SPHERICAL MEANS AND PINNED DISTANCE SETS

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Abstract. We use mixed norm estimates for the spherical averaging operator to obtain some results concerning pinned distance sets.

1. Introduction

Let E be a Borel subset of $\mathbb{R}^d$ and suppose $x \in \mathbb{R}^d$. Define

$$D_x(E) = \{|x-y| : y \in E\}.$$

Then $D_x(E)$ is called a *pinned distance set* of E. If the Hausdorff dimension $\dim E$ of E is equal to β and if $0 \le \tau < 1$, one may consider the possibility of estimates of the form

$$\dim\{x \in \mathbb{R}^d : \dim D_x(E) < \tau\} \le \alpha \tag{1.1}$$

where $\alpha = \alpha(d, \beta, \tau)$. For example, Theorem 8.3 in [5] shows that if $0 \le \tau \le \min(1, \beta)$ then one may take $\alpha = d + \tau - \max(1, \beta)$. One purpose of this note is to point out a relationship between estimates (1.1) and certain mixed norm estimates for the spherical averaging operator S defined for nice functions f on $\mathbb{R}^d$ and for $x \in \mathbb{R}^d$, $r > 0$ by

$$Sf(x,r) = \int_{S^{d-1}} f(x - r\sigma)\, d\sigma.$$

Here $d\sigma$ indicates integration with respect to Lebesgue measure on the unit sphere S^{d-1} in $\mathbb{R}^d$. The mixed norm estimates we have in mind will be of the form

$$\Big(\int_{\mathbb{R}^d} \big(\int_{r_0}^{R_0} |Sf(x,r)|^s\, dr\big)^{q/s} d\lambda(x)\Big)^{1/q} \lesssim |||f||| \tag{1.2}$$

where λ is an α-dimensional measure on $\mathbb{R}^d$ and where $|||\cdot|||$ stands for either an L^p or a Sobolev norm on $\mathbb{R}^d$.

To describe the relationship between estimates (1.1) and (1.2) requires a little background. We begin by repeating some material from [2]: for $\rho > 0$,

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let K_ρ be a kernel defined on $\mathbb{R}^d$ by

$$K_\rho(x) = |x|^{-\rho} 1_{B(0,r(d))}(x) \tag{1.3}$$

where $r = r(d)$ is a positive parameter. Suppose that the Borel probability measure ν on $\mathbb{R}^d$ is a γ-dimensional measure in the sense that $\nu\big(B(x,\delta)\big) \lesssim \delta^\gamma$ for all $x \in \mathbb{R}^d$ and $\delta > 0$. If $\rho < \gamma$ it follows that

$$\nu * K_\rho \in L^\infty(\mathbb{R}^d).$$

Also

$$\nu * K_\rho \in L^1(\mathbb{R}^d)$$

so long as $\rho < d$. Thus, for $\epsilon > 0$ and $1 < p < \infty$, we have

$$\nu * K_\rho \in L^p(\mathbb{R}^d),\ \rho = \gamma + \frac{1}{p}(d-\gamma) - \epsilon \tag{1.4}$$

by interpolation. The following lemma from [2] is a weak converse of this observation.

Lemma 1.1. *If* (1.4) *holds with* $\epsilon = 0$ *and* $1 < p < \infty$*, then* ν *is absolutely continuous with respect to Hausdorff measure of dimension* $\gamma - \epsilon'$ *for any* $\epsilon' > 0$*. Thus the support of* ν *has Hausdorff dimension at least* γ*.*

Returning to the relationship between (1.1) and (1.2), suppose that ν is a Borel probability measure on $E \subset \mathbb{R}^d$. For each $x \in \mathbb{R}^d$ define the probability measure ν_x on $D_x(E)$ by

$$\int_{[0,\infty)} g\, d\nu_x = \int_E g(|x-e|)\, d\nu(e).$$

The proof of the next lemma will be given in §3.

Lemma 1.2. *Suppose* ν *is as above,* $\rho > d-1$*, and* $0 < r_0 < R_0$*. Then, given* $r(1)$ *in* (1.3)*, it is possible to choose* $r(d)$ *(in* (1.3)*) so that we have the estimate*

$$\nu_x * K_{\rho+1-d}\,(r) \lesssim S(\nu * K_\rho)(x,r)\ (x \in \mathbb{R}^d,\ r_0 \le r \le R_0). \tag{1.5}$$

Here, then, is a rough sketch (we have neglected ϵ's and various other details) of the argument which shows how estimates (1.2) for $|||\cdot||| = \|\cdot\|_p$ can imply estimates (1.1). (This argument is analogous to one in [2] for orthogonal projections.) Suppose that $E \subset \mathbb{R}^d$ carries a β-dimensional probability measure ν. Then $\nu * K_\rho \in L^p(\mathbb{R}^d)$ for ρ given by

$$\rho = \beta + \frac{1}{p}(d-\beta).$$

If (1.2) holds for $|||\cdot||| = \|\cdot\|_p$, then (1.5) shows that $\nu_x * K_{\rho+1-d} \in L^s([r_0, R_0])$ for λ almost all $x \in \mathbb{R}^d$. It will then follow from Lemma 1.1 that for τ given by

$$\rho + 1 - d = \tau + \frac{1}{s}(1-\tau)$$

the support of ν_x has dimension at least τ for λ almost all x. Since ν_x is supported on $D_x(E)$ and since λ is an arbitrary α-dimensional measure, (1.1) follows.

The remainder of this note is organized as follows: §2 contains the statements of our results and §3 contains their proofs.

2. Results

Estimates such as (1.2) are examples of what Wolff [6] called "$L^p \to L^q$ inequalities for the wave equation relative to fractal measures" - see also [1], [3], [4]. The following example, an estimate for a "fractal spherical maximal function", is an easy consequence of Theorem 4_S in [4].

Theorem 2.1. *Suppose that λ is a nonnegative compactly-supported Borel measure on $\mathbb{R}^d$ and suppose that for some $\alpha \in (0,(d-1)/2)$ the measure λ satisfies an estimate*

$$\lambda\big(B(x,\delta)\big) \lesssim \delta^\alpha \tag{2.1}$$

for $x \in \mathbb{R}^d$, $\delta > 0$. Then, for $\epsilon > 0$, $q < 2$, and $0 < r_0 < R_0 < \infty$, there is the estimate

$$\Big(\int_{\mathbb{R}^d} \big(\sup_{r_0<r<R_0} |Sf(x,r)|\big)^q \, d\lambda(x)\Big)^{1/q} \lesssim \|f\|_{W^{2,(1-\alpha)/2-\epsilon}} \ . \tag{2.2}$$

The constant implicit in (2.2) depends on the size of the support of λ, the constant implicit in (2.1), ϵ, q, r_0, and R_0.

The next estimate, of form (1.1), is a consequence of Theorem 2.1:

Theorem 2.2. *Suppose that the compact set $E \subset \mathbb{R}^d$ satisfies $\dim E = \beta$ for some $\beta > 0$. Suppose that $0 < \tau < 1$ and that $2\tau + (d-1)/2 < \beta < 2\tau + d - 1$. Then*

$$\dim\{x \in \mathbb{R}^d : \dim D_x(E) < \tau\} \leq 2\tau - \beta + d - 1.$$

The proof of Theorem 4_S in [4] depends on Fourier analysis. Theorem 2.3 below is proved by modifying a measure-theoretic argument previously used in [2] and [3].

Theorem 2.3. *Suppose λ is a compactly supported probability measure on $\mathbb{R}^d$ and, for any fixed $0 < r_0 < R_0 < \infty$, consider the mixed norm estimate*

$$\Big(\int_{\mathbb{R}^2} \big(\int_{r_0}^{R_0} |Sf(x,r)|^s \, dr\big)^{q/s} d\lambda(x)\Big)^{1/q} \lesssim \|f\|_p. \tag{2.3}$$

(a) Suppose $d = 2$ and $\alpha > 1/2$. Suppose that λ satisfies the Frostman condition

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x_1 - x_2|^\alpha} \, d\lambda(x_2) \, d\lambda(x_1) \leq C < \infty. \tag{2.4}$$

Then (2.3) holds when

$$\Big(\frac{1}{p}, \frac{1}{q}, \frac{1}{s}\Big) = t\Big(\frac{1}{2}, \frac{1}{2\alpha}, \frac{1}{4}\Big) + (1-t)\Big(1, 0, 1\Big), \ 0 \leq t < 1.$$

(b) Suppose $d = 2$ and $0 < \alpha < \alpha' < 1/2$. Suppose that λ satisfies the condition

$$\lambda\big(B(x,\delta)\big) \le C\delta^{\alpha'} \ (x \in \mathbb{R}^2, \delta > 0). \tag{2.5}$$

Then (2.3) *holds where*

$$\Big(\frac{1}{p},\frac{1}{q},\frac{1}{s}\Big) = t\Big(\frac{1}{1+2\alpha},\frac{1}{1+2\alpha},\frac{1-\alpha}{1+2\alpha}\Big) + (1-t)\Big(1,0,1\Big),\ 0 \le t < 1. \tag{2.6}$$

(c) Suppose $d > 2$ and $0 < \alpha < \alpha' < 1$. Suppose that λ satisfies the condition

$$\lambda\big(B(x,\delta)\big) \le C\delta^{\alpha'} \ (x \in \mathbb{R}^d, \delta > 0). \tag{2.7}$$

Then (2.3) *holds where*

$$\Big(\frac{1}{p},\frac{1}{q},\frac{1}{s}\Big) = t\Big(\frac{1}{1+\alpha},\frac{1}{1+\alpha},\frac{1-\alpha}{1+\alpha}\Big) + (1-t)\Big(1,0,1\Big),\ 0 \le t < 1. \tag{2.8}$$

The implied constant in (2.3) depends on r_0, R_0, the size of the support of λ, C from (2.4), (2.5), or (2.7), and t. There is an analogue of (c) of Theorem 2.3 for $\alpha > 1$, but it is essentially subsumed by Theorem 2.2. A consequence of Theorem 2.3 is the following analogue of Theorem 1.2 in [2]:

Theorem 2.4. *Suppose $E \subset \mathbb{R}^d$ has Hausdorff dimension β.*
(a) If $d = 2$ and $\beta > 1/2$, then

$$\dim\{x \in \mathbb{R}^2 : \dim D_x(E) < (2\beta - 1)/3\} = 0.$$

(b) If $d > 2$ and $\beta > d - 2$, then

$$\dim\{x \in \mathbb{R}^2 : \dim D_x(E) < (\beta + 2 - d)/2\} = 0.$$

3. Proofs

Proof of Lemma 1.2: Suppose $\delta \in (0, r_0/2)$. Write $\sigma_{x,r}$ for normalized Lebesgue measure on $\{y \in \mathbb{R}^d : |x - y| = r\}$. Then if $r_0 \le r \le R_0$ we have

$$\nu_x([r-\delta, r+\delta]) = \int_E 1_{[r-\delta,r+\delta]}(|x-e|)\, d\nu(e) \approx$$
$$\langle \delta^{1-d} 1_{B(0,\delta)} * \sigma_{x,r}, \nu\rangle = \langle \sigma_{x,r}, \delta^{1-d} 1_{B(0,\delta)} * \nu\rangle = S(\delta^{1-d} 1_{B(0,\delta)} * \nu)(x,r),$$

where the implied constant depends on R_0. If $r(1) \le r_0/4$ and

$$K_{\rho+1-d}(x) = |x|^{-(\rho+1-d)} 1_{B(0,r(1))}(x),\ x \in \mathbb{R}$$

then

$$\nu_x * K_{\rho+1-d}(r) \approx \sum_{2^{-j} \le r(1)} 2^{(\rho+1-d)j} \nu_x([r - 2^{-j}, r + 2^{-j}]) =$$
$$\sum_{2^{-j} \le r(1)} 2^{(\rho+1-d)j} S\big((2^{-j})^{-(d-1)} 1_{B(0,2^{-j})} * \nu\big)(x,r) \lesssim S(K_\rho * \nu)(x,r),$$

so long as $r(d) \gtrsim r(1)$.

□

Proof of Theorem 2.1: As mentioned in §2, the proof of Theorem 2.1 is consequence of the following result from [4]:

Theorem 3.1. *Suppose μ is a nonnegative Borel measure on a compact subset of $\mathbb{R}^d \times (0,\infty)$ and suppose that, for some $\alpha \in \big(0,(d-1)/2\big)$, μ satisfies the estimate*

$$\mu\big(\{(x,r)\in\mathbb{R}^d\times(0,\infty) : |x-x'|+|r-r'|<\delta\}\big) \lesssim \delta^\alpha \tag{3.1}$$

for all $(x',r')\in\mathbb{R}^d\times(0,\infty)$ and $\delta>0$. Then, for $\epsilon>0$,

$$\|Sf\|_{L^{2,\infty}_\mu} \lesssim \|f\|_{W^{2,(1-\alpha)/2+\epsilon}}. \tag{3.2}$$

Since μ is compactly supported, $\mathrm{supp}(\mu) \subset B(0,M)\times[r_0,R_0]$ for some $M>0$ and $0<r_0<R_0<\infty$. The proof of Theorem 3.1 shows that the constant implicit in (3.2) is, for fixed M, r_0, R_0, α, and ϵ, bounded by a function of the constant implicit in (3.1). Suppose that λ is as in the statement of Theorem 2.1 and $x\mapsto r(x)$ is any Borel function from the compact support of λ into $[r_0,R_0]$. If the measure μ on $\mathbb{R}^d\times(0,\infty)$ is defined by

$$\int_{\mathbb{R}^d\times(0,\infty)} f\,d\mu = \int_{\mathbb{R}^d} f\big(x,r(x)\big)\,d\lambda(x)$$

then the hypothesis (2.1) on λ ensures that the measures μ satisfy (3.1) uniformly in the choice of $r(x)$. Thus

$$\|Sf\big(\cdot,r(\cdot)\big)\|_{L^{2,\infty}_\lambda} \lesssim \|f\|_{W^{2,(1-\alpha)/2+\epsilon}}$$

with implicit constant independent of $r(x)$. That is enough to establish (2.2). □

Proof of Theorem 2.2: Define α by $\alpha = 2\tau-\beta+d-1$ and let λ be any measure satisfying (2.1). It will be enough to show that if $\tilde F\subset\mathbb{R}^d$ is any compact set with $\lambda(\tilde F)>0$ then there is a compact subset F of $\tilde F$ with $\lambda(F)>0$ such that $\dim D_x(E)\geq\tau$ for λ-a.a. $x\in F$.

Fix a small $\epsilon>0$ and set $\beta'=\beta-\epsilon$. Since $\dim E=\beta$ there is a probability measure ν on E with

$$\int_{\mathbb{R}^d} \frac{|\hat\nu(\xi)|^2}{|\xi|^{d-\beta'}}\,d\xi<\infty. \tag{3.3}$$

Now fix $\tilde F$ with $\lambda(\tilde F)>0$ as above and choose $0<r_0<R_0$ (in (1.2)), $r(1)$ (in the definition of the one-dimensional kernel K_ρ), and $F\subset\tilde F$ with $\lambda(F)>0$ such that $\nu_x([r_0+r(1),R_0-r(1)])>0$ for λ-a.a. $x\in F$. Choose $r(d)$ in the definition of the d-dimensional kernel K_ρ so that (1.5) holds. Since the d-dimensional K_ρ satisfies $|\hat K_\rho(\xi)|\lesssim|\xi|^{-d+\rho}$ it follows from (3.3) that $K_\rho*\nu\in W^{2,(1-\alpha)/2-\epsilon/2}$ if

$$\rho=(\alpha+\beta+d-1)/2.$$

Then Theorem 2.1 and Lemma 1.2 show that for any $s < \infty$ we have, for the one-dimensional kernel $K_{(\alpha+\beta+1-d)/2}$, that

$$\nu_x * K_{(\alpha+\beta+1-d)/2} \in L^s(r_0, R_0)$$

for λ-a.a. $x \in F$. It follows from Lemma 1.1 (with $d = 1$) and the fact that ν_x is supported on $D_x(E)$ that

$$\dim D_x(E) \geq \frac{(\alpha+\beta+1-d)/2 - 1/s}{1-1/s} = \frac{\tau - 1/s}{1-1/s}$$

for every $s < \infty$ for λ-a.a. $x \in F$. Letting $s \to \infty$, we have $\dim D_x(E) \geq \tau$ for λ-a.a. $x \in F$ as desired. $\square$

Proof of Theorem 2.3: It is clear that (2.3) holds when $p = 1$, $q = \infty$, and $s = 1$. Thus it is enough to prove a restricted weak type version of the result that would correspond to (2.3) with $t = 1$. So suppose $E \subset \mathbb{R}^d$ and $S1_E\,(x, r) \geq \mu$ for

$$(x, r) \in F = \{(x, r) : x \in A \subset \mathbb{R}^d,\ r \in T_x \subset [r_0, R_0]\}$$

where the one-dimensional measure $|T_x|$ satisfies $B \leq |T_x| \leq 2B$ for some positive B. We will show that if $1/p$, $1/q/$, $1/s$ correspond to $t = 1$ then the inequality

$$\mu\, \lambda(A)^{1/q} B^{1/s} \lesssim |E|^{1/p} \tag{3.4}$$

holds. The strategy is to estimate $|E|$ from below. If $x \in \mathbb{R}^d$ let

$$E_x = \{x + r\omega \in E : r \in T_x,\ \omega \in S^{d-1}\}.$$

Then, for $x_1, \ldots, x_N \in A$ we have

$$|E| \geq \sum_{j=1}^{N} |E_{x_j}| - \sum_{1 \leq j < k \leq N} |E_{x_j} \cap E_{x_k}|. \tag{3.5}$$

It follows from our assumptions that if $x \in A$ then

$$|E_x| \gtrsim B\mu. \tag{3.6}$$

To make use of this by way of the estimate (3.5) we will need an upper bound for $|E_{x_1} \cap E_{x_2}|$. Thus we will begin the proof of (a) by establishing the following (two-dimensional) estimate

$$|E_{x_1} \cap E_{x_2}| \lesssim \frac{B^{3/2}}{|x_1 - x_2|^{1/2}} \tag{3.7}$$

(where the implied constant depends on r_0 and the size of the support of λ).
Proof of (3.7): Assume without loss of generality that the x_i lie on the x-axis and abuse notation by writing $x_i = (x_i, 0)$. Consider the transformation of $\mathbb{R}^2$ defined by

$$(y_1, y_2) \mapsto \big((x_1 - y_1)^2 + y_2^2, (x_2 - y_1)^2 + y_2^2\big) \doteq (r_1^2, r_2^2).$$

We will establish (3.7) by showing that

$$\int_{T_{x_2}}\int_{T_{x_1}}\Big|\frac{\partial(y_1,y_2)}{\partial(r_1,r_2)}\Big|\,dr_1\,dr_2\lesssim\frac{B^{3/2}}{|x_1-x_2|^{1/2}}\ .$$

Since $r_0\le r_1,r_2\le R_0$ it is enough to show that

$$\int_{T_{x_2}}\int_{T_{x_1}}\Big|\frac{\partial(y_1,y_2)}{\partial(r_1^2,r_2^2)}\Big|\,dr_1\,dr_2\lesssim\frac{B^{3/2}}{|x_1-x_2|^{1/2}}\tag{3.8}$$

holds. A computation shows that

$$\Big|\frac{\partial(y_1,y_2)}{\partial(r_1^2,r_2^2)}\Big|=\frac{1}{|y_2|\,|x_1-x_2|}.$$

Now $|y_2|\,|x_1-x_2|$ is just twice the area of the triangle with vertices $(x_1,0),(x_2,0),(y_1,y_2)$. Write θ for the angle between the x-axis and the line through $(x_1,0)$ and (y_1,y_2) and Δ for $|x_1-x_2|$. From

$$r_1\cos\theta=\frac{r_1^2+\Delta^2-r_2^2}{2\Delta}$$

it follows that

$$2|y_2|\,|x_1-x_2|=2\Delta r_1\sin\theta=\sqrt{\big(r_2^2-(r_1-\Delta)^2\big)\big(r_2^2-(r_1+\Delta)^2\big)}\ .$$

Since $r_1,r_2\ge r_0$ we can estimate

$$\sqrt{\big(r_2^2-|r_1-\Delta|^2\big)\big(r_2^2-|r_1+\Delta|^2\big)}\gtrsim\sqrt{\big|\big(r_2-|r_1-\Delta|\big)\big(r_2-|r_1+\Delta|\big)\big|}\ .$$

Therefore (3.8) will follow from the estimate

$$\int_{T_{x_1}}\int_{T_{x_2}}\frac{1}{\sqrt{\big|\big(r_2-|r_1-\Delta|\big)\big(r_2-|r_1+\Delta|\big)\big|}}\,dr_1\,dr_2\lesssim\frac{B^{3/2}}{\Delta^{1/2}}\ .$$

Since $B\le|T(x_i)|\le 2B$, a scaling argument shows that this will follow from

$$\int_{T_1}\int_{T_2}\frac{1}{\sqrt{\big|\big(r_2-|r_1-1|\big)\big(r_2-|r_1+1|\big)\big|}}\,dr_1\,dr_2\lesssim B^{3/2}.\tag{3.9}$$

where $B\le|T_i|\le 2B$ and $T_i\subset(\eta,\infty)$. Here $\eta>0$ depends on r_0 and an upper bound for $\Delta=|x_1-x_2|$ and so on the size of the support of λ. Also the implied constant in (3.9) depends on η. To see (3.9), note that, since $r_1>\eta$, at least one of the numbers

$$r_2-|r_1-1|,\ r_2-|r_1+1|$$

must have absolute value $\gtrsim\eta$ and so

$$\int_{T_1}\int_{T_2}\frac{1}{\sqrt{\big|\big(r_2-|r_1-1|\big)\big(r_2-|r_1+1|\big)\big|}}\,dr_1\,dr_2\lesssim$$

$$C(\eta)\int_{T_1}\int_{T_2}\Big(\frac{1}{\sqrt{\big|r_2-|r_1-1|\big|}}+\frac{1}{\sqrt{\big|r_2-|r_1+1|\big|}}\Big)\,dr_2\,dr_1.$$

Now (3.9) follows since the inner integral is $\lesssim |T_2|^{1/2}$. $\square$

We return to the project of bounding $|E|$ from below. If $x_1, \dots, x_N$ are in A, then we estimate, using (3.5), (3.6), and (3.7), that

$$|E| \geq |\cup_j E_{x_j}| \geq NB\mu - CB^{3/2} \sum_{1 \leq j < k \leq N} \frac{1}{|x_j - x_k|^{1/2}}. \tag{3.10}$$

We need to choose the $x_j \in A$ in order to control the sum

$$\sum_{1 \leq j < k \leq N} \frac{1}{|x_j - x_k|^{1/2}}$$

and we will do this by choosing the x_j independently from the probability space

$$\Big(A, \frac{\lambda}{\lambda(A)}\Big).$$

We reason

$$\mathbb{E}\Big(\frac{1}{|x_j - x_k|^{1/2}}\Big) = \frac{1}{\lambda(A)^2} \int_A \int_A \frac{1}{|x_j - x_k|^{1/2}} \, d\lambda(x_j)\, d\lambda(x_k) \leq$$

$$\frac{1}{\lambda(A)^2} \Big(\int_A \int_A 1 \, d\lambda(x_j)\, d\lambda(x_k) \Big)^{1 - 1/2\alpha} \Big(\int_A \int_A \frac{1}{|x_j - x_k|^{\alpha}} \, d\lambda(x_j)\, d\lambda(x_k) \Big)^{1/2\alpha} \leq \frac{C}{\lambda(A)^{1/\alpha}},$$

where we have used the hypotheses $\alpha > 1/2$ and

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x_j - x_k|^{\alpha}} \, d\lambda(x_j)\, d\lambda(x_k) < \infty.$$

Therefore

$$\mathbb{E}\Big(\sum_{1 \leq j < k \leq N} \frac{1}{|x_j - x_k|^{1/2}} \Big) \lesssim \frac{N^2}{\lambda(A)^{1/\alpha}}$$

and so there are $x_1, \dots, x_N \in A$ for which (3.10) gives

$$|E| \geq |\cup_j E_{x_j}| \geq NB\mu - \frac{CN^2 B^{3/2}}{\lambda(A)^{1/\alpha}}.$$

An appropriate choice of N then yields

$$\mu^2 B^{1/2} \lambda(A)^{1/\alpha} \lesssim |E|, \tag{3.11}$$

which is equivalent to (3.4). (The N here will be $\gtrsim 1$ so long as

$$\mu B^{-1/2} \lambda^{1/\alpha} \gtrsim 1.$$

If this fails it is easy to see that (3.11) holds anyway.)

The proofs of (b) and (c) follow the same general strategy. For (b) we will start with (3.10) but use the following lemma to choose the points x_j.

Lemma 3.2. *Suppose* $0 < \alpha < \alpha' < \gamma \leq d$ *and suppose that* λ *is a nonnegative compactly supported Borel measure on* $\mathbb{R}^d$ *satisfying the estimate*

$$\lambda\big(B(x, \delta)\big) \leq C\, \delta^{\alpha'} \ (x \in \mathbb{R}^d, \delta > 0). \tag{3.12}$$

Then it is possible to choose $x_1, \ldots, x_N \in A$ *such that*

$$\sum_{1\le i<j\le N} \frac{1}{|x_i - x_j|^\gamma} \lesssim \lambda(A)^{-\gamma/\alpha} N^{1+\gamma/\alpha}, \tag{3.13}$$

where the implied constant depends on $\alpha, \alpha', \gamma, C$*, and the size of the support of* λ*.*

Proof of Lemma 3.2: For $k = 2, \ldots N$, set

$$\eta_k = c\Big(\frac{\lambda(A)}{k}\Big)^{1/\alpha}$$

where c is a small positive constant which will depend only on α, α', the size of the support of λ, and C from (3.12): if $x_1, \ldots x_{N-1} \in A$, $1 \le k \le N-1$, and if we define

$$A(x_1, \ldots, x_k) = \{x \in A : |x - x_j| \ge \eta_j,\ 1 \le j \le k\},$$

then c should be chosen small enough to guarantee that

$$\lambda\big(A(x_1, \ldots, x_k)\big) \ge \lambda(A)/2. \tag{3.14}$$

Now define

$$\tilde A_N = \{(x_1, \ldots, x_N) \in A^N : x_1 \in A,\ x_2 \in A(x_1), \ldots,\ x_N \in A(x_1, \ldots, x_{N-1})\}$$

and define a probability measure $\tilde\lambda$ on $\tilde A_N$ by

$$\langle f, \tilde\lambda\rangle = \frac{1}{\lambda(A)} \int_A \frac{1}{\lambda\big(A(x_1)\big)} \int_{A(x_1)} \cdots$$
$$\frac{1}{\lambda\big(A(x_1, \ldots x_{N-1})\big)} \int_{A(x_1,\ldots,x_{N-1})} f(x_1, \ldots x_N)\, d\lambda(x_N) \cdots d\lambda(x_2)\, d\lambda(x_1).$$

If $j < k$, taking an expectation with respect to $\tilde\lambda$ gives

$$\mathbb{E}\Big(\frac{1}{|x_j - x_k|^\gamma}\Big) \lesssim \eta_j^{\alpha-\gamma} \lambda(A)^{-2} \int_{A\times A} \frac{1}{|x_j - x_k|^\alpha} d(\lambda\times\lambda)(x_k, x_j) \lesssim$$
$$\eta_j^{\alpha-\gamma}\lambda(A)^{-1} \lesssim \lambda(A)^{-\gamma/\alpha} j^{\gamma/\alpha - 1},$$

where we have used (3.12) and $\alpha' > \alpha$. Thus

$$\mathbb{E}\Big(\sum_{1\le j<k\le N} \frac{1}{|x_j - x_k|^\gamma}\Big) \lesssim \lambda(A)^{-\gamma/\alpha} \sum_{1\le j<k\le N} j^{\gamma/\alpha-1} \lesssim \lambda(A)^{-\gamma/\alpha} N^{1+\gamma/\alpha}.$$

□

We return to the proof of (b). Taking $\gamma = 1/2$, Lemma 2.3 shows that we can choose $x_1, \ldots, x_N \in A$ with

$$\sum_{1\le j<k\le N} \frac{1}{|x_j - x_k|^{1/2}} \lesssim \lambda(A)^{-1/2\alpha} N^{1+1/2\alpha}.$$

This time (3.10) gives

$$|E| \geq |\cup_j E_{x_j}| \geq NB\mu - \frac{CN^{1+1/2\alpha}B^{3/2}}{\lambda(A)^{1/2\alpha}}$$

and an optimal choice of N gives

$$\mu^{1+2\alpha}B^{1-\alpha}\lambda(A) \lesssim |E|. \tag{3.15}$$

(Again, N here will be $\gtrsim 1$ so long as $\mu B^{-1/2}\lambda(A)^{1/2\alpha} \gtrsim 1$, and if this fails it is easy to see that (3.15) holds anyway.) This completes the proof of (b).

The proof of (c) is again very similar. In this case the analogue of (3.7) is the estimate

$$|E_{x_1} \cap E_{x_2}| \lesssim \frac{B^2}{|x_1 - x_2|}, \tag{3.16}$$

which we will establish below. With (3.16) and the choice $\gamma = 1$ in Lemma 3.2, (3.5) leads to

$$|E| \geq |\cup_j E_{x_j}| \geq NB\mu - \frac{CN^{1+1/\alpha}B^2}{\lambda(A)^{1/\alpha}}.$$

One then proceeds as in the proof of (b). We omit the details here and will conclude the proof of Theorem 2.3 by sketching a proof of (3.16). Our starting point for this is the well known (see, e.g., Lemma 1 in [3]) estimate

$$|A_\delta(x_1, r_1) \cap A_\delta(x_2, r_2)| \lesssim \frac{\delta^2}{\delta + |x_1 - x_2| + |r_1 - r_2|}, \tag{3.17}$$

valid for annuli in $\mathbb{R}^d$ so long as $d \geq 3$. Assume that δ is small and that the subsets $S(x_i) \subset [r_0, R_0]$ can be written as disjoint unions of 2δ-length intervals:

$$S(x_i) = \cup_{j=1}^{J}[r_j^i - \delta, r_j^i + \delta].$$

Then since $|S(x_i)| \approx B$ we have $J\delta \approx B$. Now

$$E_{x_i} \subset \cup_{j=1}^{J} A_\delta(x_i, r_j^i)$$

so, by (3.17),

$$|E_{x_1} \cap E_{x_2}| \leq \sum_{j_1, j_2} |A_\delta(x_1, r_{j_1}^1) \cap A_\delta(x_2, r_{j_2}^2)| \lesssim \sum_{j_1, j_2} \frac{\delta^2}{\delta + |x_1 - x_2| + |r_{j_1}^1 - r_{j_2}^2|}.$$

For fixed j_1 a rearrangement argument and $J\delta \approx B$ show that

$$\sum_{j_2} \frac{\delta}{\delta + |x_1 - x_2| + |r_{j_1}^1 - r_{j_2}^2|} \lesssim \int_{|x_1 - x_2|}^{|x_1 - x_2| + B} \frac{1}{r}\, dr.$$

Thus, using $J\delta \approx B$ again,

$$|E_{x_1} \cap E_{x_2}| \lesssim B \log(1 + B/|x_1 - x_2|),$$

more than enough to establish (3.16).

$\square$

Proof of Theorem 2.4: The proof is analogous to the proof of Theorem 2.2. We begin with (a). Fix an arbitrary $\eta > 0$. We will show that

$$\dim\{x \in \mathbb{R}^2 : \dim D_x(E) < (2\beta - 1)/3 - \eta\} = 0.$$

Fix α and α' with $0 < \alpha < \alpha' < 1/2$. Suppose that λ is compactly supported and satisfies (2.5). It is enough to show that if $\tilde{F} \subset \mathbb{R}^2$ is compact and $\lambda(\tilde{F}) > 0$, then there is some compact F with $F \subset \tilde{F}$, $\lambda(F) > 0$, and $\dim D_x(E) > (2\beta - 1)/3 - \eta$ for λ-a.a. $x \in F$.

Fix a small $\epsilon > 0$. Set $\beta' = \beta - \epsilon$. Suppose p, q, and s are defined by (2.6) with $t = 1 - \epsilon$, then put

$$\rho = \beta' + \frac{1}{p}(2 - \beta') - \epsilon,$$

and finally define τ by

$$\rho - 1 = \tau + \frac{1}{s}(1 - \tau).$$

Observe that if ϵ were 0 then we would have $\tau = (2\beta - 1)/3$. Thus if ϵ is small enough, we do have $\tau > (2\beta - 1)/3 - \eta$. Since $\dim E = \beta$ there is a probability measure ν on E satisfying

$$\nu\big(B(x, \delta)\big) \lesssim \delta^{\beta'} \ (x \in \mathbb{R}^2,\ \delta > 0).$$

Now let $\tilde{F}$ be as above and then choose r_0, R_0, $r(1)$, $r(2)$, and F as in the proof of Theorem 2.2. It follows from the discussion before Lemma 1.1 and the definition of ρ that $\nu * K_\rho \in L^p(\mathbb{R}^2)$. It then follows, as in the proof of Theorem 2.2 but using (b) of Theorem 2.3 instead of Theorem 2.1, that $\dim D_x(E) \geq \tau$ for λ-a.a. $x \in F$.

The proof of (b) of Theorem 2.4 uses (c) of Theorem 2.3 instead of (b) of that theorem and is otherwise analogous to the argument above.

□

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