RIESZ ENERGY ON SELF-SIMILAR SETS

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Abstract. We investigate properties of minimal $N$-point Riesz $s$-energy on fractal sets of non-integer dimension, as well as asymptotic behavior of $N$-point configurations that minimize this energy. For $s$ bigger than the dimension of the set $A$, we constructively prove a negative result concerning the asymptotic behavior (namely, its nonexistence) of the minimal $N$-point Riesz $s$-energy of $A$, but we show that the asymptotic exists over reasonable sub-sequences of $N$. Furthermore, we give a short proof of a result concerning asymptotic behavior of configurations that minimize the discrete Riesz $s$-energy.

Keywords: Best-packing points, Cantor sets, Equilibrium configurations, Minimal discrete energy, Riesz potentials

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1. Introduction

The minimal energy problem originates from potential theory, where for a compact set $A \subset \mathbb{R}^p$ and a lower semicontinuous kernel $K$ defined on $A \times A$, it is required to find

$$I_K(A) := \inf_{\mu} \int K(x,y) d\mu(x) d\mu(y),$$

where the infimum is taken over all probability measures supported on $A$; moreover, we are interested in the measure that attains this infimum. In this paper we focus on the Riesz $s$-kernels $K_s(x,y) := |x-y|^{-s}$. It is convenient to discretize the measure on which the value $I_K(A)$ is achieved; for this purpose, we consider the discrete Riesz $s$-energy problem. Namely, for every integer $N \geq 2$ we define

$$E_s(A,N) := \inf_{\omega_N} E_s(\omega_N),$$

where the infimum is taken over all $N$-point sets $\omega_N = \{x_1, \ldots, x_N\} \subset A$, and

$$E_s(\omega_N) := \sum_{i \neq j} |x_i - x_j|^{-s}, \quad N = 2, 3, 4, \ldots$$

Since the kernel $K_s$ is lower semicontinuous, the infimum is always attained.

In general, asymptotics of energy functionals arising from pairwise interaction in discrete subsets has been the subject of a number of studies \cite{14, 13, 10, 6}; it has also been considered for random point configurations \cite{7} and in the context of random processes \cite{1, 2}. The interest in such functionals is primarily motivated by applications in physics and modeling of particle interactions, as well as by the connections to geometric measure theory.

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If $d$ is the Hausdorff dimension of $A$ and $s < d$, then there is a unique measure $\mu_{s,A}$ for which the infimum in \((1)\) is achieved, and the configurations that attain the infimum in \((2)\) resemble $\mu_{s,A}$ in the weak$^*$ sense (for the precise definition, see below). When $s > d$, we have $I_{H^s}(A) = \infty$, as the integral in the RHS is infinite on all measures $\mu$ supported on $A$. However, for “good” sets $A$ (for example, $d$-rectifiable sets) with integer dimension $d$, the configurations attaining \((2)\) resemble a certain special measure, namely, the uniform measure on $A$.

More precisely, for a configuration $\omega_N = \{x_i : 1 \leq i \leq N\} \subset A$ we define the (empirical) probability measure

$$
\nu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i},
$$

and we shall identify the two. Then, as summarized in the Poppy-seed bagel theorem (PSB), see Theorem A under some regularity requirements on the set $A$, any sequence $\{\omega_N : \#\omega_N = N, \mathcal{E}_s(A,N) = E_s(\omega_N)\}$ converges to the normalized $d$-dimensional Hausdorff measure $H_d(A \cap \cdot)/H_d(A)$ on $A$. Moreover, for such sets $A$, the following limit exists:

\begin{equation}
\lim_{N \to \infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}.
\end{equation}

On the other hand, it has been established \cite[Proposition 2.6]{4} that for a class of self-similar fractals $A$ with $\dim_H A = d$, the limit of $\mathcal{E}_s(A,N)/N^{1+s/d}$ does not exist for $s$ large enough. Using this observation, \cite{8} gives an example of a set $A$ and a sequence of optimal configurations for $\mathcal{E}_s(A,N)$ without a weak$^*$ limit.

In view of the above, it is natural to ask what can be said about weak$^*$ cluster points of $\{\nu_N : N \geq 2\}$ in the case when the underlying set $A$ is not $d$-rectifiable; a characterization of the cluster points of $\{\mathcal{E}_s(A,N)/N^{1+s/d} : N \geq 2\}$ is likewise of interest.

The following section contains formal definitions and the necessary prerequisites; Section 3 gives an overview of previously established results, both in the case of a rectifiable and a non-rectifiable set $A$. Sections 4 and 5 contain the formulations of the main theorems and their proofs, respectively.

2. Self-similarity and open set condition

We shall be working with subsets of the Euclidean space $\mathbb{R}^p$, using bold typeface for its elements: $x \in \mathbb{R}^p$. An open ball of radius $r$, centered at $x$, will be denoted by $B(x,r)$. The $d$-dimensional Hausdorff measure of a Borel set $A$ will be denoted by $H_d(A)$.

A pair of sets $A^{(1)}, A^{(2)}$ will be called metrically separated if $|x - y| \geq \sigma > 0$ whenever $x \in A^{(1)}$ and $y \in A^{(2)}$. Recall that a similitude $\psi : \mathbb{R}^p \to \mathbb{R}^p$ can be written as

$$
\psi(x) = rO(x) + z
$$

for an orthogonal matrix $O \in O(p)$, a vector $z \in \mathbb{R}^p$, and a contraction ratio $0 < r < 1$. The following definition can be found in \cite{10}.

\textbf{Definition 2.1.} A compact set $A \subset \mathbb{R}^p$ is called a self-similar fractal with similitudes $\{\psi_m\}_{m=1}^{M}$ with contraction ratios $r_m$, $1 \leq m \leq M$ if

$$
A = \bigcup_{m=1}^{M} \psi_m(A),
$$

where the union is disjoint\footnote{One also considers self-similar fractals where the union is not disjoint — these are harder to deal with.}.

We say that $A$ satisfies the open set condition if there exists a bounded open set $V \subset \mathbb{R}^p$ such that

$$
\bigcup_{m=1}^{M} \psi_m(V) \subset V,
$$

where the sets in the union are disjoint.
For a self-similar fractal $A$, it is known [11, 19] that its Hausdorff dimension $\dim_H A = d$ where $d$ is such that
\begin{equation}
\sum_{m=1}^{M} r_m^d = 1.
\end{equation}
It will further be used that if $A$ is a self-similar fractal satisfying the open set condition, then there holds $0 < \mathcal{H}_d(A) < \infty$ and $A$ is $d$-regular with respect to $\mathcal{H}_d$, that is, there exists a positive constant $c$, such that for every $r, 0 < r \leq \text{diam}(A)$, and every $x \in A$,
\begin{equation}
c^{-1}r^d \leq \mathcal{H}_d(A \cap B(x, r)) \leq cr^d.
\end{equation}

3. OVERVIEW OF PRIOR RESULTS

Recall the standard definition of the weak* convergence: given a countable sequence $\{\mu_N : N \geq 1\}$ of probability measures supported on $A$ and another probability measure $\mu$,
\begin{equation}
\mu_N \rightharpoonup \mu, \quad N \to \infty \iff \int_A f(x) d\mu_N(x) \to \int_A f(x) d\mu(x), \quad N \to \infty,
\end{equation}
for every $f \in C(A)$. (Limits along nets are not necessary, as in this context weak* topology is metrizable.) We shall say that a sequence of discrete sets converges to a certain measure if the corresponding sequence of counting measures converges to it.

The set $A$ is said to be $d$-rectifiable if it is the image of a compact subset of $\mathbb{R}^d$ under a Lipschitz map. Furthermore, we say that $A$ is $(\mathcal{H}_d, d)$-rectifiable, if
\begin{equation}
A = A^{(0)} \cup \bigcup_{k=0}^{\infty} A^{(k)},
\end{equation}
where for $k \geq 1$ each $A^{(k)}$ is $d$-rectifiable and $\mathcal{H}_d(A^{(0)}) = 0$.

We begin by discussing results dealing with the Riesz energy, both in the rectifiable and non-rectifiable contexts. To formulate the PSB theorem, suppose $s > d$ for simplicity; the case of $s = d$ is similar, but requires stronger assumptions on the set $A$. We write $\mathcal{M}_d(A)$ for the $d$-dimensional Minkowski content of the set $A$ [12, 3.2.37–39].

**Theorem A** (Poppy-seed bagel theorem, [13, 5]). If the set $A$ is $(\mathcal{H}_d, d)$-rectifiable for $s > d$ and $\mathcal{H}_d(A) = \mathcal{M}_d(A)$, then
\begin{equation}
\lim_{N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} = \frac{C_{s,d}}{\mathcal{H}_d(A)^{s/d}},
\end{equation}
and every sequence $\{\tilde{\omega}_N : N \geq 2\}$ achieving the above limit converges weak* to the uniform probability measure on $A$:
\begin{equation}
\frac{1}{N} \sum_{\# \tilde{\omega}_N} \delta_{\tilde{\omega}_N} \rightharpoonup \frac{\mathcal{H}_d(A \cap \cdot)}{\mathcal{H}_d(A)}.
\end{equation}

The smoothness assumptions on $A$ in the above theorem are essential for existence of the limit of $E_s(A, N)/N^{1+s/d}$. Let $\{\tilde{\omega}_N \subset A : \# \tilde{\omega}_N = N, N \in \mathbb{N}\}$ be a sequence of configurations such that
\begin{equation}
\lim_{N \to \infty} \frac{E_s(\tilde{\omega}_N)}{N^{1+s/d}} = \liminf_{N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} =: g_{s,d}(A),
\end{equation}
and similarly, $\{\varpi_N \subset A : \# \varpi_N = N, N \in \mathbb{N}\}$ a sequence for which
\begin{equation}
\lim_{N \to \infty} \frac{E_s(\varpi_N)}{N^{1+s/d}} = \limsup_{N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} =: \tilde{g}_{s,d}(A).
\end{equation}
In the notation of (7)-(8), the result about the non-existence of $\lim_{N \to \infty} E_s(A, N)/N^{1+s/d}$ from [4] that was mentioned in the introduction can be stated as follows.
Proposition 3.1. For a self-similar fractal $A$ with contraction ratios $r_1 = \ldots = r_m$, there exists an $S_0 > 0$ such that for every $s > S_0$,

$$0 < g_{s,d}(A) < \bar{g}_{s,d}(A) < \infty.$$ 

We remark that in the proof of Proposition 5.1, the number $S_0$ was not obtained constructively. In Theorem 4.1, we give a formula for $S_0$. The behavior of the sets $\omega_N$ that attain $E_s(A,N)$ in the non-rectifiable case is still not fully characterized. The following proposition, taken from [8], is the only known negative result so far.

Proposition 3.2. Assume that the two d-regular compact sets $A^{(1)}$, $A^{(2)}$ are metrically separated and are such that $A^{(1)}$ is a self-similar fractal with equal contraction ratios and $g_{s,d}(A^{(1)}) = \bar{g}_{s,d}(A^{(1)})$. Then for any sequence of minimizers $\{\hat{\omega}_N \subset A : \#\hat{\omega}_N = N, E_s(\hat{\omega}_N) = E_s(A,N)\}$, the corresponding sequence of measures

$$\tilde{\nu}_N = \frac{1}{N} \sum_{\# \omega_N} \delta_{\omega}$$

does not have a weak* limit.

In view of these two propositions, it is remarkable that the local properties of minimizers of $E_s$ are fully preserved on self-similar fractals. Indeed, $d$-regularity of $A$ can be readily used to obtain that any sequence of minimizers of $E_s$ has the optimal orders of separation and covering. The following result was proved in [15].

Proposition 3.3. If $A \subset \mathbb{R}^d$ is a compact $d$-regular set, $\{\hat{\omega}_N : N \geq 1\}$ a sequence of configurations minimizing $E_s$ with $\hat{\omega}_N = \{\hat{x}_i : 1 \leq i \leq N\}$, then there exist a constant $C_1 > 0$ such that for any $1 \leq i < j \leq N$,

$$|\hat{x}_i - \hat{x}_j| \geq C_1 N^{-1/d}, \quad N \geq 2,$$

and a constant $C_2 > 0$ such that for any $y \in A$,

$$\min_i |y - \hat{x}_i| \leq C_2 N^{-1/d}, \quad N \geq 2.$$

The closest one comes to an analog of the PSB theorem for self-similar fractals is the following proposition [8]. Note that we give a simpler proof of [11] for the case when $A_0 = A$ in Theorem 4.1.

Proposition 3.4. Suppose $A_0$ is a self-similar fractal satisfying the open set condition and $s > d$; fix a compact $A \subset A_0$.

1. If $\{\omega_N : N \in \mathbb{N}\}$, is a sequence of configurations for which

$$\lim_{N \to \infty} E_s(\omega_N) = g_{s,d}(A),$$

then the corresponding sequence of empirical measures converges weak*:

$$\nu_N \rightharpoonup \frac{H_d(\cdot \cap A)}{H_d(A)}, \quad \mathbb{N} \ni N \to \infty.$$ 

2. There holds

$$g_{s,d}(A) = \frac{g_{s,d}(A_0) H_d(A_0)^{s/d}}{H_d(A)^{s/d}},$$

and

$$\bar{g}_{s,d}(A) = \frac{\bar{g}_{s,d}(A_0) H_d(A_0)^{s/d}}{H_d(A)^{s/d}}.$$ 

We finish this section with another relevant result on fractal sets. In [11] it was shown that, as $s \to \infty$, there is a strong connection between the $s$-energy $E_s(A)$ and the best-packing constant

$$\delta(A,N) := \sup_{x \in A} \min_{y \neq x} |x - y|.$$
The main theorem of [18] is given in terms of the function \( N(\delta) := \max\{n : \delta(A,n) \geq \delta\} \). Our Theorem 4.3 gives an analog of the second part of this theorem for the minimal discrete energy.

**Theorem B.** Suppose \( A \) is a self-similar fractal of dimension \( d \) satisfying the open set condition with contraction ratios \( r_1, \ldots, r_m \).

1. If the additive group generated by \( \log r_1, \ldots, \log r_m \) is dense in \( \mathbb{R} \), then there exists a constant \( C \) such that
   \[
   \lim_{N \to \infty} N^{1/d} \frac{\delta(A,N)}{N^{1+s/d}} = C.
   \]
2. If the additive group generated by \( \log r_1, \ldots, \log r_M \) coincides with the lattice \( h\mathbb{Z} \) for some \( h > 0 \), then
   \[
   \lim_{N \to \infty} N^{1/d} \delta(A,N) = C_0,
   \]
   where the limit is taken over a subsequence \( \delta \to 0 \) with \( \{\frac{1}{h} \log \delta\} = \theta \).

4. **Main results**

In accordance with the prior notation, we write \( \omega_N = \{x_i : 1 \leq i \leq N\} \) for the sequence of configurations with the lowest asymptotics (i.e., such that (7) holds), and
\[
\nu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}, \quad N \in \mathbb{N}.
\]
As described above, generally the limit of \( E_s(A,N) / N^{1+s/d} \), \( N \to \infty \) does not necessarily exist. It is still possible to characterize the behavior of the sequence \( \{\omega_N : N \in \mathbb{N}\} \). The following result first appeared in [3]; we give an independent and a more direct proof.

**Theorem 4.1.** Let \( A \subset \mathbb{R}^p \) be a compact self-similar fractal satisfying the open set condition, and \( \dim_H A = d < s \). If \( \{\omega_N : N \in \mathbb{N}\} \), is a sequence of configurations for which
\[
\lim_{M \in \mathbb{N} \to \infty} \frac{E_s(\omega_N)}{N^{1+s/d}} = g_{s,d}(A),
\]
then the corresponding sequence of empirical measures converges weak*:
\[
\nu_N \overset{\text{weak*}}{\longrightarrow} h_d(\cdot) := \frac{\mathcal{H}_d(\cdot \cap A)}{\mathcal{H}_d(A)}, \quad \mathbb{N} \ni N \to \infty.
\]

When the similitudes \( \{\psi_m\}_{m=1}^M \) fixing \( A \) all have the same contraction ratio, it is natural to expect some additional symmetry of minimizers, associated with the \( M \)-fold scale symmetry of \( A \). Similarly, since the energy of interactions between particles is at most of order \( N^2 \), see proof of Lemma 5.1 below, we expect that by acting with \( \{\psi_m\}_{m=1}^M \) on a minimizer \( \tilde{\omega}_N \) with \( N \) large, we obtain a near-minimizer with \( MN \) elements. This heuristic is made rigorous in the following theorem.

**Theorem 4.2.** Let \( A \subset \mathbb{R}^p \) be a self-similar fractal, fixed under \( M \) similitudes with the same contraction ratio, and \( \mathfrak{M} = \{M^k n : k \geq 1\} \). Then the following limit exists
\[
\lim_{M \in \mathbb{N} \to \infty} \frac{E_s(A,N)}{N^{1+s/d}}.
\]

The previous theorem can be further extended. We shall need some notation first. For a sequence \( \mathfrak{M} \), let
\[
\{\mathfrak{M}\} := \lim_{M \in \mathbb{N} \to \infty} \{\log_M N\},
\]
where \( \{\cdot\} \) in the RHS denotes the fractional part, and
\[
E_s(\mathfrak{M}) := \lim_{M \in \mathbb{N} \to \infty} \frac{E_s(A,N)}{N^{1+s/d}},
\]
if the corresponding limit exists.
Theorem 4.3. If $A$ is a self-similar fractal with equal contraction ratios, and two sequences $\mathcal{G}_1, \mathcal{G}_2 \subset \mathbb{N}$ are such that

\begin{equation}
\{\mathcal{G}_1\} = \{\mathcal{G}_2\},
\end{equation}

then

\begin{equation}
E_s(\mathcal{G}_1) = E_s(\mathcal{G}_2).
\end{equation}

In particular, the limits in (11) exist. Moreover, the function $g_{s,d} : \mathcal{G} \mapsto E_s(\mathcal{G})$ is continuous on $[0,1]$.

In the case of equal contraction ratios, the argument in the proof of Theorem 4.2 can be further used to make the result of Proposition 3.1 more precise.

Theorem 4.4. Let $A \subset \mathbb{R}^p$ be a self-similar fractal, fixed under $M$ similitudes with the same contraction ratio $r$, and write $\sigma := \min\{\|x - y\| : x \in A_i, y \in A_j, i \neq j\}$. If $R := \frac{\sigma}{1 + r^d} < 1$, then for every value of $s$ such that

\begin{equation}
s \geq \max\{2d, \log_4 R [2M(M + 1)]\},
\end{equation}

there holds

\begin{equation}
0 < g_{s,d}(A) < \bar{g}_{s,d}(A) < \infty.
\end{equation}

The proof of this theorem requires an estimate for the value of $E_s(A,M)$, which results in the condition $R < 1$. When $E_s(A,M)$ can be computed explicitly, a similar conclusion can also be obtained for sets that do not necessarily satisfy $R < 1$, as in the following.

Corollary 4.5. If $A$ is the ternary Cantor set and $s > 3 \dim_H A = 3 \log_3 2$, then

\begin{equation}
0 < g_{s,d}(A) < \bar{g}_{s,d}(A) < \infty.
\end{equation}

5. Proofs

The key to proving Theorem 4.1 is that the hypersingular Riesz energy grows faster than $N^2$. We shall need this property in the following form.

Lemma 5.1. Let a pair of compact sets $A^{(1)}, A^{(2)} \subset \mathbb{R}^p$ be metrically separated; let further $\{\omega_N \subset A : N \in \mathcal{G}\}$ be a sequence for which the limits

\begin{equation}
\lim_{\mathcal{G} \ni N \to \infty} \frac{\#(\omega_N \cap A^{(i)})}{N} = \beta^{(i)}, \quad i = 1, 2.
\end{equation}

exist. Then

\begin{equation}
\liminf_{\mathcal{G} \ni N \to \infty} \frac{E_s(\omega_N)}{N^{1 + s/d}} \geq \left(\beta^{(1)}\right)^{1 + s/d} \liminf_{\mathcal{G} \ni N \to \infty} \frac{E_s(\omega_N \cap A^{(1)})}{\#(\omega_N \cap A^{(1)})^{1 + s/d}} + \left(\beta^{(2)}\right)^{1 + s/d} \liminf_{\mathcal{G} \ni N \to \infty} \frac{E_s(\omega_N \cap A^{(2)})}{\#(\omega_N \cap A^{(2)})^{1 + s/d}}.
\end{equation}

Proof. We observe that with $\sigma = \text{dist} (A^{(1)}, A^{(2)})$,

\begin{equation}
\left|E_s(\omega_N) - \left(E_s(\omega_N \cap A^{(1)}) + E_s(\omega_N \cap A^{(2)})\right)\right| = \sum_{x_i \in A^{(1)}, x_j \in A^{(2)}} |x_i - x_j|^{-s} \leq \sigma^{-s} N^2,
\end{equation}

and use the definition of $\beta^{(i)}$, $i = 1, 2$, to obtain the desired equality. \qed
This is particularly useful for self-similar fractals satisfying the open set property. Consider such a fractal \( A \); since \( \psi_m(V), 1 \leq m \leq M \), are pairwise disjoint for an open set \( V \) containing \( A \), there exists a \( \sigma > 0 \) such that \( \text{dist} (\psi_i(A), \psi_j(A)) \geq \sigma \) for \( i \neq j \). Following [17], we will write

\[
A_{m_1 \ldots m_l} := \psi_{m_1} \circ \ldots \circ \psi_{m_l}(A), \quad 1 \leq m_i \leq M, \quad l \geq 1.
\]

Then \( \text{dist} (A_{m_1 \ldots m_l}, A'_{m'_1 \ldots m'_l}) \geq r_{m_1} \ldots r_{m_k} \sigma \), where \( k = \min\{i : m_i \neq m'_i\} \), so for a fixed \( M \) in the expression

\[
A = \bigcup_{m_1 \ldots m_l=1}^M A_{m_1 \ldots m_l}
\]

not only the union is disjoint, but also the sets \( A_{m_1 \ldots m_l} \) are metrically separated. The following lemma is technical, and we give its proof for the convenience of the reader.

**Lemma 5.2.** If \( \{\mu_N : N \in \mathcal{R}\} \) is a sequence of probability measures on the set \( A \), which for every \( l \geq 1 \) satisfies

\[
\lim_{N \to \infty} \mu_N(A_{m_1 \ldots m_l}) = \mu(A_{m_1 \ldots m_l}), \quad 1 \leq m_1, \ldots, m_l \leq M,
\]

for another probability measure \( \mu \) on \( A \), then

\[
\mu_N \rightharpoonup \mu, \quad \forall N \to \infty.
\]

**Proof.** Fix an \( f \in C(A) \); since \( A \) is compact, \( f \) is uniformly continuous on \( A \). For a fixed \( \varepsilon > 0 \), there exists an \( L_0 \in \mathbb{N} \) such that \( |f(x) - f(y)| < \varepsilon \) whenever \( x, y \in A_{m_1 \ldots m_l} \) for any \( l \geq L_0 \) and any set of indices \( 0 \leq m_1, \ldots, m_l \leq M \); this is possible due to

\[
\text{diam}(A_{m_1 \ldots m_l}) \leq r_{m_1} \ldots r_{m_l} \text{diam}(A) \leq \left( \max_{1 \leq m \leq M} r_m \right)^l \text{diam}(A).
\]

Fix an \( l \geq L_0 \) until the end of this proof, then pick an \( N_0 \in \mathcal{R} \) so that for every \( N \geq N_0 \), there holds

\[
|\mu_N(A_{m_1 \ldots m_l}) - \mu(A_{m_1 \ldots m_l})| < \varepsilon / M^l, \quad 1 \leq m_1, \ldots, m_l \leq M.
\]

Finally, let us write \( f_{m_1 \ldots m_l} := \min_{A_{m_1 \ldots m_l}} f(x) \) for brevity. Then for \( N \geq N_0 \),

\[
\left| \int_A f(x) d\mu_N(x) - \int_A f(x) d\mu(x) \right| \leq \sum_{m_1, \ldots, m_l=1}^M \left| \int_{A_{m_1 \ldots m_l}} (f(x) - f_{m_1 \ldots m_l}) d\mu_N(x) - \int_{A_{m_1 \ldots m_l}} (f(x) - f_{m_1 \ldots m_l}) d\mu(x) \right|
\]

\[+ \sum_{m_1, \ldots, m_l=1}^M \left| (\mu_N(A_{m_1 \ldots m_l}) - \mu(A_{m_1 \ldots m_l})) f_{m_1 \ldots m_l} \right| \leq 2\varepsilon + \varepsilon M^l |
\]

where the estimate for the first sum used that both \( \mu_N \) and \( \mu \) are probability measures. This proves the desired statement. \( \square \)

Note that the converse is also true; since the sets \( A_{m_1 \ldots m_l} \) are metrically separated, convergence \( \mu_N \rightharpoonup \mu \) of measures supported on \( A \) immediately implies (by Urysohn’s lemma) \( \mu_N(A_{m_1 \ldots m_l}) \to \mu(A_{m_1 \ldots m_l}) \) for all \( l \geq 1 \) and all indices \( 1 \leq m_1, \ldots, m_l \leq M \).

The proof of the following statement follows a well-known approach [15, 17, Theorem 2], and can be considered standard.

**Proposition 5.3.** If \( A \) is a compact \( d \)-regular set, then \( 0 < g_{s,d}(A) \leq g_{s,d}(A) < \infty \).
The above proposition can be somewhat strengthened, to obtain uniform upper and lower bounds on
\[
\frac{E_s(\omega_N)}{N^{1+s/d}}, \quad N \geq 2;
\]
furthermore, each bound requires only one of the inequalities in \([4]\). In addition, for any sequence of configurations \(\omega_N, N \in \mathfrak{N}\), with
\[
\lim_{\mathfrak{N} \ni N \to \infty} \frac{E_s(\omega_N)}{N^{1+s/d}} < \infty,
\]
every weak* cluster point of \(\nu_N, N \in \mathfrak{N}\), must be absolutely continuous with respect to \(H_d\) on \(A\). Lastly, we will need the following standard estimate.

**Corollary 5.4.** Suppose \(A\) is a compact \(d\)-regular set, \(\omega_N = \{\mathbf{x}_i : 1 \leq i \leq N\} \subset A\), and \(s > d\). Then the minimal point energy of \(\omega_N\) is bounded by:
\[
\min_{\mathbf{x} \in A} \sum_{j=1}^N |\mathbf{x} - \mathbf{x}_j|^{-s} \leq CN^{s/d},
\]
where \(C\) depends only on \(A, s, d\).

**Proof of Theorem 4.1.** In view of the weak* compactness of probability measures in \(A\), to establish existence of the weak* limit of \(\nu_N, N \in \mathfrak{N}\), it suffices to show that any cluster point of \(\nu_N, N \in \mathfrak{N}\) in the weak* topology is \(h_d\) which is defined in \([4]\) (see \([3]\) Proposition A.2.7). To that end, consider a subsequence of \(\mathfrak{N}\) for which the empirical measures \(\nu_N\) converge to a cluster point \(\mu\); for simplicity we shall use the same notation \(\mathfrak{N}\) for this subsequence.

As discussed above, \(\nu_N(A_{m_1 \ldots m_L}) \to \mu(A_{m_1 \ldots m_L}), \mathfrak{N} \ni N \to \infty\); this ensures that the quantities
\[
\beta_m := \mu(A_m) = \lim_{N \ni N \to \infty} \frac{\nu_N(A_m)}{\mathfrak{N}} = \lim_{N \ni N \to \infty} \frac{\#(\omega_N \cap A_m)}{N}, \quad m = 1, \ldots, M,
\]
are well-defined. From \([4]\), separation of \(\{A_m\}\), and Lemma 5.1 follows
\[
g_{s,d}(A) = \sum_{m=1}^M \lim_{N \ni N \to \infty} \frac{E_s(\omega_N \cap A_m)}{N^{1+s/d}} \geq \sum_{m=1}^M \beta^{1+s/d}_m \lim_{N \ni N \to \infty} \frac{E_s(\omega_N \cap A_m)}{\#(\omega_N \cap A_m)^{1+s/d}} \geq \sum_{m=1}^M \beta^{1+s/d}_m R_m^{-s/d} g_{s,d}(A).
\]
Consider the RHS in the last inequality. As a function of \(\{\beta_m\}\), it satisfies the constraint \(\sum_m \beta_m = 1\); note also that by the defining property \([4]\) of \(d\), there holds \(\sum_m R_m = 1\) with \(R_m := r_m^d, 1 \leq m \leq M\). We have
\[
(13) \quad g_{s,d}(A) \geq \inf \left\{ \sum_{m=1}^M \beta^{1+s/d}_m R_m^{-s/d} : \sum_{m=1}^M \beta_m = 1 \right\} g_{s,d}(A).
\]
Level sets of the function \(\sum_m \beta^{1+s/d}_m R_m^{-s/d}\) are convex, so the infimum is attained and unique; it is easy to check that the solution is at \(\beta_m = R_m = r_m^d, 1 \leq m \leq M\), and the minimal value is 1. Indeed, the corresponding Lagrangian is
\[
L(\beta_1, \ldots, \beta_M, \lambda) := \sum_{m=1}^M \beta^{1+s/d}_m R_m^{-s/d} - \lambda \sum_{m=1}^M \beta_m,
\]
hence
\[
\nabla L(\beta_m) = (1 + s/d) \left( \frac{\beta_m}{R_m} \right)^{s/d} - \lambda, \quad 1 \leq m \leq M,
\]
and it remains to use \(\beta_m \geq 0, 1 \leq m \leq M\), and \(\sum_m R_m = 1\); to conclude \(\beta_m = R_m, 1 \leq m \leq M\).
Since $0 < g_{s,d}(A) < \infty$ by Lemma 5.3 from [13] it follows
\[ \beta_m = r_m^d, \quad m = 1, \ldots, M. \]

Note that this argument shows also
\[ \lim_{N \to \infty} \frac{E_s(\omega_N \cap A_m)}{\#(\omega_N \cap A_m)^{1+1/d}} = g_{s,d}(A), \]
so the above can be repeated recursively for sets $A_{m_1 \ldots m_l}$. Namely, for every $l \geq 1$ and $1 \leq m, m_1, \ldots, m_l \leq M$,
\[ \mu(A_{m_1 \ldots m_l}) := \beta_{m_1 \ldots m_l} = r_m^d \beta_{m_1 \ldots m_l}. \]
Observe further that $h_d$ satisfies
\[ h_d(A_{m_1 \ldots m_l}) = r_m^d h_d(A_{m_1 \ldots m_l}) \]
by definition, so by Lemma 5.2 follows that every weak cluster point of $\gamma_N$, $N \in \mathbb{N}$, is $h_d$, as desired. □

**Proof of Theorem 4.2.** Note that setting equal contraction ratios $r_1 = \ldots = r_m = r$ in [4] gives $r^{-s} = M^{s/d}$. Consider the set function
\[ \psi : x \mapsto \bigcup_{m=1}^M \psi_m(x), \quad x \in A, \]
and denote
\[ \psi(\omega_N) := \bigcup_{\omega \in \omega_N} \psi(\omega). \]
It follows from the open set condition that the union above is metrically separated; as before, we denote the separation distance by $\sigma$. Observe that the definition of a similitude implies $\#(\psi(\omega_N)) = M \#(\omega_N)$. We then have for any configuration $\omega_N$, $N \geq 2$,
\[ E_s(A, MN) \leq E_s(\psi(\omega_N)) \leq M r^{-s} E_s(\omega_N) + \sigma^{-s} N^2 M^2 \]
and repeated application of the second inequality yields
\[ E_s(A, M^kN) \leq E_s(\psi^{(k-1)}(\omega_N)) \leq M^{1+s/d} E_s(\psi^{(k-1)}(\omega_N)) + \sigma^{-s} (M^{k-1}N)^2 M^2 \]
\[ \leq (M^2)^{1+s/d} E_s(\psi^{(k-2)}(\omega_N)) + M^{1+s/d} \sigma^{-s} (M^{k-2}N)^2 M^2 + \sigma^{-s} (M^{k-1}N)^2 M^2 \]
\[ \leq \ldots \]
\[ \leq (M^k)^{1+s/d} E_s(\omega_N) + \sigma^{-s} N^2 \sum_{l=1}^k (M^{l-1})^{1+s/d} (M^{k-l})^2 M^2. \]
Estimating the geometric series in the last inequality, we obtain
\[ E_s(A, M^kN) \leq (M^k)^{1+s/d} E_s(\omega_N) + \sigma^{-s} N^2 M^{2k+1-s/d} \sum_{l=1}^k M^{l(s/d-1)} \]
\[ \leq (M^k)^{1+s/d} E_s(\omega_N) + \sigma^{-s} N^2 M^{2k+1-s/d} \frac{M^{k+1}(s/d-1) - 1}{M^s(d-1) - 1} \]
\[ \leq (M^k)^{1+s/d} E_s(\omega_N) + \frac{N^{1-s/d}}{\sigma^s(M^s/d-1)} \left( M^k N \right)^{(1+s/d)}. \]
(14)
Let now $\varepsilon > 0$ fixed; find $\omega_{N_0}$ such that $N_0 \in \mathbb{N}$ and
\[ \frac{E_s(\omega_{N_0})}{N_0^{1+s/d}} \leq \liminf_{M \geq N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} + \varepsilon, \]
Lemma 5.5. If \( \mathcal{M} \subset \mathbb{N} \) is a sequence such that the limit
\[
\lim_{\mathcal{M} \ni N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}}
\]
equals 0, then for any sequence of integers \( \mathcal{N} \subset \mathbb{N} \) with \( |\mathcal{N}(k)| \) majorized by \( \mathcal{M} \) and satisfying \( |\mathcal{N}(k)| = o(\mathcal{M}(k)) \), \( k \to \infty \), there holds
\[
\frac{E_s(A, N)}{N^{1+s/d}} = \lim_{\mathcal{M} \ni N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}},
\]
in view of \( \mathcal{N}(k) = o(\mathcal{M}(k)) \). Similarly,
\[
\limsup_{\mathcal{M} \ni N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} \leq \lim_{\mathcal{M} \ni N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}}.
\]
For the converse estimates, use Corollary 5.4 to conclude that for every \( N \in \mathbb{N} \) there holds
\[
E_s(A, N + 1) \leq E_s(A, N) + C N^{s/d},
\]
Applying this inequality \( \mathcal{N}(k) \) times to \( \mathcal{M}(k) \), we obtain
\[
E_s[A, (\mathcal{M} + \mathcal{N})(k)] \leq E_s(A, \mathcal{M}(k)) + \mathcal{N}(k) C (\mathcal{M}(k) + \mathcal{N}(k))^{s/d},
\]
which yields
\[
\limsup_{\mathcal{M} \ni N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} \leq \lim_{\mathcal{M} \ni N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}}.
\]
Finally, applying Corollary 5.4 \( \mathcal{M}(k) \) times to \( \mathcal{M}(k) - \mathcal{N}(k) \) gives
\[
E_s[A, \mathcal{M}(k)] \leq E_s[A, (\mathcal{M} - \mathcal{N})(k)] + \mathcal{N}(k) C (\mathcal{M}(k) - \mathcal{N}(k))^{s/d},
\]
whence, using that \( \mathcal{N}(k) = o(\mathcal{M}(k)) \), \( k \to \infty \),
\[
\liminf_{\mathcal{M} \ni N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} \geq \lim_{\mathcal{M} \ni N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}}.
\]
Combining (16) with (18) and (17) with (19), we get the desired result. \( \square \)

The proof of the previous lemma implies the following.
Corollary 5.6. If $\mathcal{M}, \mathcal{N} \subseteq \mathbb{N}$ are a pair of sequences such that

$$\mathcal{N}(k) \leq \theta \mathcal{M}(k), \quad k \geq 1,$$

then

$$\liminf_{N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} \geq \liminf_{\mathcal{M} \supseteq N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} \left( \frac{1}{1+\theta} \right)^{1+s/d},$$

and

$$\limsup_{N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} \leq \limsup_{\mathcal{M} \supseteq N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} + C \theta,$$

where $C$ is the same as in Corollary 5.4.

Proof of Theorem 5.2. To show that $g_{s,d}(\cdot)$ is well-defined, it is necessary to verify that (i) existence of the limit $\{\mathcal{M}\}$ implies that of the limit $E_s(\mathcal{M})$, and (ii) the value of $E_s(\mathcal{M})$ is uniquely defined by $\{\mathcal{M}\}$. To this end, fix a pair of sequences $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathbb{N}$ such that $\{\mathcal{M}_1\} = \{\mathcal{M}_2\}$.

First assume that $\mathcal{M}_1, \mathcal{M}_2$ are multiples of (a subset of) the geometric series, that is, $\mathcal{M}_i = \{M^k n_i : k \in \mathbb{N}\}, i = 1, 2$. Observe that (11) implies $\{\log_M n_1\} = \{\log_M n_2\}$ and let definiteness $n_2 \geq n_1$; then $n_2 = M^{k_0} n_1$ for some integer $k_0 \geq 1$. It follows that $\mathcal{M}_1 \subset \mathcal{M}_0$, $i = 1, 2$, with $\mathcal{M}_0 = \{M^k n_0 : k \geq 1\}$. By Theorem 4.2, the limit

$$\lim_{\mathcal{M}_0 \supseteq N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}}, \quad i = 1, 2,$$

exists, so it must be that the limits over subsequences of $\mathcal{M}_0$

$$\lim_{\mathcal{M}_i \supseteq N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}}, \quad i = 1, 2,$$

also exist and are equal, so the function $g_{s,d}(\cdot)$ is well-defined on the subset of $[0, 1]$ of all the sequences $\mathcal{M}$ with $\mathcal{M} = \{M^k n : k \in \mathbb{N}\}$.

Now let $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathbb{N}$ be arbitrary. Denote the common value of the limit $a := \{\mathcal{M}_i\}$, $i = 1, 2$. We shall assume for definiteness that $a \in (0, 1)$; the case of $a = 1$ can be handled similarly. In order to bound $\mathcal{M}_i$ between two sequences of the type $\{M^k n_i : k \in \mathbb{N}\}$, discussed above, fix an $\epsilon > 0$ such that $a + 2\epsilon < 1$, and find an $N_0 \in \mathbb{N}$, for which

$$\{\log_M n_i\} - a < \epsilon, \quad N_0 \leq N_i \in \mathcal{M}_i, \quad i = 1, 2.$$

By the choice of $\epsilon$, the above equation gives $\{\log_M n_1\} = \{\log_M n_2\}$ when $N_0 \leq N_i \in \mathcal{M}_i$. Now let $n_i, i = 1, 2$ be such that

$$a - 2\epsilon \leq \{\log_M n_1\} \leq a - \epsilon$$

$$a + \epsilon \leq \{\log_M n_2\} \leq a + 2\epsilon.$$

Replacing one of $n_i, i = 1, 2$, with its multiple, if necessary, we can guarantee that $0 < \log_M n_2 - \log_M n_1 < 4\epsilon$. Consider a pair of sequences $\mathcal{M}_1 = \{M^k n_i : k \geq \log_M N_0\}, i = 1, 2$; observe that by the above argument, limits

$$E_s(\mathcal{M}_1) =: L_1, \quad i = 1, 2,$$

along $\mathcal{M}_1, i = 1, 2$, both exist, and the inequality

$$\mathcal{M}_1(k) \leq N_i \leq \mathcal{M}_2(k), \quad k = \log_M N_i, \quad N_0 \leq N_i \in \mathcal{M}_i, i = 1, 2,$$

holds. By the definition of $E_s$ and due to (20)–(21),

$$\limsup_{\mathcal{M}_i \supseteq N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} \leq \limsup_{k \to \infty} \frac{E_s(A, M^k n_2)}{(M^k n_1)^{1+s/d}} = \left( \frac{n_2}{n_1} \right)^{1+s/d} L_2, \quad i = 1, 2,$$

and

$$\liminf_{\mathcal{M}_i \supseteq N \to \infty} \frac{E_s(A, N)}{N^{1+s/d}} \geq \liminf_{k \to \infty} \frac{E_s(A, M^k n_1)}{(M^k n_2)^{1+s/d}} = \left( \frac{n_1}{n_2} \right)^{1+s/d} L_1, \quad i = 1, 2.$$
Combining the last two inequalities gives
\[
\left( \frac{n_1}{n_2} \right)^{1+s/d} L_1 \leq \liminf_{\eta_0, N \to \infty} E_s(A, N) \leq \limsup_{\eta_0, N \to \infty} E_s(A, N) \leq \left( \frac{n_2}{n_1} \right)^{1+s/d} L_2,
\]
so it suffices to show that \( L_2 \) can be made arbitrarily close to \( L_1 \) by taking \( \varepsilon \to 0 \). The latter follows from Corollary \( \[5.6 \] \) and the choice of \( n_i, i = 1, 2; \)
\[
0 \leq \frac{\tilde{N}_2(k) - \tilde{N}_1(k)}{\tilde{N}_1(k)} = \frac{n_2}{n_1} - 1 \leq M^{4\varepsilon} - 1. \]
Taking \( \varepsilon \to 0 \) shows both that \( E_s(\Omega_1) = E_s(\Omega_2) \), and that these two limits exist. The function \( g_{s,d} : [0, 1] \to (0, \infty) \) is therefore well-defined. Note that repeating the above argument for \( |\{\Omega_1\} - \{\Omega_2\}| < \varepsilon \) for a fixed positive \( \varepsilon \) gives a bound on \( |E_s(\Omega_1) - E_s(\Omega_2)| \), which implies that \( g_{s,d} \) is continuous. This completes the proof. \( \square \)

**Proof of Theorem 4.4.** Assume without loss of generality that the diameter of the set \( A \) satisfies
\[
diam(A) = 1.
\]
Denote the minimal value of the Riesz \( s \)-energy on \( M \) points on \( A \) by \( E_s, M := E_s(A, M) \); recall also that \( \sigma \) is the lower bound on the distance between \( A_i, A_j \) when \( i \neq j \). With this assumption, the last inequality in \( [14] \) with \( N = M \) gives
\[
\begin{align*}
E_s(A, M^{k+1}) & \leq M^{k(1+s/d)} E_s, M + \sigma^{-s} M^{1-s/d} M^{(k+1)(1+s/d)} \\
& \leq M^{k(1+s/d)} \sigma^{-s} M^2 + \sigma^{-s} M^{2} \frac{M^2}{M^{s/d} - 1} M^{(k+1)(1+s/d)} \\
& = M^{k(1+s/d)} \sigma^{-s} M^2 \left( 1 + \frac{1}{M^{s/d} - 1} \right) \\
& = M^{k(1+s/d)} \sigma^{-s} M^{s/d - 1}. 
\end{align*}
\]
On the other hand, consider a configuration \( \omega_{M^{k+1}+M^k} \). The set \( A \) is partitioned by the \( M^{k+1} \) subsets
\[
A_{m_1 \ldots m_{k+1}}, \quad 1 \leq m_1, \ldots, m_{k+1} \leq M,
\]
so by the pigeonhole principle, for at least \( M^k \) pairs \( i \neq j \), the points \( x_i, x_j \in \omega_{M^{k+1}+M^k} \) belong to the same subset \( A_{m_1 \ldots m_{k+1}} \). Writing \( r \) for the common contraction ratio of the defining similitudes \( \{\psi_m : 1 \leq m \leq M\} \) preserving the set \( A \), we have
\[
diam(A_{m_1 \ldots m_{k+1}}) = r^{k+1} \diam(A) = r^{k+1}.
\]
Configuration \( \omega_{M^{k+1}+M^k} \) was chosen arbitrarily, so it follows,
\[
E_s(A, M^{k+1} + M^k) \geq M^{k(r^{k+1})^{-s}} = M^k(M^{s/d} + 1) M^{k+1} = M^{s/d}(M^k)^{1+s/d},
\]
where we used that \( r^{-s} = M^{s/d} \) when all the contraction ratios are equal. Combining equations \( [22] \) \( [23] \) gives
\[
\frac{g_{s,d}(A)/\bar{g}_{s,d}(A)}{\limsup_{k \to \infty} E_s(A, M^{k+1})/\liminf_{k \to \infty} E_s(A, M^{k+1} + M^k)/(M^{k+1} + M^k)^{1+s/d}} \leq \frac{1}{M^{s/d} - 1} \left( \frac{1}{M(1 + 1/M)^{1+s/d}} \right) \\
= \frac{M(1 + 1/M)^{1+s/d}}{\sigma^s(M^{s/d} - 1)}.
\]
After substituting $1/M = r^d$, the last inequality can be rewritten as

$$g_{s,d}(A)/\bar{g}_{s,d}(A) \leq \left(\frac{r(1+r^d)^{1/d}}{\sigma}\right)^s \cdot \frac{M^{s/d-1}}{M^{s/d-1}-1} \cdot M(M+1)$$

$$= R^s \cdot \frac{M^{s/d-1}}{M^{s/d-1}-1} \cdot M(M+1).$$

Note that the second factor in the above equation is less than 2 when $s > 2d$ holds (since $M \geq 2$); for an $R < 1$, choosing the Riesz exponent as in (12) makes the RHS less than 1, as desired.

**Proof of Corollary 4.5.** The proof repeats that of Theorem 4.4, except for the simplified expression for $\mathcal{E}_{s,M} = \mathcal{E}_{s,2} = 1$. Equations (22)–(23) become

$$\mathcal{E}_s(A, 2^{k+1}) = 2^{(k+1)(1+s/d)} \cdot \frac{1}{2^{2(s/d-1)-1}},$$

$$\mathcal{E}_s(A, 2^{k+1} + 2^{k}) \geq 2^{s/d}(2^k)^{1+s/d},$$

respectively. Finally, from

$$g_{s,d}(A)/\bar{g}_{s,d}(A) \leq \frac{2(3/2)^{1+s/d}}{2^{2(s/d-1)-1}} = \left(\frac{3}{4}\right)^{s/d} \cdot \frac{2^{s/d-1}}{2^{s/d-1}-1} \cdot \frac{3}{2}.$$

The RHS is a decreasing function of $s$ and is less than 1 for $s \geq 3d = 3\dim_H A = 3\log_2 2$, which completes the proof. □

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