2-stratifold groups have solvable Word Problem

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To Professor Maria Teresa Lozano on the occasion of her 70th birthday

Abstract

2-stratifolds are a generalization of 2-manifolds in that there are disjoint simple closed curves where several sheets meet. We show that the word problem for fundamental groups of 2-stratifolds is solvable.

1 Introduction

Simple stratified spaces arise in Topological Data Analysis [2], [9]. A related class of 2-complexes, called 2-*foams*, has been defined and studied by Khovanov [8] and Carter [3]. A special class of stratified spaces, called 2-*stratifolds* have been introduced and some of their properties have been studied in [4], [5], [6] and similar spaces, called *multibranched surfaces*, have been investgated in [11]. A 2-stratifold X is a compact space with empty 0-stratum and empty boundary and contains a collection of finitely many disjoint simple closed curves, the components of the 1-stratum X^1 of X, such that $X - X^1$ is a 2-manifold and a neighborhood of each interval contained in X^1 consists

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of $n \geq 3$ sheets (the precise definition is given in section 2). A 2-stratifold is essentially determined by its associated labelled graph. In [4] it is shown that a simply connected 2-stratifold is homotopy equivalent to a wedge of 2-spheres and the simply connected 2-stratifolds whose graph is a linear tree are classified. Furthermore an efficient algorithm (in terms of the associated graph) is developed for deciding whether a trivalent 2-stratifold (where a neighborhood of each component C of X^1 consists of 3 sheets) is simply-connected and in [5] an efficient algorithm is given for deciding whether a given 2-stratifold is homotopy equivalent to S^2.

Very few 2-stratifolds occur as spines of closed 3-manifolds. For example, fundamental groups of 3-manifolds are residually finite, but there are simple 2-stratifolds with non-residually finite fundamental group. Since 3-manifold groups have solvable word problem ([1]), the question arises whether this is true for 2-stratifold groups. The main goal of this paper is to prove that this is indeed the case.

2 Fundamental group of a graph of groups

In this section we show that the word problem is solvable for fundamental groups of certain graphs of groups. A similar result for graphs of groupoids has been obtained by [7]. Our proof for the graph of groups is more direct, using Serre's normal form. We first describe the fundamental group of a graph of groups (G, Γ) following Serre [14].

A graph of groups (G, Γ) consists of a graph Γ with vertex set $vert\Gamma$ and (oriented) edge set $edge\Gamma$, an associated group G_v to each $v \in vert\Gamma$ and a group G_e to each $e \in edge\Gamma$ such that $G_e = G_{\bar{e}}$, where $\bar{e}$ is the inverse edge of e. (If $e \in \Gamma$, then $\bar{e} \in \Gamma$, $\bar{\bar{e}} = e$, $e \neq \bar{e}$ and the initial edge $o(e) = t(\bar{e})$, the terminal edge of $\bar{e}$). For each $e \in edge\Gamma$ with terminal vertex $t(e)$ there is monomorphisms $\delta_{t(e)} : G_e \to G_{t(e)}$.

The group $F(G, \Gamma)$ is generated by the groups G_v ($v \in vert\Gamma$) and $edge\Gamma$, subject to the relations

$\bar{e} = e^{-1}$ and $e\delta_{t(e)}(a)e^{-1} = \delta_{o(e)}(a)$, for each edge $e \in edge\Gamma$ with initial edge $o(e)$ and terminal edge $t(e)$ and $a \in G_e$.

For a fixed vertex v_0, the fundamental group $\pi_1(G, \Gamma, v_0)$ of the graph of groups (G, Γ) is the subgroup of $F(G, \Gamma)$ generated by all words

$$\omega = r_0 e_1 r_1 e_2 \dots e_n r_n$$

where $v_0 \overset{e_1}{-} v_1 \overset{e_2}{-} v_2 - \cdots \overset{e_n}{-} v_n$ is an edge path with initial and terminal vertex $v_0 = v_n$ (i.e. a cycle based at v_0) and $r_i \in G_{v_i}$.

The word $\omega = r_0 e_1 r_1 e_2 \dots e_n r_n$ of length n is reduced, if
for $n = 0$, $r_0 \neq 1 \in G_{v_0}$;
for $n \geq 2$, $r_i \notin \delta_{t(e_i)}(G_{e_i})$, for each index i such that $e_{i+1} = \bar{e}_i$ (backtracking at vertex v_i).

Serre proves ([14] Theorem 11):

If $\omega \in \pi_1(G, \Gamma, v_0)$ is a reduced word then $\omega \neq 1$ in $\pi_1(G, \Gamma, v_0)$.

Theorem 1. *Let (G, Γ) be a graph of groups with finite graph Γ. Suppose that*
(i) The word problem for each vertex group G_v and each edge group G_e is solvable.
(ii) For each edge e of G, the membership problem with respect to $\delta_{t(e)}(G_e)$ is solvable in $G_{t(e)}$.
Then $\pi_1(G, \Gamma, v_0)$ has a solvable word problem.

Proof. Let $g \in \pi_1(G, \Gamma, v_0)$ be represented by $\omega = r_0 e_1 r_1 e_2 \dots e_n r_n$, a word of length n.

If $n = 0$ then $g = 1$ if and only if $r_0 = 1$ in G_{v_0} and by (i) we can effectively decide whether this is the case.
If $n = 1$ then ω is reduced and so $g \neq 1$.

If $n \geq 2$ we check if there is backtracking at v_i. If there is no backtracking at each $i = 1, \dots, n-1$, then ω is reduced and $g \neq 1$.
If there is backtracking at v_i then by (ii) we can effectively check whether $r_i \in \delta_{t(e_i)}(G_{e_i})$. If this is the case we find $a \in G_{e_i}$ such that $\delta_{t(e_i)}(a) = r_i$. Then $e_i r_i e_{i+1} = \delta_{o(e_i)}(a) \in G_{v_{i-1}}$ and we replace $\omega = \dots r_{i-1} e_i r_i e_{i+1} r_{i+1} \dots$

by $\omega' = \dots (r_{i-1}\delta_{o(e_i)}(a) r_{i+1}) \dots$, a word of length $n-2$ which represents the same $g \in \pi_1(G, \Gamma, v_0)$.

Therefore we can effectively decide whether the word ω of length $n \geq 2$ representing g is reduced and, if ω is not reduced, effectively find a word of length $n-2$ representing the same g.

$\square$

3 The graph of a 2-stratifold.

We first review the definition of a 2-stratifold X and its associated graph G_X given in [4]. A 2-*stratifold* is a compact, Hausdorff space X that contains a closed (possibly disconnected) 1-manifold X^1 as a closed subspace with the following property: Each point $x \in X^1$ has a neighborhood homeomorphic to $\mathbb{R} \times CL$, where CL is the open cone on L for some (finite) set L of cardinality > 2 and $X - X^1$ is a (possibly disconnected) open 2-manifold.

A component $C \approx S^1$ of X^1 has a regular neighborhood $N(C) = N_\pi(C)$ that is homeomorphic to $(Y \times [0,1])/(y,1) \sim (h(y), 0)$, where Y is the closed cone on the discrete space $\{1, 2, ..., d\}$ and $h : Y \to Y$ is a homeomorphism whose restriction to $\{1, 2, ..., d\}$ is the permutation $\pi : \{1, 2, ..., d\} \to \{1, 2, ..., d\}$. The space $N_\pi(C)$ depends only on the conjugacy class of $\pi \in S_d$ and therefore is determined by a partition of d. A component of $\partial N_\pi(C)$ corresponds then to a summand of the partition determined by π. Here the neighborhoods $N(C)$ are chosen sufficiently small so that for disjoint components C and C' of X_1, $N(C)$ is disjoint from $N(C')$. The components of $\overline{N(C) - C}$ are called the *sheets* of $N(C)$.

The associated labelled graph $G = G_X$ of a given 2-stratifold $X = X_G$ is a bipartite graph with black vertices and labelled white vertices and edges. The white vertices w of G_X are the components W of $M := \overline{X - \cup_j N(C_j)}$ where C_j runs over the components of X^1; the black vertices b_j are the C_j's. An edge e corresponds to a component S of ∂M; it joins a white vertex w corresponding to W with a black vertex b corresponding to C_j if $S = W \cap N(C_j)$. Note that the number of boundary components of W is the number of adjacent edges of W.

The label assigned to a white vertex W is its genus g; the label of an edge e is an integer m, where $|m|$ is the summand of the partition π corresponding to the component $S \subset \partial N_\pi(C)$ and the sign of m is determined by an orientation of C_j and S. (Here we use Neumann's [12] convention of assinging negative genus g to nonorientable surfaces; for example the genus g of the projective plane or the Moebius band is -1, the genus of the Klein bottle is -2). Note that the partition π of a black vertex is determined by the labels of the adjacent edges. If all white vertices have labels $g < 0$ or if G is a tree, then the labeled graph determines X uniquely.

4 Natural presentation of $\pi_1(X_G)$

In this section we describe a natural presentation for the fundamental group of a 2-stratifold X. First we fix a notation.

$X = X_G$ is a 2- stratifold with associated bipartite graph $G = G_X$.

$N(C_{b_j})$ is a regular neighborhood of C_{b_j}, a component of $X^{(1)}$ corresponding to the black vertex b_j of G_X

W_i is a component of $M = \overline{X - \cup_j N(C_j)}$ corresponding to the white vertex w_i of G_X

c_{ijk} are the components of $W_i \cap N(C_j)$ corresponding to the edges e_{ijk} of G_X

For a given white vertex w, the compact 2-manifold W has conveniently oriented boundary curves $c_1, \ldots, c_p$ such that

(*) $\qquad \pi_1(W) = \langle c_1, \ldots, c_p, y_1, \ldots, y_n : c_1 \cdots c_p \cdot q = 1 \rangle$

where $q = [y_1, y_2] \ldots [y_{2g-1}, y_{2g}]$, if W is orientable of genus g and $n = 2g$, $q = y_1^2 \ldots y_n^2$, if W is non-orientable of genus $-n$.

Let $\mathcal{B}$ be the set of black vertices, $\mathcal{W}$ the set of white vertices and choose a fixed maximal tree T of G.

We choose orientations of the black vertices and of all boundary components of M such that all labels of edges in T are positive.

Then $\pi_1(X_G)$ has a natural presentation with

generators: $\{b\}_{b\in\mathcal{B}}$
$\{c_1,\ldots,c_p,y_1,\ldots,y_n\}$, one set for each $w\in\mathcal{W}$, as in $(*)$
$\{t_i\}$, one t_i for each edge $c_i\in G-T$ between w and b

and relations: $c_1\cdots c_p\cdot q=1$, one for each $w\in\mathcal{W}$, as in $(*)$
$b^m=c_i$, for each edge $c_i\in T$ between w and b with label $m\geq 1$ (corresponding to $W\cap N(C_b)$)
$t_i^{-1}c_it_i=b^{m_i}$, for each edge $c_i\in G-T$ between w and b with label $m_i\in\mathbb{Z}$.

5 The graph of groups of X_G

Let $X=X_G$ be the 2-stratifold associated to the labeled graph G; we assume a maximal tree T of G is given and the labels of edges of T are positive, so the labeling is unique. We first define a graph of CW-complexes as in [13], with underlying graph that of G.

For a black vertex b representing a singular oriented circle C_b, let $o(b)$ be the order of C_b in $\pi_1(X_G)$. Note that, if e is an edge joining a black vertex b to a white vertex w and the label of e is m, then e represents an oriented circle c of ∂W whose order in $\pi_1(X_G)$ is $k=o(b)/(o(b),m)$. Here $(o(b),m)$ denotes the greatest common divisor of $o(b)$ and m. (If $o(b)=0$, then $(o(b),m)=m$).

Construct a space $\hat{X}$ from X by attaching disks as follows:

If b is a black vertex of order $o(b)\geq 1$, attach a 2-cell d_b to C_b with degree $o(b)$ (i.e. attach a disk under the attaching map $z\to z^{o(b)}$). If e is an edge joining b to w with label m and $o(b)\geq 1$, attach to c a 2-cell d_e with degree $k=o(b)/(o(b),m)$. If $o(b)=0$, do not attach 2-cells). Note that

$$\pi(\hat{X})=\pi(X_G).$$

The graph of spaces associated to $\hat{X}$ has the same underlying graph as G_X, with vertices $\hat{X}_b$, $\hat{X}_w$, and edges $\hat{X}_e$, defined as follows:

$\hat{X}_b$: For a black vertex b of G, $\hat{X}_b = N(C_b) \cup d_b \cup (\cup d_e)$, where e runs over the edges having b as an endpoint.

$\hat{X}_e$: For a white vertex w of G let $\hat{X}_w$, $W \cup (\cup d_e)$, where e runs over the edges incident to w. (Recall that there is one such edge for each boundary curve c of W).

$\hat{X}_e$: For an edge e joining b to w, $\hat{X}_e = (\hat{X}_b \cap \hat{X}_w)$. (If $o(b) \geq 1$ this is a pseudo-projective plane of degree $k = o(b)/(o(b), m)$).

Since $\hat{X}_b$, $\hat{X}_w$ and $\hat{X}_e$ are path-connected and the inclusion-induced homomorphisms $\pi_1(\hat{X}_e) \to \pi_1(X_b)$ and $\pi_1(\hat{X}_e) \to \pi_1(\hat{X}_w)$ are injective, this graph of spaces determines a graph of groups $\mathcal{G} = \{G_b, G_e, G_w\}$ (with the same underlying graph as G_X). The vertex groups are $G_b = \pi_1(\hat{X}_b)$ and $G_w = \pi_1(\hat{X}_w)$, the edge groups are $G_e = \pi_1(\hat{X}_e)$, the monomorphisms $\delta : G_e \to G_b$ (resp. $G_e \to G_w$ are induced by inclusion. Then (see for example [13],[14])

$$\pi_1\mathcal{G} \cong \pi_1(\hat{X})$$

Note that the groups G_b of the black vertices and the groups G_e of the edges are cyclic. For a white vertex w with edges $e_1, \ldots e_p$ labelled $m_1, \ldots m_p$ with associated edge space $X_w = W \cup_{i=1}^r d_{e_i}$ we have

$$G_w = \pi_1(\hat{X}_w) = \langle c_1, \ldots, c_p, y_1, \ldots, y_n : c_1 \cdots c_p \cdot q = 1\,,\, c_1^{k_1} = \cdots = c_1^{k_r} = 1 \ (1 \leq r \leq p) \,\rangle.$$

If all $k_i \geq 2$ and $r = p$ then G_w is an F-group ([10] p. 126-127), otherwise it is a free product of cyclic groups.

6 The Word Problem for Fundamental groups of 2-stratifolds

It is well-known that free groups have solvable membership problem with respect to cyclic subgroups. More generally it follows from the Proposition below, which is Corollary 4.16 of [1], that free products of cyclic groups have solvable membership problems.

Proposition 1. *Solvability of the membership problem is preserved under taking free products.*

We are interested in the membership problem of free products with amalgamation with respect to cyclic groups and give an elementary proof of the following

Lemma 1. *Let $G = A *_C B$. Assume that C has solvable membership problem with respect to cyclic subgroups and A and B have solvable membership problem with respect to the subgroup C. Then G has solvable membership problem with respect to cyclic subgroups.*

Proof. For $g, g' \in A$ or B we can decide whether $g(g')^{-1}$ is in C. Therefore, given $g \in G$ and a fixed choice of right coset representatives of C in A (resp. in B) we can effectively find the (unique) reduced normal form $w = g_1 \dots g_n c$ of g, where $g_i \in A$ or B are the chosen representatives of the right cosets $g_i C$, $c \in C$, and g_i, g_{i+1} are in different subgroups A, B, for $i = 1, \dots, n-1$. The length of g is $l(g) = l(w) = n$. In particular, $l(w) = 0$ iff $g \in C$. Also, if w is not cyclically reduced (i.e. g_1 and g_n are in the same subgroup A or B), then we can effectively reduce w to a cyclically reduced word.

Let $t \in G$ of length $l(t) \geq 0$ generate an infinite cyclic subgroup $\langle t \rangle \subset G$ and let $g \in G$. Now $w \in \langle t \rangle$ if and only if $w = t^k$ for some $|k| \geq 1$. Since $w \in \langle t \rangle$ iff $w^{-1} \in \langle t \rangle$ we may assume $k \geq 1$. If $l(t) = 0$ then $l(w) = 0$ and the result follows since C has solvable membership problem with respect to cyclic groups. Thus assume $l(t) \geq 1$.

If the word t is cyclically reduced then $l(t^k) = kl(t)$. Thus there is a unique k such that $l(w) = kl(t)$ and we can effectively check whether the reduced words w and t^k agree.

If t is not cyclically reduced then $t = uru^{-1}$ for some reduced word r and cyclically reduced word r. Then $w \in \langle t \rangle$ iff $u^{-1}wu = r^k$ for some k. We effectively find the reduced word w' representing $u^{-1}wu$ and (by the above argument) effectively determine whether $w' = r^k$. □

Corollary 1. *Let G be a free product of cyclic groups or a free product of two such groups amalgamated over a cyclic group. Then the membership problem with respect to cyclic subgroups is solvable.*

Proof. This follows from Proposition 1 and Lemma 1. □

Corollary 2. *F-groups have solvable membership problem with respect to cyclic subgroups.*

Proof. Let $G = \langle c_1, \dots, c_p, y_1, \dots, y_n : c_1 \cdots c_p \cdot q = 1\, ,\, c_1^{k_1} = \cdots = c_1^{k_p} = 1 \rangle$ be an F-group.

If $p \geq 1$, let $A = \langle c_1, \dots, c_p : c_1^{k_1} = \cdots = c_1^{k_p} = 1 \rangle$, $B = \langle y_1, \dots, y_n : \rangle$, C the infinite cyclic group generated by $c_1 \cdots c_p$ in A resp. by q in B. Then $G = A *_C B$.

If $p = 0$ then G is the fundamental group of a closed surface of genus g. If $g \neq 1, -2$, then G can be similarly written as a free product of two free groups with amalgamation over a cyclic group. If $g = 1, -2$ the result is trivial (in this case every element of G has a normal form of length ≤ 2). □

Theorem 2. *The fundamental group of a* 2*-stratifold has solvable word problem.*

Proof. From section 5 we know that $\pi(X_G) \cong \pi_1 \mathcal{G}$ for a graph of groups where the edge groups and black vertex groups are cyclic and the white vertex groups are F-groups or free products of finitely many cyclic groups. By Lemma 1 and Corollary 2 all these groups have solvable membership problem with respect to finite cyclic subgroups. Now the Theorem follows from Theorem 1.

□

7 Some Consequences

Corollary 3. *There is an algorithm to decide whether or not $\pi(X_G)$ is abelian.*

Proof. $\pi(X_G)$ is abelian if and only if $[x_i, x_j] = 1$ for $1 \leq i < j \leq n]$, where $x_1, \dots, x_n$ generate $\pi(X_G)$. Since the word problem is solvable, we can decide whether this is true. □

A 0-terminal edge of X_G is an edge $b \overset{m}{-} w$, where w is a terminal white vertex of genus 0. The following deals with a special case of the order problem.

Corollary 4. *Let $b \overset{m}{-} w$ be a* 0*-terminal edge of G_X. One can calculate the (finite) order o of b in $\pi(X_G)$.*

Proof. o is one of the (finitely many) divisors of the finite nonzero labels of $b - w$. The Corollary follows since $\pi(X_G)$ has solvable word problem. □

In [4] and [6] we obtained for certain classes of 2-stratifolds X_G (namely those with a linear graph G_X or those that are trivalent) an *efficient* algorithm to decide if X_G is simply connected. These algorithms can be read off from the labelled graph G_X. For the general case we do not yet have an *efficient* algorithm, but we now see that there is an algorithm:

Corollary 5. *There is an algorithm to decide whether or not X_G is simply-connected.*

Proof. If S is a finite set of generators of $\pi = \pi_1(X_G)$, the $\pi = 1$ if and only if $s = 1$ in π for every $s \in S$. Since π has solvable word problem one can decide if every s in S is 1. □

In [4] it was shown that a necessary condition for a 2-stratifold X_G to be simply-connected is that G_X is a tree, all white vertices are of genus 0, and all terminal edges are white. If there is an efficient algorithm for the order problem in Corollary 4 then this result may be used in obtaining an efficient algorithm in Corollary 5 as follows:

If G_X is not a tree or if some white vertex has nonzero genus, or if there is a black terminal vertex, then $\pi(X_G) \neq 1$. Otherwise apply repeatedly the following "pruning":

Calculate the order o in $\pi(X_G)$ of b where $b - w$ is a 0-terminal edge ; if o is not 1 then X_G is not simply-connected; if $o = 1$ delete b, w and all edges incident to b from X_G. Each component G_i of the resulting graph (the "pruned" graph) corresponds to a 2-stratifold X_{G_i}, and since $o(b) = 1$ in $\pi(X_G)$ it follows that X_G is simply-connected if and only if each X_{G_i} is simply-connected. Then X_G is 1-connected if and only if we eventually obtain a graph with no edges.

Corollary 6. *One can decide whether or not X_G is homotopically equivalent to a wedge of n 2-spheres and, if so, calculate n.*

Proof. In [5] it was shown that a simply-connected 2-stratifold X_G is homotopy equivalent to a wedge of 2-spheres and moreover if n_b (resp. n_w) denotes the number of black (resp. white) vertices of G_X, then X_G is homotopy equivalent to a wedge of $n_w - n_b$ 2-spheres. Now the Corollary follows from Corollary 5 and, if $\pi(X_G) = 1$, then $n = n_w - n_b$. □

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