n$ complex matrices. If S is a set of s coordinates in this space, i.e., a set of entries of $n \times n$ matrices, we denote by L_S the subspace of \mathbb{P}^{n^2-1} spanned by S , and we consider the projection $\pi_S : \mathbb{P}^{n^2-1} \dashrightarrow \mathbb{P}^{n^2-s-1}$ centered at L_S . We depict S by highlighting its entries in an $n \times n$ grid:

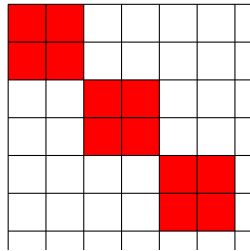


The problem we approach in this paper is the computation of the degree of the closure of the image via π_S of the locus $\sigma_{n,r}$ of matrices of rank $\leq r$.

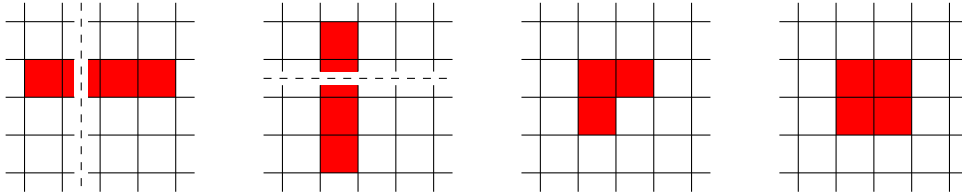
The original motivation for this problem stems from complexity theory. In [GHIL13], following L. Valiant (cf. [Val77]), the *r-rigidity* of an $n \times n$ matrix M is defined to be the smallest s such that there exists a set S of s entries for which $\pi_S(M) \in \overline{\pi_S(\sigma_{n,r})}$. There is interest in finding explicit equations for the set of matrices with given rigidity; these may be used to give lower bounds for the rigidity of a given matrix, with applications to the complexity of performing explicit operations such as matrix multiplication. We address the reader to [GHIL13] for a thorough discussion of this motivation, and for a lucid explanation of the algebro-geometric aspects of this question. One idea introduced in [GHIL13] is that the *degree* $d_{n,r,S}$ of the projection $\overline{\pi_S(\sigma_{n,r})}$ (or equivalently of the join of L_S and $\sigma_{n,r}$) gives useful information on polynomial equations for matrices with given rigidity. A method for the computation of $d_{n,r,S}$ by successive projections from the entries in S is introduced in [GHIL13] and used to obtain several concrete results. An explicit computation of the value of $d_{n,r,S}$ for sets S with no two elements on the same row or column (i.e., up to permutations, consisting of diagonal entries) is also presented in [GHIL13], subject to a conjectural statement of the behavior of taking joins and tangent spaces in the situation arising in this case. See §4.4 for a more precise statement of this conjecture.

Our main purpose in this paper is to illustrate a different method for the computation of $d_{n,r,S}$, based on Fulton-MacPherson intersection theory. (The precise definition we will

adopt for $d_{n,r,S}$ is given in §2.2; it is the degree of $\overline{\pi_S(\sigma_{n,r})}$ when the projection is birational to its image.) The computation is reduced to the computation of the *Segre class* of the intersection $L_S \cap \sigma_{n,r}$ in $\sigma_{n,r}$, and this in turn may be expressed in terms of a Segre class involving the standard resolution of $\sigma_{n,r}$. Computing these Segre classes in complete generality appears to be a very challenging problem, and it would be worthwhile, for this application as well as other applications of Segre classes, to develop techniques capable of performing such computations. We carry this out for sets S which can be constructed from ‘blocks’ of certain types. For example, we give a formula for the case in which no two elements of S lie in the same column or row of the matrix. In particular, this verifies the formula for diagonal S mentioned above, thereby (indirectly) proving the conjectural statement from [GHIL13] on which that depends. Among other cases we consider, we include some for which entries of S may lie on the same row and the same column as other entries in S . However, the complexity of the formulas increases rapidly. The largest case covered by our results consists of the ‘thickened’ diagonal:



See §5.6 to get a sense of how involved the formulas become, even for such comparatively mild sets S . In general, we can deal with sets S consisting (up to permutations) of blocks with arbitrarily many entries in a row or a column, three adjacent entries, and 2×2 squares.



The Segre class computations needed to deal with these cases are streamlined by the use of a ‘twisted’ version of Segre classes (Definition 2.8) closely related to a class introduced by Steven Kleiman and Anders Thorup ([KT96]). The needed facts about this notion are summarized in Lemmas 2.10, 4.2, and 4.3. The tool that allows us to combine blocks of entries into larger sets S stems from Lemma 4.3, which is a variation on a formula for Segre classes of *splayed intersections* from [AF], and a formula for joins, Lemma 4.2, which is an ingredient in the proof (given elsewhere) of the rationality of a ‘zeta function’ recording the behavior of Segre classes of ind-limits of projective schemes ([Alu]).

For any set S of entries in an $n \times n$ matrix, and any rank r , we define by means of the twisted Segre class a class $\Sigma_{n,r,S}$ in the Chow group of the Grassmannian $G(n-r, n)$, which carries the information of the degree $d_{n,r,S}$. This is the *Grassmann class* of S w.r.t. r , Definition 2.12. We prove that if S consists of two blocks S_1, S_2 (in the sense specified in §4.1), then the corresponding Grassmann classes satisfy a multiplicativity property:

$$(1 - \Sigma_{n,r,S}) = (1 - \Sigma_{n,r,S'}) (1 - \Sigma_{n,r,S''})$$

(Theorem 4.5). This multiplicative structure organizes the numbers $d_{n,r,S}$ to some extent, but is only visible at the level of the corresponding Grassmann classes. Analyzing the structure of these classes further appears to be an interesting project.

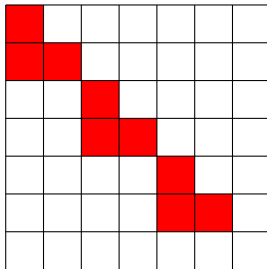


FIGURE 1

We compute $\Sigma_{n,r,S}$ explicitly for the elementary blocks listed above. Once $\Sigma_{n,r,S}$ is known for a set S , the degree $d_{n,r,S}$ may be computed as the intersection product $\int c(\mathcal{S}^\vee)^n \cap \Sigma_{n,r,S}$ in the Grassmannian (Theorem 2.13). We express our results in terms of these intersection numbers; the Macaulay2 package Schubert2 ([GSS⁺]) can perform such computations at impressive speed. (Use of both Macaulay2 and Schubert2 was essential to us in this work.) Of course Macaulay2 ([GS]) is capable of performing the degree computation directly; however, computation times become prohibitively long via such direct methods when the matrices reach even modest size. Computations performed by implementing the results given here are much faster. For example, let S consist of six diagonal and three subdiagonal entries in a 7×7 matrix (Figure 1). Computing the degree of the projection of $\sigma_{7,2}$ (which happens to equal 13395; this is a dimension 23 subvariety of \mathbb{P}^{39}) took us about 20 minutes by a ‘direct’ method using Macaulay2 on a quad-core ($4 \times 3.4\text{GHz}$) computer, and about .004 seconds on the same equipment by implementing in Schubert2 the formulas proven here. (This example is covered by the formula given in Theorem 5.8.) Thus, the results proven here allow for much more extensive experimentation with these numbers.

It would be interesting to perform the intersection products explicitly and obtain closed formulas for $d_{n,r,S}$ for all the cases considered here. We do this in some cases, for example, when no two elements of S are in the same row or column; this is the case relevant to the conjecture from [GHIL13] mentioned above.

After compiling the results of many explicit computations, we noticed that many of the numbers $d_{n,r,S}$ appear in the existing literature in contexts far removed from the enumerative geometry of rank conditions; we owe this surprising (to us) observation to perusal of [Slo14]. For example, there is a persistent appearance of series reproducing the numbers of *Kekulé structures* of many different types of *benzenoid hydrocarbons* ([CG88]). We ignore whether the combinatorial literature on these numbers links them directly to intersection products in Grassmannian. It would seem worthwhile to establish such a direct connection, and understand why these numbers of interest in chemistry should arise as degrees of projections of rank loci. The numbers also appear in other contexts, for reasons that are equally mysterious to us. See §3.5, §4.6, Remark 5.7, Example 5.11.

It would be straightforward to extend the results presented here to rank loci in the space \mathbb{P}^{mn-1} of $m \times n$ matrices.

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2. PRELIMINARIES

2.1. We work over an algebraically closed field κ . We identify the space of $n \times n$ matrices with entries in κ , up to scalars, with \mathbb{P}^{n^2-1} ; the entries of the matrices are homogeneous coordinates in this projective space. For a set S of $s \geq 1$ entries, L_S will denote the linear subspace of \mathbb{P}^{n^2-1} spanned by S .

We let $\sigma_{n,r} \subseteq \mathbb{P}^{n^2-1}$ denote the set of $n \times n$ matrices of rank $\leq r$. It is often notationally more convenient to work with the corank $k = n - r$; we will let $\tau_{n,k} = \sigma_{n,n-k}$. Thus, $\tau_{n,k}$ consists of the set of $n \times n$ matrices A such that $\dim \ker A \geq k$. The algebraic set $\tau_{n,k}$ is defined (with its reduced structure) by the ideal generated by the $n - k + 1$ minors of $A \in \mathbb{P}^{n^2-1}$. A simple dimension count shows that $\dim \tau_{n,k} = n^2 - k^2 - 1$. Also,

$$(1) \quad \deg \tau_{n,k} = \prod_{i=0}^{k-1} \frac{\binom{n+i}{k}}{\binom{k+i}{k}}$$

(Example 14.4.14 in [Ful84]).

The subspace L_S defines a projection $\pi_S : \mathbb{P}^{n^2-1} \rightarrow \mathbb{P}^{n^2-s-1}$. The closure of $\pi_S(\tau_{n,k})$ is an irreducible subvariety $\tau_{n,k,S}$ of \mathbb{P}^{n^2-s-1} . Briefly stated, our goal is the computation of the degree of this subvariety. We begin by formulating this task more precisely.

2.2. Let $T_{n,k,S}$ be any variety mapping properly birationally to $\tau_{n,k}$ and resolving the indeterminacies of the restriction of π_S to $\tau_{n,k}$:

$$\begin{array}{ccccc} \mathbb{P}^{n^2-1} & \xleftarrow{\quad} & \tau_{n,k} & \xrightarrow{\quad \pi_S|_{\tau_{n,k}} \quad} & \tau_{n,k,S} & \xrightarrow{\quad} & \mathbb{P}^{n^2-s-1} \\ & & \swarrow & & \nearrow & & \\ & & T_{n,k,S} & & & & \end{array}$$

For example, we can take $T_{n,k,S}$ to be the blow-up of $\tau_{n,k}$ along the intersection $L_S \cap \tau_{n,k}$. We let $\tilde{\pi}_S$ be the induced regular map $T_{n,k,S} \rightarrow \tau_{n,k,S}$. Note that the general fiber of $\tilde{\pi}_S$ is finite if and only if so is the fiber of $\tau_{n,k,S}$ (away from L_S), and $\deg \tilde{\pi}_S$ equals the number of points in the general fiber of $\pi_S|_{\tau_{n,k}}$ and not on L_S in that case.

Let h be the hyperplane class in \mathbb{P}^{n^2-s-1} .

Definition 2.1. We let

$$d_{n,r,S} := \int_{T_{n,k,S}} (\tilde{\pi}_S^* h)^{n^2-k^2-1}$$

where $k = n - r$.

Lemma 2.2. *The number $d_{n,r,S}$ equals $f \cdot \deg \tau_{n,k,S}$, where f is the degree of $\tilde{\pi}_S$ (and $f = 0$ if the general fiber of $\tilde{\pi}_S$ has positive dimension). In particular, $d_{n,r,S}$ is independent of the chosen variety $T_{n,k,S}$.*

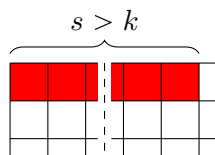
Proof. The first point is immediate from the definition, and the second point follows since f (if $\neq 0$) equals the number of points in a general fiber of $\pi_S|_{\tau_{n,k}}$ in the open set $\mathbb{P}^{n^2-1} \setminus L_S$. Alternately, the second point follows from the projection formula, since two resolutions $T_{n,k,S}$ are dominated by a third one. \square

Remark 2.3. The number $d_{n,r,S}$ is evidently invariant under permutations of the rows or columns of the matrix. \square

Our main goals in this note are to present a method for the computation of $d_{n,r,S}$ and to apply this method to several concrete cases. In principle the method can be applied to more general situations than those that will be considered explicitly here, but the complexity of the needed calculations increases rapidly.

Remark 2.4. When $\tilde{\pi}_S$ is birational onto its image $\tau_{n,k,S}$, then $d_{n,r,S}$ simply equals the degree of $\tau_{n,k,S}$ as a subvariety of \mathbb{P}^{n^2-s-1} . This degree also equals the degree of the cone over $\tau_{n,k,S}$ with vertex L_S , that is, the *join* of L_S and $\tau_{n,k}$. The language of joins is preferred in [GHIL13], and is essentially equivalent to the one used here. \square

Remark 2.5. There are of course situations in which $d_{n,r,S}$ does *not* equal the degree of $\tau_{n,k,S}$. For example, if the general fiber of $\pi_S|_{\tau_{n,k}}$ is positive dimensional, then $\dim \tau_{n,k,S} < n^2 - k^2 - 1$, and $d_{n,r,S} = 0$. This is necessarily the case if $s > k^2$, but can occur in more interesting situations. For example, the general fiber of $\pi_S|_{\tau_{n,k}}$ is positive dimensional for S consisting of $s > k = n - r$ entries in one row:



This could be verified by geometric means, but it is an immediate consequence of the formulas obtained in this note, since $d_{n,r,S}$ will turn out to vanish in this case (Example 3.5). Thus, $d_{n,r,S}$ (as defined in Definition 2.1) carries interesting geometric information, and we choose to focus on it rather than on the degree of $\tau_{n,k,S}$ proper. \square

Remark 2.6. It may also occur that the general fiber of $\tilde{\pi}_S$ has dimension zero, but degree larger than 1. For example, our formulas will give that $d_{7,4,S} = 2$ for the ladder configuration shown in Figure 1. In this case $s = 9 = k^2$, so the projection of $\sigma_{7,4}$ must be generically 2-to-1 onto $\mathbb{P}^{n^2-s-1} = \mathbb{P}^{39}$. Again this information is immediately accessible from $d_{7,4,S} = 2$ (and not from $\deg \tau_{7,4,S} = 1$). For an instance of the same phenomenon in which the projection is not dominant, see Example 4.11.

It would be interesting to have a criterion indicating when $\deg \tilde{\pi}_S = 1$. According to Lemma 5.4.1 in [GHIL13], this is the case if e.g. S is a subset of the diagonal and $s < (n-r)^2$.

Even for diagonal S , the degree of $\deg \tilde{\pi}_S$ is not 1 in general if $s = (n-r)^2$: for example, it is 2 for $r = 2$, $n = s = 2^2 = 4$ and 42 for $r = 6$, $n = s = 3^2 = 9$ (since $d_{4,2,S} = 2$, $d_{9,6,S} = 42$ if S is the full diagonal, see Example 4.9). In fact, numerical experimentation indicates that this degree is *independent of n* for S consisting of $(n-r)^2$ diagonal elements, a fact for which we do not have a proof. \square

2.3. We will treat the problem of computing $d_{n,r,S}$ as a problem of *excess intersection*. Let $N = \dim \tau_{n,k} = n^2 - k^2 - 1$. The intersection of N general hyperplanes h_1, \dots, h_N in \mathbb{P}^{n^2-s-1} meets $\tau_{n,k,S}$ at reduced points by Bertini, and it follows that $d_{n,r,S}$ equals the number of points of intersection of $\tilde{\pi}_S^{-1}(h_1), \dots, \tilde{\pi}_S^{-1}(h_N)$. The hyperplanes h_i correspond to general hyperplanes H_i of \mathbb{P}^{n^2-1} containing L_S , and it follows that the intersection

$$H_1 \cap \dots \cap H_N \cap \tau_{n,k}$$

consists of $d_{n,r,S}$ isolated reduced points and of a scheme supported on $L_S \cap \tau_{n,k}$. The intersection number

$$(2) \quad \int H_1 \cdots H_N \cdot \tau_{n,k}$$

equals $\deg \tau_{n,k}$ (given in (1)); therefore

$$d_{n,r,S} = \prod_{i=0}^{k-1} \frac{\binom{n+i}{k}}{\binom{k+i}{k}} - \text{contribution of } (L_S \cap \tau_{n,k}) \text{ to the intersection product (2)}.$$

The challenge consists therefore of evaluating the contribution of the (*a priori* positive dimensional) locus $L_S \cap \tau_{n,k}$ to the intersection product of $\tau_{n,k}$ and N general hyperplanes containing L_S . By Fulton-MacPherson intersection theory, this contribution may be expressed in terms of a Segre class.

Lemma 2.7. *Let H be the class of a hyperplane in \mathbb{P}^{n^2-1} . Then*

$$(3) \quad d_{n,r,S} = \prod_{i=0}^{k-1} \frac{\binom{n+i}{k}}{\binom{k+i}{k}} - \int (1 + H)^{n^2-k^2-1} \cap s(L_S \cap \tau_{n,k}, \tau_{n,k}) \quad .$$

Here $s(L_S \cap \tau_{n,k}, \tau_{n,k})$ is the *Segre class* in the sense of [Ful84], Chapter 4. In (3), $L_S \cap \tau_{n,k}$ is taken with its natural (not necessarily reduced!) scheme structure, i.e., the one given by the sum of the ideals of L_S and $\tau_{n,k}$.

Proof. Notice that $L_S \cap \tau_{n,k}$ is the base scheme of the linear system cut out on $\tau_{n,k}$ by the hyperplanes containing L_S . Then apply Proposition 4.4 in [Ful84]. \square

The problem shifts then to the computation of the Segre class $s(L_S \cap \tau_{n,k}, \tau_{n,k})$.

2.4. In order to obtain more manageable formulas, we will use Lemma 2.7 in a slightly modified version, ‘twisting’ the needed Segre class by means of an operation introduced in [Alu94].

Definition 2.8. Let Z be a subscheme of a variety V , and let \mathcal{L} be a line bundle on Z . We let

$$(4) \quad s(Z, V)^{\mathcal{L}} := c(\mathcal{L})^{-1} \cap (s(Z, V) \otimes_V \mathcal{L}) \quad .$$

The notation used in the right-hand side of (4) is defined in §2 of [Alu94], to which we address the reader for further details (which will be immaterial here). In practice, $s(Z, V)^{\mathcal{L}}$ is obtained by capping by $c(\mathcal{L})^{-(c+1)}$ the piece of $s(Z, V)$ of codimension c in V , for all c .

Remark 2.9. The class introduced in Definition 2.8 is closely related to the *twisted Segre class* studied in [KT96].

We collect here a few facts concerning this notion, that we will need in the next sections.

Lemma 2.10. (1) *If Z is regularly embedded in V , then*

$$s(Z, V)^{\mathcal{L}} = c(\mathcal{L})^{-1} c(N_Z V \otimes \mathcal{L})^{-1} \cap [Z] \quad .$$

(2) *If $\pi : \tilde{V} \rightarrow V$ is a proper birational morphism, and $\tilde{\mathcal{L}}$ is the pull-back to $\pi^{-1}(Z)$ of the line bundle \mathcal{L} , then*

$$s(Z, V)^{\mathcal{L}} = \pi_* \left(s(\pi^{-1}(Z), \tilde{V})^{\tilde{\mathcal{L}}} \right) \quad .$$

(3) *If V is nonsingular, then the class*

$$c(TV|_Z \otimes \mathcal{L}) \cap s(Z, V)^{\mathcal{L}}$$

only depends on Z and \mathcal{L} .

(4) *With notation as in §2.3,*

$$d_{n,r,S} = \prod_{i=0}^{k-1} \frac{\binom{n+i}{k}}{\binom{k+i}{k}} - \int s(L_S \cap \tau_{n,k}, \tau_{n,k})^{\mathcal{O}(-H)} \quad .$$

Full proofs of these facts will be given elsewhere. (1), (2) are formal consequence of the definition, the standard formula for Segre classes of regularly embedded schemes (see Chapter 4 of [Ful84]), and of the birational invariance of Segre classes (Proposition 4.2 (a) in [Ful84]). (3) follows from Example 4.2.6 (a) in [Ful84]. (4) is again a formal consequence of the definition and of Lemma 2.7.

By (4), the problem of computing $d_{n,r,S}$ is reduced to the computation of the class $s(L_S \cap \tau_{n,k}, \tau_{n,k})^{\mathcal{O}(-H)}$. We will use (2) to relate this computation to one in an ambient nonsingular variety and (3) to change this ambient nonsingular variety to a more convenient one. In particularly simple situations, (1) will suffice to compute the needed class.

We will need somewhat more sophisticated facts concerning $s(Z, V)^{\mathcal{L}}$ in §4.1 (Lemma 4.2, Lemma 4.3).

2.5. By Lemma 2.10 (4), our objective consists of the computation of $s(L_S \cap \tau_{n,k}, \tau_{n,k})^{\mathcal{O}(-H)}$; by (2) in the same lemma, we may replace $\tau_{n,k}$ with any birational model $\hat{\tau}_{n,k}$. We choose for $\hat{\tau}_{n,k}$ the standard resolution of $\tau_{n,k}$ as a projective bundle over a Grassmannian.

Briefly, $\hat{\tau}_{n,k}$ consists of pairs $(K, \varphi) \in G(k, n) \times \mathbb{P}^{n^2-1}$ where $K \in G(k, n)$ is a k -dimensional subspace of κ^n , and φ is a matrix such that $K \subseteq \ker \varphi$ (and hence $\text{rk } \varphi \leq r = n - k$). More intrinsically, consider the tautological sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}^n \longrightarrow \mathcal{Q} \longrightarrow 0$$

over the Grassmannian $G(k, n)$. The space $\text{Hom}(\kappa^n, \kappa^n)$ of matrices defines the trivial bundle $\text{Hom}(\mathcal{O}^n, \mathcal{O}^n)$ over $G(k, n)$. For $K \in G(k, n)$, the homomorphisms whose kernel contains K correspond to those $\varphi : \kappa^n \rightarrow \kappa^n$ that are induced from a homomorphism on the quotient κ^n/K . Globalizing, the $\mathbb{P}^{n(n-k)-1}$ -bundle $\hat{\tau}_{n,k}$ on $G(k, n)$ whose fiber over K parametrizes matrices φ for which $K \subseteq \ker \varphi$ is the projectivization of

$$\text{Hom}(\mathcal{Q}, \mathcal{O}^n) \cong (\mathcal{Q}^\vee)^n \quad .$$

The embedding $\mathcal{Q}^\vee \subseteq \mathcal{O}^n$ induces an inclusion

$$\hat{\tau}_{n,k} \subseteq \mathbb{P}\mathcal{O}^{n^2} = G(k, n) \times \mathbb{P}^{n^2-1} \quad ;$$

the projection to \mathbb{P}^{n^2-1} maps $\hat{\tau}_{n,k}$ onto $\tau_{n,k}$:

$$\begin{array}{ccccc} & & \hat{\tau}_{n,k} \hookrightarrow G(k, n) \times \mathbb{P}^{n^2-1} & & \\ & \rho \swarrow & \searrow \nu & \searrow & \\ G(k, n) & & \tau_{n,k} \hookrightarrow \mathbb{P}^{n^2-1} & & \end{array}$$

Both $\tau_{n,k}$ and $\hat{\tau}_{n,k}$ are projective, and every $\varphi \in \tau_{n,k}$ with rank *exactly* $r = n - k$ has a well-defined kernel $K \in G(k, n)$; thus $\nu : \hat{\tau}_{n,k} \rightarrow \tau_{n,k}$ is a proper birational morphism, resolving the singularities of $\tau_{n,k} = \sigma_{n,r}$. (In particular $\dim \tau_{n,k} = \dim G(k, n) + n(n - k) - 1 = k(n - k) + n(n - k) - 1 = n^2 - k^2 - 1$ as stated in §2.1.)

As in the diagram, we let $\rho : \hat{\tau}_{n,k} = \mathbb{P}((\mathcal{Q}^\vee)^n) \rightarrow G(k, n)$ be the structure map. Note that the universal bundle $\mathcal{O}(1)$ on $\mathbb{P}((\mathcal{Q}^\vee)^n)$ is the restriction of the hyperplane bundle $\mathcal{O}(H)$ from the second factor in $G(k, n) \times \mathbb{P}^{n^2-1}$.

The following observation will be useful in concrete calculations:

Lemma 2.11.

$$\deg \sigma_{n,r} = \deg \tau_{n,k} = \prod_{i=0}^{k-1} \frac{\binom{n+i}{k}}{\binom{k+i}{k}} = \int_{G(k,n)} c(\mathcal{S}^\vee)^n \quad .$$

Proof. The numerical value for the degree was already recalled in §2.1. To see that this is the degree of $c(\mathcal{S}^\vee)^n$, we may argue as follows. By definition,

$$\deg \tau_{n,k} = \int H^{\dim \tau_{n,k}} \cap [\tau_{n,k}] = \int \sum_i H^i \cap [\tau_{n,k}] \quad ;$$

by the projection formula (applied to ν), this equals

$$\int \sum_i c_1(\mathcal{O}(1))^i \cap [\mathbb{P}((\mathcal{Q}^\vee)^n)] \quad ,$$

since as observed above H pulls back to $c_1(\mathcal{O}(1))$ on $\hat{\tau}_{n,k} = \mathbb{P}((\mathcal{Q}^\vee)^n)$. Since degrees are preserved by proper push-forwards, the degree in turn equals

$$\int \rho_* \left(\sum_i c_1(\mathcal{O}(1))^i \cap [\mathbb{P}((\mathcal{Q}^\vee)^n)] \right) = \int s((\mathcal{Q}^\vee)^n) \cap [G(k,n)]$$

by definition of Segre class of a vector bundle. By the Whitney formula applied to the (dual of the) tautological sequence we have $s(\mathcal{Q}^\vee) = c(\mathcal{S}^\vee)$, and this concludes the proof. \square

2.6. Applying Lemma 2.10 (2) and (4), we have

$$(5) \quad d_{n,r,S} = \prod_{i=0}^{k-1} \frac{\binom{n+i}{k}}{\binom{k+i}{k}} - \int s(\nu^{-1}(L_S \cap \tau_{n,k}), \hat{\tau}_{n,k})^{\mathcal{O}(-H)}$$

where of course H denotes the pull-back of the hyperplane class to $\hat{\tau}_{n,k}$.

Now let $\hat{L}_S = G(k,n) \times L_S \cong G(k,n) \times \mathbb{P}^{s-1}$, and note that $\nu^{-1}(L_S \cap \tau_{n,k}) = \hat{L}_S \cap \hat{\tau}_{n,k}$.

$$\begin{array}{ccccc} \hat{\tau}_{n,k} & \hookrightarrow & G(k,n) \times \mathbb{P}^{n^2-1} & \longleftarrow & \hat{L}_S = G(k,n) \times L_S \\ \rho \swarrow & & \nu \searrow & & \searrow \\ G(k,n) & & \tau_{n,k} & \hookrightarrow & \mathbb{P}^{n^2-1} \longleftarrow L_S \end{array}$$

Denote by $\pi_1 : \hat{L}_S \rightarrow G(k,n)$ the projection. We note that the push-forward π_{1*} acts by reading off the coefficient of H^{s-1} . The information needed in order to compute the degrees $d_{n,r,S}$ is captured by the following class in the Chow group of the Grassmannian $G(k,n)$.

Definition 2.12. The *Grassmann class* $\Sigma_{n,r,S}$ associated with S (w.r.t. r) is the push-forward

$$\Sigma_{n,r,S} := \pi_{1*} \left(s(\hat{L}_S \cap \hat{\tau}_{n,k}, \hat{L}_S)^{\mathcal{O}(-H)} \right)$$

of the ‘twisted’ Segre class $s(\hat{L}_S \cap \hat{\tau}_{n,k}, \hat{L}_S)^{\mathcal{O}(-H)}$, where $k = n - r$.

Note that we are taking the Segre class *in* \hat{L}_S in this definition (rather than *in* $\hat{\tau}_{n,k}$). The following result will be our main tool in the rest of the paper.

Theorem 2.13. *With notation as above,*

$$(6) \quad d_{n,r,S} = \int_{G(n-r,n)} c(\mathcal{S}^\vee)^n \cap (1 - \Sigma_{n,r,S}) \quad .$$

Proof. By Lemma 2.10 (3),

$$s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{\tau}_{n,k})^{\mathcal{O}(-H)} = \frac{c(T\widehat{L}_S \otimes \mathcal{O}(-H))}{c(T\widehat{\tau}_{n,k} \otimes \mathcal{O}(-H))} \cap s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)}$$

The Chern classes of both bundles appearing on the right are evaluated by standard Euler sequences. Since \widehat{L}_S is a trivial \mathbb{P}^{s-1} -bundle over $G(k, n)$, we have the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(H)^s \longrightarrow T\widehat{L}_S \longrightarrow TG(k, n) \longrightarrow 0$$

(omitting the evident pull-back). Tensoring by $\mathcal{O}(-H)$ we get the exact sequence

$$0 \longrightarrow \mathcal{O}(-H) \longrightarrow \mathcal{O}^s \longrightarrow T\widehat{L}_S \otimes \mathcal{O}(-H) \longrightarrow TG(k, n) \otimes \mathcal{O}(-H) \longrightarrow 0$$

from which

$$c(T\widehat{L}_S \otimes \mathcal{O}(-H)) = c(\mathcal{O}(H))^{-1} c(TG(k, n) \otimes \mathcal{O}(-H)) \quad .$$

Tensoring the Euler sequence for $\mathbb{P}((\mathcal{Q}^\vee)^n)$ by $\mathcal{O}(-H)$ (and omitting pull-backs and restrictions by a common abuse of language) gives likewise

$$0 \longrightarrow \mathcal{O}(-H) \longrightarrow (\mathcal{Q}^\vee)^n \longrightarrow T\widehat{\tau}_{n,k} \otimes \mathcal{O}(-H) \longrightarrow TG(k, n) \otimes \mathcal{O}(-H) \longrightarrow 0$$

from which

$$c(T\widehat{\tau}_{n,k} \otimes \mathcal{O}(-H)) = c(\mathcal{O}(H))^{-1} c(\mathcal{Q}^\vee)^n c(TG(k, n) \otimes \mathcal{O}(-H)) \quad .$$

Therefore

$$\frac{c(T\widehat{L}_S \otimes \mathcal{O}(-H))}{c(T\widehat{\tau}_{n,k} \otimes \mathcal{O}(-H))} = \frac{1}{c(\mathcal{Q}^\vee)^n} = c(\mathcal{S}^\vee)^n \quad ,$$

and (5) gives

$$d_{n,r,S} = \prod_{i=0}^{k-1} \frac{\binom{n+i}{k}}{\binom{k+i}{k}} - \int_{\widehat{L}_S} c(\mathcal{S}^\vee)^n \cap s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)}$$

Since \int is not affected by proper push-forwards,

$$d_{n,r,S} = \prod_{i=0}^{k-1} \frac{\binom{n+i}{k}}{\binom{k+i}{k}} - \int_{G(k,n)} c(\mathcal{S}^\vee)^n \cap \pi_{1*} \left(s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)} \right)$$

and by Lemma 2.11 we then have

$$d_{n,r,S} = \int_{G(k,n)} c(\mathcal{S}^\vee)^n \left(1 - \pi_{1*} \left(s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)} \right) \right) \quad ,$$

which is the statement. \square

In view of expression (6) obtained in Theorem 2.13, the key ingredient in the calculation of $d_{n,r,S}$ in our approach is the computation of the Grassmann class $\Sigma_{n,r,S}$ determined by the choice of n, r, S . In the rest of the paper we will compute this class explicitly in several template cases, and prove a result relating $\Sigma_{n,r,S}$ to the Grassmann classes of smaller sets, provided these form ‘blocks’ for S . Once $\Sigma_{n,r,S}$ is obtained, tools such as Schubert2 can perform the degree computation required by (6) and obtain numerical values for $d_{n,r,S}$ in many cases of reasonable size. In some cases (particularly when S is ‘diagonal’) we will be able to perform the degree computation explicitly in general.

dualizing, we get a morphism $\mathcal{O}^n \rightarrow \mathcal{O}(1) = \mathcal{O}(H)$. Let T be the kernel of this morphism, so with notation as above, the hyperplane T_φ is the fiber of T over φ , and $\widehat{L}_S \cap \widehat{\tau}_{k,n}$ is the Grassmann bundle $G(k, T)$ over $\mathbb{P}^{\ell-1}$. By construction, $G(k, T)$ is the zero-scheme of the section of $\mathcal{S}^\vee \otimes \mathcal{O}(H) = \text{Hom}(\mathcal{S}, \mathcal{O}(H))$ determined by composition:

$$\mathcal{S} \longrightarrow \mathcal{O}^n \longrightarrow \mathcal{O}(H) \quad .$$

This section is regular, as the above local coordinate computation shows. \square

Corollary 3.2. *Let S consist of ℓ entries in a row. We have*

$$s(\widehat{L}_S \cap \widehat{\tau}_{k,n}, \widehat{L}_S)^{\mathcal{O}(-H)} = c(\mathcal{O}(-H))^{-1} c(\mathcal{S}^\vee)^{-1} \cap [\widehat{L}_S \cap \widehat{\tau}_{k,n}] \quad .$$

As a class in $\widehat{L}_S = G(k, n) \times \mathbb{P}^{\ell-1}$,

$$(8) \quad s(\widehat{L}_S \cap \widehat{\tau}_{k,n}, \widehat{L}_S)^{\mathcal{O}(-H)} = c(\mathcal{O}(-H))^{-1} c(\mathcal{S}^\vee)^{-1} c_{\text{top}}(\mathcal{S}^\vee \otimes \mathcal{O}(H)) \quad .$$

Proof. By Lemma 3.1, $\widehat{L}_S \cap \widehat{\tau}_{k,n}$ is regularly embedded in \widehat{L}_S , with class $c_{\text{top}}(\mathcal{S}^\vee \otimes \mathcal{O}(H))$ and normal bundle $\mathcal{S}^\vee \otimes \mathcal{O}(H)$. The statement then follows from Lemma 2.10 (1). \square

Corollary 3.3. *Let S consist of ℓ entries in a row. Then*

$$(9) \quad \Sigma_{n,r,S} = c(\mathcal{S}^\vee) (c_{n-r-\ell+1}(\mathcal{S}^\vee) + \cdots + c_{n-r}(\mathcal{S}^\vee)) \quad .$$

Proof. By definition, $\Sigma_{n,r,S} = \pi_{1*} s(\widehat{L}_S \cap \widehat{\tau}_{k,n}, \widehat{L}_S)^{\mathcal{O}(-H)}$. Recall that π_1 denotes the projection $\widehat{L}_S \rightarrow G(k, n)$, and it acts by collecting the coefficient of $H^{\ell-1}$. We have

$$(10) \quad c_{\text{top}}(\mathcal{S}^\vee \otimes \mathcal{O}(H)) = \sum_{i=0}^k H^i \cdot c_{k-i}(\mathcal{S}^\vee)$$

(p. 55 in [Ful84]), and it follows that the coefficient of $H^{\ell-1}$ in

$$c(\mathcal{O}(-H))^{-1} c_{\text{top}}(\mathcal{S}^\vee \otimes \mathcal{O}(H)) = \sum_{j \geq 0} H^j \cdot \sum_{i=0}^k H^i \cdot c_{k-i}(\mathcal{S}^\vee)$$

equals $c_k(\mathcal{S}^\vee) + \cdots + c_{k-\ell+1}(\mathcal{S}^\vee)$. Therefore

$$\begin{aligned} \Sigma_{n,r,S} &= \pi_{1*} (c(\mathcal{S}^\vee)^{-1} c(\mathcal{O}(-H))^{-1} \cap c_{\text{top}}(\mathcal{S}^\vee \otimes \mathcal{O}(H))) \\ &= c(\mathcal{S}^\vee)^{-1} (c_{k-\ell+1}(\mathcal{S}^\vee) + \cdots + c_k(\mathcal{S}^\vee)) \end{aligned}$$

as stated. \square

3.2. The information obtained in Corollary 3.3 suffices by Theorem 2.13 to complete the computation of $d_{n,r,S}$. Here S consists of ℓ entries in a row; we denote the number $d_{n,r,S}$ by $d_{n,r|\ell}$, since the row and the location of the entries in that row are immaterial.

Theorem 3.4. *With notation as above,*

$$(11) \quad d_{n,r|\ell} = \int_{G(n-r,n)} c(\mathcal{S}^\vee)^{n-1} \sum_{i=0}^{n-r-\ell} c_i(\mathcal{S}^\vee) \quad .$$

Proof. Corollary 3.3 yields

$$\begin{aligned} c(\mathcal{S}^\vee)^n (1 - \Sigma_{n,r,S}) &= c(\mathcal{S}^\vee)^n (1 - c(\mathcal{S}^\vee)^{-1} \cap (c_{k-\ell+1}(\mathcal{S}^\vee) + \cdots + c_k(\mathcal{S}^\vee))) \\ &= c(\mathcal{S}^\vee)^{n-1} (c(\mathcal{S}^\vee) - (c_{k-\ell+1}(\mathcal{S}^\vee) + \cdots + c_k(\mathcal{S}^\vee))) \\ &= c(\mathcal{S}^\vee)^{n-1} (c_0(\mathcal{S}^\vee) + \cdots + c_{k-\ell}(\mathcal{S}^\vee)) \quad , \end{aligned}$$

which gives the statement by Theorem 2.13. \square

Example 3.5. • $d_{n,r|0} = \int_{G(k,n)} c(\mathcal{S}^\vee)^n = \deg \sigma_{n,r}$ (Lemma 2.11).

- For $\ell = 1$, (11) gives

$$\begin{aligned} \int_{G(n-r,n)} c(\mathcal{S}^\vee)^{n-1} \sum_{i=0}^{n-r-1} c_i(\mathcal{S}^\vee) &= \int_{G(n-r,n)} c(\mathcal{S}^\vee)^n - \int_{G(n-r,n)} c(\mathcal{S}^\vee)^{n-1} \cdot c_{n-r}(\mathcal{S}^\vee) \\ &= \int_{G(n-r,n)} c(\mathcal{S}^\vee)^n - \int_{G(n-r,n-1)} c(\mathcal{S}^\vee)^{n-1} \\ &= \deg \sigma_{n,r} - \deg \sigma_{n-1,r-1} \\ &= \prod_{i=0}^{n-r-1} \frac{\binom{n+i}{n-r}}{\binom{n-r+i}{n-r}} - \prod_{i=0}^{n-r-1} \frac{\binom{n-1+i}{n-r}}{\binom{n-r+i}{n-r}}. \end{aligned}$$

Therefore, the multiplicity of $\sigma_{n,r}$ at a matrix of rank 1 is the degree of $\sigma_{n-1,r-1}$. This is clear for e.g., the determinant hypersurface $\sigma_{n,n-1}$, since the tangent cone to $\sigma_{n,n-1}$ at a rank-1 matrix is a cone over the determinant $\sigma_{n-1,n-2}$. In general, this degree computation can be used to show that the tangent cone to $\sigma_{n,r}$ at a rank-1 matrix is a cone over $\sigma_{n-1,r-1}$.

- $d_{n,r|n-r} = \int_{G(n-r,n)} c(\mathcal{S}^\vee)^{n-1}$. This degree may also be evaluated explicitly, yielding $\prod_{i=0}^{n-r-2} \frac{\binom{n+i}{n-r}}{\binom{n-r+i}{n-r}}$.

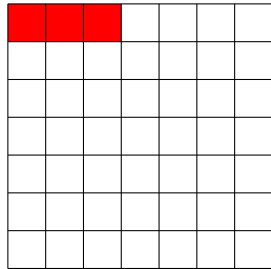
- For $\ell > n - r$, $d_{n,r|\ell} = 0$. Indeed, $\sum_{i=0}^{n-r-\ell} c_i(\mathcal{S}^\vee) = 0$ trivially in this case. This fact was already mentioned in Remark 2.5.

The following Macaulay2 script using Schubert2 ([GSS⁺]):

```
needsPackage("Schubert2")
onerow = (n,r,l) -> (
  G = flagBundle({n-r,r});
  (S,Q) = bundles G;
  Sd = dual S;
  integral(chern(Sd)^(n-1)*sum(0..n-r-1, i-> chern(i,Sd)))
)
```

produces a function `onerow` which evaluates (11) given the input of the dimension n of the ambient space, the rank r , and the number ℓ of entries of S (which must all be in the same row).

Example 3.6. The degrees of the joins of the locus of 7×7 matrices of rank $\leq r$ with the space spanned by 3 entries in the top row are as follows:



$$d_{7,r|3} = \begin{cases} 896 & r = 1 \\ 15582 & r = 2 \\ 11172 & r = 3 \\ 490 & r = 4 \\ 0 & r = 5 \\ 0 & r = 6 \\ 0 & r = 7 \end{cases}$$

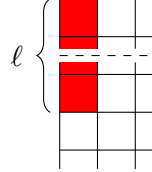
These may be computed quickly using the script given above. For example,

```
i2 : time onerow(7,3,3)
-- used 0.00669973 seconds
o2 = 11172
```

on our computing equipment. (By comparison, a verification of the table for $n = 7$, $\ell = 3$ shown above took nearly twenty minutes on the same computer, running a more straightforward algorithm for degrees of projections in Macaulay2.)

3.3. By symmetry, $d_{n,r,S} = d_{n,r,S^\dagger}$, where S^\dagger is the transpose configuration of entries from S . Thus, (11) must also be the degree of the projection of the rank- r locus from a set of entries lying in the same *column*. Performing an independent computation for this number from this different viewpoint leads to an identity in the Chow ring of the Grassmannian.

Let S consist of ℓ entries in one column, e.g.:



We use notation as above: L_S is the span of the entries of S , \widehat{L}_S denotes $G(k, n) \times L_S \cong G(k, n) \cong \mathbb{P}^{\ell-1}$, and $\widehat{\tau}_{n,k}$ is the standard resolution of $\tau_{n,k} = \sigma_{n,n-k}$. The Grassmann class $\Sigma_{n,r,S}$ (Definition 2.12) needed in order to apply Theorem 2.13 is obtained from $s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)}$, where H is the pull-back of the hyperplane class from the $\mathbb{P}^{\ell-1}$ factor.

Lemma 3.7. $\widehat{L}_S \cap \widehat{\tau}_{n,k} \cong G(k, n-1) \times \mathbb{P}^{\ell-1}$.

Proof. The intersection $\widehat{L}_S \cap \widehat{\tau}_{n,k}$ consists of pairs (K, φ) such that $K \subseteq \ker \varphi$. Since φ is concentrated in one column, $\ker \varphi$ consists of a fixed hyperplane, and the statement follows, at least set-theoretically. A simple coordinate computation shows that this description in fact holds scheme-theoretically. \square

Corollary 3.8. Let S consist of ℓ entries in a column. As a class in \widehat{L}_S ,

$$(12) \quad s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)} = c(\mathcal{O}(-H))^{-1} c(\mathcal{S}^\vee \otimes \mathcal{O}(-H))^{-1} c_k(\mathcal{S}^\vee) \quad .$$

Proof. This follows from Lemma 3.7 and Lemma 2.10 (1), since $G(k, n-1) \times \mathbb{P}^{\ell-1}$ is the zero-scheme of a regular section of \mathcal{S}^\vee . \square

The reader should compare (12) with the similar but different expression (8) for S concentrated in a row.

Corollary 3.9. Let S consist of ℓ entries in a column. Then

$$(13) \quad \Sigma_{n,r,S} = \sum_{i=0}^r \binom{\ell-1+n-r+i}{\ell-1} c_i(\mathcal{Q}^\vee) c_{n-r}(\mathcal{S}^\vee) \quad .$$

Proof. Tensoring the dual of the universal sequence over $G(k, n)$ by $\mathcal{O}(-H)$ gives an exact sequence

$$0 \longrightarrow \mathcal{Q}^\vee \otimes \mathcal{O}(-H) \longrightarrow \mathcal{O}(-H)^{n-1} \longrightarrow \mathcal{S}^\vee \otimes \mathcal{O}(-H) \longrightarrow 0$$

from which $c(\mathcal{S}^\vee \otimes \mathcal{O}(-H))^{-1} = (1-H)^{-n} c(\mathcal{Q}^\vee \otimes \mathcal{O}(-H))$. Substituting in (12) and applying the formula for the Chern class of a tensor product ([Ful84], p. 55) gives

$$s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)} = \sum_{i=0}^{n-k} \frac{c_i(\mathcal{Q}^\vee) c_k(\mathcal{S}^\vee)}{(1-H)^{k+i+1}} \quad .$$

The Grassmann class is obtained by pushing this forward to $G(k, n)$, which amounts to computing the coefficient of $H^{\ell-1}$, with the stated result. \square

Applying Theorem 2.13 gives the degree:

Theorem 3.10. *If S consists of ℓ elements in a column, then with $k = n - r$:*

$$(14) \quad d_{n,r,S} = \int_{G(k,n)} c(\mathcal{S}^\vee)^n \left(1 - \sum_{i=0}^r \binom{\ell-1+k+i}{\ell-1} c_i(\mathcal{Q}^\vee) c_k(\mathcal{S}^\vee) \right) .$$

3.4. As mentioned above, the expression obtained in Theorem 3.10 must agree with the expression given for $d_{n,r|\ell}$ in Theorem 3.4. In other words, if $\Sigma'_{n,r|\ell}$, $\Sigma''_{n,r|\ell}$ denote the classes (9), (13) obtained for S consisting of ℓ elements of a row, resp., a column, then necessarily

$$\int_{G(k,n)} c(\mathcal{S}^\vee)^n \cdot \Sigma'_{n,r|\ell} = \int_{G(k,n)} c(\mathcal{S}^\vee)^n \cdot \Sigma''_{n,r|\ell} .$$

This equality translates into identities in the Chow ring of the Grassmannian, for all choices of n, r, ℓ . They are nontrivial, in the sense that in general the classes $\Sigma'_{n,r|\ell}$, $\Sigma''_{n,r|\ell}$ differ. In fact, even the dimensions of these classes need not agree.

Example 3.11. For $n = 3, r = 1, \ell = 2$, $\Sigma'_{n,r|\ell}$ and $\Sigma''_{n,r|\ell}$ are classes in $G(2, 3) \cong \mathbb{P}^2$. The reader can verify that $\Sigma'_{3,1|2}$ is the hyperplane class h , while $\Sigma''_{3,1|2}$ equals $3h^2$.

$$\begin{array}{|c|c|c|} \hline \color{red}{\blacksquare} & \color{red}{\blacksquare} & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \quad \Sigma'_{3,1|2} = h \quad ; \quad \begin{array}{|c|c|c|} \hline \color{red}{\blacksquare} & \square & \square \\ \hline \color{red}{\blacksquare} & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \quad \Sigma''_{3,1|2} = 3h^2$$

With this notation, $c(\mathcal{S}^\vee)^3 = (1 + h + h^2)^3 = 1 + 3h + 6h^2$, so that

$$\int c(\mathcal{S}^\vee)^3 \cdot \Sigma'_{3,1|2} = 3 = \int c(\mathcal{S}^\vee)^3 \cdot \Sigma''_{3,1|2}$$

as expected. J

We can use the equality of degrees to obtain information on $d_{n,r|\ell}$. It is straightforward to write (14) in terms of an intersection degree in $G(k, n-1)$:

$$(15) \quad d_{n,r,S} = \prod_{i=0}^{k-1} \frac{\binom{n+i}{k}}{\binom{k+i}{k}} - \sum_{i=0}^{n-k-1} \binom{\ell-1+k+i}{\ell-1} \int_{G(k,n-1)} c(\mathcal{S}^\vee)^n c_i(\mathcal{Q}^\vee) .$$

where \mathcal{Q}' denotes the universal quotient bundle on $G(k, n-1)$. The expression in (15) must evaluate $d_{n,r|\ell}$, and hence agree with (11). Formula (15) and the degree computation

$$(16) \quad \int_{G(k,n-1)} c(\mathcal{S}^\vee)^n \cdot c_i(\mathcal{Q}') = \frac{\binom{n}{k+1} \cdots \binom{n+k-1}{k+1}}{\binom{k+1}{k+1} \cdots \binom{2k}{k+1}} \cdot \frac{\binom{n-k-1}{i} \binom{i+k-1}{i}}{\binom{2k+i}{i}} ,$$

(which is likely well-known) yield the following explicit expression for $d_{n,r|\ell}$:

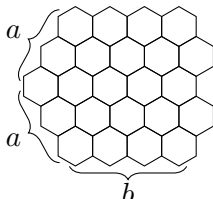
$$(17) \quad d_{n,r|\ell} = \prod_{j=0}^{k-1} \frac{\binom{n+j}{k}}{\binom{k+j}{k}} - \prod_{j=0}^{k-1} \frac{\binom{n+j}{k+1}}{\binom{k+1+j}{k+1}} \sum_{i=0}^{n-k-1} (-1)^i \binom{\ell-1+k+i}{\ell-1} \frac{\binom{n-k-1}{i} \binom{i+k-1}{i}}{\binom{2k+i}{i}}$$

for $\ell \leq k = n - r$. (The script given in §3.2 confirms this expression for dozens of examples.) A reader more versed in Schubert calculus than this writer could verify this formula rigorously (by checking (16)) or provide more directly an explicit expression computing (11).

3.5. Here is a table displaying $d_{a+b,a|b-2}$ for $a = 1, \dots, 10$ and $b = 2, 3$:

	$a = 1$	2	3	4	5	6	7	8	9	10
$b = 2$	6	20	50	105	196	336	540	825	1210	1716
3	19	155	805	3136	9996	27468	67320	150645	313027	611611

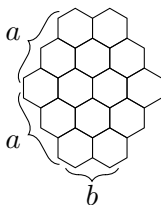
This table matches the one found on page 188 of [CG88] for the values of $K(D^j(a, b))$, the number of *Kekulé structures for benzenoid hydrocarbons* in an ‘oblate pentagon’ configuration:



(For $b > 3$, the values of $K(D^j(a, b))$ and $d_{a+b,a|b-2}$ as above do not match.) Several other series of numbers of Kekulé structures may be expressed as numbers $d_{n,r|\ell}$. For example, the second row in the above table is one of four nonzero possibilities for $d_{n,n-3|\ell}$:

	$n = 3$	4	5	6	7	8	9	10
$\ell = 0$	1	20	175	980	4116	14112	41580	108900
1	1	19	155	805	3136	9996	27468	67320
2	1	16	110	490	1666	4704	11592	25740
3	1	10	50	175	490	1176	2520	4950

These agree with the numbers of Kekulé structures for benzenoids forming ‘5-tier dihedral hexagons, oblate pentagons, intermediate pentagons’, and ‘4-tier centrosymmetric hexagons’ ([CG88], p. 167 no. 1, 2, 3, and p. 166 no. 1, respectively). We are not aware of any direct connection between results in the extensive literature on Kekulé structures and the enumerative geometry of varieties expressing rank conditions. Is every degree $d_{n,r,S}$ the number of Kekulé structures for a benzenoid hydrocarbon molecule? The degrees $\deg \sigma_{n,r}$ are all Kekulé numbers, for dihedral hexagons: with notation as on p. 108 of [CG88], $K\{O(a, b)\}$ equals $d_{a+b,b|0} = \deg \sigma_{a+b,b}$.



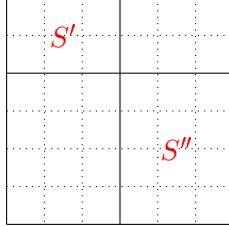
There are several other apparent cameo appearances of these numbers in the literature, which it would be interesting to explain more conceptually. The $\ell = 0$ row in the above table (i.e., $\deg \sigma_{n,n-3}$) reproduces the dimension $\dim V^{(n-3)}$ of the *distinguished module of dimension 20 of a Lie algebra in the subexceptional series* as per the $a = 2$ case in Theorem 7.2 of [LM06]. The $\ell = 3$ row agrees with the dimension $H_5(w)$ of the *space of semiinvariants of weight w , $k[\text{Mat}(2, 5)]_{(w)}^{SL(2)}$* , as computed in [Muk03], p. 238.

4. BUILDING S FROM BLOCKS

4.1. In this section we will use the information obtained in Corollary 3.3 and Corollary 3.9 to compute $d_{n,r,S}$ in the more interesting cases in which S is built from entries in separate

rows and columns. For example, this will give $d_{n,r,S}$ when S consists of a collection of entries (i_s, j_s) such that all j_s are distinct, i.e., such that no two entries in S are in the same column; this will in particular cover the case in which S is a subset of the diagonal of a matrix.

In general, assume S (possibly after a permutation of rows and columns) consists of two blocks S', S'' with no overlapping rows or columns:



We will prove a relation at the level of Grassmann classes:

$$(1 - \Sigma_{n,r,S}) = (1 - \Sigma_{n,r,S'}) (1 - \Sigma_{n,r,S''})$$

(Theorem 4.5) which will allow us to analyze the cases mentioned above; in fact, it will lead to a direct generalization of the results obtained in §3. This multiplicative property organizes the degrees $d_{n,r,S}$, but is invisible at the numerical level.

The proof of Theorem 4.5 relies on more sophisticated properties of Segre classes, which we state in this subsection.

First, we need to deal with a generalization of the join construction. Consider two disjoint subspaces $\mathbb{P}^m, \mathbb{P}^{M-m-1}$ of a projective space \mathbb{P}^M , and let I be a homogeneous ideal, defining a subscheme Z of \mathbb{P}^m . Let $\check{Z} \subseteq \mathbb{P}^M$ be the subscheme of \mathbb{P}^M defined by a set of generators of I . Geometrically, \check{Z} is the join of Z and \mathbb{P}^{M-m-1} in \mathbb{P}^M (i.e., the cone over Z with vertex \mathbb{P}^{M-m-1}).

Since Z and \mathbb{P}^{M-m-1} are disjoint, the projection of Z from \mathbb{P}^{M-m-1} is isomorphic to Z itself, and it follows that the degree of \check{Z} equals the degree Z . Segre classes are a more refined invariant than degree, so a relation between $s(Z, \mathbb{P}^m)$ and $s(\check{Z}, \mathbb{P}^M)$ is bound to be subtler. For example, the Segre class of \check{Z} depends on the specific ideal I chosen to define Z .

Example 4.1. Let $m = 1$ and $M = 2$, so we are dealing with the simple case of joining a point in the plane with a subscheme of a line. The ideals (x_0, x_1) and (x_0^2, x_0x_1) define the same subscheme Z in \mathbb{P}^1 (i.e., a reduced point); but the Segre classes of the corresponding joins, defined by (x_0, x_1) and (x_0^2, x_0x_1) in \mathbb{P}^2 , differ (due to the embedded component at the origin in the second case).

In fact we need to deal with a somewhat more general situation, for which Z is a subscheme of a product $V \times \mathbb{P}^m$. We assume that Z is defined by a section of a vector bundle $\mathcal{E} \otimes \mathcal{O}(H)$, where \mathcal{E} is the pull-back of a vector bundle from V , and H denotes the hyperplane class. (This condition will be automatically verified for all the examples we will consider.) For $M > m$, we let $\check{Z} = Z \vee (V \times \mathbb{P}^{M-m-1})$ be the subscheme of $V \times \mathbb{P}^M$ defined by the ‘pull-back’ of the section to $V \times \mathbb{P}^M$. Since this join operation \vee clearly preserves rational equivalence, we can extend it to the Chow group and define a homomorphism $\alpha \mapsto \alpha \vee (V \times \mathbb{P}^{M-m-1})$ from $A_* Z$ to $A_* \check{Z}$.

Lemma 4.2. *With the above notation,*

$$(18) \quad s(\check{Z}, V \times \mathbb{P}^M)^{\mathcal{O}(-H)} \\ = s(Z, V \times \mathbb{P}^m)^{\mathcal{O}(-H)} \vee (V \times \mathbb{P}^{M-m-1}) + (1 + H + H^2 + \cdots) \cap [V \times \mathbb{P}^{M-m-1}] \quad .$$

This is a particular case of a rationality result for Segre classes of ind-schemes, and its proof is discussed elsewhere ([Alu]). This result will allow us to control the behavior of the relevant classes as the chosen set S of entries increases in size. In fact, (18) is our main reason for choosing to work with the class introduced in Definition 2.8: a relation for ordinary Segre classes may be obtained from (18), but is more complicated. As a rule, formulas for the twisted classes in the concrete situations that are considered in this note are simpler than formulas for ordinary Segre classes (this is already the case for the formula obtained in Corollary 3.2).

A second result we will need concerns *splayed* subschemes. Two subschemes X, Y of a nonsingular variety V are *splayed* if at each point of their intersection there are analytic coordinates $(x_1, \dots, x_a, y_1, \dots, y_b)$ for V such that X may be defined by an ideal generated by functions in the coordinates x_i and Y by an ideal generated by functions in the coordinates y_j .

Lemma 4.3. *Let X and Y be splayed in V , and let \mathcal{L} be a line bundle on V . Then*

$$(19) \quad s(X \cap Y, V)^{\mathcal{L}} = c(\mathcal{L}) \cap (s(X, V)^{\mathcal{L}} \cdot s(Y, V)^{\mathcal{L}})$$

in the Chow group of $X \cap Y$.

Proof. By Lemma 3.1 in [AF],

$$s(X \cap Y, V) = s(X, V) \cdot s(Y, V) \quad .$$

The stated formula follows immediately from this result, Definition 2.8, and the evident fact that \otimes_V preserves intersection products in V . \square

4.2. Now assume S consists of two blocks S', S'' , as in §4.1. Since $S' \subseteq S$, we have an embedding $L_{S'} \subseteq L_S$, and hence an embedding $\widehat{L}_{S'} \subseteq \widehat{L}_S$. Similarly, $\widehat{L}_{S''} \subseteq \widehat{L}_S$. For any $Z' \subseteq \widehat{L}_{S'}$, we can consider the join \check{Z}' of Z' and $\widehat{L}_{S''}$ in \widehat{L}_S , and similarly we let \check{Z}'' be the join of Z'' and $\widehat{L}_{S'}$, for every subscheme $Z'' \subseteq \widehat{L}_{S''}$.

Lemma 4.4. *Let $Z' = \widehat{L}'_{S'} \cap \widehat{\tau}_{n,k}$, $Z'' = \widehat{L}'_{S''} \cap \widehat{\tau}_{n,k}$. Then with notation as above:*

- Both \check{Z}', \check{Z}'' are zeros of sections of bundles $\mathcal{E} \otimes \mathcal{O}(H)$, with \mathcal{E} a pull-back from $G(k, n)$;
- $\widehat{L}_S \cap \widehat{\tau}_{n,k} = \check{Z}' \cap \check{Z}''$ (scheme-theoretically); and
- \check{Z}' and \check{Z}'' are splayed.

Proof. Denote elements of $\widehat{L}_S = G(k, n) \times L_S$ by pairs (C, A) , where $C = (c_{ij})$ is a $k \times n$ matrix whose rows span the given element of $G(k, n)$, and A is an $n \times n$ matrix with entries in S . The condition that $(C, A) \in \widehat{L}_S \cap \widehat{\tau}_{n,k}$ means that the row-span of C is contained in the kernel of A ; thus, the ideal defining $\widehat{L}_S \cap \widehat{\tau}_{n,k}$ is generated by the entries of $A \cdot C^t$.

Under the current assumption on S , A is a block matrix:

$$A = \left(\begin{array}{c|c} A' & 0 \\ \hline 0 & A'' \end{array} \right) \quad ;$$

and we can split C^t horizontally into matrices C', C'' with entries c_{ij} with $j \in J', j \in J''$ resp., so that the product $A \cdot C^t$ is in fact

$$\left(\begin{array}{c|c} A' & 0 \\ \hline 0 & A'' \end{array} \right) \cdot \begin{pmatrix} C' \\ C'' \end{pmatrix} = \begin{pmatrix} A' \cdot C' \\ A'' \cdot C'' \end{pmatrix} \quad .$$

With this notation, \check{Z}' is defined by the vanishing of the entries of $A' \cdot C'$, and \check{Z}'' by the vanishing of the entries of $A'' \cdot C''$.

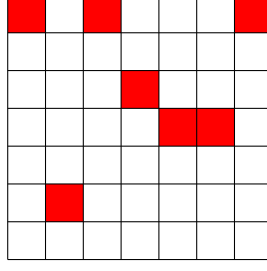


FIGURE 2

The statement follows. The first point is clear. For the second point, the ideal of $\widehat{L}_S \cap \widehat{\tau}_{n,k}$ is indeed the sum of the ideals of \check{Z}' and \check{Z}'' . The third point holds since generators for the ideals of \check{Z}' , \check{Z}'' are expressed in different sets of coordinates (in every local chart). \square

Theorem 4.5. *Assume that S consists of blocks S_1, \dots, S_m . Then*

$$(1 - \Sigma_{n,r,S}) = \prod_{i=1}^m (1 - \Sigma_{n,r,S_i})$$

in $G(n-r, n)$.

Proof. By an immediate induction it suffices to prove the case $m = 2$, so we may assume S is the union of two blocks S', S'' as above. The needed formula follows then from Lemma 4.4 and the technical Lemmas 4.2 and 4.3. Indeed, since $\widehat{L}_S \cap \widehat{\tau}_{n,k}$ is the splayed intersection of \check{Z}' and \check{Z}'' ,

$$s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)} = (1 - H) \cdot s(\check{Z}', \widehat{L}_S)^{\mathcal{O}(-H)} \cdot s(\check{Z}'', \widehat{L}_S)^{\mathcal{O}(-H)}$$

by Lemma 4.3; and $s(\check{Z}', \widehat{L}_S)^{\mathcal{O}(-H)}$, $s(\check{Z}'', \widehat{L}_S)^{\mathcal{O}(-H)}$ may be expressed in terms of joins by using Lemma 4.2, so that

$$\begin{aligned} s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)} &= (1 - H) \left(s(Z', \widehat{L}_{S'})^{\mathcal{O}(-H)} \vee [\widehat{L}_{S''}] + (1 - H)^{-1} \cap [\widehat{L}_{S''}] \right) \\ &\quad \cdot \left(s(Z'', \widehat{L}_{S''})^{\mathcal{O}(-H)} \vee [\widehat{L}_{S'}] + (1 - H)^{-1} \cap [\widehat{L}_{S'}] \right) . \end{aligned}$$

By definition of Grassmann class of a set of entries (Definition 2.12),

$$s(Z', \widehat{L}_{S'})^{\mathcal{O}(-H)} = \left(\alpha'_0 + \alpha'_1 H + \dots + \alpha'_{s'-2} H^{s'-2} + \Sigma_{n,r,S'} H^{s'-1} \right) \cap [\widehat{L}_{S'}]$$

where s' is the size of S' and the α'_i are (pull-backs of) classes in $G(k, n)$; and similarly for Z'' . These same expressions hold for the joins (but capping against $[\widehat{L}_S]$). The Grassmann class $\Sigma_{n,r,S}$ is therefore given by the coefficient of $H^{s'+s''-1}$ in

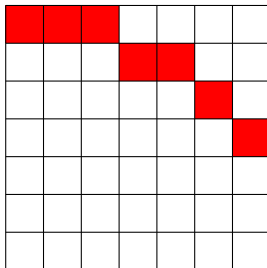
$$(1 - H) \left(\alpha'_0 + \dots + \Sigma_{n,r,S'} H^{s'-1} + \frac{H^{s'}}{1 - H} \right) \cdot \left(\alpha''_0 + \dots + \Sigma_{n,r,S''} H^{s''-1} + \frac{H^{s''}}{1 - H} \right) .$$

This is $-\Sigma_{n,r,S'} \Sigma_{n,r,S''} + \Sigma_{n,r,S'} + \Sigma_{n,r,S''}$, and the statement follows immediately. \square

4.3. As an illustration of the use of Theorem 4.5, we consider the case in which *no two of the s entries in S are in the same column*. We can describe S as the union of the subsets S_i of entries in the i -th row of the matrix; and we let ℓ_i be the cardinality of S_i . Thus S consists of ℓ_1 entries in the first row, ℓ_2 entries in the second row, etc., subject to the condition that no two entries are in the same column. We have $\ell_1 + \dots + \ell_n = s \leq n$.

The number $d_{n,r,S}$ clearly only depends on the numbers ℓ_1, \dots, ℓ_n , not on the specific set S of entries; so we can use the notation $d_{n,r,S} = d_{n,r|\ell_1, \ell_2, \dots}$. Since permutations of the ℓ_i 's do not affect the degree $d_{n,r,S}$, we can in fact list the ℓ_i 's in any order and omit trailing zeros. Thus, this notation is compatible with the notation used in §3. The degree for rank r and the configuration shown in Figure 2 can be denoted $d_{7,r|3,2,1,1}$.

Of course, up to a permutation of the columns and rows, such a configuration consists of blocks:



so that this case is covered by Theorem 4.5 and the computation for individual rows carried out in §3.1 and §3.2.

Corollary 4.6. *For S described by (ℓ_1, \dots, ℓ_n) as above,*

$$1 - \Sigma_{n,r,S} = \prod_{i=1}^n c(\mathcal{Q}^\vee) (c_0(\mathcal{S}^\vee) + \dots + c_{k-\ell_i}(\mathcal{S}^\vee)) \quad .$$

This follows immediately from Theorem 4.5 and Corollary 3.3. Note that for $\ell_i = 0$, the contribution is

$$c(\mathcal{Q}^\vee) (c_0(\mathcal{S}^\vee) + \dots + c_k(\mathcal{S}^\vee)) = c(\mathcal{Q}^\vee)c(\mathcal{S}^\vee) = 1 \quad ,$$

so that such rows do not affect the Grassmann class (as should be expected).

Theorem 4.7.

$$(20) \quad d_{n,r|\ell_1, \dots, \ell_n} = \int_{G(n-r,n)} \prod_{i=1}^n (c_0(\mathcal{S}^\vee) + \dots + c_{n-r-\ell_i}(\mathcal{S}^\vee)) \quad .$$

This statement generalizes directly the case in which S is concentrated in one row, i.e., Theorem 3.4. (Indeed, factors corresponding to $\ell_i = 0$ reproduce $c(\mathcal{S}^\vee)$.) Note that the pretty multiplicative structure underlying (20) is invisible at the level of the degrees $d_{n,r|\ell_1, \dots, \ell_n}$.

Proof. This now follows immediately from Theorem 2.13 and Corollary 4.6. \square

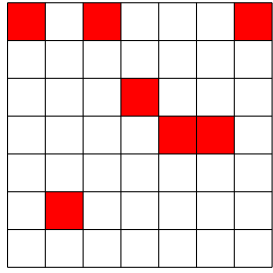
It is straightforward to adapt the Macaulay2 script given in §3.2 and implement the formula (20) given in Theorem 4.7:

```
needsPackage("Schubert2")
manyrows = (n,r,li) -> (
  m=#li;
  G = flagBundle({n-r,r});
  (S,Q) = bundles G;
  Sd = dual S;
  integral(chern(Sd)^(n-m)*product(0..m-1,
    j-> sum(0..n-r-li_j, i-> chern(i,Sd))))
)
```

For example,

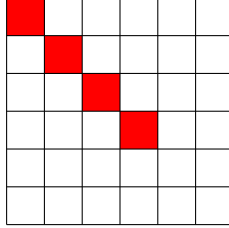
`manyrows(7, 3, {3, 2, 1, 1})`

returns 5990, the degree of the projection of the locus of 7×7 matrices of rank ≤ 3 from the set S depicted in Figure 2. This script took 0.023 seconds using our computing equipment to compile the data for all ranks for this set S :



$$d_{7,r|3,2,1,1} = \begin{cases} 887 & r = 1 \\ 13957 & r = 2 \\ 5990 & r = 3 \\ 35 & r = 4 \\ 0 & r = 5 \\ 0 & r = 6 \\ 0 & r = 7 \end{cases}$$

4.4. It is clear from (20) that $d_{n,r|\ell_1,\dots,\ell_n} = 0$ unless $\ell_i \leq n - r$ for all i . (Geometrically, this signals that the projection contracts the locus, cf. Remark 2.5.) We do not have a (conjectural) formula such as (17) computing $d_{n,r,S}$ as a combination of binomial coefficients in the more general case considered here; it is probably possible to obtain such a formula, or at least to conjecture one based on numerical experimentation. We will only deal explicitly with the case in which *no two elements of S lie in the same row or column*, i.e., all $\ell_i = 0$ or 1. Up to a permutation of rows and column, S may be realized as a set of s diagonal entries.



The following result will also answer a question raised in [GHIL13].

Theorem 4.8. For $s = 1, \dots, n$,

$$d_{n,r|1^s} = \sum_{j=0}^s \binom{s}{j} (-1)^j \deg \sigma_{n-j,r-j} = \sum_{j=0}^s \binom{s}{j} (-1)^j \prod_{i=0}^{n-r-1} \frac{\binom{n-j+i}{n-r}}{\binom{n-r+i}{n-r}}$$

Proof. We have $\ell_i = 1$ for s rows, and $\ell_i = 0$ for the remaining $n - s$ rows. According to Theorem 4.7,

$$\begin{aligned} d_{n,r|1^s} &= \int_{G(n-r,n)} c(\mathcal{S}^\vee)^{n-s} (c(\mathcal{S}^\vee) - c_k(\mathcal{S}^\vee))^s \\ &= \sum_{j=0}^s \binom{s}{j} (-1)^j \int_{G(n-r,n)} c(\mathcal{S}^\vee)^{n-j} c_k(\mathcal{S}^\vee)^j . \end{aligned}$$

Since $c_k(\mathcal{S}^\vee)^j$ is the class of $G(n - r, n - j)$ in $G(n - r, n)$, this gives

$$d_{n,r|1^s} = \sum_{j=0}^s \binom{s}{j} (-1)^j \int_{G(n-r,n-j)} c(\mathcal{S}^\vee)^{n-j} = \sum_{j=0}^s \binom{s}{j} (-1)^j \deg \sigma_{n-j,r-j} ,$$

by Lemma 2.11. □

Theorem 4.8 confirms the equality stated in Theorem 5.4.7 of [GHIL13]. In this statement, the equality is conditional on (and in fact proven to be equivalent to) the following conjectural statement. For $x \in S$, let $S' = S \setminus \{x\}$. Denote by $TC_x Z$ the tangent cone to Z at x . Conjecture 5.4.6 in [GHIL13] states:

Let S be such that no two elements of S lie in the same row or column and let $x \in S$. Then $TC_x J(\sigma_{n,r}, L_S) = J(TC_x \sigma_{n,r}, L_{S'})$.

(Here J denotes join.) As Theorem 4.8 verifies the equality in Theorem 5.4.7 of *loc. cit.*, it implies this statement.

Example 4.9. If S consists of $s = (n - r)^2$ diagonal elements, then the projection of $\sigma_{n,r}$ from L_S is dominant onto \mathbb{P}^{n^2-s-1} . The projection is generically d -to-1, with

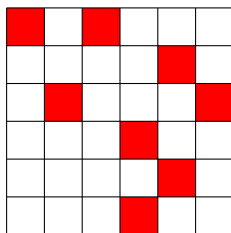
$$d = d_{n,r|1^{(n-r)^2}} = \sum_{j=0}^{(n-r)^2} \binom{(n-r)^2}{j} (-1)^j \prod_{i=0}^{n-r-1} \frac{\binom{n-j+i}{n-r}}{\binom{n-r+i}{n-r}}.$$

Thus

$$d = 1, 2, 42, 24024, 701149020, \dots \quad s \geq 1$$

for $s = (n - r)^2$. Remarkably, this degree appears to be independent of n , for $n \geq s^2$: for example, the projections of both $\sigma_{25,20}$ and $\sigma_{125,120}$ are generically 701149020-to-1 onto \mathbb{P}^{599} , \mathbb{P}^{15599} , respectively. (We do not have a conceptual explanation for this experimental observation.)

4.5. Assume next that *no element of S is on both the same row and the same column as other elements of S* . For example:

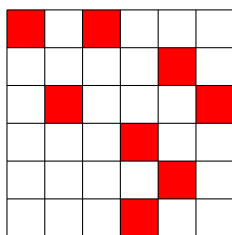


The degrees $d_{n,r,S}$ may be computed for such configurations by using Theorem 4.5. Indeed, up to permutations of rows and columns, S may be described as a block matrix with blocks consisting of subsets of rows or subsets of columns. We describe S by the lists (ℓ_1, ℓ_2, \dots) of lengths of rows and (m_1, m_2, \dots) of lengths of columns, and write $d_{n,r|\ell_1, \dots, \ell_a | m_1, \dots, m_b}$ for the corresponding degrees. For example, the degrees for the configuration shown above would be denoted $d_{6,r|2,2|2,2}$.

Theorem 4.10.

$$d_{n,r|\ell_1, \dots, \ell_a | m_1, \dots, m_b} = \int_{G(n-r,n)} c(\mathcal{S}^\vee)^{n-a} \cdot \prod_{j=1}^a \left(\sum_{i=0}^{n-r-\ell_j} c_i(\mathcal{S}^\vee) \right) \cdot \prod_{j=1}^b \left(1 - \sum_{i=0}^r \binom{m_j - 1 + n - r + i}{m_j - 1} c_i(\mathcal{Q}^\vee) c_{n-r}(\mathcal{S}^\vee) \right).$$

Example 4.11. For the example shown above:



$$d_{6,r|2,2|2,2} = \begin{cases} 228 & r = 1 \\ 734 & r = 2 \\ 8 & r = 3 \\ 0 & r = 4 \\ 0 & r = 5 \\ 0 & r = 6 \end{cases}$$

An explicit Macaulay2 computation shows that the projection of $\sigma_{6,3}$ ($\dim_{6,3} = 6^2 - 3^2 - 1 = 26$) from L_S is a hypersurface of degree 4 in \mathbb{P}^{27} . Since $d_{6,r|2,2|2,2} = 8$, the projection map must be generically 2-to-1 in this case. \square

4.6. As in the case in which S is concentrated in one row (cf. §3.5), the numbers $d_{n,r,S}$ computed in this section may have significance in other contexts, and it would be interesting to explore conceptual connections suggested by these apparent coincidences. For example, the numbers

$$d_{n,n-3|1^5} = 85, 295, 771, 1681, 3235, 5685, 9325, \dots \quad (n \geq 5)$$

form the *Ehrhart series for the matroid K_4* according to [Slo14]. The numbers $d_{n,n-3|1,1}$, $d_{n,n-3|1,2}$, $d_{n,n-3|2,2}$, etc. all appear to be Kekulé numbers for certain benzenoids (§3.5). For example,

$$d_{n,n-3|2,3} = 25, 65, 140, 266, 462, 750, 1155, \dots \quad (n \geq 5)$$

reproduce the Kekulé numbers for benzenoid hydrocarbons in the *chevron* configuration $Ch(2, 3, n - 3)$, cf. [CG88], p. 166. The numbers

$$d_{n,n-3|2|2} = 12, 60, 200, 525, 1176, 2352, 4320, \dots \quad (n \geq 4)$$

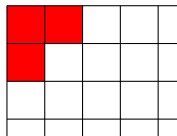
and

$$d_{n,n-3|2|3} = 6, 20, 50, 105, 196, 336, 540, \dots \quad (n \geq 4)$$

are likewise Kekulé numbers ([CG88], p. 233 #11 and p. 165 I1).

5. INCREASING THE COMPLEXITY OF S

5.1. The cases studied in §4.3 and ff. are still very special from the point of view of the initial motivation mentioned in the introduction. The smallest S not covered by these cases is of the following type:



where one element is on the same row *and* column as other elements of S . In this section we indicate how this case may be treated by the same technique used in §3 and §4. Theorem 4.5 can then be used to obtain $d_{n,r,S}$ for S consisting of blocks including this configuration as well as those studied in §4. In §5.6 we will further extend the discussion to blocks consisting of 2×2 squares. Not surprisingly, as the complexity of S increases so do the technical subtleties needed to evaluate the relevant Segre classes.

5.2. Let S consist of the entries $(1, 1)$, $(1, 2)$, $(2, 1)$, as in the above figure. We will adopt the set-up in previous sections: $L_S \cong \mathbb{P}^2$ is the span of the three entries in S as a subspace of \mathbb{P}^{n^2-1} ; $\widehat{L}_S = G(k, n) \times L_S$; and $\widehat{\tau}_{n,k} \subseteq G(k, n) \times \mathbb{P}^{n^2-1}$ denotes the standard resolution of the locus $\tau_{n,k} = \sigma_{n,n-k}$ of matrices with bounded rank. The main information is carried by the Grassmann class of S , a projection of the Segre class of $\widehat{L}_S \cap \widehat{\tau}_{n,k}$ in \widehat{L}_S (Theorem 2.13).

It is easy to provide a qualitative description of $\widehat{L}_S \cap \widehat{\tau}_{n,k}$. By definition (§2.5) this is the locus of pairs $(K, \varphi) \in \widehat{L}_S = G(k, n) \times L_S$ such that $K \subseteq \ker \varphi$. We may use coordinates a_{11}, a_{12}, a_{21} for $L_S \cong \mathbb{P}^2$, corresponding to entries in the matrix

$$\varphi = \begin{pmatrix} a_{11} & a_{12} & 0 & \dots \\ a_{21} & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

There are then three possibilities for $\ker \varphi$:

- If $a_{12}a_{21} \neq 0$, then $\text{rk } \varphi = 2$, and $E := \ker \varphi$ is the codimension 2 subspace defined by the vanishing of the first two components: $x_1 = x_2 = 0$;
- If $a_{12} = 0$, defining a line $L_1 \subseteq L_S$, then $\text{rk } \varphi = 1$, and $F := \ker \varphi$ is the hyperplane defined by $x_1 = 0$;
- If $a_{21} = 0$, defining a line $L_2 \subseteq L_S$ then $\text{rk } \varphi = 1$, and $F_\varphi := \ker \varphi$ is the hyperplane with equation $a_{11}x_1 + a_{12}x_2 = 0$.

Correspondingly, $\widehat{L}_S \cap \widehat{\tau}_{n,k}$ must consist of three components:

- $X = G(k, E) \times L_S \cong G(k, n-2) \times \mathbb{P}^2$; $\text{codim}_X L_S = 2k$.
- $W_1 = G(k, F) \times L_1 \cong G(k, n-1) \times \mathbb{P}^1$; $\text{codim}_{W_1} L_S = k+1$.
- W_2 , a Grassmann bundle over $L_2 \cong \mathbb{P}^1$; the fiber over $\varphi \in L_2$ is $G(k, F_\varphi)$; $\text{codim}_{W_2} L_S = k+1$.

Lemma 5.1. • *The scheme $\widehat{L}_S \cap \widehat{\tau}_{n,k}$ consists of the reduced union of X, W_1, W_2 ;*

- *The varieties X, W_1, W_2 are nonsingular. As classes in $\widehat{L}_S \cong G(k, n) \times \mathbb{P}^2$,*

$$\begin{aligned} s(X, \widehat{L}_S)^{\mathcal{O}(-H)} &= \frac{c_k(\mathcal{S}^\vee)^2}{(1-H)c(\mathcal{S}^\vee \otimes \mathcal{O}(-H))^2} \\ s(W_1, \widehat{L}_S)^{\mathcal{O}(-H)} &= \frac{H \cdot c_k(\mathcal{S}^\vee)}{(1-H)c(\mathcal{S}^\vee \otimes \mathcal{O}(-H))} \\ s(W_2, \widehat{L}_S)^{\mathcal{O}(-H)} &= \frac{H \cdot c_k(\mathcal{S}^\vee \otimes \mathcal{O}(H))}{(1-H)c(\mathcal{S}^\vee)}. \end{aligned}$$

where H is the pull-back of the hyperplane class from \mathbb{P}^2 , and \mathcal{S} denotes the (pull-back of the) universal subbundle from $G(k, n)$.

Proof. The first statement may be verified by a coordinate computation analogous to the one performed in the proof of Lemma 3.1. Working in local coordinates for \widehat{L}_S , e.g., in the chart

$$\left(C = \begin{pmatrix} c_{11} & \dots & c_{1,n-k} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{k1} & \dots & c_{k,n-k} & 0 & \dots & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & a_{12} & 0 & \dots \\ a_{21} & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right)$$

equations for $\widehat{L}_S \cap \widehat{\tau}_{k,n}$ are $A \cdot C^t = 0$, i.e., generators for the ideal of $\widehat{L}_S \cap \widehat{\tau}_{k,n}$ in \widehat{L}_S in these coordinates are

$$c_{11} + a_{12}c_{12}, \dots, c_{k1} + a_{12}c_{k2}, a_{21}c_{11}, \dots, a_{21}c_{k1} \quad .$$

The first point follows at once.

The second point follows from the description of X, W_1, W_2 preceding the statement, from Lemma 2.10 (1), and from computations of normal bundles analogous to those performed in the proof of Corollary 3.2. \square

Remark 5.2. The scheme $\widehat{L}_S \cap \widehat{\tau}_{n,k}$ may in fact be described as the zero-scheme of a section of $(\mathcal{S}^\vee \oplus \mathcal{S}^\vee) \otimes \mathcal{O}(H)$. \square

5.3. The fact that the information collected in Lemma 5.1 is *not* enough in itself to compute $s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)}$ is a good illustration of the subtleties involved in the notion of Segre class; the result that follows involves substantially more technical considerations. The classes presented in Lemma 5.1 should at least serve to make the general shape of the formula (21) given below look plausible. For example, since both the classes for W_1 and W_2 are multiples of H , one should expect the part of $s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)}$ that is *not* a multiple of H to agree with $s(X, \widehat{L}_S)^{\mathcal{O}(-H)}$ modulo H :

$$s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)} = \frac{c_k(\mathcal{S}^\vee)^2}{c(\mathcal{S}^\vee)^2} + H(\dots) \quad .$$

On the other hand, since the top-dimensional components of $\widehat{L}_S \cap \widehat{\tau}_{n,k}$ are W_1 and W_2 , one should expect the dominant terms in the Segre class to be

$$\begin{aligned} H \cdot (c_k(\mathcal{S}^\vee) + c_k(\mathcal{S}^\vee \otimes \mathcal{O}(H))) &= H \cdot (c_k(\mathcal{S}^\vee) + c_k(\mathcal{S}^\vee) + H \cdot c_{k-1}(\mathcal{S}^\vee) + \dots) \\ &= 2c_k(\mathcal{S}^\vee)H + c_{k-1}(\mathcal{S}^\vee)H^2 \end{aligned}$$

(note that $H^3 = 0$ since H is the pull-back of the hyperplane in \mathbb{P}^2). These features are indeed present in (21).

Proposition 5.3.

$$(21) \quad s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)} = \frac{c_k^2}{c^2} + \left(\frac{2c_k}{c} - \frac{c_k^2}{c^2} \right) H + \left(\frac{k c_k + c_{k-1}}{c} + \frac{c_k^2 - \sum_{i=0}^k i c_i c_k}{c^2} \right) H^2$$

where $c = c(\mathcal{S}^\vee)$, $c_i = c_i(\mathcal{S}^\vee)$.

Proof. We may blow-up \widehat{L}_S along $W_1 \cup W_2$; this produces a variety \widetilde{L} with an exceptional divisor (consisting of two components), and one may verify that the (scheme-theoretic) inverse image of $\widehat{L}_S \cap \widehat{\tau}_{n,k}$ in \widetilde{L} consists of the exceptional divisor and of a residual scheme \widetilde{X} that may be verified to equal the proper transform of X . Applying Fulton's residual intersection formula (Proposition 9.2 in [Ful84], in the form given in [Alu94], Proposition 3) and Lemma 2.10 (2) one obtains the stated formula. The notationally rather demanding details are left to the reader. \square

Corollary 5.4. For S as above, and $k = n - r$,

$$\Sigma_{n,r,S} = \frac{k c_k + c_{k-1}}{c} + \frac{c_k^2 - \sum_{i=0}^k i c_i c_k}{c^2}$$

where $c = c(\mathcal{S}^\vee)$, $c_i = c_i(\mathcal{S}^\vee)$.

5.4. Theorem 2.13 now yields a computation of the degrees:

Theorem 5.5. For $S = \{(1, 1), (1, 2), (2, 1)\}$, the degrees $d_{n,r,S}$ of the projections of the rank loci are given by

$$(22) \quad \int_{G(k,n)} c^{n-2} \left(c^2 - (kc_k + c_{k-1})c - (c_k^2 - \sum_{i=0}^k ic_i c_k) \right)$$

with $k = n - r$, and where $c = c(\mathcal{S}^\vee)$, $c_i = c_i(\mathcal{S}^\vee)$.

Example 5.6. For $n = 7$:

	$d_{7,r,S} = \begin{cases} 912 & r = 1 \\ 17303 & r = 2 \\ 15218 & r = 3 \\ 1001 & r = 4 \\ 6 & r = 5 \\ 0 & r = 6 \\ 0 & r = 7 \end{cases}$
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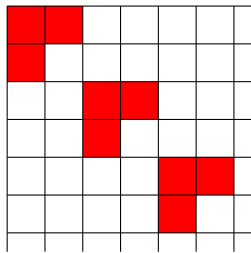
(Compiling this table took 0.06 seconds on our computing equipment, *vs.* about 15 minutes by ordinary degree calculations. The case $n = 8$ takes 0.13 seconds applying Theorem 5.5, giving degrees 3418, 217007, 592956, 118188, 2548, 7, 0, 0 for $r = 1, \dots, 8$. This is already beyond the reach of a straightforward implementation of a more direct method, at least on equipment available to us.)

Remark 5.7. It appears that $d_{n,n-2,S} = n - 1$ (as can probably be readily verified using (22)), while

$$d_{n,n-3,S} = 1, 14, 84, 330, 1001, 2548, 5712, \dots \quad (n \geq 3)$$

are the dimensions $\dim V^{(n-3)}$ appearing in the case $a = 1$ of Theorem 7.2 in [LM06], on Lie algebras in the subexceptional series. These numbers also arise in enumerating paths in the plane (cf. [Slo14]), and as Kekulé numbers for benzenoid hydrocarbons configured as prolate pentagons $D^i(3, n - 2)$, [CG88], p. 183.

5.5. Using Theorem 4.5, we can now compute the degrees $d_{n,r,S}$ for S obtained by including blocks with the corner shape analyzed above. For example, S could consist of several such ‘corners’, provided that rows and columns of distinct configurations do not overlap.



Theorem 5.8. Suppose $S = \cup_{i=1}^g \{(a_i, b_i), (a_i, b_i + 1), (a_i + 1, b_i)\}$, where the different groups have no overlapping rows or columns. Then

$$d_{n,r,S} = \int_{G(k,n)} c^{n-2g} \left(c^2 - (kc_k + c_{k-1})c - (c_k^2 - \sum_{i=0}^k ic_i c_k) \right)^g$$

with $k = n - r$, and where $c = c(\mathcal{S}^\vee)$, $c_i = c_i(\mathcal{S}^\vee)$.

Proof. This follows from Theorem 4.5 and Corollary 5.4. \square

Example 5.9. For $n = 7, g = 3$, and after a permutation of the rows of the matrix:

$$d_{7,r,S} = \begin{cases} 888 & r = 1 \\ 13395 & r = 2 \\ 4078 & r = 3 \\ 2 & r = 4 \\ 0 & r = 5 \\ 0 & r = 6 \\ 0 & r = 7 \end{cases}$$

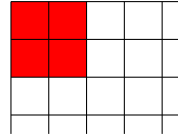
The $r = 2$ case was mentioned in the introduction, and the $r = 4$ case in Remark 2.6.

Example 5.10. By the same token, one can include all blocks for which we have computed Grassmann classes. For instance:

$$d_{7,r,S} = \begin{cases} 886 & r = 1 \\ 12967 & r = 2 \\ 3102 & r = 3 \\ 0 & r = 4 \\ 0 & r = 5 \\ 0 & r = 6 \\ 0 & r = 7 \end{cases}$$

The degrees are obtained by applying Theorem 4.5 together with the computation of the relevant classes $\Sigma_{n,r,S}$ in Corollary 3.3, 3.9, and 5.4.

5.6. As a last example we consider a set S consisting of entries in a 2×2 square: for instance, $S = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.



The determination of $\widehat{L}_S \cap \widehat{\tau}_{n,k}$ as a subscheme of $\widehat{L}_S = G(k, n) \times \mathbb{P}^3$ is very similar to the one reviewed for the case considered in §5.2. This locus consists of two components: a codimension $2k$ component dominating \mathbb{P}^3 , and a codimension $k + 1$ component dominating the determinantal quadric Q given by $a_{11}a_{22} - a_{12}a_{21} = 0$. The Segre class may be computed by blowing up this latter locus and using residual intersection and Lemma 2.10 (2), as in §5.2. The computation yields the class

$$s(\widehat{L}_S \cap \widehat{\tau}_{n,k}, \widehat{L}_S)^{\mathcal{O}(-H)} = \frac{[G(k, n-2) \times \mathbb{P}^3]}{(1+H)c(\mathcal{S}^\vee \otimes \mathcal{O}(H))^2} + \frac{c_k(\mathcal{S}^\vee \otimes \mathcal{O}(h_1))[G(k, n) \times Q]}{(1-H)(1+H)c(\mathcal{S}^\vee \otimes \mathcal{O}(-h_2))} \\ + \frac{\left(\sum_{j=0}^2 \sum_{i=0}^k \binom{i}{j+1} (1-H)^{i-j} c_{k-i}(\mathcal{S}^\vee \otimes \mathcal{O}(h_1))(2H)^j \right) [G(k, n-2) \times Q]}{(1-H)(1+H)c(\mathcal{S}^\vee \otimes \mathcal{O}(-h_2))c(\mathcal{S}^\vee \otimes \mathcal{O}(H))^2},$$

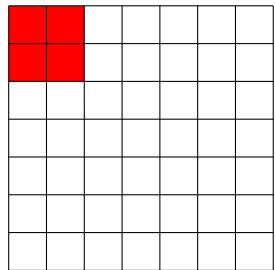
where H denotes the pull-back of the hyperplane class in \mathbb{P}^3 and of its restriction to Q , and h_1, h_2 are pull-backs of the classes from the two rulings on Q .

The Grassmann class of a 2×2 square can be computed from this expression:

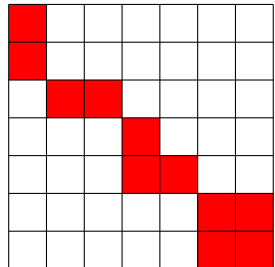
$$\begin{aligned} \Sigma_{n,r,S} = & - \sum_{i=0}^r \sum_{j=0}^r \binom{2k+i+j+3}{3} c_i(\mathcal{Q}^\vee) c_j(\mathcal{Q}^\vee) c_k(\mathcal{S}^\vee)^2 + \sum_{j=0}^r c_j(\mathcal{Q}^\vee) (2c_k(\mathcal{S}^\vee) \\ & + (k+j)c_{k-1}(\mathcal{S}^\vee)) + \sum_{i=0}^k \sum_{u=0}^r \sum_{v=0}^r \sum_{w=0}^r \left(2 \binom{i}{3} - 2(k-u+v+w+1) \binom{i}{2} \right) \\ & + (k-u+v+w+2)(2k+v+w+1) \binom{i}{1} \Big) c_u(\mathcal{Q}^\vee) c_v(\mathcal{Q}^\vee) c_w(\mathcal{Q}^\vee) c_{k-i}(\mathcal{S}^\vee) c_k(\mathcal{S}^\vee)^2 . \end{aligned}$$

Theorem 2.13 can then be used to compute $d_{n,r,S}$ for sets S including blocks consisting of 2×2 squares.

Example 5.11. For $n = 7$:



$$d_{7,r,S} = \begin{cases} 887 & r = 1 \\ 14701 & r = 2 \\ 9478 & r = 3 \\ 371 & r = 4 \\ 1 & r = 5 \\ 0 & r = 6 \\ 0 & r = 7 \end{cases}$$

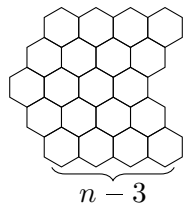


$$d_{7,r,S} = \begin{cases} 861 & r = 1 \\ 10701 & r = 2 \\ 1424 & r = 3 \\ 0 & r = 4 \\ 0 & r = 5 \\ 0 & r = 6 \\ 0 & r = 7 \end{cases}$$

For S consisting of a single 2×2 square, the degrees $d_{n,n-3,S}$:

$$d_{n,n-3,S} = 1, 10, 46, 146, 371, 812, 1596, 2892, \dots \quad (n \geq 3)$$

agree with the Kekulé numbers for benzenoid hydrocarbons in the symmetric ‘chevron’ configuration $Ch(3, n-3)$, cf. [CG88], p. 120.



For S consisting of *three* blocks of 2×2 squares, the numbers

$$d_{n,n-4,S} = 105, 336, 825, 1716, 3185, 5440, \dots \quad (n \geq 6)$$

agree with $d_{2n+1, 2n-2|2|3}$, cf. §4.6, and are in particular Kekulé numbers for hexagonal configurations of benzenoids.

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MATHEMATICS DEPARTMENT, FLORIDA STATE UNIVERSITY, TALLAHASSEE FL 32306, U.S.A.
E-mail address: aluffi@math.fsu.edu