PROJECTIVE DUALITY AND A CHERN-MATHER INVOLUTION

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Abstract. We observe that linear relations among Chern-Mather classes of projective varieties are preserved by projective duality. We deduce the existence of an explicit involution on a part of the Chow group of projective space, encoding the effect of duality on Chern-Mather classes. Applications include Plücker formulae, constraints on self-dual varieties, generalizations to singular varieties of classical formulas for the degree of the dual and the dual defect, formulas for the Euclidean distance degree, and computations of Chern-Mather classes and local Euler obstructions for cones.

1. Introduction

The conormal space of a projective variety $V \subseteq \mathbb{P}^n$ is intimately related with both the dual variety $V^\vee \subseteq \mathbb{P}^{n\vee}$ and intersection-theoretic invariants of V, specifically the Chern-Mather class of V. Not surprisingly, formulas for Chern-Mather classes arise in the study of dual varieties, and invariants associated with the latter may be expressed in terms of the former. The main result of this paper is a very direct expression of this kinship: we will prove that the (push-forward to projective space of the) Chern-Mather class of $V^\vee$ may be obtained by applying an explicit involution to the Chern-Mather class of V. While straightforward, this fact carries a remarkable amount of information: we will present applications to Plücker formulae, self-dual varieties, formulas for the dual defect of a variety, computations of Chern-Mather classes and local Euler obstructions for cones, and more.

The main result of this note admits several equivalent formulations. Let $V \subseteq \mathbb{P}^n$ be a projective variety over an algebraically closed field of characteristic 0. The Chern-Mather class of V, introduced by R. MacPherson ([Mac74]), is a generalization to arbitrary varieties of the notion of total Chern class of the tangent bundle of a nonsingular variety. This note will be concerned with the push-forward to $\mathbb{P}^n$ of the Chern-Mather class; we denote this push-forward by $c_{\mathrm{Ma}}(V)$, and let $c^-_{\mathrm{Ma}}(V) = (-1)^{\dim V} c_{\mathrm{Ma}}(V)$. The first formulation expresses a basic compatibility property of this signed Chern-Mather class under projective duality.

Theorem 1.1. *Let $V_1, \dots, V_r \subsetneq \mathbb{P}^n$ be proper closed subvarieties and $a_1, \dots, a_r$ integers such that*

$$\sum_{k=1}^{r} a_k \, c^-_{Ma}(V_k) = 0$$

in $A_*\mathbb{P}^n$. Then

$$\sum_{k=1}^{r} a_k \, c^-_{Ma}(V_k^\vee) = 0$$

in $A_*\mathbb{P}^{n\vee}$.

This result implies that we can define a 'duality' map on rational equivalence classes $\alpha \in A^*\mathbb{P}^n$ supported in dimension $< n$: write α as a linear combination of signed Chern-Mather classes of proper subvarieties, $\alpha = \sum_i a_i \, c^-_{\mathrm{Ma}}(V_i)$; and then define $\alpha^\vee$

to be $\sum_i a_i\, c^-_{\rm Ma}(V_i^\vee)$. This operation is well-defined by Theorem 1.1, maps $c^-_{\rm Ma}(V)$ to $c^-_{\rm Ma}(V^\vee)$, and is uniquely determined by this property. The other formulations of the result aim at making this duality map more explicit and effectively computable.

For the second formulation, let H denote the hyperplane class and note that the classes

$$(1.1)\qquad \{\mathbb{P}^k\} := c^-_{\rm Ma}(\mathbb{P}^k) = (-1)^k(1+H)^{k+1}H^{n-k}\cap[\mathbb{P}^n] \quad , \quad k=0,\dots,n$$

form an additive basis of $A_*\mathbb{P}^n$, so that every rational equivalence class may be written as a combination of these classes.

Theorem 1.2. *Let $V \subsetneq \mathbb{P}^n$ be a projective variety. If*

$$c^-_{Ma}(V) = \sum_{i=0}^{n-1} a_i\,\{\mathbb{P}^i\} \quad ,$$

then

$$c^-_{Ma}(V^\vee) = \sum_{i=0}^{n-1} a_{n-1-i}\,\{\mathbb{P}^i\} \quad .$$

The integers a_i may be expressed as certain explicit combinations of the degrees of the components of $c_{\rm Ma}(V)$: $a_i = \sum_{j=i}^{\dim V}\binom{j+1}{i+1}(-1)^{\dim V-j}c_{\rm Ma}(V)_j$ (see §2.3 and Proposition 3.13). In fact, a_i equals the i-th 'rank', or 'polar class', of V (Remark 2.7). Theorem 1.2 amounts to an alternative viewpoint on the very classical theory of these ranks: the ranks are interpreted here as the coefficients of the signed Chern-Mather class in an additive basis for the Chow group of the ambient variety (projective space) obtained by considering signed Chern-Mather classes of certain distinguished subvarieties (linear subspaces). It is tempting to guess that a similar approach may yield tools analogous to the ranks in more general settings. Also, the ranks are essentially obtained by applying to the Chern-Mather class of a variety a change-of-basis matrix whose entries are the coefficients of the (signed) Chern classes of the distinguished subvarieties in terms of their fundamental classes. The square of this 'Chern matrix' is the identity (Proposition 2.8), a simple but intriguing fact that is likely just a facet of a substantially more general result, and which seems (to us) less apparent from the classical point of view on ranks.

The third formulation of the main result amounts to an explicit form of the duality introduced above. For a fixed n, consider the following $\mathbb{Z}$-linear map defined on polynomials $p(t)\in\mathbb{Z}[t]$:

$$p\mapsto \mathcal{J}_n(p),\quad p(t)\mapsto p(-1-t)-p(-1)\big((1+t)^{n+1}-t^{n+1}\big) \quad .$$

It is immediately verified that if $\deg p\le n$, then $\deg\mathcal{J}_n(p)\le n$; and that $\mathcal{J}_n^2(p)=p$ for polynomials $p(t)\in(t)$.

Theorem 1.3. *Let $V \subsetneq \mathbb{P}^n$ be a projective variety. If*

$$c^-_{Ma}(V) = q(H)\cap[\mathbb{P}^n]$$

for a polynomial q of degree $\le n$ in the hyperplane class H in $\mathbb{P}^n$, then

$$c^-_{Ma}(V^\vee) = \mathcal{J}_n(q)\cap[\mathbb{P}^{n\vee}] \quad ,$$

where $\mathcal{J}_n(q)$ is viewed as a polynomial in the hyperplane class in $\mathbb{P}^{n\vee}$.

The equivalence of Theorems 1.1—1.3 is verified in §2.1; and then Theorem 1.1 is proven in §2.2. Corollary 2.3, proven along the way, has a direct application to the *Euclidean distance degree,* and this is presented in §2.4 as a generalization of a result from [DHOS].

Other applications illustrating the use of the main theorem are presented in §3. With some exceptions, these results are standard; our main goal here is to illustrate that the simple 'Chern-Mather involution' defined above leads to very efficient proofs of these facts, often providing at the same time a straightforward generalization to the singular case. In §3.1 we recover a general form of the classical Plücker formula for plane curves, and give a transparent derivation of B. Teissier's generalization of this formula from a result of R. Piene on Chern-Mather classes. (This is essentially a re-packaging of Piene's own proof in [Pie88]. Also see [Kle77, p. 356-7] for an equally transparent derivation of a generalization of this formula.) In §3.2 we obtain constraints for a variety to be self-dual; for example, we show (Proposition 3.9) that the singular locus of a self-dual hypersurface of degree d in $\mathbb{P}^n$ has dimension $\geq \frac{n-3}{2}$. This simple statement may be new—we were not able to find it in the literature. Proposition 3.13 in §3.3 recovers and generalizes to singular varieties the Katz-Kleiman-Holme formula for the 'dual defect' and degree of $V^\vee$. The formula we obtain is equivalent to, but somewhat leaner than, the generalization given in [MT07] (see Remark 3.15 for a comparison).

In §3.4 (Propositions 3.16, 3.17, and 3.20) we obtain formulas for the Chern-Mather class of a cone and for the local Euler obstruction along the vertex of a cone; these formulas also appear to be new, although they would likely be straightforward consequences of other known results, such as the expression of the local Euler obstruction as an alternating sum of multiplicities of polar varieties from [LT81]. For instance, we prove that if V is a cone over a variety W, then the local Euler obstruction of V at a point p on the vertex equals

$$\mathrm{Eu}_V(p) = \sum_{j=0}^{\dim W} (-1)^j c_{\mathrm{Ma}}(W)_j$$

where $c_{\mathrm{Ma}}(W)_j$ denotes the degree of the j-dimensional component of the Chern-Mather class of W. Questions of this type do not involve duality in their formulation, but duality offers an effective tool to address them, relying directly on the Chern-Mather involution. The proofs given here are direct and self-contained.

An alternative treatment of the connection between the Chern-Mather classes of a projective variety V and of its dual $V^\vee$ (including the more general case of higher order duals in Grassmannians) may be found in [EOY97] (Theorem 4.5), with focus on the relation between the corresponding constructible functions via the *topological Radon transform* studied by L. Ernström, cf. [Ern94]. (We should also mention the formalism of [KS94, §9.7], particularly the 'Fourier transform' of p. 403 and ff.) The approach taken in this note is different: here we decompose the Chern-Mather class of a variety as a linear combination of Chern-Mather classes of *a priori* unrelated varieties, such as linear subspaces; no such decomposition holds at the level of constructible functions. Working directly at the level of classes results in a coarser theory, but affords a greater level of flexibility, which plays a key role in the applications presented in §3. The involution introduced here is also a Radon-type transformation, but it is different from either the 'homological Radon transformation' or the 'homological Verdier Radon transformation' of [EOY97, §3]. It would be interesting to study precise relationships between these notions.

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2. Proofs

2.1. **Theorems 1.1—1.3 are equivalent.** The fact that Theorem 1.1 implies Theorem 1.2 follows immediately from the fact that the dual of a dimension i linear subspace of $\mathbb{P}^n$ is a dimension $n-1-i$ linear subspace of $\mathbb{P}^{n\vee}$: if $c^-_{\mathrm{Ma}}(V)=\sum_{i=0}^{n-1} a_i\{\mathbb{P}^i\}$, then $c^-_{\mathrm{Ma}}(V)-\sum_{i=0}^{n-1} a_i c^-_{\mathrm{Ma}}(\mathbb{P}^i)=0$, so $c^-_{\mathrm{Ma}}(V^\vee)-\sum_{i=0}^{n-1} a_i c^-_{\mathrm{Ma}}(\mathbb{P}^{n-1-i})=0$ by Theorem 1.1, and Theorem 1.2 follows.

To see that Theorem 1.2 implies Theorem 1.3, observe that $\{\mathbb{P}^k\}=\pi_k(H)\cap[\mathbb{P}^n]$ with $\pi_k(H)=(-1)^k\left((1+H)^{k+1}H^{n-k}-H^{n+1}\right)$, a polynomial in H of degree $\le n$. If $c^-_{\mathrm{Ma}}(V)=q(H)\cap[\mathbb{P}^n]$ as in the statement of Theorem 1.3, and $c^-_{\mathrm{Ma}}(V)=\sum_i a_i\{\mathbb{P}^i\}$, then $q(H)=\sum_i a_i\pi_i(H)$. Therefore $\mathcal{J}_n(q)=\sum_i a_i\mathcal{J}_n(\pi_i)$ by linearity, while by Theorem 1.2

$$c^-_{\mathrm{Ma}}(V^\vee)=\sum_i a_i\{\mathbb{P}^{n-i-1}\}\quad .$$

It suffices then to observe that $\mathcal{J}_n(\pi_k)\cap[\mathbb{P}^n]=\{\mathbb{P}^{n-k-1}\}$, that is,

$$\mathcal{J}_n\left((-1)^k\left((1+H)^{k+1}H^{n-k}-H^{n+1}\right)\right)=(-1)^{n-k-1}\left((1+H)^{n-k}H^{k-1}-H^{n+1}\right)\quad ,$$

and this is an immediate consequence of the definition of $\mathcal{J}_n$.

Finally, to verify that Theorem 1.3 implies Theorem 1.1, assume as in the statement of Theorem 1.1 that $\sum_{k=1}^r a_k\, c^-_{\mathrm{Ma}}(V_k)=0$. If $q_k\in(H)\subseteq\mathbb{A}_*(\mathbb{P}^n)$ are the polynomials of degree $\le n$ corresponding to the classes $c^-_{\mathrm{Ma}}(V_k)$, we have

$$\sum_{k=1}^r a_k\, c^-_{\mathrm{Ma}}(V_k)=0\implies\sum_{k=1}^r a_k\, q_k=0\implies\sum_{k=1}^r a_k\,\mathcal{J}_n(q_k)=0$$

by linearity, and this implies $\sum_{k=1}^r a_k\, c^-_{\mathrm{Ma}}(V_k^\vee)=0$ according to Theorem 1.3, yielding Theorem 1.1.

2.2. **Proof of Theorem 1.1.** We work over an algebraically closed field of characteristic 0. This restriction on the characteristic is commonly adopted in dealing with characteristic classes of singular varieties, and it is further needed for Lemma 2.1 below.

The (total) *Chern-Mather* class of V is defined by R. MacPherson as the push-forward of the Chern class of the tautological bundle of the Nash blow-up of V, [Mac74, §2]. By construction the tautological bundle restricts to the pull-back of the tangent bundle over the nonsingular part V_{reg} of V, and it follows that the Chern-Mather class restricts to $c(TV_{\mathrm{reg}})\cap[V_{\mathrm{reg}}]$ on V_{reg}. In particular, the Chern-Mather class agrees with $c(TV)\cap[V]$ if V is nonsingular. MacPherson's theory of Chern classes for singular varieties is based on a natural trasformation defined by associating the Chern-Mather class with the *local Euler obstruction,* cf. [Mac74, §3]. (The local

Euler obstruction was independently introduced and studied by M. Kashiwara in the context of index theorems for holonomic $\mathscr{D}$-modules, [Kas73].)

We will use an alternative formulation, given by C. Sabbah in terms of the *conormal variety,* [Sab85, §1.2.1]. For V a proper closed subvariety of $\mathbb{P}^n$, we denote by $\mathbb{P}(C^*_V\mathbb{P}^n) \subseteq \mathbb{P}(T^*\mathbb{P}^n)$ the projective conormal variety of V, obtained as the Zariski closure of the projectivization of the conormal bundle of the nonsingular part $V_{\rm red}$ of V. Letting $g : \mathbb{P}(C^*_V\mathbb{P}^n) \to V \hookrightarrow \mathbb{P}^n$ be the projection followed by the inclusion in $\mathbb{P}^n$, the Chern-Mather class of V pushes forward to the class

$$c_{\rm Ma}(V) = (-1)^{n-1-\dim V} c(T\mathbb{P}^n) \cap g_* \left(c(\mathcal{O}(1))^{-1} \cap [\mathbb{P}(C^*_V\mathbb{P}^n)]\right)$$

in $\mathbb{A}_*\mathbb{P}^n$ (cf. [PP01, §1]). Here, $\mathcal{O}(1)$ is the restriction of the universal quotient bundle over $\mathbb{P}(T^*\mathbb{P}^n)$. If $V \overset{\iota}{\hookrightarrow} \mathbb{P}^n$ is a *nonsingular* variety, then $c_{\rm Ma}(V) = \iota_*(c(TV) \cap [V])$.

As in §1 we introduce a sign according to the dimension of V and let

$$c^-_{\rm Ma}(V) = (-1)^{n-1} c(T\mathbb{P}^n) \cap g_* \left(c(\mathcal{O}(1))^{-1} \cap [\mathbb{P}(C^*_V\mathbb{P}^n)]\right) \in \mathbb{A}_*\mathbb{P}^n \quad .$$

Now $\mathbb{P}(T^*\mathbb{P}^n)$ may be realized as the incidence correspondence $I \subseteq \mathbb{P}^n \times \mathbb{P}^{n\vee}$, a divisor of class $H + h$ (denoting by H, resp., h the hyperplane classes in $\mathbb{P}^n$, resp., $\mathbb{P}^{n\vee}$, and their pull-backs). It is easy to verify that in this realization the universal bundle $\mathcal{O}(1)$ agrees with the normal bundle to I in $\mathbb{P}^n \times \mathbb{P}^{n\vee}$. Further, the projective conormal variety $\mathbb{P}(C^*_V\mathbb{P}^n)$ may then be identified with the closure of the correspondence

$$\Phi_V := \overline{\{(p, H) \,|\, p \in V_{\rm reg} \text{ and } H \supseteq \mathbb{T}_pV\}} \quad .$$

Therefore we may write

$$c^-_{\rm Ma}(V) = (-1)^{n-1}(1 + H)^{n+1} \cap \pi_{1*} \left(\frac{1}{1 + H + h} \cap [\Phi_V]\right) \tag{2.1}$$

where π_{1*} is the restriction of the projection onto the first factor $\mathbb{P}^n \times \mathbb{P}^{n\vee} \to \mathbb{P}^n$.

Switching factors gives the canonical isomorphism

$$\mathbb{P}^n \times \mathbb{P}^{n\vee} \cong \mathbb{P}^{n\vee} \times \mathbb{P}^n \cong \mathbb{P}^{n\vee} \times \mathbb{P}^{n\vee\vee} \quad , \tag{*}$$

and we recall the following classical result, which holds under our blanket characteristic 0 hypothesis. (In fact, this reflexivity holds as soon as the projection $\Phi_V \to V^\vee$ is generically smooth, cf. [Kle86, p. 169].)

Lemma 2.1. *Via the identification (*),* $\Phi_V = \Phi_{V^\vee}$.

Proof. See [Kle86] or e.g., [Tev05, Theorem 1.7]. □

Let then $\Phi := \Phi_V = \Phi_{V^\vee}$. The class of Φ in $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ determines $c^-_{\rm Ma}(V)$ by (2.1), and $c^-_{\rm Ma}(V^\vee)$ by the same token:

$$c^-_{\rm Ma}(V^\vee) = (-1)^{n-1}(1 + h)^{n+1} \cap \pi_{2*} \left(\frac{1}{1 + H + h} \cap [\Phi]\right) \tag{2.2}$$

where π_2 is the restriction of the projection onto the second factor $\mathbb{P}^n \times \mathbb{P}^{n\vee} \to \mathbb{P}^{n\vee}$. In order to prove Theorem 1.1 it suffices to prove that, conversely, $[\Phi]$ is determined by the class $c^-_{\rm Ma}(V)$ and depends linearly on this class.

This is a general fact, with no direct relation to Chern-Mather classes.

Proposition 2.2. *Let* $\gamma = \sum_{i=0}^{n-1} \gamma_i \, [\mathbb{P}^i]$ *be a class in* $A_*\mathbb{P}^n$*. Then the unique class* Γ *in* $A_{n-1}I$ *such that*

$$\gamma = (-1)^{n-1}(1 + H)^{n+1} \cap \pi_{1*} \left(\frac{1}{1 + H + h} \cap \Gamma\right)$$

is

$$\Gamma = \sum_{i=0}^{n-1} (-1)^i \gamma_i \, (H+h)^i H^{n-i} \cap [I] \quad .$$

Proof. Since $I \cong \mathbb{P}T^*\mathbb{P}^n$, every class $\Gamma \in A_{n-1}I$ may be written uniquely as

$$\Gamma = \sum_{i=0}^{n-1} a_i \, c_1(\mathcal{O}(1))^i \cdot H^{n-i} \cap [I] \quad ,$$

by [Ful84, Theorem 3.3 (b)]. Therefore, as a class in $\mathbb{P}^n \times \mathbb{P}^{n\vee}$,

$$\Gamma = \sum_{i=0}^{n-1} a_i \, (H+h)^{i+1} \cdot H^{n-i} \cap [\mathbb{P}^n \times \mathbb{P}^{n\vee}] \quad .$$

It follows that the push-forward

$$\pi_{1*}\left(\frac{1}{1+H+h} \cap \Gamma\right)$$

is the coefficient of h^n in the series expansion of

$$\sum_{i=0}^{n-1} a_i \, \frac{(H+h)^{i+1} \cdot H^{n-i}}{1+H+h}$$

and this is easily computed to be $\sum_{i=0}^{n-1} a_i \, \frac{(-1)^{n-i-1} H^{n-i}}{(1+H)^{n+1}}$. Therefore

$$(-1)^{n-1}(1+H)^{n+1} \cap \pi_{1*}\left(\frac{1}{1+H+h} \cap \Gamma\right) = \sum_{i=0}^{n-1} (-1)^i a_i \, H^{n-i} \quad .$$

Equating this class with γ gives the statement. □

Theorem 1.1 is an easy consequence of Proposition 2.2.

Proof of Theorem 1.1. Assume $\sum_{k=1}^r a_k \, c^-_{\mathrm{Ma}}(V_k) = 0$. By Proposition 2.2,

$$\sum_{k=1}^{r} a_k \, [\Phi_k] = 0$$

where $\Phi_k = \Phi_{V_k} = \Phi_{V_k^\vee}$. By (2.2) this implies that $\sum_{k=1}^r a_k \, c^-_{\mathrm{Ma}}(V_k^\vee) = 0$, completing the proof. □

The explicit relation between the Chern-Mather class of V and the conormal cycle $\Phi = \Phi_V$ is the following.

Corollary 2.3. *Let V be a proper closed subvariety of $\mathbb{P}^n$. Then the class of the conormal variety of V in $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ is given by*

$$(2.3) \qquad [\Phi_V] = \sum_{j=0}^{n-1} (-1)^{\dim V + j} c_{Ma}(V)_j \, (H+h)^{j+1} H^{n-j} \cap [\mathbb{P}^n \times \mathbb{P}^{n\vee}]$$

where $c_{Ma}(V)_j$ denotes the degree of the j-dimensional component of the Chern-Mather class of V in $\mathbb{P}^n$: $c_{Ma}(V) = \sum_{j=1}^{n-1} c_{Ma}(V)_j \, [\mathbb{P}^j]$.

Proof. This follows from (2.1) and Proposition 2.2. (Note the correction in the sign of the coefficients to account for the difference between $c_{\mathrm{Ma}}(V)$ and $c^-_{\mathrm{Ma}}(V)$.) □

2.3. **The $\{\mathbb{P}^i\}$ coefficients.** It is occasionally useful to extract explicit expressions for the coefficients a_i appearing in Theorem 1.2. It is in fact a simple matter to obtain Poincaré duals of the classes $\{\mathbb{P}^i\}$, and this yields expressions of the coefficients for any given class α as intersection degrees of α with certain explicit classes in $\mathbb{P}^n$.

Lemma 2.4. *For $i = 0, \dots, n$ let*

$$\{\mathbb{P}^i\}^* = \frac{(-1)^i}{(1+H)^{i+2}}[\mathbb{P}^{n-i}] \quad .$$

Then for all $0 \le i, j \le n$, $\int \{\mathbb{P}^i\}^* \cdot \{\mathbb{P}^j\} = \delta_{ij}$.

Proof. The claim is that

$$\int \frac{(-1)^i H^i}{(1+H)^{i+2}} \cap \{\mathbb{P}^j\} = \delta_{ij}$$

for $0 \le i, j \le n$, i.e., that the degree of H^n in the expansion of

$$\frac{(-1)^i H^i}{(1+H)^{i+2}} \cdot (-1)^j (1+H)^{j+1} H^{n-j} = (-1)^{i+j}(1+H)^{j-i-1} H^{n+i-j}$$

is the Kronecker delta δ_{ij}; and this is immediate. □

Proposition 2.5. *Let $\alpha = \sum_j \alpha_j [\mathbb{P}^j] \in A_* \mathbb{P}^n$, and assume $\alpha = \sum_i a_i \{\mathbb{P}^i\}$. Then*

$$a_i = \int \frac{(-1)^i H^i}{(1+H)^{i+2}} \cap \alpha = \sum_{j=i}^{\dim V} \binom{j+1}{i+1} (-1)^j \alpha_j \quad . \tag{2.4}$$

Proof. By Lemma 2.4,

$$a_i = \sum_{j=0}^{n-1} a_j \delta_{ij} = \sum_{j=0}^{n-1} a_j \int \{\mathbb{P}^i\}^* \cdot \{\mathbb{P}^j\} = \int \{\mathbb{P}^i\}^* \cdot \alpha \quad ,$$

giving the first equality. The second equality is obtained by computing the coefficient of H^n in $\frac{(-1)^i H^i}{(1+H)^{i+2}} \sum_j \alpha_j H^j$. □

Example 2.6. Let $V \overset{\iota}{\hookrightarrow} \mathbb{P}^n$ be a nonsingular hypersurface of degree d. Then

$$c_{\mathrm{Ma}}^-(V) = (-1)^{n-1} \iota_* c(TV) \cap [V] = \sum_{i=0}^{n-1} d(d-1)^{n-1-i} \{\mathbb{P}^i\} \quad .$$

Indeed,

$$\iota_* c(TV) \cap [V] = \frac{(1+H)^{n+1}\, dH}{1+dH} \cap [\mathbb{P}^n] \quad , \tag{2.5}$$

so by Proposition 2.5 the coefficient of $\{\mathbb{P}^i\}$ in $c_{\mathrm{Ma}}^-(V)$ equals the coefficient of H^n in the expansion of

$$(-1)^{n-1} \frac{(-1)^i H^i}{(1+H)^{i+2}} \cdot \frac{(1+H)^{n+1}\, dH}{1+dH} = (-1)^{n-i-1} \frac{(1+H)^{n-1-i}\, dH^{i+1}}{1+dH} \quad ,$$

i.e., the coefficient of H^{n-i-1} in

$$(-1)^{n-i-1} d\, \frac{(1+H)^{n-1-i}}{1+dH} \quad .$$

By [AF95], this equals the coefficient of H^{n-i-1} in

$$(-1)^{n-i-1} d\,(1+(1-d)H)^{n-1-i} \quad ,$$

and this is $d(d-1)^{n-1-i}$.

More generally, we see that if V is a hypersurface in $\mathbb{P}^n$ and $\dim \operatorname{Sing} V \le r$, and $c_{\mathrm{Ma}}^- = \sum_{i=0}^{n-1} a_i\{\mathbb{P}^i\}$, then

$$a_i = d(d-1)^{n-1-i} \quad \text{for } i = r+1, \dots, n-1.$$

Indeed, the Chern-Mather class of V restricts to $c(TV_{\mathrm{reg}}) \cap [V_{\mathrm{reg}}]$ on the nonsingular part V_{reg}, so it agrees with the Chern class of the virtual tangent bundle of V (given by the right-hand side of (2.5)) in dimension $> r$. ⌟

For instance, this implies that the dual of a nonsingular hypersurface V of degree $d > 1$ is a hypersurface of degree $d(d-1)^{n-1}$: indeed, the degree of $V^\vee$ is the coefficient of $\{\mathbb{P}^{n-1}\}$ in $c_{\mathrm{Ma}}^-(V^\vee)$, hence (by Theorem 1.2) the coefficient of $\{\mathbb{P}^0\}$ in $c_{\mathrm{Ma}}^-(V)$, hence (by Example 2.6) it equals $d(d-1)^{n-1}$.

Remark 2.7. In terms of conormal spaces, the equality $c_{\mathrm{Ma}}^-(V) = \sum_i a_i\{\mathbb{P}^i\}$ is equivalent to $[\Phi_V] = \sum_i a_i[\Phi_{\mathbb{P}^i}]$. Now $\Phi_{\mathbb{P}^i} = \mathbb{P}^i \times \mathbb{P}^{n-1-i}$ as a subvariety of $\mathbb{P}^n \times \mathbb{P}^n$; thus

$$[\Phi_V] = a_0 H^n h + \dots + a_{n-1} H h^n \quad .$$

Classically, the a_i are known as the *ranks* (or *polar classes*) of V. ⌟

By Proposition 2.5, the ranks of a projective variety $V \subsetneq \mathbb{P}^n$ may be obtained from $c_{\mathrm{Ma}}^-(V) = \sum_j \alpha_j[\mathbb{P}^j]$ by multiplying the $n \times n$ matrix M with entries

$$M_{ij} = (-1)^{j-1}\binom{j}{i}$$

by the column vector $(\alpha_0, \dots, \alpha_{n-1})^t$. We can now observe that this matrix is *also* the matrix expressing the coefficients α_j of the signed $c_{\mathrm{Ma}}^-(V)$ in terms of the ranks a_i, that is, the $n \times n$ matrix whose (i,j) entry is the coefficient of $[\mathbb{P}^{i-1}]$ in $c_{\mathrm{Ma}}^-(\mathbb{P}^{j-1})$. In other words, this 'matrix of Chern coefficients' M satisfies the following identity.

Proposition 2.8. *For all n, $M^2 = \mathrm{Id}_n$.*

Proof. For all $1 \le i, j \le n$,

$$\sum_{k=1}^n (-1)^{k-1}(-1)^{j-1}\binom{i}{k}\binom{k}{j} = \sum_{k=1}^n (-1)^{k-j}\binom{i-j}{k-j}\binom{i}{j} = \delta_{ij}\binom{i}{j} = \delta_{ij}$$

as needed. □

This intriguing fact appears to admit a nontrivial generalization to setting of flag varieties, which will be discussed elsewhere.

2.4. **The Euclidean distance degree.** The *Euclidean distance degree* of a variety is the number of critical points of the squared distance to a general point outside the variety. It is proven in [DHOS, Theorem 5.4] that if the incidence variety $\Phi = \Phi_V$, i.e., the projective conormal variety $\mathbb{P}(C_V^*\mathbb{P}^n)$, does not meet the diagonal $\Delta(\mathbb{P}^n) \subseteq \mathbb{P}^n \times \mathbb{P}^n$, then the Euclidean distance degree of V equals the sum

$$\delta_0(V) + \dots + \delta_{n-1}(V) \quad , \tag{2.6}$$

where $\delta_i(V)$ are the polar degrees of V (and hence equal the coefficients a_i computed in §2.3, cf. Remark 2.7):

$$[\Phi_V] = \delta_0(V) H^n h + \dots + \delta_{n-1}(V) H h^n \quad .$$

The quantity (2.6) is easy to compute in term of the Chern-Mather class of V, by means of Corollary 2.3.

Proposition 2.9. *Let V be a proper subvariety of $\mathbb{P}^n$ of dimension m. Then $\delta_i(V) = 0$ for $i > m$, and*

$$\delta_0(V) + \cdots + \delta_m(V) = \sum_{j=0}^{m} (-1)^{m+j} c_{Ma}(V)_j \, (2^{j+1} - 1) \quad ,$$

where $c_{Ma}(V)_j$ is the degree of the j-dimensional component of the Chern-Mather class of V in $\mathbb{P}^n$.

Proof. Formally set $H = h = 1$ in the formula for $[\Phi_V]$ obtained in Corollary 2.3. (The fact that $H^{n+1} = 0$ accounts for the '-1'.) The vanishing of $\delta_i(V)$ for $i > m$ is also immediate from (2.3). □

Proposition 2.9 generalizes to arbitrarily singular varieties the formula appearing in Theorem 5.8 in [DHOS]: in the singular case, Chern-Mather classes play in this formula the same role as ordinary Chern classes in the nonsingular case. The right-hand side of this formula computes the Euclidean distance degree of any (nonsingular or otherwise) closed subvariety $V \subsetneq \mathbb{P}^n$ such that Φ_V does not meet the diagonal $\Delta(\mathbb{P}^n) \subseteq \mathbb{P}^n \times \mathbb{P}^{n\vee}$.

Other general formulas for the Euclidean distance degree in terms of degrees of polar classes are given in [Pie15, §5]. The precise relation between polar classes and Chern-Mather classes is studied in [Pie88].

3. Examples and applications

3.1. **Plücker formulae.** Several formulas of Plücker type may be recast in terms of Theorem 1.3; we recover the classical form of the Plücker formulae for plane curves as an illustration.

Let C be a reduced, irreducible plane curve of degree $d \geq 2$, and let $p_1, \dots, p_r$ be the singular points of C. Let m_i, resp., μ_i be the multiplicity, resp., Milnor number of C at p_i. A standard computation gives

$$c^{-}_{\mathrm{Ma}}(C) = -(dH + (3d - d^2 + \sum_i (\mu_i + m_i - 1))H^2) \cap [\mathbb{P}^2] \quad . \tag{3.1}$$

(Proof: The Chern-Mather class of C is the image of the local Euler obstruction Eu_C via MacPherson's natural transformation. Since the local Euler obstruction of a curve equals the multiplicity function, we have $\mathrm{Eu}_C = 1\!\!1_C + \sum_i (m_i - 1) 1\!\!1_{p_i}$, and therefore

$$c_{\mathrm{Ma}}(C) = d[\mathbb{P}^1] + \chi(C)[\mathbb{P}^0] + \sum_i (m_i - 1)[p_i] = (dH + (\chi(C) + \sum_i (m_i - 1))H^2) \cap [\mathbb{P}^2]$$

after push-forward to $\mathbb{P}^2$. The Euler characteristic of C equals the Euler characteristic of a nonsingular curve of degree d corrected by the presence of the singular points; each points contributes by its Milnor number, and this gives the stated formula.)

Let $\rho_i = \mu_i + m_i - 1$. For example, $\rho_i = 2$ if p_i is a node and $\rho_i = 3$ if p_i is a cusp.

Proposition 3.1. *With notation as above:*

$$c^{-}_{Ma}(C^\vee) = -\left((d(d-1) - \sum_i \rho_i)H + (d(2d-3) - 2\sum_i \rho_i)H^2 \right) \cap [\mathbb{P}^2] \quad . \tag{3.2}$$

Proof. Apply the involution in Theorem 1.3 to (3.1). □

As an immediate consequence, we recover a (well-known) general form of the classical Plücker formula for plane curves.

Corollary 3.2. $\deg C^\vee = d(d-1) - \sum_i \rho_i$.

The coefficient of H^2 in (3.2) also gives some information. Let $q_1, \dots, q_s$ be the singular points of $C^\vee$, and $\rho_j^\vee$, $j = 1, \dots, s$, the corresponding ρ numbers. Then (3.2) implies that

$$\sum_j \rho_j^\vee = R^2 - (2d^2 - 2d - 1)R + d^3(d-2)$$

where $R = \sum_i \rho_i$. The same conclusion may be drawn by applying biduality. For example, if C is nonsingular, so that $R = 0$, and C has b bitangents and f inflection points (counted with appropriate multiplicities) this shows that $2b + 3f = d^3(d-2)$.

More generally, R. Piene evaluates the contribution to the Chern-Mather class of a hypersurface X in $\mathbb{P}^n$ due to isolated singularities p_i ([Pie88, p. 25]) :

$$c_{\text{Ma}}(X) = c_{\text{Ma}}(X_g) + \sum_i (-1)^n (\mu_X(p_i) + \mu_{X\cap \underline{H}_i}(p_i))[\mathbb{P}^0] \tag{3.3}$$

where X_g is a nonsingular hypersurface of the same degree as X (so that $c_{\text{Ma}}(X_g)$ equals the push-forward of the ordinary total Chern class of X_g) and $\underline{H}_i$ is a general hyperplane through p_i. Therefore, with notation as in §1 we have

$$c^-_{\text{Ma}}(X) = c^-_{\text{Ma}}(X_g) - \sum_i (\mu_X(p_i) + \mu_{X\cap \underline{H}_i}(p_i))\{\mathbb{P}^0\}$$

and Theorem 1.1 implies

$$c^-_{\text{Ma}}(X^\vee) = c^-_{\text{Ma}}(X_g^\vee) - \sum_i (\mu_X(p_i) + \mu_{X\cap \underline{H}_i}(p_i))\{\mathbb{P}^{n-1}\} \quad .$$

Reading off the terms of dimension $n-1$ in this identity (cf. Example 2.6) yields

$$\deg(X^\vee) = d(d-1)^{n-1} - \sum_i (\mu_X(p_i) + \mu_{X\cap \underline{H}_i}(p_i)) \quad , \tag{3.4}$$

that is, B. Teissier's generalized Plücker formula [Tei80, II.3], [Pie88, p. 16]. Of course the nontrivial input here is Piene's generalization (3.3) of the basic formula (3.1); we are simply observing that Theorem 1.1 makes the derivation of (3.4) from (3.3) particularly transparent. Teissier has shown that $\mu_X(p) + \mu_{X\cap \underline{H}_i}(p)$ equals the multiplicity of the Jacobian ideal at p (in fact this is an ingredient in Piene's proof of (3.3)), and S. Kleiman generalized the Teissier-Plücker formula to arbitrary varieties with isolated singularities by replacing this multiplicity with the Buchsbaum-Rim multiplicity of the Jacobian matrix, [Kle94, Theorem 2]. It would be interesting to interpret Kleiman's formula as a computation of a Chern-Mather class; for complete intersections, the Buchsbaum-Rim multiplicity at p equals $\mu_X(p) + \mu_{X\cap \underline{H}_i}(p)$ (see [Kle99, (3.3.1)]).

The same invariants appearing in the Plücker formula for curves allow us to generalize another result from [DHOS], by applying Proposition 2.9 to expression (3.1) (or equivalently (3.2)).

Corollary 3.3. *Let $C \subseteq \mathbb{P}^n$ be a reduced curve of degree d. Assume that C meets the isotropic quadric transversally, and let $R = \sum_i \rho_i$ be the invariant associated with a general plane projection of C. Then the Euclidean distance degree of C is given by*

$$\textit{EDdegree}(C) = d^2 - R \quad .$$

Here we also use the fact that the Chern-Mather class is preserved by a general projection ([Pie88, Corollaire, p. 20]; and cf. [DHOS, Corollary 6.1]). The 'isotropic quadric' in $\mathbb{P}^n$ is the quadric with equation $x_0^2 + \cdots + x_n^2 = 0$. For a discussion of the requirement that C meets it transversally see [DHOS], particularly the comment following Theorem 5.4. Corollary 3.3 generalizes [DHOS, Corollary 5.9] to singular curves.

3.2. **Self-dual varieties.** A variety $V \subseteq \mathbb{P}^n$ is *self-dual* if there exists an isomorphism $\mathbb{P}^n \to \mathbb{P}^{n\vee}$ restricting to an isomorphism $V \to V^\vee$. Theorem 1.3 implies immediately that the Chern-Mather class of a self-dual plane curve is determined by its degree.

Proposition 3.4. *If C is a self-dual curve of degree d in $\mathbb{P}^2$, then*

$$c_{Ma}(C) = d[\mathbb{P}^1] + d[\mathbb{P}^0] \quad .$$

Proof. If $c_{\mathrm{Ma}}(C) = (dH+eH^2)\cap[\mathbb{P}^2]$ and C is self-dual, then $\mathscr{J}_2(dH+eH^2) = dH+eH^2$ by Theorem 1.3. By definition, $\mathscr{J}_2(dH + eH^2) = (2d - e)H + (3d - 2e)H^2$. Thus necessarily $d = e$ for self-dual curves. □

Applying Proposition 2.9, we see that the Euclidean distance degree of a self-dual curve of degree d (meeting the isotropic conic transversally) is $2d$. There are self-dual curves of all degrees $d \geq 2$: for all $1 \leq k \leq d-1$, the curve $y^d = x^k z^{d-k}$ is self-dual.

Analogous results can be obtained e.g., for surfaces in $\mathbb{P}^3$.

Proposition 3.5. *Let $S \subseteq \mathbb{P}^3$ be a self-dual surface of degree d. Then*

$$c_{Ma}(S) = d[\mathbb{P}^2] + e[\mathbb{P}^1] + 2(e-d)[\mathbb{P}^0]$$

for some integer e.

Proof. According to Theorem 1.3, if $c_{\mathrm{Ma}}(S) = (dH + eH^2 + fH^3) \cap [\mathbb{P}^3]$, then

$$c_{\mathrm{Ma}}(S^\vee) = ((3d - 2e + f)H + (6d - 5e + 3f)H^2 + (4d - 4e + 3f)H^3) \cap [\mathbb{P}^3]$$

(if $3d - 2e + f \neq 0$). Equating these two expressions determines f. □

If S has isolated singularities, then necessarily $e = -d(d-4)$, so this class equals

$$c_{\mathrm{Ma}}(S) = d[\mathbb{P}^2] - d(d-4)[\mathbb{P}^1] - 2d(d-3)[\mathbb{P}^0] \quad . \tag{3.5}$$

Examples include the nonsingular quadric, the cubic surface $x_0^3 = x_1x_2x_3$, and the Kummer quartic surface ([GH78, p. 784]). According to (3.5), any self-dual quartic surface with isolated singularities must have Chern-Mather class equal to $4[\mathbb{P}^2] - 8[\mathbb{P}^0]$. (Another candidate is a quartic surface with 7 ordinary double points and 3 singularities of type A_3 constructed in [Nar83].) This is a direct consequence of Theorem 1.3, and does not require any specific knowledge of these surfaces. Similarly, no further work is needed to establish the following result.

Corollary 3.6. *The Euclidean distance degree of a self-dual surface of degree d with isolated singularities in $\mathbb{P}^3$, transversal to the isotropic quadric, is $d^2 + d$.*

(Use Proposition 2.9.) With notation as in Proposition 3.5, the Euclidean distance degree of a self-dual surface with arbitrary singularities in $\mathbb{P}^3$ is $5d - e$. The Euclidean distance degree of a general nonsingular surface of degree d in $\mathbb{P}^3$ is $d^3 - d^2 + d$.

For more refined considerations, we can again invoke Piene's result (3.3): an isolated singularity p corrects the Chern-Mather class of S by $\rho(p) = \mu_S(p) + \mu_{S\cap\underline{H}}(p)$, where $\underline{H}$ is a general plane through p. For example, $\rho = 2$ for an ordinary double point.

Proposition 3.7. *Let $S \subseteq \mathbb{P}^3$ be a self-dual surface of degree d with isolated singularities $p_1, \dots, p_s$. Then $\sum_j \rho(p_j) = d^2(d-2)$.*

Proof. We have

$$c_{\mathrm{Ma}}(S) = d[\mathbb{P}^2] + d(4-d)[\mathbb{P}^1] + (d(d^2-4d+6) - \sum_j \rho(p_j))[\mathbb{P}^0]$$

by (3.3) (i.e., [Pie88, p. 25]). Comparing with (3.5), we see that

$$\sum_j \rho(p_j) = 2d(d-3) + d(d^2-4d+6) = d^2(d-2)$$

as stated. □

Corollary 3.8. *A self-dual surface in $\mathbb{P}^3$ of degree d with only ordinary double points as singularities must have $d^2(d-2)/2$ singular points. (In particular, d is necessarily even.)*

The Kummer surface is an example.

Analogous results may be obtained in higher dimension, by the same method. For hypersurfaces, one finds the following constraint.

Proposition 3.9. *The singular locus of a self-dual hypersurface of degree $d \geq 3$ in $\mathbb{P}^n$ has dimension $\geq \frac{n-3}{2}$.*

Proof. Let X be a self-dual hypersurface of degree d, and assume that $\dim \operatorname{Sing} X < \frac{n-3}{2}$; that is, $\dim \operatorname{Sing} X \leq \frac{n-1}{2} - 2$ if n is odd, and $\dim \operatorname{Sing} X \leq \frac{n}{2} - 2$ if n is even. We will show that this forces $d \leq 2$. As we saw in Example 2.6, under the stated conditions on $\operatorname{Sing} X$ we must have

$$\begin{aligned} c^{-}_{\mathrm{Ma}}(X) = d\{\mathbb{P}^{n-1}\} &+ d(d-1)\{\mathbb{P}^{n-2}\} + \cdots \\ &+ d(d-1)^{\frac{n-1}{2}-1}\{\mathbb{P}^{\frac{n-1}{2}+1}\} + d(d-1)^{\frac{n-1}{2}}\{\mathbb{P}^{\frac{n-1}{2}}\} + d(d-1)^{\frac{n-1}{2}+1}\{\mathbb{P}^{\frac{n-1}{2}-1}\} \\ &+ a_{\frac{n-1}{2}-2}\{\mathbb{P}^{\frac{n-1}{2}-2}\} + \cdots + a_0\{\mathbb{P}^0\} \end{aligned}$$

if n is odd, and

$$\begin{aligned} c^{-}_{\mathrm{Ma}}(X) = d\{\mathbb{P}^{n-1}\} &+ d(d-1)\{\mathbb{P}^{n-2}\} + \cdots \\ &+ d(d-1)^{\frac{n}{2}-1}\{\mathbb{P}^{\frac{n}{2}}\} + d(d-1)^{\frac{n}{2}}\{\mathbb{P}^{\frac{n}{2}-1}\} \\ &+ a_{\frac{n}{2}-2}\{\mathbb{P}^{\frac{n}{2}-2}\} + \cdots + a_0\{\mathbb{P}^0\} \end{aligned}$$

if n is even, for some integers a_i. If X is self-dual, by Theorem 1.2 necessarily $a_i = d(d-1)^i$ and

$$\begin{cases} d(d-1)^{\frac{n-1}{2}-1} = d(d-1)^{\frac{n-1}{2}+1} & \text{if } n \text{ is odd,} \\ d(d-1)^{\frac{n}{2}-1} = d(d-1)^{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$$

Since

$$d(d-1)^{\frac{n-1}{2}+1} = d(d-1)^{\frac{n-1}{2}-1} \iff d^2(d-1)^{\frac{n-1}{2}-1}(d-2) = 0$$

and

$$d(d-1)^{\frac{n}{2}} = d(d-1)^{\frac{n}{2}-1} \iff d(d-1)^{\frac{n}{2}-1}(d-2) = 0$$

we see that this cannot occur for $d \geq 3$. □

For example, the singular locus of a self-dual hypersurface of degree $d \geq 3$ in $\mathbb{P}^6$ or $\mathbb{P}^7$ has dimension at least 2. One example in $\mathbb{P}^7$ is the *Coble quartic,* which is singular along a 3-dimensional Kummer variety ([Pau02]). Applying Theorem 1.3 shows that the Chern-Mather class of the Coble quartic must be

$$4[\mathbb{P}^6]+16[\mathbb{P}^5]+48[\mathbb{P}^4]+(R+32)[\mathbb{P}^3]+(4R-136)[\mathbb{P}^2]+(6R-288)[\mathbb{P}^1]+(4R-208)[\mathbb{P}^0]$$

for some integer R. (What is R?) The hypersurface $x_0^n = x_1 \cdots x_n$ is self-dual for every n, and its singular locus has dimension $n-3$.

In arbitrary codimension, one simple constraint holds for proper subvarieties of *even* dimensional projective space. Let $c_{\mathrm{Ma}}(V)_j$ denote the degree of the j-dimensional component of $c_{\mathrm{Ma}}(V)$.

Proposition 3.10. *Let V be a self-dual subvariety of $\mathbb{P}^n$ with n even. Then $\sum_{j=0}^{\dim V}(-1)^j c_{Ma}(V)_j = 0$.*

Proof. Let $c_{\mathrm{Ma}}(V) = q(H) \cap [\mathbb{P}^n]$ with $\deg q(H) \leq n$. Then $\sum_{j=0}^{\dim V}(-1)^j c_{\mathrm{Ma}}(V)_j = (-1)^n q(-1)$. By self-duality and Theorem 1.3, $\mathcal{J}_n(q) = q$. It is immediately seen that $\mathcal{J}_n(q)(-1) = (-1)^{n+1}q(-1)$, and the result follows. □

3.3. **Dual defect and the Katz-Kleiman-Holme formula.** The (dual) *defect* of a variety $V \subseteq \mathbb{P}^n$ is $\operatorname{def} V := n-1-\dim V^\vee$: so that the defect of V is 0 when $V^\vee$ is a hypersurface. It is known that nonlinear nonsingular complete intersections have 0 defect ([Kle77, p. 362]; and see [Ein86, Proposition 3.1]). In the nonsingular case, the Katz-Kleiman-Holme formula [Tev05, Theorem 6.2] shows that the defect is determined by the total Chern class of V. Theorem 1.3 implies that in general the defect of V is determined by its Chern-Mather class, and gives an effective way to compute the defect and degree of $V^\vee$.

Proposition 3.11. *Let V be a proper closed subvariety of $\mathbb{P}^n$, and let $c_{Ma}(V) = q(H) \cap [\mathbb{P}^n]$ for a polynomial q of degree $\leq n$ in the hyperplane class H in $\mathbb{P}^n$. Then* $\operatorname{codim} V^\vee$ *equals the order of vanishing of $\mathcal{J}_n(q)$ at 0, and* $\deg V^\vee$ *equals the absolute value of its trailing coefficient.*

Proof. This is an immediate consequence of Theorem 1.3, since $\mathcal{J}_n((-1)^{\dim V}q) = (-1)^{\dim V}\mathcal{J}_n(q)$. □

Example 3.12. (Cf. [MT07, Example 4.2].) Consider the hypersurface X of $\mathbb{P}^4$ with equation

$$x_0^2x_1 + x_0x_2x_4 + x_3x_4^2 = 0$$

(this threefold and its dual are studied in [FP01, p. 78–81]). The algorithm in [Har] may be used to compute the Chern-Mather class of X from this equation:

$$c_{\mathrm{Ma}}(X) = (3H + 8H^2 + 9H^3 + 6H^4) \cap [\mathbb{P}^4] \quad ,$$

so that $c^-_{\mathrm{Ma}}(X) = -(3H + 8H^2 + 9H^3 + 6H^4) \cap [\mathbb{P}^4]$. Applying $\mathcal{J}_4$, we get

$$c^-_{\mathrm{Ma}}(X^\vee) = (3H^2 + 5H^3 + 4H^4) \cap [\mathbb{P}^4]$$

and we conclude that $X^\vee$ is a cubic surface. ⌟

The same result may be recast in terms closer to the Katz-Kleiman-Holme formula, by using Theorem 1.2. If $c^-_{\mathrm{Ma}}(V) = \sum_{i=0}^{n-1} a_i\{\mathbb{P}^i\}$, then $c^-_{\mathrm{Ma}}(V^\vee) = \sum_{i=0}^{n-1} a_{n-1-i}\{\mathbb{P}^i\}$ by Theorem 1.2, and it follows that $\operatorname{def} V$ equals the minimum i such that $a_i \neq 0$, and $\deg V^\vee = a_{\operatorname{def} V}$. The ranks a_i were computed in Proposition 2.5, and we get the following consequence

Proposition 3.13. *Let V be a proper closed subvariety of $\mathbb{P}^n$, and let*

$$a_i = \sum_{j=i}^{\dim V} \binom{j+1}{i+1} (-1)^{\dim V - j} c_{Ma}(V)_j \tag{3.6}$$

where $c_{Ma}(V)_j$ is the degree of the j-dimensional component of $c_{Ma}(V)$. Then $\operatorname{def} V$ *equals the minimum i such that $a_i \neq 0$, and* $\deg V^\vee = a_{\operatorname{def} V}$.

In the nonsingular case, this expression may be found in [Hol88, formula (3)]. In this reference it is stated that 'The full content of formula (3) does not seem to have an immediate generalization to the singular case'. Proposition 3.13 provides such a generalization. The first full treatment of the singular case was provided by R. Piene ([Pie78]), who should be credited with the realization that, in dealing with such issues, Chern-Mather classes play the same role in the singular case as ordinary Chern classes in the nonsingular case (cf. [Pie88]).

Example 3.14. Consider the (singular) surface S with equation

$$16x_0^2x_3^3 - 8x_0x_1^2x_3^2 + 36x_0x_1x_2^2x_3 - 27x_0x_2^4 + x_1^4x_3 - x_1^3x_2^2 = 0$$

in $\mathbb{P}^3$. Again using the algorithm in [Har], we get

$$c_{\mathrm{Ma}}(S) = c^-_{\mathrm{Ma}}(S) = 5[\mathbb{P}^2] + 11[\mathbb{P}^1] + 7[\mathbb{P}^0] \quad .$$

From (3.6) we get $a_0 = 7 - 2\cdot 11 + 3\cdot 5 = 0$, $a_1 = -11 + 3\cdot 5 = 4$, $a_2 = 5$, and we conclude that the dual of S is a quartic curve. ⌟

Remark 3.15. A formula for the degree of the dual variety at the same level of generality as Proposition 3.13, and also using Chern-Mather classes, is given in [MT07, Theorem 3.4]: according to this reference,

$$\deg V^\vee = (-1)^{\dim V + r + 1} \sum_{j=0}^{r-1} \binom{r+1}{j} (r-j) \int_V \frac{H^j}{(1+H)^{r+1}} \cap c_{\mathrm{Ma}}(V) \tag{3.7}$$

where $r = \operatorname{codim} V^\vee = \operatorname{def} V + 1$. At first blush this does not appear to be equivalent to the statement of Proposition 3.13: for example, if $\dim V = 3$ and $\operatorname{def} V^\vee = 2$, then with $c_i := c_{\mathrm{Ma}}(V)_i$ we have

$$a_2 = -c_2 + 4c_3$$

according to (3.6), while the right-hand side of (3.7) equals

$$-3c_0 + 4c_1 - 4c_2 + 4c_3$$

for $r = 3$. However, it can be shown that *if* $a_0 = \cdots = a_{r-2} = 0$, *then* the right-hand side of (3.7) does equal a_{r-1}. With $\dim V = 3$, we have

$$a_0 = -c_0 + 2c_1 - 3c_2 + 4c_3 \quad , \quad a_1 = c_1 - 3c_2 + 6c_3 \quad ,$$

and indeed $-3c_0 + 4c_1 - 4c_2 + 4c_3 = -c_2 + 4c_3$ if $a_0 = a_1 = 0$.

Expression (3.6) appears to be somewhat more efficient than (3.7), since it only involves terms of the Chern-Mather class of dimension $\geq \operatorname{def} V$. ⌟

3.4. **Chern-Mather classes of cones and local Euler obstruction at cone points.** Fix a subspace $\mathbb{P}^m$ of $\mathbb{P}^n$, with $m < n$, and a complementary subspace $\Lambda = \mathbb{P}^{n-m-1}$. Let W be a proper subvariety of $\mathbb{P}^m$, and let V be the cone over W with vertex Λ. Duality considerations allow us to express the Chern-Mather class of V in terms of the Chern-Mather class for W, by means of the involution introduced in §1.

Proposition 3.16. *Let q_W, resp., q_V be polynomials of degree less than m resp., n such that*

$$c_{Ma}(W) = q_W(H) \cap [\mathbb{P}^m] \quad , \quad c_{Ma}(V) = q_V(H) \cap [\mathbb{P}^n] \quad .$$

Then

$$q_V = \mathcal{J}_n((-H)^{n-m} \cdot \mathcal{J}_m(q_W)) \quad .$$

The hypothesis that the vertex Λ be contained in a complementary subspace can be relaxed; this will be pointed out at the end of the section.

Proof. The subspace dual to $\Lambda \cong \mathbb{P}^{n-m-1}$ in $\mathbb{P}^{n\vee}$ is naturally identified with $\mathbb{P}^{m\vee}$, and it is easy to see that $V^\vee$ coincides with $W^\vee$ in this subspace. (See e.g., [Tev05, Theorem 1.23] for the case $m = n-1$, from which the general case is easily derived.) By Theorem 1.3,

$$c^-_{\mathrm{Ma}}(V^\vee) = \mathcal{J}_n((-1)^{\dim V} q_V) \cap [\mathbb{P}^{n\vee}]$$

and

$$c^-_{\mathrm{Ma}}(W^\vee) = \mathcal{J}_m((-1)^{\dim W} q_W) \cap [\mathbb{P}^{m\vee}] = (-1)^{\dim W} H^{n-m} \cdot \mathcal{J}_m(q_W) \cap [\mathbb{P}^{n\vee}] \quad .$$

Since these two classes coincide and $\mathcal{J}_n$ is an involution,

$$q_V = \mathcal{J}_n^2(q_V) = (-1)^{\dim V - \dim W} \mathcal{J}_n(H^{n-m} \cdot \mathcal{J}_m(q_W)) \quad .$$

The statement follows, since $\dim V - \dim W = n - m$. □

Explicitly,

$$q_V(H) = (1+H)^{n-m} q_W(H) + (-1)^m q_W(-1)\, H^{m+1} \left((1+H)^{n-m} - H^{n-m}\right) \quad . \tag{3.8}$$

There are analogous formulas expressing the *Chern-Schwartz-MacPherson* (CSM) class of V in terms of the class of the CSM of W, and more generally the CSM class of the join of two subvarieties W_1, W_2 in complementary subspaces in terms of the CSM classes of W_1 and W_2 ([AM11, Theorem 3.13]; also cf. [BFK90]). Putting these pieces of information together yields the following formula for the local Euler obstruction of a cone singularity.

Proposition 3.17. *Let V be a cone over a subvariety $W \subseteq \mathbb{P}^m$ with vertex Λ in a complementary subspace (as above). Let $p \in \Lambda$. Then the local Euler obstruction* $\mathrm{Eu}_V(p)$ *equals*

$$\mathrm{Eu}_V(p) = \sum_{j=0}^{\dim W} (-1)^j c_{Ma}(W)_j$$

where $c_{Ma}(W)_j$ denotes the degree of the j-dimensional component of $c_{Ma}(W)$.

Again, it is not necessary to require Λ to be contained in a complementary subspace; see Proposition 3.20. Proposition 3.17 is a consequence of Proposition 3.16, by means of a lemma which seems worth stating explicitly. In the situation described above, let $\pi : \mathbb{P}^n \smallsetminus \Lambda \to \mathbb{P}^m$ be the projection, and let φ be a constructible function defined on $\mathbb{P}^m$. Let $\pi^*\varphi$ be the constructible function on $\mathbb{P}^n$ defined by

$$\pi^*\varphi(p) = \begin{cases} 0 & \text{if } p \in \Lambda \\ \varphi(\pi(p)) & \text{if } p \notin \Lambda \end{cases}$$

Also, let q_φ, resp., $q_{\pi^*\varphi}$ be the polynomials of degrees $\le m$, resp., $\le n$ such that

$$c_*(\varphi) = q_\varphi(H) \cap [\mathbb{P}^m] \quad , \quad c_*(\pi^*\varphi) = q_{\pi^*\varphi}(H) \cap [\mathbb{P}^n] \quad .$$

Here c_* is MacPherson's natural transformation [Mac74]. For example $c_*(1\!\!1_W) = c_{\mathrm{SM}}(W)$ and $c_*(\mathrm{Eu}_W) = c_{\mathrm{Ma}}(W)$ (cf. §2.2).

Lemma 3.18. *With the above notation,*

$$q_{\pi^*\varphi}(H) = (1+H)^{n-m} q_\varphi(H) \quad . \tag{3.9}$$

Proof. By linearity we may assume that $\varphi = 1\!\!1_W$ for a subvariety W of $\mathbb{P}^m$; and therefore $\pi^*\varphi = 1\!\!1_V - 1\!\!1_\Lambda$, where V is the cone over W as above. By [AM11, Theorem 3.13], the polynomial corresponding to $c_{\mathrm{SM}}(V) = c_*(1\!\!1_V)$ equals

$$((q_\varphi(H) + H^{m+1})(1+H)^{n-m})(1+H)^{n-m} - H^{n+1} \quad .$$

It follows that $q_{\pi^*\varphi}$ is obtained from this polynomial by subtracting the polynomial for $1\!\!1_\Lambda$, i.e., $H^{m+1}((1+H)^{n-m} - H^{n-m})$, and this yields the stated formula. □

A good interpretation (and, with due care, an alternative argument) for identity (3.9) is that MacPherson's natural transformation preserves products ([Kwi92], [Alu06]), and the factors $1+H$ may be viewed as the c_{SM} class of a linearly embedded affine line.

Proof of Proposition 3.17. Recall that $c_{\mathrm{Ma}}(V) = c_*(\mathrm{Eu}_V)$ (§2.2). The cone V is the union of the vertex Λ and the inverse image $\pi^{-1}(W)$. The restriction of Eu_V to $\pi^{-1}(W)$ equals $\pi^*\,\mathrm{Eu}_W$; this follows for example from [Mac74, p. 426, 3.]. The restriction of Eu_V to Λ is constant, so it equals $\mathrm{Eu}_V(p)$. Thus

$$\mathrm{Eu}_V = \pi^*\,\mathrm{Eu}_W + \mathrm{Eu}_V(p)\, 1\!\!1_\Lambda \quad ,$$

and therefore

$$c_{\mathrm{Ma}}(V) = c_*(\pi^*\,\mathrm{Eu}_W) + \mathrm{Eu}_V(p)\, c_*(1\!\!1_\Lambda) \quad . \tag{3.10}$$

Now $\Lambda \cong \mathbb{P}^{n-m-1} \subseteq \mathbb{P}^n$, therefore

$$c_*(1\!\!1_\Lambda) = c(T\mathbb{P}^{n-m-1}) \cap [\Lambda] = \big((1+H)^{n-m} - H^{n-m}\big)\, H^{m+1} \cap [\mathbb{P}^n] \quad .$$

On the other hand, by Lemma 3.18,

$$c_*(\pi^*\,\mathrm{Eu}_W) = (1+H)^{n-m} q_W(H) \quad .$$

Therefore, identity (3.10) states that

$$q_V(H) = (1+H)^{n-m} q_W(H) + \mathrm{Eu}_V(p)\, H^{m+1}\, \big((1+H)^{n-m} - H^{n-m}\big) \quad .$$

Comparing this identity with (3.8) proves that $\mathrm{Eu}_V(p) = (-1)^m q_W(-1)$. Finally, $q_W(H) = \sum_j c_{\mathrm{Ma}}(W)_j H^{m-j}$, hence

$$(-1)^m q_W(-1) = (-1)^m \sum_{j\ge 0} c_{\mathrm{Ma}}(W)_j (-1)^{m-j} = \sum_{j\ge 0} (-1)^j c_{\mathrm{Ma}}(W)_j$$

as needed. □

Example 3.19. The local Euler obstruction at the vertex of a cone in $\mathbb{P}^n$ over a nonsingular curve of degree d and genus g in $\mathbb{P}^{n-1}$ equals $2-2g-d$. Indeed, the Chern-Mather class of a nonsingular curve equals the Chern class of its tangent bundle, so it pushes forward to $d[\mathbb{P}^1] + (2-2g)[\mathbb{P}^0]$, and the formula follows from Proposition 3.17.

If the curve is a plane curve, then $2-2g = 3d - d^2$, so the local Euler obstruction at the vertex of a cone in $\mathbb{P}^3$ over a nonsingular plane curve of degree d equals $2d - d^2$, in agreement with [Mac74, p. 426, 2.]. ⌟

It is natural to ask whether the hypothesis that the vertex Λ be in a complementary subspace may be relaxed. In the situation presented at the beginning of this subsection, the codimension of W is bounded below by $\dim \Lambda + 2$. It turns out that this is the only requirement needed for the results, provided that Λ is general.

Proposition 3.20. *Let $W \subseteq \mathbb{P}^n$ be a closed subvariety of codimension $\geq r \geq 2$, and let $\Lambda \cong \mathbb{P}^{r-2} \subseteq \mathbb{P}^n$ be a general subspace. Let V be the cone over W with vertex Λ. Also, let q_W, resp., q_V be polynomials of degree less than $n-r-1$, resp., n such that*

$$c_{Ma}(W) = H^{r-1}\, q_W(H) \cap [\mathbb{P}^n] \quad , \quad c_{Ma}(V) = q_V(H) \cap [\mathbb{P}^n] \quad .$$

Then

$$q_V = \mathcal{J}_n((-H)^{r-1} \cdot \mathcal{J}_{n-r+1}(q_W))$$

and $\mathrm{Eu}_V(p) = \sum_{j=0}^{\dim W} (-1)^j c_{Ma}(W)_j$ *for all* $p \in \Lambda$.

Proof. The Chern-Mather class is preserved under general projections ([Pie88, Corollaire, p. 20]). Therefore, $c_{\mathrm{Ma}}(W) = c_{\mathrm{Ma}}(\pi(W))$ as classes in $\mathbb{P}^n$, where $\pi : \mathbb{P}^n \smallsetminus \Lambda \to \mathbb{P}^{n-\dim\Lambda-1}$ is the projection ($W \cap \Lambda = 0$ by dimension considerations, since Λ is general). Since the cone over W with vertex Λ equals the cone over $\pi(W)$ with vertex Λ, we may then replace W with $\pi(W)$. This reduces the general statement to the case in which Λ is a complementary subspace, so the stated formulas follow from Propositions 3.16 and 3.17. □

Example 3.21. Let C be a twisted cubic in $\mathbb{P}^3$, p a general point of $\mathbb{P}^3$, and let V be the cone over C with vertex at p. Then Proposition 3.20 gives $c_{\mathrm{Ma}}(V) = 3[\mathbb{P}^2] + 5[\mathbb{P}^1] + [\mathbb{P}^0]$ (which is confirmed by an explicit computation performed with the algorithm in [Har]) and $\mathrm{Eu}_V(p) = -1$. ⌟

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