RIESZ ENERGY ON SELF-SIMILAR SETS

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ABSTRACT. We investigate properties of minimal N-point Riesz s-energy on fractal sets of non-integer dimension, as well as asymptotic behavior of N-point configurations that minimize this energy. For s bigger than the dimension of the set A, we constructively prove a negative result concerning the asymptotic behavior (namely, its nonexistence) of the minimal N-point Riesz s-energy of A, but we show that the asymptotic exists over reasonable sub-sequences of N. Furthermore, we give a short proof of a result concerning asymptotic behavior of configurations that minimize the discrete Riesz s-energy.

Keywords: Best-packing points, Cantor sets, Equilibrium configurations, Minimal discrete energy, Riesz potentials

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1. INTRODUCTION

The minimal energy problem originates from potential theory, where for a compact set $A \subset \mathbb{R}^p$ and a lower semicontinuous kernel K defined on $A \times A$, it is required to find

$$I_K(A) := \inf_{\mu} \int K(x,y)\mathrm{d}\mu(x)\mathrm{d}\mu(y), \tag{1}$$

where the infimum is taken over all probability measures supported on A; moreover, we are interested in the measure that attains this infimum. In this paper we focus on the *Riesz s-kernels* $K_s(x,y) := |x-y|^{-s}$. It is convenient to discretize the measure on which the value $I_K(A)$ is achieved; for this purpose, we consider the *discrete Riesz s-energy problem*. Namely, for every integer $N \geqslant 2$ we define

$$\mathcal{E}_s(A,N) := \inf_{\omega_N} E_s(\omega_N), \tag{2}$$

where the infimum is taken over all N-point sets $\omega_N = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subset A$, and

$$E_s(\omega_N) := \sum_{i \neq j} |\boldsymbol{x}_i - \boldsymbol{x}_j|^{-s}, \qquad N = 2,3,4,\dots$$

Since the kernel K_s is lower semicontinuous, the infimum is always attained.

In general, asymptotics of energy functionals arising from pairwise interaction in discrete subsets has been the subject of a number of studies [14, 13, 10, 6]; it has also been considered for random point configurations [7] and in the context of random processes [1, 2]. The interest in such functionals is primarily motivated by applications in physics and modeling of particle interactions, as well as by the connections to geometric measure theory.

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If d is the Hausdorff dimension of A and $s < d$, then there is a unique measure $\mu_{s,A}$ for which the infimum in (1) is achieved, and the configurations that attain the infimum in (2) resemble $\mu_{s,A}$ in the weak* sense (for the precise definition, see below). When $s > d$, we have $I_{K_s}(A) = \infty$, as the integral in the RHS is infinite on all measures μ supported on A. However, for "good" sets A (for example, d-rectifiable sets) with integer dimension d, the configurations attaining (2) resemble a certain special measure, namely, the uniform measure on A.

More precisely, for a configuration $\omega_N = \{\boldsymbol{x}_i : 1 \leqslant i \leqslant N\} \subset A$ we define the (empirical) probability measure

$$\nu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\boldsymbol{x}_i},$$

and we shall identify the two. Then, as summarized in the *Poppy-seed bagel theorem* (PSB), see Theorem A, under some regularity requirements on the set A, any sequence $\{\tilde{\omega}_N : \#\tilde{\omega}_N = N, \mathcal{E}_s(A,N) = E_s(\tilde{\omega}_N)\}$ converges to the normalized d-dimensional Hausdorff measure $\mathcal{H}_d(A \cap \cdot)/\mathcal{H}_d(A)$ on A. Moreover, for such sets A, the following limit exists:

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}. \tag{3}$$

On the other hand, it has been established [4, Proposition 2.6] that for a class of self-similar fractals A with $\dim_H A = d$, the limit of $\mathcal{E}_s(A,N)/N^{1+s/d}$ does not exist for s large enough. Using this observation, [8] gives an example of a set A and a sequence of optimal configurations for $\mathcal{E}_s(A,N)$ without a weak* limit.

In view of the above, it is natural to ask what can be said about weak* cluster points of $\{\nu_N : N \geqslant 2\}$ in the case when the underlying set A is not d-rectifiable; a characterization of the cluster points of $\{\mathcal{E}_s(A,N)/N^{1+s/d} : N \geqslant 2\}$ is likewise of interest.

The following section contains formal definitions and the necessary prerequisites; Section 3 gives an overview of previously established results, both in the case of a rectifiable and a non-rectifiable set A. Sections 4 and 5 contain the formulations of the main theorems and their proofs, respectively.

2. Self-similarity and open set condition

We shall be working with subsets of the Euclidean space $\mathbb{R}^p$, using bold typeface for its elements: $\boldsymbol{x} \in \mathbb{R}^p$. An open ball of radius r, centered at $\boldsymbol{x}$, will be denoted by $B(\boldsymbol{x}, r)$. The d-dimensional Hausdorff measure of a Borel set A will be denoted by $\mathcal{H}_d(A)$.

A pair of sets $A^{(1)}$, $A^{(2)}$ will be called *metrically separated* if $|\boldsymbol{x} - \boldsymbol{y}| \geqslant \sigma > 0$ whenever $\boldsymbol{x} \in A^{(1)}$ and $\boldsymbol{y} \in A^{(2)}$. Recall that a *similitude* $\psi : \mathbb{R}^p \to \mathbb{R}^p$ can be written as

$$\psi(\boldsymbol{x}) = rO(\boldsymbol{x}) + \boldsymbol{z}$$

for an orthogonal matrix $O \in \mathcal{O}(p)$, a vector $\boldsymbol{z} \in \mathbb{R}^p$, and a contraction ratio $0 < r < 1$. The following definition can be found in [16].

Definition 2.1. *A compact set $A \subset \mathbb{R}^p$ is called a self-similar fractal with similitudes $\{\psi_m\}_{m=1}^M$ with contraction ratios r_m, $1 \leqslant m \leqslant M$ if*

$$A = \bigcup_{m=1}^{M} \psi_m(A),$$

where the union is disjoint[1].

We say that A satisfies the open set condition if there exists a bounded open set $V \subset \mathbb{R}^p$ such that

$$\bigcup_{m=1}^{M} \psi_m(V) \subset V,$$

where the sets in the union are disjoint.

[1]One also considers self-similar fractals where the union is not disjoint — these are harder to deal with

For a self-similar fractal A, it is known [11, 19] that its Hausdorff dimension $\dim_H A = d$ where d is such that

$$\sum_{m=1}^{M} r_m^d = 1. \tag{4}$$

It will further be used that if A is a self-similar fractal satisfying the open set condition, then there holds $0 < \mathcal{H}_d(A) < \infty$ and A is *d-regular* with respect to $\mathcal{H}_d$; that is, there exists a positive constant c, such that for every r, $0 < r \leqslant \operatorname{diam}(A)$, and every $\boldsymbol{x} \in A$,

$$c^{-1} r^d \leqslant \mathcal{H}_d(A \cap B(\boldsymbol{x}, r)) \leqslant c r^d. \tag{5}$$

3. Overview of prior results

Recall the standard definition of the weak* convergence: given a countable sequence $\{\mu_N : N \geqslant 1\}$ of probability measures supported on A and another probability measure μ,

$$\mu_N \xrightarrow{*} \mu,\ N \to \infty \quad \Longleftrightarrow \quad \int_A f(\boldsymbol{x}) d\mu_N(\boldsymbol{x}) \longrightarrow \int_A f(\boldsymbol{x}) d\mu(\boldsymbol{x}),\ N \to \infty,$$

for every $f \in C(A)$. (Limits along nets are not necessary, as in this context weak* topology is metrizable.) We shall say that a sequence of discrete sets *converges* to a certain measure if the corresponding sequence of counting measures converges to it.

The set A is said to be *d-rectifiable* if it is the image of a compact subset of $\mathbb{R}^d$ under a Lipschitz map. Furthermore, we say that A is *$(\mathcal{H}_d, d)$-rectifiable*, if

$$A = A^{(0)} \cup \bigcup_{k=0}^{\infty} A^{(k)}, \tag{6}$$

where for $k \geqslant 1$ each $A^{(k)}$ is d-rectifiable and $\mathcal{H}_d(A^{(0)}) = 0$.

We begin by discussing results dealing with the Riesz energy, both in the rectifiable and non-rectifiable contexts. To formulate the PSB theorem, suppose $s > d$ for simplicity; the case of $s = d$ is similar, but requires stronger assumptions on the set A. We write $\mathcal{M}_d(A)$ for the d-dimensional Minkowski content of the set A [12, 3.2.37–39].

Theorem A (Poppy-seed bagel theorem, [13, 5])**.** *If the set A is $(\mathcal{H}_d, d)$-rectifiable for $s > d$ and $\mathcal{H}_d(A) = \mathcal{M}_d(A)$, then*

$$\lim_{N\to\infty} \frac{\mathcal{E}(A, N)}{N^{1+s/d}} = \frac{C_{s,d}}{\mathcal{H}_d(A)^{s/d}},$$

and every sequence $\{\tilde{\omega}_N : N \geqslant 2\}$ achieving the above limit converges weak to the uniform probability measure on A:*

$$\frac{1}{N} \sum_{\tilde{\boldsymbol{x}} \in \tilde{\omega}_N} \delta_{\tilde{\boldsymbol{x}}} \xrightarrow{*} \frac{\mathcal{H}_d(A \cap \cdot)}{\mathcal{H}_d(A)}.$$

The smoothness assumptions on A in the above theorem are essential for existence of the limit of $\mathcal{E}(A, N)/N^{1+s/d}$. Let $\{\underline{\omega}_N \subset A : \#\underline{\omega}_N = N, N \in \underline{\mathfrak{N}}\}$ be a sequence of configurations such that

$$\lim_{\underline{\mathfrak{N}} \ni N \to \infty} \frac{E_s(\underline{\omega}_N)}{N^{1+s/d}} = \liminf_{N\to\infty} \frac{\mathcal{E}_s(A, N)}{N^{1+s/d}} =: \underline{g}_{s,d}(A), \tag{7}$$

and similarly, $\{\overline{\omega}_N \subset A : \#\underline{\omega}_N = N, N \in \overline{\mathfrak{N}}\}$ a sequence for which

$$\lim_{\overline{\mathfrak{N}} \ni N \to \infty} \frac{E_s(\overline{\omega}_N)}{N^{1+s/d}} = \limsup_{N\to\infty} \frac{\mathcal{E}_s(A, N)}{N^{1+s/d}} =: \bar{g}_{s,d}(A). \tag{8}$$

In the notation of (7)-(8), the result about the non-existence of $\lim_{N\to\infty} \mathcal{E}_s(A, N)/N^{1+s/d}$ from [4] that was mentioned in the introduction can be stated as follows.

Proposition 3.1. *For a self-similar fractal A with contraction ratios $r_1 = \ldots = r_m$, there exists an $S_0 > 0$ such that for every $s > S_0$,*

$$0 < \underline{g}_{s,d}(A) < \bar{g}_{s,d}(A) < \infty.$$

We remark that in the proof of Proposition 3.1, the number S_0 was not obtained constructively. In Theorem 4.4 we give a formula for S_0. The behavior of the sets ω_N that attain $\mathcal{E}_s(A, N)$ in the non-rectifiable case is still not fully characterized. The following proposition, taken from [8], is the only known negative result so far.

Proposition 3.2. *Assume that the two d-regular compact sets $A^{(1)}$, $A^{(2)}$ are metrically separated and are such that $A^{(1)}$ is a self-similar fractal with equal contraction ratios and $\underline{g}_{s,d}(A^{(2)}) = \bar{g}_{s,d}(A^{(2)})$. Then for any sequence of minimizers $\{\tilde{\omega}_N \subset A : \#\tilde{\omega}_N = N, E_s(\tilde{\omega}_N) = \mathcal{E}_s(A, N)\}$, the corresponding sequence of measures*

$$\tilde{\nu}_N = \frac{1}{N} \sum_{\tilde{\boldsymbol{x}} \in \tilde{\omega}_N} \delta_{\tilde{\boldsymbol{x}}}$$

does not have a weak limit.*

In view of these two propositions, it is remarkable that the local properties of minimizers of E_s are fully preserved on self-similar fractals. Indeed, d-regularity of A can be readily used to obtain that any sequence of minimizers of E_s has the optimal orders of separation and covering. The following result was proved in [15]:

Proposition 3.3. *If $A \subset \mathbb{R}^p$ is a compact d-regular set, $\{\tilde{\omega}_N : N \geqslant 1\}$ a sequence of configurations minimizing E_s with $\tilde{\omega}_N = \{\tilde{\boldsymbol{x}}_i : 1 \leqslant i \leqslant N\}$, then there exist a constant $C_1 > 0$ such that for any $1 \leqslant i < j \leqslant N$,*

$$|\tilde{\boldsymbol{x}}_i - \tilde{\boldsymbol{x}}_j| \geqslant C_1 N^{-1/d}, \qquad N \geqslant 2,$$

and a constant $C_2 > 0$ such that for any $\boldsymbol{y} \in A$,

$$\min_i |\boldsymbol{y} - \tilde{\boldsymbol{x}}_i| \leqslant C_2 N^{-1/d}, \qquad N \geqslant 2.$$

The closest one comes to an analog of the PSB theorem for self-similar fractals is the following proposition [3]. Note that we give a simpler proof of (1) for the case when $A_0 = A$ in Theorem 4.1.

Proposition 3.4. *Suppose A_0 is a self-similar fractal satisfying the open set condition and $s > d$; fix a compact $A \subset A_0$.*

(1) *If $\{\underline{\omega}_N : N \in \underline{\mathfrak{N}}\}$, is a sequence of configurations for which*

$$\lim_{\underline{\mathfrak{N}} \ni N \to \infty} \frac{E_s(\underline{\omega}_N)}{N^{1+s/d}} = \underline{g}_{s,d}(A),$$

then the corresponding sequence of empirical measures converges weak:*

$$\underline{\nu}_N \xrightarrow{\ *\ } \frac{\mathcal{H}_d(\cdot \cap A)}{\mathcal{H}_d(A)}, \qquad \underline{\mathfrak{N}} \ni N \to \infty.$$

(2) *There holds*

$$\underline{g}_{s,d}(A) = \frac{\underline{g}_{s,d}(A_0)\mathcal{H}_d(A_0)^{s/d}}{\mathcal{H}_d(A)^{s/d}}$$

and

$$\bar{g}_{s,d}(A) = \frac{\bar{g}_{s,d}(A_0)\mathcal{H}_d(A_0)^{s/d}}{\mathcal{H}_d(A)^{s/d}}.$$

We finish this section with another relevant result on fractal sets. In [4] it was shown that, as $s \to \infty$, there is a strong connection between the s-energy $\mathcal{E}_s(A)$ and the *best-packing constant*

$$\delta(A, N) := \sup_{\omega_N} \min_{i \neq j} |\boldsymbol{x}_i - \boldsymbol{x}_j|.$$

The main theorem of [18] is given in terms of the function $N(\delta) := \max\{n : \delta(A,n) \geqslant \delta\}$. Our Theorem 4.3 gives an analog of the second part of this theorem for the minimal discrete energy.

Theorem B. *Suppose A is a self-similar fractal of dimension d satisfying the open set condition with contraction ratios $r_1, \dots, r_m$.*

(1) *If the additive group generated by $\log r_1, \dots, \log r_m$ is dense in $\mathbb{R}$, then there exists a constant C such that*
$$\lim_{N\to\infty} N^{1/d}\delta(A,N) = \lim_{\delta\to 0} N(\delta)^{1/d}\delta = C.$$
(2) *If the additive group generated by $\log r_1, \dots, \log r_M$ coincides with the lattice $h\mathbb{Z}$ for some $h > 0$, then*
$$\lim N(\delta)^{1/d}\delta = C_\theta,$$
where the limit is taken over a subsequence $\delta \to 0$ with $\left\{\frac{1}{h}\log\delta\right\} = \theta$.

4. Main results

In accordance with the prior notation, we write $\underline{\omega}_N = \{\underline{\boldsymbol{x}}_i : 1 \leqslant i \leqslant N\}$ for the sequence of configurations with the lowest asymptotics (i.e., such that (7) holds), and
$$\underline{\nu}_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\underline{\boldsymbol{x}}_i}, \quad N \in \underline{\mathfrak{N}}.$$
As described above, generally the limit of $\mathcal{E}_s(A,N)/N^{1+s/d}$, $N \to \infty$ does not necessarily exist. It is still possible to characterize the behavior of the sequence $\{\underline{\omega}_N : N \in \underline{\mathfrak{N}}\}$. The following result first appeared in [3]; we give an independent and a more direct proof.

Theorem 4.1. *Let $A \subset \mathbb{R}^p$ be a compact self-similar fractal satisfying the open set condition, and $\dim_H A = d < s$. If $\{\underline{\omega}_N : N \in \underline{\mathfrak{N}}\}$, is a sequence of configurations for which*
$$\lim_{\underline{\mathfrak{N}}\ni N\to\infty}\frac{E_s(\underline{\omega}_N)}{N^{1+s/d}} = \underline{g}_{s,d}(A),$$
then the corresponding sequence of empirical measures converges weak:*
$$\underline{\nu}_N \xrightarrow{\ *\ } h_d(\cdot) := \frac{\mathcal{H}_d(\cdot\cap A)}{\mathcal{H}_d(A)}, \qquad \underline{\mathfrak{N}} \ni N \to \infty. \tag{9}$$

When the similitudes $\{\psi_m\}_{m=1}^M$ fixing A all have the same contraction ratio, it is natural to expect some additional symmetry of minimizers, associated with the M-fold scale symmetry of A. Similarly, since the energy of interactions between particles in different $A^{(m)}$ is at most of order N^2, see proof of Lemma 5.1 below, we expect that by acting with $\{\psi_m\}_{m=1}^M$ on a minimizer $\tilde{\omega}_N$ with N large, we obtain a near-minimizer with MN elements. This heuristic is made rigorous in the following theorem.

Theorem 4.2. *Let $A \subset \mathbb{R}^p$ be a self-similar fractal, fixed under M similitudes with the same contraction ratio, and $\mathfrak{M} = \{M^k n : k \geqslant 1\}$. Then the following limit exists*
$$\lim_{\mathfrak{M}\ni N\to\infty}\frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}.$$

The previous theorem can be further extended. We shall need some notation first. For a sequence $\mathfrak{N}$, let
$$\{\mathfrak{N}\} := \lim_{\mathfrak{N}\ni N\to\infty}\{\log_M N\},$$
where $\{\cdot\}$ in the RHS denotes the fractional part, and
$$E_s(\mathfrak{N}) := \lim_{\mathfrak{N}\ni N\to\infty}\frac{\mathcal{E}_s(A,N)}{N^{1+s/d}},$$
if the corresponding limit exists.

Theorem 4.3. *If A is a self-similar fractal with equal contraction ratios, and two sequences $\mathfrak{N}_1, \mathfrak{N}_2 \subset \mathbb{N}$ are such that*

$$\{\mathfrak{N}_1\} = \{\mathfrak{N}_2\}, \tag{10}$$

then

$$E_s(\mathfrak{N}_1) = E_s(\mathfrak{N}_2). \tag{11}$$

In particular, the limits in (11) exist. Moreover, the function $g_{s,d} : \{\mathfrak{N}\} \mapsto E_s(\mathfrak{N})$ is continuous on $[0,1]$.

In the case of equal contraction ratios, the argument in the proof of Theorem 4.2, can be further used to make the result of Proposition 3.1 more precise.

Theorem 4.4. *Let $A \subset \mathbb{R}^p$ be a self-similar fractal, fixed under M similitudes with the same contraction ratio r, and write $\sigma := \min\{\|\boldsymbol{x} - \boldsymbol{y}\| : \boldsymbol{x} \in A_i,\ \boldsymbol{y} \in A_j, i \neq j\}$. If*

$$R := \frac{r}{\sigma}(1 + r^d)^{1/d} < 1,$$

then for for every value of s such that

$$s \geqslant \max\left\{2d,\ \log_{1/R}[2M(M+1)]\right\}, \tag{12}$$

there holds

$$0 < \underline{g}_{s,d}(A) < \bar{g}_{s,d}(A) < \infty.$$

The proof of this theorem requires an estimate for the value of $\mathcal{E}_s(A, M)$, which results in the condition $R < 1$. When $\mathcal{E}_s(A, M)$ can be computed explicitly, a similar conclusion can also be obtained for sets that do not necessarily satisfy $R < 1$, as in the following.

Corollary 4.5. *If A is the ternary Cantor set and $s > 3 \dim_H A = 3 \log_3 2$, then*

$$0 < \underline{g}_{s,d}(A) < \bar{g}_{s,d}(A) < \infty.$$

5. Proofs

The key to proving Theorem 4.1 is that the hypersingular Riesz energy grows faster than N^2. We shall need this property in the following form.

Lemma 5.1. *Let a pair of compact sets $A^{(1)}, A^{(2)} \subset \mathbb{R}^p$ be metrically separated; let further $\{\omega_N \subset A : N \in \mathfrak{N}\}$ be a sequence for which the limits*

$$\lim_{\mathfrak{N} \ni N \to \infty} \frac{\#(\omega_N \cap A^{(i)})}{N} = \beta^{(i)}, \quad i = 1, 2.$$

exist. Then

$$\liminf_{\mathfrak{N} \ni N \to \infty} \frac{E_s(\omega_N)}{N^{1+s/d}} \geqslant \left(\beta^{(1)}\right)^{1+s/d} \liminf_{\mathfrak{N} \ni N \to \infty} \frac{E_s(\omega_N \cap A^{(1)})}{\#(\omega_N \cap A^{(1)})^{1+s/d}} + \left(\beta^{(2)}\right)^{1+s/d} \liminf_{\mathfrak{N} \ni N \to \infty} \frac{E_s(\omega_N \cap A^{(2)})}{\#(\omega_N \cap A^{(2)})^{1+s/d}}.$$

Proof. We observe that with $\sigma = \operatorname{dist}(A^{(1)}, A^{(2)})$,

$$\left| E_s(\omega_N) - \left(E_s(\omega_N \cap A^{(1)}) + E_s(\omega_N \cap A^{(2)}) \right) \right| = \sum_{\substack{\boldsymbol{x}_i \in A^{(1)} \\ ,\boldsymbol{x}_j \in A^{(2)}}} |\boldsymbol{x}_i - \boldsymbol{x}_j|^{-s} \leqslant \sigma^{-s} N^2,$$

and use the definition of $\beta^{(i)}$, $i = 1, 2$, to obtain the desired equality. □

This is particularly useful for self-similar fractals satisfying the open set property. Consider such a fractal A; since $\psi_m(V)$, $1 \leqslant m \leqslant M$, are pairwise disjoint for an open set V containing A, there exists a $\sigma > 0$ such that $\operatorname{dist}(\psi_i(A), \psi_j(A)) \geqslant \sigma$ for $i \neq j$. Following [11], we will write

$$A_{m_1 \dots m_l} := \psi_{m_1} \circ \dots \circ \psi_{m_l}(A), \qquad 1 \leqslant m_i \leqslant M, \quad l \geqslant 1.$$

Then $\operatorname{dist}(A_{m_1 \dots m_l}, A_{m'_1 \dots m'_l}) \geqslant r_{m_1} \dots r_{m_k} \sigma$, where $k = \min\{i : m_i \neq m'_i\}$, so for a fixed M in the expression

$$A = \bigcup_{m_1, \dots, m_l = 1}^{M} A_{m_1 \dots m_l}$$

not only the union is disjoint, but also the sets $A_{m_1 \dots m_l}$ are metrically separated. The following lemma is technical, and we give its proof for the convenience of the reader.

Lemma 5.2. *If $\{\mu_N : N \in \mathfrak{N}\}$ is a sequence of probability measures on the set A, which for every $l \geqslant 1$ satisfies*

$$\lim_{\mathfrak{N} \ni N \to \infty} \mu_N(A_{m_1 \dots m_l}) = \mu(A_{m_1 \dots m_l}), \qquad 1 \leqslant m_1, \dots, m_l \leqslant M,$$

for another probability measure μ on A, then

$$\mu_N \xrightarrow{*} \mu, \qquad \mathfrak{N} \ni N \to \infty.$$

Proof. Fix an $f \in C(A)$; since A is compact, f is uniformly continuous on A. For a fixed $\varepsilon > 0$, there exists an $L_0 \in \mathbb{N}$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A_{m_1, \dots, m_l}$ for any $l \geqslant L_0$ and any set of indices $0 \leqslant m_1, \dots, m_l \leqslant M$; this is possible due to

$$\operatorname{diam}(A_{m_1, \dots, m_l}) \leqslant r_{m_1} \dots r_{m_l} \operatorname{diam}(A) \leqslant \left(\max_{1 \leqslant m \leqslant M} r_m \right)^l \operatorname{diam}(A).$$

Fix an $l \geqslant L_0$ until the end of this proof, then pick an $N_0 \in \mathfrak{N}$ so that for every $N \geqslant N_0$, there holds

$$|\mu_N(A_{m_1 \dots m_l}) - \mu(A_{m_1 \dots m_l})| < \varepsilon / M^l, \qquad 1 \leqslant m_1, \dots, m_l \leqslant M.$$

Finally, let us write $f_{m_1 \dots m_l} := \min_{A_{m_1 \dots m_l}} f(x)$ for brevity. Then for $N \geqslant N_0$,

$$\begin{aligned} &\left| \int_A f(x) d\mu_N(x) - \int_A f(x) d\mu(x) \right| \\ &\qquad \leqslant \sum_{m_1, \dots, m_l = 1}^{M} \left| \int_{A_m} (f(x) - f_{m_1 \dots m_l}) d\mu_N(x) - \int_{A_m} (f(x) - f_{m_1 \dots m_l}) d\mu(x) \right| \\ &\qquad + \sum_{m_1, \dots, m_l = 1}^{M} |(\mu_N(A_m) - \mu(A_m)) f_{m_1 \dots m_l}| \\ &\qquad \leqslant 2\varepsilon + \varepsilon \|f\|_\infty, \end{aligned}$$

where the estimate for the first sum used that both μ_N and μ are probability measures. This proves the desired statement. □

Note that the converse is also true: since the sets $A_{m_1, \dots, m_l}$ are metrically separated, convergence $\mu_N \xrightarrow{*} \mu$ of measures supported on A immediately implies (by Urysohn's lemma) $\mu_N(A_{m_1 \dots m_l}) \to \mu(A_{m_1 \dots m_l})$ for all $l \geqslant 1$ and all indices $1 \leqslant m_1, \dots, m_l \leqslant M$.

The proof of the following statement follows a well-known approach [15, 17, Theorem 2], and can be considered standard.

Proposition 5.3. *If A is a compact d-regular set, then $0 < \underline{g}_{s,d}(A) \leqslant \bar{g}_{s,d}(A) < \infty$.*

The above proposition can be somewhat strengthened, to obtain uniform upper and lower bounds on

$$\frac{\mathcal{E}_s(\omega_N)}{N^{1+s/d}}, \qquad N \geqslant 2;$$

furthermore, each bound requires only one of the inequalities in (5). In addition, for any sequence of configurations ω_N, $N \in \mathfrak{N}$, with

$$\lim_{\mathfrak{N}\ni N\to\infty} \frac{E_s(\omega_N)}{N^{1+s/d}} < \infty,$$

every weak* cluster point of ν_N, $N \in \mathfrak{N}$, must be absolutely continuous with respect to $\mathcal{H}_d$ on A. Lastly, we will need the following standard estimate.

Corollary 5.4. *Suppose A is a compact d-regular set, $\omega_N = \{\boldsymbol{x}_i : 1 \leqslant i \leqslant N\} \subset A$, and $s > d$. Then the minimal point energy of ω_N is bounded by:*

$$\min_{\boldsymbol{x}\in A} \sum_{j=1}^{N} |\boldsymbol{x} - \boldsymbol{x}_j|^{-s} \leqslant CN^{s/d},$$

where C depends only on A, s, d.

Proof of Theorem 4.1. In view of the weak* compactness of probability measures in A, to establish existence of the weak* limit of $\underline{\nu}_N$, $N \in \underline{\mathfrak{N}}$, it suffices to show that any cluster point of $\underline{\nu}_N$, $N \in \underline{\mathfrak{N}}$, in the weak* topology is h_d which is defined in (9) (see [9, Proposition A.2.7]). To that end, consider a subsequence of $\underline{\mathfrak{N}}$ for which the empirical measures $\underline{\nu}_N$ converge to a cluster point μ; for simplicity we shall use the same notation $\underline{\mathfrak{N}}$ for this subsequence.

As discussed above, $\underline{\nu}_N(A_{m_1\ldots m_l}) \to \mu(A_{m_1\ldots m_l})$, $\underline{\mathfrak{N}} \ni N \to \infty$; this ensures that the quantities

$$\underline{\beta}_m := \mu(A_m) = \lim_{\underline{\mathfrak{N}}\ni N\to\infty} \underline{\nu}_N(A_m) = \lim_{\underline{\mathfrak{N}}\ni N\to\infty} \frac{\#(\underline{\omega}_N \cap A_m)}{N}, \qquad m = 1, \ldots, M,$$

are well-defined. From (7), separation of $\{A_m\}$, and Lemma 5.1 follows

$$\begin{aligned} \underline{g}_{s,d}(A) &= \sum_{m=1}^{M} \lim_{\underline{\mathfrak{N}}\ni N\to\infty} \frac{E_s(\underline{\omega}_N \cap A_m)}{N^{1+s/d}} \geqslant \sum_{m=1}^{M} \underline{\beta}_m^{1+s/d} \liminf_{\underline{\mathfrak{N}}\ni N\to\infty} \frac{E_s(\underline{\omega}_N \cap A_m)}{\#(\underline{\omega}_N \cap A_m)^{1+s/d}} \\ &\geqslant \sum_{m=1}^{M} \underline{\beta}_m^{1+s/d} r_m^{-s} \underline{g}_{s,d}(A). \end{aligned}$$

Consider the RHS in the last inequality. As a function of $\{\underline{\beta}_m\}$, it satisfies the constraint $\sum_m \underline{\beta}_m = 1$; note also that by the defining property (4) of d, there holds $\sum_m R_m = 1$ with $R_m := r_m^d$, $1 \leqslant m \leqslant M$. We have

$$\underline{g}_{s,d}(A) \geqslant \inf \left\{ \sum_{m=1}^{M} \beta_m^{1+s/d} R_m^{-s/d} : \sum_{m=1}^{M} \beta_m = 1 \right\} \underline{g}_{s,d}(A). \tag{13}$$

Level sets of the function $\sum_m \beta_m^{1+s/d} R_m^{-s/d}$ are convex, so the infimum is attained and unique; it is easy to check that the solution is at $\beta_m = R_m = r_m^d$, $1 \leqslant m \leqslant M$, and the minimal value is 1. Indeed, the corresponding Lagrangian is

$$L(\beta_1, \ldots, \beta_M, \lambda) := \sum_{m=1}^{M} \beta_m^{1+s/d} R_m^{-s/d} - \lambda \sum_{m=1}^{M} \beta_m,$$

hence

$$\nabla L_{\beta_m} = (1 + s/d) \left(\frac{\beta_m}{R_m} \right)^{s/d} - \lambda, \quad 1 \leqslant m \leqslant M,$$

and it remains to use $\beta_m \geqslant 0$, $1 \leqslant m \leqslant M$, and $\sum_m R_m = 1$, to conclude $\beta_m = R_m$, $1 \leqslant m \leqslant M$.

Since $0 < \underline{g}_{s,d}(A) < \infty$ by Lemma 5.3, from (13) it follows

$$\underline{\beta}_m = r_m^d, \qquad m = 1, \dots, M.$$

Note that this argument shows also

$$\lim_{\mathfrak{N} \ni N \to \infty} \frac{E_s(\underline{\omega}_N \cap A_m)}{(\#(\underline{\omega}_N \cap A_m))^{1+s/d}} = \underline{g}_{s,d}(A),$$

so the above can be repeated recursively for sets $A_{m_1 \dots m_l}$. Namely, for every $l \geqslant 1$ and $1 \leqslant m, m_1, \dots, m_l \leqslant M$,

$$\mu(A_{m m_1 \dots m_l}) =: \underline{\beta}_{m m_1 \dots m_l} = r_m^d \, \underline{\beta}_{m_1 \dots m_l}.$$

Observe further that h_d satisfies

$$h_d(A_{m m_1 \dots m_l}) = r_m^d h_d(A_{m_1 \dots m_l})$$

by definition, so by Lemma 5.2 follows that every weak* cluster point of $\underline{\nu}_N$, $N \in \underline{\mathfrak{N}}$, is h_d, as desired. □

Proof of Theorem 4.2. Note that setting equal contraction ratios $r_1 = \dots = r_m = r$ in (4) gives $r^{-s} = M^{s/d}$. Consider the set function

$$\psi : \boldsymbol{x} \mapsto \bigcup_{m=1}^{M} \psi_m(\boldsymbol{x}), \quad \boldsymbol{x} \in A,$$

and denote

$$\psi(\omega_N) := \bigcup_{\boldsymbol{x} \in \omega_N} \psi(x).$$

It follows from the open set condition that the union above is metrically separated; as before, we denote the separation distance by σ. Observe that the definition of a similitude implies $\#(\psi(\omega_N)) = M\#(\omega_N)$. We then have for any configuration ω_N, $N \geqslant 2$,

$$\begin{aligned} \mathcal{E}_s(A, MN) &\leqslant E_s(\psi(\omega_N)) \leqslant M r^{-s} E_s(\omega_N) + \sigma^{-s} N^2 M^2 \\ &= M^{1+s/d} E_s(\omega_N) + \sigma^{-s} N^2 M^2, \end{aligned}$$

and repeated application of the second inequality yields

$$\begin{aligned} \mathcal{E}_s(A, M^k N) &\leqslant E_s[\psi(\psi^{(k-1)}(\omega_N))] \leqslant M^{1+s/d} E_s(\psi^{(k-1)}(\omega_N)) + \sigma^{-s}(M^{k-1}N)^2 M^2 \\ &\leqslant (M^2)^{1+s/d} E_s(\psi^{(k-2)}(\omega_N)) + M^{1+s/d}\sigma^{-s}(M^{k-2}N)^2 M^2 + \sigma^{-s}(M^{k-1}N)^2 M^2 \\ &\leqslant \dots \\ &\leqslant (M^k)^{1+s/d} E_s(\omega_N) + \sigma^{-s} N^2 \sum_{l=1}^{k} (M^{l-1})^{1+s/d} (M^{k-l})^2 M^2. \end{aligned}$$

Estimating the geometric series in the last inequality, we obtain

$$\begin{aligned} \mathcal{E}_s(A, M^k N) &\leqslant (M^k)^{1+s/d} E_s(\omega_N) + \sigma^{-s} N^2 M^{2k+1-s/d} \sum_{l=1}^{k} M^{l(s/d-1)} \\ &\leqslant (M^k)^{1+s/d} E_s(\omega_N) + \sigma^{-s} N^2 M^{2k+1-s/d} \frac{M^{(k+1)(s/d-1)} - 1}{M^{s/d-1} - 1} \\ &\leqslant (M^k)^{1+s/d} E_s(\omega_N) + \frac{N^{1-s/d}}{\sigma^s (M^{s/d-1} - 1)} \left(M^k N\right)^{(1+s/d)}. \end{aligned} \tag{14}$$

Let now $\varepsilon > 0$ fixed; find ω_{N_0} such that $N_0 \in \mathfrak{M}$ and

$$\frac{E_s(\omega_{N_0})}{N_0^{1+s/d}} \leqslant \liminf_{\mathfrak{M} \ni N \to \infty} \frac{\mathcal{E}_s(A, N)}{N^{1+s/d}} + \varepsilon,$$

and in addition, $N_0^{1-s/d} < \varepsilon\sigma^s(M^{s/d-1} - 1)$. Then by (14) we have

$$\frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \leqslant \frac{E_s(\omega_{N_0})}{N_0^{1+s/d}} + \varepsilon \leqslant \liminf_{\mathfrak{M}\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} + 2\varepsilon, \qquad \mathfrak{M} \ni N \geqslant N_0.$$

This proves the desired statement. □

In the following lemma we write $\mathfrak{N}(k), k \in \mathbb{N}$, to denote the k-th element of the sequence $\mathfrak{N} \subset \mathbb{N}$; we say that $\mathfrak{N}$ is *majorized* by a sequence $\mathfrak{M}$, if the inequality $\mathfrak{N}(k) < \mathfrak{M}(k)$ holds for every $k \geqslant 1$.

Lemma 5.5. *If $\mathfrak{M} \subset \mathbb{N}$ is a sequence such that the limit*

$$\lim_{\mathfrak{M}\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}$$

exists, then for any sequence of integers $\mathfrak{N} \subset \mathbb{Z}$ with $|\mathfrak{N}(k)|$ majorized by $\mathfrak{M}$ and satisfying $|\mathfrak{N}(k)| = o(\mathfrak{M}(k))$, $k \to \infty$, there holds

$$\lim_{(\mathfrak{M}+\mathfrak{N})\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} = \lim_{\mathfrak{M}\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}, \tag{15}$$

where the addition $\mathfrak{M}+\mathfrak{N}$ is performed elementwise.

Proof. First, observe that by passing to subsequences of $\mathfrak{M}$ and $\mathfrak{N}$, it suffices to assume $\mathfrak{N}(k) \geqslant 0$ and to show(15) for $\mathfrak{M}+\mathfrak{N}$ and $\mathfrak{M}-\mathfrak{N}$. If $\mathfrak{N}(k) \geqslant 0$, we have by the definition of $\mathcal{E}_s$,

$$\mathcal{E}_s[A, (\mathfrak{M}+\mathfrak{N})(k)] \geqslant \mathcal{E}_s(A, \mathfrak{M}(k))$$

Thus

$$\begin{aligned} \liminf_{(\mathfrak{M}+\mathfrak{N})\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} &\geqslant \lim_{k\to\infty} \frac{\mathcal{E}_s(A,\mathfrak{M}(k))}{(\mathfrak{M}(k)+\mathfrak{N}(k))^{1+s/d}} \\ &= \lim_{k\to\infty} \frac{\mathcal{E}_s(A,\mathfrak{M}(k))}{(\mathfrak{M}(k))^{1+s/d}} \left(\frac{\mathfrak{M}(k)}{\mathfrak{M}(k)+\mathfrak{N}(k)}\right)^{1+s/d} = \lim_{\mathfrak{M}\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}, \end{aligned} \tag{16}$$

in view of $\mathfrak{N}(k) = o(\mathfrak{M}(k))$. Similarly,

$$\limsup_{(\mathfrak{M}-\mathfrak{N})\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \leqslant \lim_{\mathfrak{M}\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}. \tag{17}$$

For the converse estimates, use Corollary 5.4 to conclude that for every $N \in \mathbb{N}$ there holds

$$\mathcal{E}_s(A, N+1) \leqslant \mathcal{E}_s(A,N) + CN^{s/d}.$$

Applying this inequality $\mathfrak{N}(k)$ times to $\mathfrak{M}(k)$, we obtain

$$\mathcal{E}_s[A, (\mathfrak{M}+\mathfrak{N})(k)] \leqslant \mathcal{E}_s(A, \mathfrak{M}(k)) + \mathfrak{N}(k)C[\mathfrak{M}(k)+\mathfrak{N}(k)]^{s/d},$$

which yields

$$\limsup_{(\mathfrak{M}+\mathfrak{N})\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \leqslant \lim_{\mathfrak{M}\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}. \tag{18}$$

Finally, applying Corollary 5.4 $\mathfrak{N}(k)$ times to $\mathfrak{M}(k) - \mathfrak{N}(k)$ gives

$$\mathcal{E}_s[A, \mathfrak{M}(k)] \leqslant \mathcal{E}_s[A, (\mathfrak{M}-\mathfrak{N})(k)] + \mathfrak{N}(k)C\mathfrak{M}(k)^{s/d},$$

whence, using that $\mathfrak{N}(k) = o(\mathfrak{M}(k))$, $k \to \infty$,

$$\liminf_{(\mathfrak{M}-\mathfrak{N})\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \geqslant \lim_{\mathfrak{M}\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}. \tag{19}$$

Combining (16) with (18) and (17) with (19), we get the desired result. □

The proof of the previous lemma implies the following.

Corollary 5.6. *If $\mathfrak{M}, \mathfrak{N} \subset \mathbb{N}$ are a pair of sequences such that*

$$\mathfrak{N}(k) \leqslant \theta\, \mathfrak{M}(k), \qquad k \geqslant 1,$$

then

$$\liminf_{(\mathfrak{M}+\mathfrak{N})\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \geqslant \liminf_{\mathfrak{M}\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \cdot \left(\frac{1}{1+\theta}\right)^{1+s/d}$$

and

$$\limsup_{(\mathfrak{M}+\mathfrak{N})\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \leqslant \limsup_{\mathfrak{M}\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} + \frac{C\theta}{1+\theta},$$

where C is the same as in Corollary 5.4.

Proof of Theorem 4.3. To show that $g_{s,d}(\cdot)$ is well-defined, it is necessary to verify that (i) existence of the limit $\{\mathfrak{N}\}$ implies that of the limit $E_s(\mathfrak{N})$, and (ii) the value of $E_s(\mathfrak{N})$ is uniquely defined by $\{\mathfrak{N}\}$. To this end, fix a pair of sequences $\mathfrak{N}_1$, $\mathfrak{N}_2 \subset \mathbb{N}$ such that $\{\mathfrak{N}_1\} = \{\mathfrak{N}_2\}$.

First assume that $\mathfrak{N}_1$, $\mathfrak{N}_2$ are multiples of (a subset of) the geometric series, that is, $\mathfrak{N}_i = \{M^k n_i : k \in \mathfrak{K}_i\}$, $i = 1, 2$. Observe that (10) implies $\{\log_M n_1\} = \{\log_M n_2\}$ and let for definiteness $n_2 \geqslant n_1$; then $n_2 = M^{k_0} n_1$ for some integer $k_0 \geqslant 1$. It follows that $\mathfrak{N}_i \subset \mathfrak{N}_0$, $i = 1, 2$, with $\mathfrak{N}_0 = \{M^k n_0 : k \geqslant 1\}$. By Theorem 4.2, the limit

$$\lim_{\mathfrak{N}_0\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}$$

exists, so it must be that the limits over subsequences of $\mathfrak{N}_0$

$$\lim_{\mathfrak{N}_i\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}, \qquad i = 1, 2,$$

also exist and are equal, so the function $g_{s,d}(\cdot)$ is well-defined on the subset of $[0, 1]$ of all the sequences $\mathfrak{N}$ with $\mathfrak{N} = \{M^k n : k \in \mathfrak{K}\}$.

Now let $\mathfrak{N}_1$, $\mathfrak{N}_2 \subset \mathbb{N}$ be arbitrary. Denote the common value of the limit $a := \{\mathfrak{N}_i\}$, $i = 1, 2$. We shall assume for definiteness that $a \in [0, 1)$; the case of $a = 1$ can be handled similarly. In order to bound $\mathfrak{N}_i$ between two sequences of the type $\{M^k n_i : k \in \mathfrak{K}_i\}$, discussed above, fix an $\varepsilon > 0$ such that $a + 2\varepsilon < 1$, and find an $N_0 \in \mathbb{N}$, for which

$$|\{\log_M N_i\} - a| < \varepsilon, \qquad N_0 \leqslant N_i \in \mathfrak{N}_i, \quad i = 1, 2. \tag{20}$$

By the choice of ε, the above equation gives $\lfloor\{\log_M N_1\}\rfloor = \lfloor\{\log_M N_2\}\rfloor$ when $N_0 \leqslant N_i \in \mathfrak{N}_i$. Now let n_i, $i = 1, 2$ be such that

$$\begin{aligned} a - 2\varepsilon \leqslant \{\log_M n_1\} \leqslant a - \varepsilon \\ a + \varepsilon \leqslant \{\log_M n_2\} \leqslant a + 2\varepsilon. \end{aligned} \tag{21}$$

Replacing one of n_i, $i = 1, 2$, with its multiple, if necessary, we can guarantee that $0 < \log_M n_2 - \log_M n_1 < 4\varepsilon$. Consider a pair of sequences $\widetilde{\mathfrak{N}}_i = \{M^k n_i : k \geqslant \lceil \log_M N_0 \rceil\}$, $i = 1, 2$; observe that by the above argument, limits

$$E_s(\widetilde{\mathfrak{N}}_i) =: L_i, \qquad i = 1, 2,$$

along $\widetilde{\mathfrak{N}}_i$, $i = 1, 2$, both exist, and the inequality

$$\widetilde{\mathfrak{N}}_1(k) \leqslant N_i \leqslant \widetilde{\mathfrak{N}}_2(k), \qquad k = \lfloor \log_M N_i \rfloor, \quad N_0 \leqslant N_i \in \mathfrak{N}_i,\ i = 1, 2,$$

holds. By the definition of $\mathcal{E}_s$, and due to (20)–(21),

$$\limsup_{\mathfrak{N}_i\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \leqslant \lim_{k\to\infty} \frac{\mathcal{E}_s(A, M^k n_2)}{(M^k n_1)^{1+s/d}} = \left(\frac{n_2}{n_1}\right)^{1+s/d} L_2, \qquad i = 1, 2,$$

and

$$\liminf_{\mathfrak{N}_i\ni N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \geqslant \lim_{k\to\infty} \frac{\mathcal{E}_s(A, M^k n_1)}{(M^k n_2)^{1+s/d}} = \left(\frac{n_1}{n_2}\right)^{1+s/d} L_1, \qquad i = 1, 2.$$

Combining the last two inequalities gives

$$\left(\frac{n_1}{n_2}\right)^{1+s/d} L_1 \leqslant \liminf_{\mathfrak{N}_i \ni N \to \infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \leqslant \limsup_{\mathfrak{N}_i \ni N \to \infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \leqslant \left(\frac{n_2}{n_1}\right)^{1+s/d} L_2,$$

so it suffices to show that L_2 can be made arbitrarily close to L_1 by taking $\varepsilon \to 0$. The latter follows from Corollary 5.6, and the choice of n_i, $i = 1, 2$:

$$0 \leqslant \frac{\widetilde{\mathfrak{N}}_2(k) - \widetilde{\mathfrak{N}}_1(k)}{\widetilde{\mathfrak{N}}_1(k)} = \frac{n_2}{n_1} - 1 \leqslant M^{4\varepsilon} - 1.$$

Taking $\varepsilon \to 0$ shows both that $E_s(\mathfrak{N}_1) = E_s(\mathfrak{N}_2)$, and that these two limits exist. The function $g_{s,d} : [0,1] \to (0,\infty)$ is therefore well-defined. Note that repeating the above argument for $|\{\mathfrak{N}_1\} - \{\mathfrak{N}_2\}| < \varepsilon$ for a fixed positive ε gives a bound on $|E_s(\mathfrak{N}_1) - E_s(\mathfrak{N}_2)|$, which implies that $g_{s,d}$ is continuous. This completes the proof. □

Proof of Theorem 4.4. Assume without loss of generality that the diameter of the set A satisfies

$$\operatorname{diam}(A) = 1.$$

Denote the minimal value of the Riesz s-energy on M points on A by $\mathcal{E}_{s,M} := \mathcal{E}_s(A, M)$; recall also that σ is the lower bound on the distance between A_i, A_j when $i \neq j$. With this assumption, the last inequality in (14) with $N = M$ gives

$$\begin{aligned}
\mathcal{E}_s(A, M^{k+1}) &\leqslant M^{k(1+s/d)} \mathcal{E}_{s,M} + \sigma^{-s} \frac{M^{1-s/d}}{(M^{s/d-1}-1)} M^{(k+1)(1+s/d)} \\
&\leqslant M^{k(1+s/d)} \sigma^{-s} M^2 + \sigma^{-s} \frac{M^2}{(M^{s/d-1}-1)} M^{k(1+s/d)} \\
&= M^{k(1+s/d)} \sigma^{-s} M^2 \left(1 + \frac{1}{M^{s/d-1}-1}\right) \\
&= M^{(k+1)(1+s/d)} \frac{\sigma^{-s}}{M^{s/d-1}-1}.
\end{aligned} \tag{22}$$

On the other hand, consider a configuration $\omega_{M^{k+1}+M^k}$. The set A is partitioned by the M^{k+1} subsets

$$A_{m_1 \ldots m_{k+1}}, \qquad 1 \leqslant m_1, \ldots, m_{k+1} \leqslant M,$$

so by the pigeonhole principle, for at least M^k pairs $i \neq j$, the points $\boldsymbol{x}_i, \boldsymbol{x}_j \in \omega_{M^{k+1}+M^k}$ belong to the same subset $A_{m_1 \ldots m_{k+1}}$. Writing r for the common contraction ratio of the defining similitudes $\{\psi_m : 1 \leqslant m \leqslant M\}$ preserving the set A, we have

$$\operatorname{diam}(A_{m_1 \ldots m_{k+1}}) = r^{k+1} \operatorname{diam}(A) = r^{k+1}.$$

Configuration $\omega_{M^{k+1}+M^k}$ was chosen arbitrarily, so it follows,

$$\mathcal{E}_s(A, M^{k+1} + M^k) \geqslant M^k (r^{k+1})^{-s} = M^k (M^{s/d})^{k+1} = M^{s/d} (M^k)^{1+s/d}, \tag{23}$$

where we used that $r^{-s} = M^{s/d}$ when all the contraction ratios are equal. Combining equations (22)–(23) gives

$$\begin{aligned}
\underline{g}_{s,d}(A) / \overline{g}_{s,d}(A) &\leqslant \limsup_{k\to\infty} \frac{\mathcal{E}_s(A, M^{k+1})}{(M^{k+1})^{1+s/d}} \Big/ \liminf_{k\to\infty} \frac{\mathcal{E}_s(A, M^{k+1}+M^k)}{(M^{k+1}+M^k)^{1+s/d}} \\
&= \frac{\sigma^{-s}}{M^{s/d-1}-1} \Big/ \frac{1}{M(1+1/M)^{1+s/d}} \\
&= \frac{M(1+1/M)^{1+s/d}}{\sigma^s (M^{s/d-1}-1)}.
\end{aligned}$$

After substituting $1/M = r^d$, the last inequality can be rewritten as

$$\underline{g}_{s,d}(A)/\bar{g}_{s,d}(A) \leqslant \left(\frac{r(1+r^d)^{1/d}}{\sigma}\right)^s \cdot \frac{M^{s/d-1}}{M^{s/d-1}-1} \cdot M(M+1) = R^s \cdot \frac{M^{s/d-1}}{M^{s/d-1}-1} \cdot M(M+1).$$

Note that the second factor in the above equation is less than 2 when $s > 2d$ holds (since $M \geqslant 2$); for an $R < 1$, choosing the Riesz exponent as in (12) makes the RHS less than 1, as desired.

□

Proof of Corollary 4.5. The proof repeats that of Theorem 4.4, except for the simplified expression for $\mathcal{E}_{s,M} = \mathcal{E}_{s,2} = 1$. Equations (22)–(23) become

$$\mathcal{E}_s(A, 2^{k+1}) = 2^{(k+1)(1+s/d)} \frac{1}{2^2(2^{s/d-1}-1)},$$
$$\mathcal{E}_s(A, 2^{k+1} + 2^k) \geqslant 2^{s/d}(2^k)^{1+s/d},$$

respectively. Finally, from

$$\underline{g}_{s,d}(A)/\bar{g}_{s,d}(A) \leqslant \frac{2(3/2)^{1+s/d}}{2^2(2^{s/d-1}-1)} = \left(\frac{3}{4}\right)^{s/d} \cdot \frac{2^{s/d-1}}{2^{s/d-1}-1} \cdot \frac{3}{2}.$$

The RHS is a decreasing function of s and is less than 1 for $s \geqslant 3d = 3\dim_H A = 3\log_3 2$, which completes the proof. □

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