2-stratifold spines of closed 3-manifolds

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Abstract

2-stratifolds are a generalization of 2-manifolds in that there are disjoint simple closed branch curves. We obtain a list of all closed 3-manifolds that have a 2-stratifold as a spine. 1 2

1 Introduction

2-stratifolds form a special class of 2-dimensional stratified spaces. A (closed with empty 0-stratum) 2-stratifold is a compact connected 2-dimensional cell complex $X$ that contains a 1-dimensional subcomplex $X^{(1)}$, consisting of branch curves, such that $X - X^{(1)}$ is a (not necessarily connected) 2-manifold. The exact definition is given in section 2. $X$ can be constructed from a disjoint union $X^{(1)}$ of circles and compact 2-manifolds $W^2$ by attaching each component of $\partial W^2$ to $X^{(1)}$ via a covering map $\psi : \partial W^2 \rightarrow X^{(1)}$, with $\psi^{-1}(x) > 2$ for $x \in X^{(1)}$. A slightly more general class of 2-dimensional stratified spaces, called multibranched surfaces and which have been defined and studied in [11], is obtained by allowing boundary curves, i.e. considering

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a covering map $\psi : \partial W' \to X^{(1)}$, where $\partial W'$ is a sub collection of the components of $\partial W^2$.

2-stratifolds arise as the nerve of certain decompositions of 3-manifolds into pieces where they determine whether the $\mathcal{G}$-category of the 3-manifold is 2 or 3 ([4]). They are related to foams, which include special spines of 3-dimensional manifolds and which have been studied by Khovanov [8] and Carter [3]. Simple 2-dimensional stratified spaces arise in Topological Data Analysis [2], [9].

Matsuzaki and Ozawa [11] show that 2-stratifolds can be embedded in $\mathbb{R}^4$. Furthermore they show that they can be embedded into some orientable closed 3-manifold if and only if their branch curves satisfy a certain regularity condition. However, the embeddings are not $\pi_1$-injective, i.e. the induced homomorphism of fundamental groups is not injective. In fact, there are many 2-stratifolds whose fundamental group is not isomorphic to a subgroup of a 3-manifold group; for example there are infinitely many 2-stratifolds with (Baumslag-Solitar) non-Hopfian fundamental groups. These can not be embedded as $\pi_1$-injective subcomplexes into 3-manifolds since 3-manifold groups are residually finite. On the other hand, every 2-manifold embeds $\pi_1$-injectively in some (Haken) 3-manifold. Since subgroups of 3-manifold groups are 3-manifold groups, the following question arises:

Question 1. Which 3-manifolds $M$ have fundamental groups isomorphic to the fundamental group of a 2-stratifold?

The fundamental group of a closed 2-manifold $S$ is isomorphic to the fundamental group of a closed 3-manifold $M$ if and only if $S$ is the 2-sphere or projective plane and $M$ is $S^3$ or $P^3$, respectively. Since $S^2$ is not a spine of $S^3$, the only closed 3-manifold with a (closed) 2-manifold spine is $P^3$. This motivates the next question:

Question 2. Which closed 3-manifolds $M$ have spines that are 2-stratifolds?

The main results of this paper are Theorem 1 which answers question 1 for closed 3-manifolds and Theorem 2, which answers question 2 by showing that a closed 3-manifold $M$ has a 2-stratifold spine if and only if $M$ is a connected sum of lens spaces, $S^2$-bundles over $S^1$, and $P^2 \times S^1$'s.

2 2-stratifolds and their graphs.

In this section we review the definitions of a 2-stratifold $X$ and its associated graph $G_X$ given in [5].
A (closed) 2-stratifold is a compact 2-dimensional cell complex $X$ that contains a 1-dimensional subcomplex $X^{(1)}$, such that $X - X^{(1)}$ is a 2-manifold ($X^{(1)}$ and $X - X^{(1)}$ need not be connected). A component $C \approx S^1$ of $X^1$ has a regular neighborhood $N(C) = N_\pi(C)$ that is homeomorphic to $(Y \times [0,1])/(y,1) \sim (h(y),0)$, where $Y$ is the closed cone on the discrete space $\{1, 2, ..., d\}$ (for $d \geq 3$) and $h : Y \to Y$ is a homeomorphism whose restriction to $\{1, 2, ..., d\}$ is the permutation $\pi : \{1, 2, ..., d\} \to \{1, 2, ..., d\}$. The space $N_\pi(C)$ depends only on the conjugacy class of $\pi \in S_d$ and therefore is determined by a partition of $d$. A component of $\partial N_\pi(C)$ corresponds then to a summand of the partition determined by $\pi$. Here the neighborhoods $N(C)$ are chosen sufficiently small so that for disjoint components $C$ and $C'$ of $X^1$, $N(C)$ is disjoint from $N(C')$.

We construct an associated bicolored graph $G = G_X$ of $X = X_G$ by letting the white vertices $w$ of $G_X$ be the components $W$ of $M := X - \cup j N(C_j)$ where $C_j$ runs over the components of $X^1$; the black vertices $b_j$ are the $C_j$'s. An edge $e$ is a component $S$ of $\partial M$; it joins a white vertex $w$ corresponding to $W$ with a black vertex $b$ corresponding to $C_j$ if $S = W \cap N(C_j)$. The number of boundary components of $W$ is the number of adjacent edges of $W$.

$G_X$ embeds naturally as a retract into $X_G$.

We label the white vertices $w$ with the genus $g$ of $W$; here we use Neumann’s [14] convention of assigning negative genus $g$ to nonorientable surfaces; for example the genus $g$ of the projective plane or the Moebius band is $-1$, the genus of the Klein bottle is $-2$. We orient all components $C_j$ and $S$ of $X^{(1)}$ and $\partial W$, resp., and assign a label $m$ to an edge $e$, where $|m|$ is the summand of the partition $\pi$ corresponding to the component $S \subset \partial N_\pi(C)$; the sign of $m$ is determined by the orientation of $C_j$ and $S$. In terms of attaching maps, $m$ is the degree of the covering map $\psi : S \to C_j$ for the corresponding components of $\partial W$ and $X^{(1)}$.

(Note that the partition $\pi$ of a black vertex is determined by the labels of its adjacent edges).
3 Structure of $\pi_1(X_G)$

In this section we obtain a natural presentation for the fundamental group of a 2-stratifld $X_G$ with associated bicolored graph $G = G_X$ and describe $\pi_1(X_G)$ as the fundamental group of a graph of groups $\mathcal{G}$ with the same underlying graph $G$.

For a given white vertex $w$, the compact 2-manifold $W$ has conveniently oriented boundary curves $s_1, \ldots, s_p$ such that

\[(*) \quad \pi_1(W) = \langle s_1, \ldots, s_p, y_1, \ldots, y_n : s_1 \cdots s_p \cdot q = 1 \rangle\]

where $q = [y_1, y_2] \cdots [y_{2g-1}, y_{2g}]$, if $W$ is orientable of genus $g$ and $n = 2g$, $q = y_1^2 \cdots y_n^2$, if $W$ is non-orientable of genus $-n$.

Let $\mathcal{B}$ be the set of black vertices, $\mathcal{W}$ the set of white vertices and choose a fixed maximal tree $T$ of $G$. Choose orientations of the black vertices and of all boundary components of $M$ such that all labels of edges in $T$ are positive.

Then $\pi_1(X_G)$ has a natural presentation with

- generators: $\{b\}_{b \in \mathcal{B}}$
  - $\{s_1, \ldots, s_p, y_1, \ldots, y_n\}$, one set for each $w \in \mathcal{W}$, as in $(*)$
  - $\{t_i\}$, one $t_i$ for each edge $c_i \in G - T$ between $w$ and $b$
- and relations: $s_1 \cdots s_p \cdot q = 1$, one for each $w \in \mathcal{W}$, as in $(*)$
  - $b^m = s_i$, for each edge $s_i \in T$ between $w$ and $b$ with label $m \geq 1$
  - $t_i^{-1}s_it_i = b^m_i$, for each edge $s_i \in G - T$ between $w$ and $b$ with label $m_i \in \mathbb{Z}$.

As an example we show in Figure 1 (the graph of) a 2-stratifld $X_G$ with $\pi_1(X_G) = \mathcal{F}$, an $F$-group as in Proposition (III)5.3 of [10], with presentation

\[(\mathcal{F}) \quad \mathcal{F} = \langle c_1, \ldots, c_p, y_1, \ldots, y_n : c_1^{m_1} \cdots c_p^{m_p}, c_1 \cdots c_p \cdot q = 1 \rangle\]

where $p, n \geq 0$, all $m_i > 1$ and $q = [y_1, y_2] \cdots [y_{2g-1}, y_{2g}]$ or $q = y_1^2 \cdots y_n^2$.

Here we have denoted the generators corresponding to the black vertices by $c_i$, rather than $b_i$, to indicate that the finite order elements correspond to attaching disks along the boundary curves of $W$. 

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The fundamental group of $X_G$ is best described as the fundamental group of a graph of groups [6].

If $\pi_1(X_G)$ has no elements of finite order, then $\pi_1(X_G)$ is the fundamental group of a graph of groups $G$, with underlying graph $G$, the groups of white vertices are the fundamental groups of the $W'$s, the groups of the black vertices and edges are (infinite) cyclic.

Elements of finite order occur when a generator $b$ of a black vertex has finite order $o(b) \geq 1$. In this case we attach 2-cells $d_b$ and $d_e$ to $C_b$, the circle corresponding to $b$, as follows: $d_b$ is attached by a map of degree $o(b)$. If $e$ is an edge joining $b$ to $w$ with label $m$, attach $d_e$ with degree $o(c) = o(b)/o(b, m)$. Letting $\hat{X}_b = N(C_b) \cup d_b \cup (\cup d_e)$, where $e$ runs over the edges having $b$ as an endpoint, $\hat{X}_w = W \cup (\cup d_e)$, where $e$ runs over the edges incident to $w$, and $\hat{X}_e = (\hat{X}_b \cap \hat{X}_w)$, for an edge $e$ joining $b$ to $w$, we obtain a graph of CW-complexes that determines a graph of groups $G$ with the same underlying graph as $G_X$.

The vertex groups are $G_b = \pi_1(\hat{X}_b)$ and $G_w = \pi_1(\hat{X}_w)$, the edge groups are $G_e = \pi_1(\hat{X}_e)$, the monomorphisms $\delta : G_e \to G_b$ (resp. $G_e \to G_w$ are induced by inclusion. Then (see for example [15],[16]) $\pi_1 G \cong \pi_1 X$.

Note that the groups $G_b$ of the black vertices and the groups $G_e$ of the edges are cyclic. For a white vertex $w$ with edges $e_1, \ldots, e_p$ labelled $m_1, \ldots m_p$ with associated vertex space $X_w = W \cup_{i=1}^p d_{e_i}$ we obtain

$G_w = \langle c_1, \ldots, c_p, y_1, \ldots, y_n : c_1 \cdots c_p \cdot q = 1, c_1^{k_1} = \cdots = c_p^{k_p} = 1 \rangle$

where $q$ is as in $(F)$, $1 \leq r \leq p$ and $k_i \geq 1$.

If all $k_i \geq 2$ and $r = p$ then $G_w$ is an $F$-group ([10] p. 126-127), otherwise
it is a free product of cyclic groups.

4 Necessary Conditions

In this section we show that a 2-stratifold group that is a closed 3-manifold group is a free product of cyclic or $\mathbb{Z} \times \mathbb{Z}_2$ groups.

First consider an $F$-group $F$ as in (F).

**Proposition 1.** ([10] Proposition (III)7.4) Let $H$ be a subgroup of an $F$-group. If $H$ has finite index then $H$ is an $F$-group. If $H$ has infinite index then $H$ is a free product of cyclic groups.

**Proposition 2.** ([10] p.132) (a) $F$ is finite non-cyclic if and only if $n = 0, p = 3$ and $(m_1, m_2, m_3) = (2, 2, m)$ ($m \geq 2$) (the dihedral group of order $2m$) or $(m_1, m_2, m_3) = (2, 3, k)$ for $k = 3, 4$ or 5 (the tetrahedral, octahedral, dodecahedral groups). In each case, $c_1$ is a non-central element of order 2.

(b) $F$ is finite cyclic if and only if $n = 0, p \leq 2$ (the 2-sphere orbifold with at most two cone points) or $n = 1, p \leq 1$ (the projective plane orbifold with at most one cone point).

**Lemma 1.** $F$ is not a non-trivial free product.

*Proof.* If $F = A \ast B$ with $A, B$ non-trivial, then $A$ and $B$ have infinite index and so, by Proposition 1, $A, B$ and $F$ are free products of cyclic groups. However, $F$ is not such a group since it contains a subgroup isomorphic to the fundamental group of an orientable closed surface of genus $\geq 1$ (see the remark after Proposition (III)7.12 in [10]).

The following remark is easy to see.

**Remark 1.** If $F \neq \mathbb{Z}_2$ then $F$ has no elements of finite order if and only if $F$ is a surface group.

**Lemma 2.** If $M$ is an orientable (not necessarily closed or compact) 3-manifold with $\pi_1(M) \cong F$ then $\pi_1(M)$ is cyclic or a surface group.

*Proof.* We may assume that $\partial M$ contains no 2-spheres. By Scott’s Core Theorem we may assume that $M$ is compact and by Lemma 1 that $M$ is irreducible.
If $\pi_1(M)$ is infinite then $M$ is aspherical (see e.g. [1]). It follows that $\pi_1(M)$ is torsion-free and from Remark 1 that $\pi_1(M)$ is a surface group.

If $\pi_1(M)$ is finite then $M$ is closed. If $\pi_1(M)$ is also non-cyclic then by Proposition 2, $\pi_1(M)$ contains a non-central element of order 2. This can not happen by Milnor [13].

We now consider a 2-stratifold $X_G$ with $\pi_1(X_G) = \pi(G)$ as in section 3.

Up to conjugacy, the only elements of finite order of $\pi_1(X_G)$ are contained in the vertex groups; they correspond to black vertices of finite order and elements of white vertices $w$ whose corresponding group in $G$ is finite. The latter are described in Proposition 2. It is also shown in [10] (proof of Proposition (III)7.12) that in an infinite F-group the only elements of finite order are the obvious ones, namely conjugates of powers of $c_1, \ldots, c_p$.

For a group $H$, denote by $QH$ be the quotient group of $H$ modulo the smallest subgroup of $H$ containing all elements of finite order of $H$.

Let $w$ be a white vertex in $G_X$. We say that $w$ is a white hole, if $w$ has label $-1$, all of its (black) neighbors have finite order and at most one of its neighbors has order $> 1$.

If $G_X$ has more than one vertex, note that $Q\pi_1(X_G)$ is obtained from $\pi_1(X_G)$ by killing the open stars of all the black vertices representing elements of finite order $\geq 1$ of $\pi_1(X_G)$ and deleting the white holes. In the example of Figure 1, when genus $g = -1$ (and so $n = 1$), $Q\pi_1(X_G) = \mathbb{Z}_2$. (Note that the white vertex of genus $-1$ is not a white hole if $p \geq 2, m_i > 1$).

**Proposition 3.** If $Q(\pi_1(X_G))$ has no elements of order 2, then $H_3(Q\pi_1(X_G)) = 0$.

**Proof.** Let $G'$ be the labelled subgraph of $G_X$ obtained by deleting the open stars of all black vertices representing elements of finite order of $\pi_1(X_G)$ and all white holes. ($\pi_1(X_0) = 1$ by definition). Let $C$ be a component of $G'$. Then $Q\pi(X_G) = L * (\pi(X_C))$, the free product of a free group $L$ with the free product of the $\pi(X_C)$ where $C$ runs over the components of $G'$.

If $C$ consists of only one (white) vertex, then $X_C$ is a closed 2-manifold, different from $P^2$, since by assumption $Q(\pi_1(X_G))$ has no elements of order 2. We may ignore the $C$'s consisting of spheres, since they do not contribute to $Q\pi(X_G)$. (A nonseparating 2-sphere only changes the rank of $L$). In
all other cases $X_C$ is the total space of a bicolored graph of spaces with white vertex spaces 2-manifolds with boundary, edge spaces circles, and black vertex spaces homotopy equivalent to circles.

Thus every vertex and edge space of $X_C$ is aspherical (with free fundamental group) of dimension $\leq 2$. By Proposition 3.6 (ii) of [15], $X_C$ is aspherical. It follows that $Q\pi_1(X_G)$ has (co)homological dimension $\leq 2$ and so $H_3(Q\pi_1(X_G)) = 0$.

The assumption that $Q(\pi_1(X_G))$ has no elements of finite order is satisfied if $\pi_1(X_G)$ is a 3-manifold group: We claim that $Q\pi_1(M)$ is torsion free if $M$ is a closed orientable 3-manifold.

For let $M = M_1 \# \ldots \# M_k$ be its prime decomposition. If $M_i$ is irreducible with infinite fundamental group, then $M_i$ is aspherical and so $\pi_1(M_i)$ is torsion free; if $M_i$ has finite fundamental group, then $Q\pi_1(M_i) = 1$. Now the claim follows since $Q\pi_1(M) = Q\pi_1(M_1) \star \cdots \star Q\pi_1(M_k)$.

**Lemma 3.** Let $M$ be a closed orientable 3-manifold with prime decomposition $M = M_1 \# \ldots \# M_k$. If $\pi_1(M) \cong \pi_1(X_G)$, then each $\pi_1(M_i)$ is infinite cyclic or finite.

**Proof.** If there is some $M_i$ with $\pi_1(M_i) \neq \mathbb{Z}$, then $M_i$ is irreducible. If $\pi_1(M_i)$ is infinite then $M_i$ is aspherical and hence $H_3(Q\pi_1(M_i)) = H_3(\pi_1(M_i)) = H_3(M_i) \neq 0$. Since $Q\pi_1(M) = Q\pi_1(M_1) \star \cdots \star Q\pi_1(M_k)$ it follows that $H_3(Q\pi_1(M)) \neq 0$, which contradicts Proposition 3.

**Lemma 4.** Let $M$ be a closed 3-manifold and suppose $\pi_1(M) = \pi_1(X_G)$. Then any finite subgroup $H$ of $\pi_1(M)$ is cyclic.

**Proof.** $\pi_1(X_G) \cong \pi_1(G)$ where $G$ is a graph of groups in which the groups of black vertices are cyclic and the groups of white vertices are $F$-groups or free products of finitely many cyclic groups. The finite group $H$ is non-splittable (i.e. not a non-trivial HHN extension or free product with amalgamation). By Corollary 3.8 and the Remark after Theorem 3.7 of [15], $H$ is a cyclic group or isomorphic to a subgroup of an $F$-group. If $H$ is not cyclic, then (since $H$ is not a non-trivial free product of cyclic groups), $H$ is itself an $F$-group by Proposition 1. Since $H$ is a 3-manifold group it follows from Lemma 2 that this case can not occur.

**Corollary 1.** Let $M$ be a closed orientable 3-manifold. If $\pi_1(M) \cong \pi_1(X_G)$, then $\pi_1(M)$ is a free product of cyclic groups.
**Proof.** This follows from Lemmas 3 and 4.

**Theorem 1.** Let $M$ be a closed 3-manifold. If $\pi_1(M) \cong \pi_1(X_G)$, then $\pi_1(M)$ is a free product of groups, where each factor is cyclic or $\mathbb{Z} \times \mathbb{Z}_2$.

**Proof.** If $M$ is orientable this is Corollary 1 (with each factor cyclic). Thus assume $M$ is non-orientable and let $p : \tilde{M} \to M$ be the 2-fold orientable cover of $M$. Then $\pi_1(\tilde{M}) = \pi_1(X_{\tilde{G}})$ for the 2-stratifolds $X_{\tilde{G}}$, which is the 2-fold cover of $X_G$ corresponding to the orientation subgroup of $\pi_1(M)$. Hence $\pi_1(\tilde{M})$ is a free product of cyclic groups.

Let $M = \tilde{M}_1 \# \ldots \# \tilde{M}_k$ be a prime decomposition of $M = M_1 \cup \ldots \cup M_k$. If $M_i$ is orientable, then $M_i$ lifts to two homeomorphic copies $\tilde{M}_{i1}, \tilde{M}_{i2}$ of $M_i$, with each $\tilde{M}_{ij}$ a factor of the prime decomposition of $\tilde{M}$ and it follows that $\pi_1(M_i)$ is cyclic.

If $\tilde{M}_i$ is non-orientable and $P^2$-irreducible, then $M_i$ lifts to $\tilde{M}_i$, where $\tilde{M}_i$ is irreducible. Then $\pi_1(\tilde{M}_i)$, being a factor of the free product decomposition of $\pi_1(\tilde{M})$, is finite cyclic, which can not occur since $\pi_1(M_i)$ is infinite.

If $\tilde{M}_i$ is non-orientable irreducible, contains $P^2$s, but is not $P^2 \times S^1$, then by Proposition (2.2) of [17], $M_i$ splits along two-sided $P^2$’s into 3-manifolds $\tilde{N}_1, \ldots, \tilde{N}_m$ such that the fundamental group of the lifts $\tilde{N}_i$ is indecomposable, torsion free and not isomorphic to $\mathbb{Z}$. Since $\pi_1(\tilde{N}_i)$ is a factor of the free product decomposition of $\pi_1(\tilde{M})$, this can not happen.

Therefore each non-orientable $M_i$ is either the $S^2$-bundle over $S^1$ or $P^2 \times S^1$, which proves the Theorem.

**5 Realizations of spines.**

Recall that a subpolyhedron $P$ of a 3-manifold $M$ is a spine of $M$, if $M - \text{Int}(B^3)$ collapses to $P$, where $B^3$ is a 3-ball in $M$.

An equivalent definition is that $M - P$ is homeomorphic to an open 3-ball (Theorem 1.1.7 of [12]).

We first construct 2-stratifolds spines of lens spaces (different from $S^3$), the non-orientable $S^2$-bundle over $S^1$, and $P^2 \times S^1$.

**Example 1.** Lens space $L(0, 1) = S^3$. 

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$S^3$ does not have a 2-stratifold spine. Otherwise such a spine $X$ would be a deformation retract of the 3-ball and therefore contractible. However there are no contractible 2-stratifolds [5].

**Example 2.** Lens spaces $L(p, q)$ with $q \neq 0, 1$.

Let $r$ be the rotation of the disk $D^2$ about its center $c$ with angle $2\pi p/q$, let $1 \in S^1 \subset D^2$ and let $x_i = r^{i-1}(1), i = 1, \ldots, q$. Let $Y \subset D^2$ be the cone of $\{x_1, \ldots, x_q\}$ with cone point $c$. Embed $Y \times I/(x_i, 0) \sim (x_{i+1}, 1)$ into the solid torus $V = D^2 \times I/(x, 0) \sim (r(x), 1)$. The punctured lens space $L(p, q)$ is obtained from $V$ by attaching a 2-handle $D \times I$ with $\partial D$ attached to the boundary curve of $(Y \times I)/\sim$. Then $L(p, q)$ deformation retracts to $(Y \times I)/\sim \cup D$, which is the 2-stratifold with one white vertex of genus 0, one black vertex, and one edge with label $q$.

**Example 3.** Lens space $L(1, 0) = S^2 \times S^1$ and non-orientable $S^2$-bundle over $S^1$.

Consider $S^2 \vee S^1$, the non-orientable $S^2$-bundle over $S^1$, as as the quotient space $q(S^2 \times I)$ under the quotient map $q : S^2 \times I \to S^2 \times S^1$ that identifies $(x, 0)$ with $(x, 1), x \in S^2$.

Let $D_0 \subset S^2 \times \{0\}$ be a disk and $B_1$ be the 3-ball $D_0 \times I \subset S^2 \times I$, let $D_1$ be the disk $B_1 \cap S^2 \times \{1\}$, let $A$ be the annulus $\partial B_1 - (\text{Int}(D_0) \cup \text{Int}(D_1))$, and let $B_2$ be the ball $S^2 \times I - \text{Int}(B_1)$, see Figure 2.

![Figure 2: $S^2 \vee S^1 - \text{Int}(B_2) \searrow S^1 \vee S^1 \cup D^2$](image.png)

Then $S^2 \times I - \text{Int}(B_2) = S^2 \times \{0\} \cup B_1 \cup S^2 \times \{1\}$ and $S^2 \vee S^1 - \text{Int}(B_2) = q(S^2 \times I - \text{Int}(B_2)) = S^2 \cup q(B_1)$, where $S^2 = q(S^2 \times \{0\}) = q(S^2 \times \{1\})$. Collapsing the ball $q(B_1)$ across the free face $q(D_1)$ onto $q(A) \cup q(D_0)$ we obtain
a collapse of $S^2 \times S^1 - \text{Int}(B_2)$ onto $(S^2 - \text{Int}(q(D_1))) \cup q(A)$, which is a Kleinbottle with a disk attached. This is a 2-stratifold $X_G$ with graph $G_X$ in Figure 3(a). (The white vertices have genus 0).

A similar construction, considering $S^2 \times S^1$ as the obvious quotient space of $q : S^2 \times I \rightarrow S^2 \times S^1$ and first isotoping the ball $B_1$ such that $D_0 \cap D_1 = \emptyset$, we obtain a collapse of $S^2 \times S^1 - \text{Int}(B_2)$ onto a torus with a disk attached. This is a 2-stratifold $X_G$ with graph $G_X$ in Figure 3(b).

Figure 3: 2-stratifold spines of punctured $S^2 \times S^1$ and $P^2 \times S^1$

**Example 4.** $P^2 \times S^1$.

For a one-sided simple closed curve $c$ in $P^2$ and a point $t_0$ in $S^1$ let $X = P^2 \times \{t_0\} \cup c \times S^1 \subset P^2 \times S^1$. Observe that the boundary of a regular neighborhood $N$ of $X$ in $P^2 \times S^1$ is a 2-sphere. Since $P^2 \times S^1$ is irreducible, $\partial N$ bounds a 3-ball $B^3$ and therefore $P^2 \times S^1 - \text{Int}(B^3) = N$, which collapses onto $X = X_G$, a 2-stratifold with graph in Figure 3(c).

**Proposition 4.** If the closed 3-manifold $M_i (i = 1, 2)$ has a 2-stratifold spine and $M$ is a connected sum of $M_1$ and $M_2$, then $M$ has a 2-stratifold spine.

**Proof.** Let $K_i$ be a 2-stratifold spine of $M_i$. Let $K_1 \vee K_2$ be obtained by identifying, in the disjoint union of $K_1$ and $K_2$ a nonsingular point of $K_1$ with a nonsingular point of $K_2$. By Lemma 1 of [7], $K_1 \vee K_2$ is a spine of $M$. Though $K_1 \vee K_2$ is not a 2-stratifold, by performing the operation explained below (replacing the wedge point by a disk) we will change $K_1 \vee K_2$ to a 2-stratifold spine $K_1 \Delta K_2$.

A 3-ball neighborhood $B^3$ of the wedge point of $K_1 \vee K_2$ intersects $K_1 \vee K_2$ in the double cone shown in Fig.4. Replace, in $K_1 \vee K_2$, $K_1 \vee K_2 \cap B^3$ by $A \cup D$, as shown in Fig. 2, where $A = S^1 \times [0, 1]$ is an horizontal cylinder, $\partial A =$
\[(K_1 \lor K_2) \cap \partial B^3, D\] is a vertical 2-disk with \(A \cap D = \partial D = S^1 \times (1/2)\). The result is a 2-stratifold \(K_1 \Delta K_2\). There is a homeomorphism from \(B^3 - (A \cup D)\) onto \(B^3 - K_1 \lor K_2\) which is the identity on the boundary (roughly collapse \(D\) to a point) and so \(M - K_1 \Delta K_2\) is homeomorphic to \(M - K_1 \lor K_2\) which is homeomorphic to \(\mathbb{R}^3\).

Therefore \(K_1 \lor K_2\) is a 2-stratifold spine of \(M\).

\[
\begin{array}{c}
\text{Figure 4: } K_1 \Delta K_2
\end{array}
\]

Now Theorem 1 together with the examples and Proposition 4 yields our main Theorem. Here we do not consider \(S^3\) to be a lens space.

**Theorem 2.** A closed 3-manifold \(M\) has a 2-stratifold as a spine if and only if \(M\) is a connected sum of lens spaces, \(S^2\)-bundles over \(S^1\), and \(P^2 \times S^1\)'s.

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**References**


