POSITIVITY OF SEGRE-MACPHERSON CLASSES

PAOLO ALUFFI, LEONARDO C. MIHALCEA, JÖRG SCHÜRMANN, AND CHANGJIAN SU

Abstract. Let X be a complex nonsingular variety with globally generated tangent bundle. We prove that the signed Segre-MacPherson (SM) class of a constructible function on X with effective characteristic cycle is effective. This observation has a surprising number of applications to positivity questions in classical situations, unifying previous results in the literature and yielding several new results. We survey a selection of such results in this paper. For example, we prove general effectivity results for SM classes of subvarieties which admit proper (semi-)small resolutions and for regular or affine embeddings. Among these, we mention the effectivity of (signed) Segre-Milnor classes of complete intersections if X is projective and an alternation property for SM classes of Schubert cells in flag manifolds; the latter result proves and generalizes a variant of a conjecture of Fehér and Rimányi. Among other applications we prove the positivity of Behrend's Donaldson-Thomas invariant for a closed subvariety of an abelian variety and the signed-effectivity of the intersection homology Chern class of the theta divisor of a non-hyperelliptic curve; and we extend the (known) non-negativity of the Euler characteristic of perverse sheaves on a semi-abelian variety to more general varieties dominating an abelian variety.

1. Introduction

1.1. In this note, X will denote a nonsingular complex variety and $Z \subseteq X$ will be a closed subvariety; here (sub)varieties are by definition irreducible and reduced. We will assume that the tangent bundle of X is globally generated. In the projective case, this is equivalent to asking that X be a projective homogeneous variety—for example a projective space, a flag manifold, or an abelian variety; but our main results will hold in the non-complete case as well. We denote by $A_*(Z)$ the Chow group of cycles on Z modulo rational equivalence, and by $\mathsf{F}(Z)$ the group of constructible functions on Z; here we allow Z to be more generally a closed reduced subscheme of X. Our general aim is to investigate the *positivity* of certain rational equivalence classes associated with the embedding of Z in X, or more generally with suitable constructible functions on Z. We generalize several known positivity results on e.g., Euler characteristics, and provide a framework leading to analogous results in a broad range of situations. Answering a conjecture of Deligne and Grothendieck, MacPherson [Mac74] constructed a group homomorphism $c_* : \mathsf{F}(Z) \to A_*(Z)$ which commutes with proper push-forwards and satisfies a normalization property: if Z is non-singular, then $c_*(1\!\!1_Z) = c(TZ) \cap [Z]$, where $c(TZ)$ is the total Chern class of Z. (MacPherson worked in homology; see [Ful84, Example 19.1.7] for the refinement of the theory to the Chow group.) If $Y \subseteq Z$ is a constructible subset, the Chern-Schwartz-MacPherson (CSM) class $c_{\mathrm{SM}}(Y) \in A_*(Z)$ is the image $c_*(1\!\!1_Y)$ of the indicator function of

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Y under MacPherson's natural transformation. Let $\varphi \in \mathsf{F}(Z)$. We will focus on the closely related Segre-MacPherson (SM) class

$$s_*(\varphi, X) := c(TX|_Z)^{-1} \cap c_*(\varphi) \in A_*(Z).$$

(The class $c(TX|_Z)$ is invertible in $A_*(Z)$, because it is of the form $1 + a$, where a is nilpotent.) In particular, we let $s_{\mathrm{SM}}(Y, X)$ denote the Segre-Schwartz-MacPherson (SSM) class $s_*(1\!\!1_Y, X) \in A_*(Z)$; note that this class depends on both Y and the ambient variety X. If Y is a subvariety of Z, then the top-degree component of $s_{\mathrm{SM}}(Y, X)$ in $A_{\dim Y}(Z)$ is the fundamental class $[\overline{Y}]$ of the closure of Y. Further, if $Y = Z$ is a *nonsingular* closed subvariety of X, then $s_{\mathrm{SM}}(Y, X) \in A_*Y$ equals the ordinary Segre class $s(Y, X)$; in general, the two classes differ (as discussed in subsection 8.2 for Y a global complete intersection in a nonsingular projective variety X). See [Alu03] and (in the equivariant case) [Ohm06] for general properties of SM classes, and [Sch17] for their compability with transversal pullbacks.

There are 'signed' versions of both c_* and s_* (respectively, c_{SM} and s_{SM}), which appear naturally when relating them to characteristic cycles. If $c_*(\varphi) = c_0 + c_1 + \dots$ is the decomposition into homogeneous components (i.e., $c_i \in A_i(Z)$) then the 'signed' class $\check{c}_*(\varphi)$ is defined by

$$\check{c}_*(\varphi) = c_0 - c_1 + c_2 - \dots \quad \text{and} \quad \check{c}_{\mathrm{SM}}(Y) := \check{c}_*(1\!\!1_Y);$$

that is, by changing the sign of each homogeneous component of odd dimension. One defines similarly the signed SM class

$$\check{s}_*(\varphi, X) := c(T^*X|_Z)^{-1} \cap \check{c}_*(\varphi) \quad \text{and} \quad \check{s}_{\mathrm{SM}}(Y, X) := \check{s}_*(1\!\!1_Y, X).$$

A basis for $\mathsf{F}(Z)$ consists of the *local Euler obstructions* Eu_Y for closed subvarieties Y of Z. In fact, the characteristic cycle of the (signed) local Euler obstruction is an irreducible Lagrangian cycle in T^*X, and from this perspective the functions Eu_Y are the 'atoms' of the theory; see equation (2.2) below. If Y is nonsingular, $\mathrm{Eu}_Y = 1\!\!1_Y$. The local Euler obstruction is a subtle and well-studied invariant of singularities (see e.g., [Mac74, BDK81, LT81, Ern94, BLS00, Beh09]). The corresponding class $c_*(\mathrm{Eu}_Z) \in A_*(Z)$ for Z a subvariety of X is the *Chern-Mather class* of Z, $c_{\mathrm{Ma}}(Z) = \nu_*(c(\tilde{T}) \cap [\tilde{Z}])$, with $\nu : \tilde{Z} \to Z$ the *Nash blow-up* of Z and $\tilde{T}$ the tautological bundle on $\tilde{Z}$ extending TZ_{reg}, cf. [Mac74] or [Ful84, Example 4.2.9]. In particular $c_{\mathrm{Ma}}(Z) = c(TZ) \cap [Z]$ if Z is nonsingular. If Z is complete, we denote by $\chi_{\mathrm{Ma}}(Z) := \chi(Z, \mathrm{Eu}_Z)$ the degree of $c_{\mathrm{Ma}}(Z)$; so $\chi_{\mathrm{Ma}}(Z)$ equals the usual topological Euler characteristic $\chi(Z)$ if Z is nonsingular and complete. We also consider the corresponding Segre-Mather class $s_{\mathrm{Ma}}(Z, X) := s_*(\mathrm{Eu}_Z, X)$ as well as the signed classes

$$\check{c}_{\mathrm{Ma}}(Z) := \check{c}_*(\mathrm{Eu}_Z); \quad \check{s}_{\mathrm{Ma}}(Z, X) := c(T^*X|_Z)^{-1} \cap \check{c}_*(\mathrm{Eu}_Z).$$

With our conventions, we get $(-1)^{\dim Z} \check{c}_{\mathrm{Ma}}(Z) = \nu_*(c(\tilde{T}^*) \cap [\tilde{Z}])$ in terms of the dual tautological bundle on the Nash blow-up (which differs by the sign $(-1)^{\dim Z}$ from the definition of the signed Chern-Mather class used in some references like [Sab85, ST10]).

The main result in this paper is the (signed) effectivity of Segre-MacPherson classes in a large class of examples. By an *effective* class we mean a class which can be represented by a nonzero, non-negative, cycle.

Theorem 1.1. *Let X be a complex nonsingular variety, and assume that the tangent bundle TX is globally generated. Let $Z \subseteq X$ be a closed subvariety of X. Then the following hold:*

(a) The class $(-1)^{\dim Z} \check{s}_{Ma}(Z, X) \in A_*(Z)$ is effective.

(b) Assume that the inclusion $U \hookrightarrow Z$ is an affine morphism, where U is a locally closed smooth subvariety of Z. Then $(-1)^{\dim U} \check{s}_{SM}(U, X) \in A_*(Z)$ is effective.

If in addition X is assumed to be complete, then the requirement that TX is globally generated is equivalent to X being a homogeneous variety; cf. e.g., [Bri12, Corollary 2.2]. Further, Borel and Remmert [BR62] (see also [Bri12, Theorem 2.6]) prove that all complete homogeneous varieties are products $(G/P) \times A$, where G is a semisimple Lie group, $P \subseteq G$ is a parabolic subgroup, and A is an abelian variety.

Theorem 1.1 is extended to more general constructible functions in Theorem 2.2, below. These theorems may be used to prove several positivity statements, unifying and generalizing analogous results from the existing literature. We list below the situations which we will highlight in this paper to illustrate applications of our methods, and the sections where these are discussed.

(a) Closed subvarieties of abelian varieties; primarily in §3.
(b) The proof of a generalization of a conjecture of Fehér and Rimányi [FR18] concerning SSM classes of Schubert cells in Grassmannians; §4.1.
(c) Complements of hyperplane arrangements; §4.2
(d) Positivity of certain Donaldson-Thomas type invariants; §5.
(e) Intersection homology Segre and Chern classes; §6.
(f) Semi-small resolutions; §8.1.
(g) Regular embeddings and Milnor classes; §8.2.
(h) Semi-abelian varieties and generalizations; §8.3.

Ultimately, these positivity statements follow from the effectivity of the associated characteristic cycles. In §7 we survey a more comprehensive list of situations in which the characteristic cycle is positive.

The proof of Theorem 1.1 will be given in §2.2 below. It is suprisingly easy, but not elementary; it is based on a classical formula by Sabbah [Sab85], calculating the (signed) CSM class of a constructible function φ in terms of its characteristic cycle $\mathrm{CC}(\varphi)$; see Theorem 2.1 below.

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Finally, this paper is dedicated to Professor William Fulton in the occasion of his 80th birthday. His interest in positivity questions arising in algebraic geometry, and his influential ideas, continue to inspire us.

2. Characteristic classes via characteristic cycles; proof of the main theorem

2.1. **Characteristic cycles.** Let X be a smooth complex variety. We recall a commutative diagram which plays a central role in seminal work of Ginzburg [Gin86]; it is largely based on results from [BB81, BK81, KT84]. We also considered this diagram in our previous work [AMSS17, §6], and we use the notation from this reference.

$$\begin{tikzcd} \mathrm{Perv}(X) \arrow[d, "\chi_{stalk}"'] & \mathrm{Mod}_{rh}(\mathcal{D}_X) \arrow[l, "\mathrm{DR}"', "\sim"] \arrow[d, "\mathrm{Char}"] \\ \mathsf{F}(X) \arrow[r, "\mathrm{CC}", "\sim"'] & L(X) \end{tikzcd} \tag{2.1}$$

Here $\mathrm{Mod}_{rh}(\mathcal{D}_X)$ denotes the Abelian category of algebraic holonomic $\mathcal{D}_X$-modules with regular singularities, and $\mathrm{Perv}(X)$ is the Abelian category of perverse (algebraically) constructible complexes of sheaves of $\mathbb{C}$-vector spaces on X; $\mathsf{F}(X)$ is the group of constructible functions on X and $L(X)$ is the group of conic Lagrangian cycles in T^*X. The functor DR is defined on $M \in \mathrm{Mod}_{rh}(\mathcal{D}_X)$ by

$$\mathrm{DR}(M) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, M)[\dim X],$$

that is, it computes the DeRham complex of a holonomic module (up to a shift), viewed as an *analytic* $\mathcal{D}_X$-module. This functor realizes the Riemann-Hilbert correspondence, and is an equivalence. We refer to e.g., [KT84, Gin86] for details. The left map χ_{stalk} computes the stalkwise Euler characteristic of a constructible complex, and the right map Char gives the characteristic cycle of a holonomic $\mathcal{D}_X$-module. The map CC is the characteristic cycle map for constructible functions; if $Z \subseteq X$ is closed and irreducible, then

$$\mathrm{CC}(\mathrm{Eu}_Z) = (-1)^{\dim Z}[T^*_Z X]; \tag{2.2}$$

here Eu_Z is the local Euler obstruction (see §1), and $T^*_Z X := \overline{T^*_{Z^{reg}} X}$ is the conormal space of Z, i.e., the closure of the conormal bundle of the smooth locus of Z. The commutativity of diagram (2.1) is shown in [Gin86] using deep $\mathcal{D}$-module techniques; it also follows from [Sch03, Example 5.3.4, p. 359–360] (even for a holonomic $\mathcal{D}$-module without the regularity requirement). Also note that the upper transformations in (2.1) factor over the corresponding Grothendieck groups, so they also apply to complexes of such $\mathcal{D}$-modules. If $f : X \to Y$ is a proper map of smooth complex varieties, there are well-defined push-forwards for each of the objects in the diagram, denoted by f_*. Furthermore, all the maps commute with proper push forwards; cf. [Gin86, Appendix]. For others proofs, see [HTT08, Proposition 4.7.5] for the transformation DR, [Sch03, §2.3] for the transformation χ_{stalk} and [Sch05, §4.6] for the transformation CC (for the transformation Char it then follows from the commutativity of diagram (2.1)).

The next result, relating characteristic cycles to (signed) CSM classes, has a long history. See [Sab85, Lemme 1.2.1], and more recently [PP01, (12)], [Sch05, §4.5], [Sch17, §3], especially diagram (3.1) in [Sch17].

Theorem 2.1. *Let X be a complex nonsingular variety, and let $Z \subseteq X$ be a closed reduced subscheme. Let $\varphi \in \mathsf{F}(Z)$ be a constructible function on Z. Then*

$$\check{c}_*(\varphi) = c(T^*X|_Z) \cap \mathrm{Segre}(\mathrm{CC}(\varphi))$$

as elements in the Chow group $A_*(Z)$ of Z. Here $\mathrm{Segre}(\mathrm{CC}(\varphi))$ *is the Segre class associated to the conic Lagrangian cycle* $\mathrm{CC}(\varphi) \subseteq T^*X|_Z$.

We recall the definition of the Segre class used in Theorem 2.1. Let $q : \mathbb{P}(T^*X|_Z \oplus 1\!\!1) \to Z$ be the projection from the restriction of the projective completion of the cotangent bundle of X. If $C \subseteq T^*X|_Z$ is a cone supported over Z, and $\overline{C}$ is the closure in $\mathbb{P}(T^*X|_Z \oplus 1\!\!1)$, the Segre class is defined by

$$\mathrm{Segre}(C) := q_*\big(\sum_{i\geq 0} c_1(\mathcal{O}_{\mathbb{P}(T^*X|_Z\oplus \mathbf{1})}(1))^i \cap [\overline{C}]\big)$$

as an element of $A_*(Z)$; see [Ful84, §4.1].

Every irreducible conic Lagrangian subvariety of T^*X is a conormal cycle $T^*_Z X$ for $Z \subseteq X$ a closed subvariety; see e.g., [HTT08, Theorem E.3.6]. From this it follows that every nontrivial characteristic cycle is a linear combination of conormal spaces:

$$\mathrm{CC}(\varphi) = \sum_Y a_Y [T^*_Y X] \tag{2.3}$$

for uniquely determined closed subvarieties Y of Z and nonzero integer coefficients a_Y. By (2.2), the coefficients a_Y are determined by the equality of constructible functions

$$0 \neq \varphi = \sum_Y a_Y (-1)^{\dim Y} \mathrm{Eu}_Y\,. \tag{2.4}$$

2.2. **Proof of the main theorem.** The following result is at the root of all applications in this note.

Theorem 2.2. *Let X be a complex nonsingular variety such that TX is globally generated. Let $Z \subseteq X$ be a closed reduced subscheme of X and let $\varphi \in \mathsf{F}(Z)$ be a constructible function on Z such that the characteristic cycle $\mathrm{CC}(\varphi) \in A_*(T^*X|_Z)$ is effective.*

Then $\check{s}_*(\varphi, X)$ is effective in $A_*(Z)$. If TX is trivial (e.g., X is an abelian variety), then $\check{c}_*(\varphi)$ is effective.

Proof. By Theorem 2.1,

$$\check{s}_*(\varphi, X) = c(T^*X|_Z)^{-1} \cap \check{c}_*(\varphi) = \mathrm{Segre}(\mathrm{CC}(\varphi)).$$

By hypothesis we have a decomposition (2.3) with positive coefficients a_Y. It follows from (2.4) that the Segre class of $\mathrm{CC}(\varphi)$ is a linear combination of Segre classes of subvarieties:

$$\mathrm{Segre}(\mathrm{CC}(\varphi)) = \sum_Y a_Y (-1)^{\dim Y} \mathrm{Segre}(\mathrm{CC}(\mathrm{Eu}_Y)) = \sum_Y a_Y (-1)^{\dim Y} \check{s}_{\mathrm{Ma}}(Y, X).$$

By definition, the top degree part of each signed Segre-Mather class $(-1)^{\dim Y} \check{s}_{\mathrm{Ma}}(Y, X)$ equals $[Y]$. Then the top degree part of $\mathrm{Segre}(\mathrm{CC}(\varphi))$ equals a positive linear combination of those fundamental classes $[Y]$ of maximal dimension, and in particular $\check{s}_*(\varphi, X) = \mathrm{Segre}(\mathrm{CC}(\varphi))$ is not zero. Since the tangent bundle TX is globally generated, it follows that the line bundle $\mathcal{O}_{\mathbb{P}(T^*X \oplus \mathbf{1})}(1)$ is globally generated, as it is a quotient of $TX \oplus 1\!\!1$. Therefore its first Chern class preserves non-negative classes. Since non-negativity is preserved by proper push-forwards, we can conclude that under the given hypotheses $\mathrm{Segre}(\mathrm{CC}(\varphi))$ is non-negative, and this completes the proof. □

Theorem 1.1 follows from Theorem 2.2:

Proof of Theorem 1.1. By Theorem 2.2 it suffices to show that the characteristic cycles for the constructible functions $(-1)^{\dim Z} \mathrm{Eu}_Z$ and $(-1)^{\dim U} 1\!\!1_U$ are effective. If $\varphi = \check{\mathrm{Eu}}_Z := (-1)^{\dim Z} \mathrm{Eu}_Z$, then $\mathrm{CC}(\varphi)$ equals the conormal cycle $[T^*_Z X]$ of Z, and it is therefore trivially effective.

Consider then $(-1)^{\dim U} 1\!\!1_U$, and let $j : U \hookrightarrow X$ be the inclusion. We use the Riemann-Hilbert correspondence (diagram (2.1)) to express the characteristic cycle. By definition,

$$(-1)^{\dim U} 1\!\!1_U = \chi_{stalk}(j_! \mathbb{C}_U[\dim U]).$$

Since U is nonsingular, the sheaf $\mathbb{C}_U[\dim U]$ is perverse, and $\mathcal{O}_U$ is the corresponding regular holonomic $\mathcal{D}_U$-module. We have $j_!\mathbb{C}_U[\dim U] = \mathrm{DR}(j_!(\mathcal{O}_U))$; since j is an affine morphism, $j_!(\mathcal{O}_U)$ is a single regular holonomic $\mathcal{D}_X$-module (with support in Z); see [HTT08, p. 95].

As pointed out in [HTT08, p. 119], the characteristic cycles of non-trivial holonomic $\mathcal{D}_X$-modules are effective, and this finishes the proof. $\square$

Remark 2.3. As the proof shows, the hypothesis that U is smooth in Theorem 1.1(ii) can be weakened, by only requiring that that $\mathbb{C}_U[\dim U]$ is a perverse sheaf. The proof of the effectivity then uses the fact that for an affine inclusion $j : U \hookrightarrow Z$, $j_!\mathbb{C}_U[\dim U]$ is a perverse sheaf on Z ([Sch03, Lemma 6.0.2, p. 384 and Theorem 6.0.4, p. 409]). We will formalize this conclusion below, in Proposition 2.4. For the case in which $U = X \smallsetminus D$ is the open complement of a hypersurface D in $Z := X$, the result also follows from [Sch03, Proposition 6.0.2, p. 404]. ⌟

2.3. **Effective characteristic cycles (I).** The applications in the rest of the paper follow from Theorem 2.2: they represent situations when the characteristic cycle $\mathrm{CC}(\varphi)$ is effective.

As pointed out above, every non-trivial characteristic cycle is a linear combination of conormal spaces (2.3), and the coefficients a_Y in a linear combination are determined by the equality of constructible functions (2.4). The characteristic cycle of $\varphi \neq 0$ is effective if and only if the coefficients a_Y are positive. In particular, this condition is intrinsic to the constructible function $\varphi \in \mathsf{F}(Z)$ and does not depend on the chosen closed embedding of Z into an ambient nonsingular variety X.

A key source of examples where $\mathrm{CC}(\varphi)$ is effective, but possibly reducible, arises as follows. Constructible functions may be associated with (regular) holonomic $\mathcal{D}$-modules and perverse sheaves $\mathcal{F} \in \mathrm{Perv}(Z)$, cf. diagram (2.1); for example, in the latter case the value of the constructible function $\varphi := \chi_{stalk}(\mathcal{F})$ at the point $z \in Z$ is the Euler characteristic $\varphi(z) = \chi(\mathcal{F}_z)$ of the stalk at z of the given complex of sheaves $\mathcal{F}$.

Proposition 2.4. *Let X be a complex nonsingular variety such that TX is globally generated, and let $Z \subseteq X$ be a closed reduced subscheme. Let $0 \neq \varphi \in \mathsf{F}(Z)$ be a non-trivial constructible function associated with a regular holonomic $\mathcal{D}_X$-module supported on Z, or (equivalently) a perverse sheaf on Z. Then $\check{s}_*(\varphi, X)$ is effective in $A_*(Z)$.*

Proof. This follows from the argument used in the proof of Theorem 2.2: the main observation is that the characteristic cycle of a non-trivial (regular) holonomic $\mathcal{D}$-module is effective; see e.g., [HTT08, p. 119]. Further, perverse sheaves correspond to regular holonomic $\mathcal{D}$-modules by means of the Riemann-Hilbert correspondence (see e.g., [HTT08, Theorem 7.2.5]), compatibly with the construction of the associated constructible functions and characteristic cycles; cf. diagram (2.1). $\square$

There are situations where the characteristic cycle associated to a constructible sheaf is known to be irreducible: examples include characteristic cycles of the intersection cohomology sheaves of Schubert varieties in the Grassmannian [BFL90], in more general minuscule spaces [BF97], of certain determinantal varieties [Zha18], and of the theta divisors in the Jacobian of a non-hyperelliptic curve [BB98]. In all such cases, $\check{s}_*(\varphi, X)$ is effective provided that TX is globally generated, by Theorem 2.2. Also note that for the varieties Z listed above, the Chern-Mather class $c_{\mathrm{Ma}}(Z)$ equals $c_{\mathrm{IH}}(Z)$, the intersection homology class defined in §6 below. This follows because in this case the characteristic cycle of $\mathcal{IC}_Z$ is irreducible, thus it must agree with the conormal cycle of Z.

In the next few sections we discuss specific applications of Theorem 2.2 for various choices of the variety X or constructible function φ. The sections are mostly logically independent of each other, and the reader may skip directly to the case of interest. The only exception are the results concerning Abelian varieties; these will be mentioned throughout this note.

A more detailed discussion on effective characteristic cycles is given in §7 below, including a more comprehensive list of constructible functions φ for which $\mathrm{CC}(\varphi)$ is effective, and operations on characteristic functions which preserve the effectivity of the corresponding characteristic cycles.

3. Abelian varieties

If X is an abelian variety, then TX is trivial. (In fact, this characterizes abelian varieties among complete varieties, cf. [Bri12, Corollary 2.3].) If TX is trivial, then for all constructible functions φ on Z the signed SM class agrees with the signed CM class: $\check{s}_*(\varphi, X) = \check{c}_*(\varphi) \in A_*(Z)$. In particular, $\check{s}_{\mathrm{Ma}}(Z, X) = \check{c}_{\mathrm{Ma}}(Z)$ for Z a subvariety of X. The following result follows then immediately from Theorems 1.1 and 2.2.

Corollary 3.1. *Let Z be a closed subvariety of a smooth variety X with TX trivial (for example, an abelian variety). Then $(-1)^{\dim Z}\check{c}_{Ma}(Z)$ is effective.*

More generally, let φ be a constructible function on Z such that $\mathrm{CC}(\varphi)$ is effective. Then $\check{c}_*(\varphi) \in A_*(Z)$ is effective.

As an example, the total Chern-Mather class $c_{\mathrm{Ma}}(\Theta)$ of the theta divisor in the Jacobian of a nonsingular curve must be signed-effective.

Corollary 3.1 implies that $\chi(Z, \varphi) \geq 0$, which also follows from [FK00, Theorem 1.3]. In particular, if Z is a closed subvariety of an abelian variety, then

$$(-1)^{\dim Z}\chi_{\mathrm{Ma}}(Z) = (-1)^{\dim Z}\chi(Z, \mathrm{Eu}_Z) \geq 0\,.$$

For nonsingular subvarieties Z, the Euler obstruction Eu_Z equals $\mathbb{1}_Z$. Then the fact that $(-1)^{\dim Z}\chi(Z) \geq 0$ is proven (in the more general semi-abelian case) in [FK00, Corollary 1.5] (also see [EGM18, (2)]). We note that the fact that $c(T^*Z) \cap [Z]$ is effective if Z is a *nonsingular* subvariety of an abelian variety X also follows immediately from the fact that T^*Z is globally generated, as it is a homomorphic image of the restriction of T^*X, which is trivial. Corollary 3.1 extends this result to *arbitrarily singular* closed subvarieties of a smooth variety X with trivial tangent bundle.

In fact, Corollary 3.1 also follows from Propositions 2.7 and 2.9 from [ST10], where explicit effective cycles representing $\check{s}_*(\varphi, X)$ in terms of suitable 'polar classes' are constructed.

4. Affine embeddings

An important family of positivity statements focus on the indicator function $\mathbb{1}_U$ of a typically nonsingular and noncompact subvariety U of $Z \subseteq X$. In this case, among our main applications is the proof of a conjecture of Fehér and Rimányi about the the effectivity of SSM classes of Schubert cells.

Throughout this section we impose the hypotheses of Theorem 1.1(b), i.e., X is a complex nonsingular variety with TX globally generated, $Z \subseteq X$ is a closed subvariety of X, and the inclusion $U \hookrightarrow Z$ is an affine morphism, with U a locally closed smooth subvariety of Z. Recall that

$$s_{\mathrm{SM}}(U, X) := c(TX|_Z)^{-1} \cap c_*(\mathbb{1}_U); \quad \check{s}_{\mathrm{SM}}(U, X) := c(T^*X|_Z)^{-1} \cap \check{c}_*(\mathbb{1}_U)$$

denote the SSM and the signed SSM classes associated to U. By Theorem 1.1 (b),

$$(-1)^{\dim U}\check{s}_{\mathrm{SM}}(U, X) \in A_*(Z)$$

is effective. If TX is trivial, then $(-1)^{\dim U}\check{c}_{\mathrm{SM}}(U)$ is effective, so in particular for X an abelian variety this implies that $(-1)^{\dim U}\chi(U) \geq 0$.

4.1. **Schubert cells in flag manifolds and a conjecture of Fehér and Rimányi.** Theorem 1.1 applies in particular if $U := X(u)^\circ$ is a Schubert cell in a flag manifold $X = G/P$, where G is a complex simple Lie group and P is a parabolic subgroup. For example, X could be a Grassmannian, or a complete flag manifold. Here $u \in W$ is a minimal length representative for its coset in W/W_P, where W is the Weyl group of G and W_P is the Weyl group of P. The Schubert *cell* $X(u)^\circ$ is defined to be BuB/P, where $B \subseteq P$ is a Borel subgroup. It is well known that $X(u)^\circ \cong \mathbb{C}^{\ell(u)}$, where $\ell(u)$ denotes the length of u. The closure $X(u)$ of $X(u)^\circ$ is the corresponding Schubert *variety,* and $X(u) = \bigsqcup_{w \leq u} X(w)^\circ$. We refer to e.g., [Bri05] for further details on these definitions. Since the inclusion $X(u)^\circ \subseteq X$ is affine, we obtain the following result.

Corollary 4.1. *Let $X(u)^\circ$ be a Schubert cell in a generalized flag manifold G/P. Then the class $(-1)^{\ell(u)}\check{s}_{SM}(X(u)^\circ, G/P) \in A_*(X(u))$ is effective.*

Recall that $A_*(G/P)$ (resp., $A_*(X(u))$) has a $\mathbb{Z}$-basis given by fundamental classes $[X(v)]$ of Schubert varieties (with $X(v) \subseteq X(u)$, i.e., $v \leq u$). With this understood, Corollary 4.1 may be rephrased as follows.

Corollary 4.2. *Let $u \in W$ and consider the Schubert expansion*

$$s_{SM}(X(u)^\circ, G/P) = \sum a(w;u)[X(w)]$$

with $a(w;u) \in \mathbb{Z}$. Then $(-1)^{\ell(u)-\ell(w)}a(w;u) \geq 0$ for all w.

A similar positivity statement was conjectured by Fehér and Rimányi in §1.5 and Conjecture 8.4 of the paper [FR18]. Their conjecture is stated for certain degeneracy loci in quiver varieties, and in the 'universal' situation where the ambient space is a vector space with a group action. The Schubert cells and varieties in the flag manifolds of Lie type A are closely related to a compactified version of such quiver loci[1]. After passing to the compactified version of the statements from [FR18], Corollary 4.2 proves the conjecture from [FR18] in the Schubert instances; see [FRW18, §6 and §7] for a comparison between the 'universal' and 'compactified' versions. A specific comparison between our calculations and those from [FR18] is included in the following example. We note that in arbitrary Lie type a description of Schubert varieties via quiver loci is not available.

Example 4.3. Let $X = \mathrm{Gr}(2,5)$ be the Grassmann manifold parametrizing subspaces of dimension 2 in $\mathbb{C}^5$. In this case one can index the Schubert cells by partitions included in the 2×3 rectangle, such that each cell has dimension equal to the number of boxes in the partition. With this notation, and using the calculation of CSM classes of Schubert cells from [AM09], one obtains the following matrix encoding Schubert expansions of SSM classes of Schubert cells:

[1] One example are the matrix Schubert varieties, regarded in the space of all matrices. The study of those of maximal rank is closely related to Schubert varieties in the Grassmannian.

$$\begin{pmatrix}
1 & -4 & 5 & 4 & -2 & -10 & 5 & 4 & -4 & 1 \\
0 & 1 & -3 & -3 & 2 & 10 & -7 & -5 & 7 & -2 \\
0 & 0 & 1 & 0 & -2 & -3 & 7 & 3 & -9 & 3 \\
0 & 0 & 0 & 1 & 0 & -3 & 2 & 2 & -3 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -3 & 0 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & -2 & 5 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

The columns, read left to right, and rows, read top to bottom, are indexed by:

$$\emptyset\,, \ydiagram{1}\,, \ydiagram{2}\,, \ydiagram{1,1}\,, \ydiagram{3}\,, \ydiagram{2,1}\,, \ydiagram{3,1}\,, \ydiagram{2,2}\,, \ydiagram{3,2}\,, \ydiagram{3,3}\ .$$

After taking duals in the 2×3 rectangle, these give the same coefficients as in equation (3) from [FR18]. (The calculations in [FR18] are done in a stable limit, therefore for our purposes one disregards partitions not included in the given rectangle.) Another example is given by the calculation of the SSM class for the partition $(3,1)$ in $\mathrm{Gr}(2,6)$:

$$s_{\mathrm{SM}}((\ydiagram{3,1})^\circ) = \ydiagram{3,1} - 3\ \ydiagram{2,1} - 4\ \ydiagram{3} + 13\ \ydiagram{2} + 5\ \ydiagram{1,1} - 22\ \ydiagram{1} + 22\ \emptyset.$$

(Here λ denotes the Schubert class indexed by λ, and λ° is the indicator function of the Schubert cell.) This is consistent with [FR18, Example 8.3]. $\lrcorner$

If the parabolic subgroup P is the Borel subgroup B, Corollary 4.2 is equivalent to the positivity of CSM classes of Schubert cells, [AMSS17, Corollary 1.4]. Indeed, in this case

$$s_{\mathrm{SM}}(X(u)^\circ, G/B) = (-1)^{\ell(u)}\check{c}_*(1\!\!1_{X(u)^\circ})$$

as shown in [AMSS17, Corollary 7.4]. This equality does not hold for more general flag manifolds G/P, and its proof relies on additional properties relating the CSM/SSM classes to Demazure-Lusztig operators from the Hecke algebra [AM16, AMSS17]. From this prospective, SSM classes appear to have a simpler behavior than the CSM classes, and one can obtain positivity-type statements for a larger class of varieties.

4.2. **Complements of hyperplane arrangements.** A typical example of an affine embedding $U \subseteq X$ is the complement $U = X \smallsetminus D$ of a hypersurface $D \subseteq Z := X$. In particular, one can consider a projective hyperplane arrangement $\mathcal{A}$ in complex projective space $X = \mathbb{P}^n$, with $A := D$ the union of hyperplanes and $U = \mathbb{P}^n \smallsetminus A$ its complement. In this particular case Theorem 1.1(b) recovers a consequence of the following result of [Alu13, Corollary 3.2]:

$$c_{\mathrm{SM}}(U) = \pi_{\widehat{\mathcal{A}}}\left(\frac{-h}{1+h}\right) \cap \left(c(T\mathbb{P}^n) \cap [\mathbb{P}^n]\right)\ .$$

Here h denotes the hyperplane class in $\mathbb{P}^n$, $\widehat{\mathcal{A}}$ is the corresponding 'central arrangement' in $\mathbb{C}^{n+1}$ with $\widehat{A}$ its union of linear hyperplanes, and $\pi_{\widehat{\mathcal{A}}}$ denotes the corresponding 'Poincaré polynomial' of $\widehat{\mathcal{A}}$ (see e.g., [Alu13, p. 1880]):

$$\pi_{\widehat{\mathcal{A}}}(t) = \sum_{k=0}^{n+1} rk\ H^k(\mathbb{C}^{n+1} \smallsetminus \widehat{A}, \mathbb{Q})t^k\ .$$

In particular the Poincaré polynomial $\pi_{\widehat{\mathcal{A}}}$ has non-negative coefficients and constant term one, and

$$(-1)^{\dim U}\check{s}_{\mathrm{SM}}(U,\mathbb{P}^n)=\pi_{\widehat{\mathcal{A}}}\left(\frac{h}{1-h}\right)\cap[\mathbb{P}^n]$$

is effective.

5. Donaldson-Thomas type invariants

Let $Z\subseteq X$ be a closed reduced subscheme of X as before. K. Behrend ([Beh09, Definition 1.4, Proposition 4.16]) defines a constructible function ν_Z and proves that if Z is proper, then the dimension-0 component of $c_*(\nu_Z)$ equals the corresponding virtual fundamental class $[Z]^{\mathrm{vir}}$, a 'Donaldson-Thomas type invariant' in the terminology of [Beh09, p. 1308].

The characteristic cycle of Behrend's constructible function ν_Z is effective because the intrinsic normal cone of Z is effective. More explicitly, ν_Z is defined as a weighted sum

$$\nu_Z:=\sum_Y(-1)^{\dim Y}\,\mathrm{mult}(Y)\cdot\mathrm{Eu}_Y\quad,$$

where the summation is over the supports Y of the components of the intrinsic normal cone of Z and $\mathrm{mult}(Y)$ is the multiplicity of the corresponding component. (Cf. [Beh09, Definition 1.4].) Since these multiplicities are positive, $\mathrm{CC}(\nu_Z)$ is effective, cf. (2.4).

Corollary 5.1. *Let X be a complex nonsingular variety with globally generated tangent bundle, and let $Z\subseteq X$ be a closed reduced subscheme. Then $\check{s}_*(\nu_Z,X)\in A_*(Z)$ is effective. If TX is trivial, then $\check{c}_*(\nu_Z)$ is effective.*

In particular for X an abelian variety this implies that $[Z]^{\mathrm{vir}}$ is non-negative and hence

$$\chi_{vir}(Z):=\chi(Z,\nu_Z)=deg([Z]^{\mathrm{vir}})\geq 0\,.$$

6. Characteristic classes from intersection cohomology

Another particular case of interest is the characteristic class of the intersection cohomology sheaf complex. If $Z\subseteq X$ is a closed subvariety, let $\mathcal{IC}_Z\in Perv(Z)$ denote the intersection cohomology complex associated to Z. This is the key example of a perverse sheaf on Z, cf. [GM83], [HTT08, Definition 8.2.13] or [Sch03, p. 385]. The associated constructible function is $\mathrm{IC}_Z:=\chi_{stalk(\mathcal{IC}_Z)}$. We let

$$c_{\mathrm{IH}}(Z):=(-1)^{\dim Z}c_*(\mathrm{IC}_Z),$$

be the *intersection homology Chern class* of Z. (Note that $c_{\mathrm{IH}}(Z)$ is an element of the Chow group of Z, not of its intersection homology.) The sign is introduced in order to ensure that $c_{\mathrm{IH}}(Z)=c(TZ)\cap[Z]$ if Z is nonsingular[2].

Similarly, if Z has a *small* resolution of singularities $f:Y\to Z$, then $\mathcal{IC}_Z\simeq Rf_*\mathbb{C}_Y[\dim Y]$ (see e.g., [Sch03, Example 6.0.9, p. 400]) so that

$$c_{\mathrm{IH}}(Z)=f_*(c(TY)\cap[Y])$$

in this case. This class corresponds to the constructible function $(-1)^{\dim Z}f_*(1\!\!1_Y)$. We also consider the signed version, $\check{c}_{\mathrm{IH}}(Z)$. For Z complete, the degree of the zero-dimensional component of $c_{\mathrm{IH}}(Z)$ equals the 'intersection homology Euler characteristic' of Z, $\chi_{\mathrm{IH}}(Z)=(-1)^{\dim Z}\chi(Z,\mathrm{IC}_Z)$. By the functoriality of c_* and χ_{stalk}, $\chi_{\mathrm{IH}}(Z)$ agrees with the intersection

[2]More generally, $c_{\mathrm{IH}}(Z)=c_{\mathrm{SM}}(Z)$ if Z is a rational homology manifold (e.g., Z has only quotient singularities). In fact a quasi-isomorphism $\mathcal{IC}_Z\simeq\mathbb{C}_Z[\dim Z]$ characterizes a rational homology manifold Z [BM83, p. 34], see also [HTT08, Proposition 8.2.21].

homology Euler characteristic defined as alternating sum of ranks of intersection homology groups, as in e.g., [EGM18].

More generally, let $f : Y \to Z$ be a proper morphism. Using that χ_{stalk} commutes with Rf_* [Sch03, §2.3] one may define $\varphi := \chi_{stalk}(Rf_*\mathbb{C}_Y) = f_*(1\!\!1_Y)$; then $c_*(\varphi) = f_*c_*(1\!\!1_Y)$ by the functoriality of c_*.

Theorem 2.2 and Proposition 2.4 imply the following result.

Corollary 6.1. *Let Z be a closed subvariety of a smooth variety X with TX globally generated. Then $\check{s}_*(\mathrm{IC}_Z, X)$ is effective. If TX is trivial (e.g., X is an abelian variety), then $(-1)^{\dim Z}\check{c}_{IH}(Z) = \check{c}_{SM}(\mathrm{IC}_Z)$ is effective.*

In particular for X an abelian variety this implies that $(-1)^{\dim Z}\chi_{\mathrm{IH}}(Z) \geq 0$, recovering [EGM18, Theorem 5.3]. In case Z has a *small resolution* $f : Y \to Z$,

$$(-1)^{\dim Z}\check{c}_{\mathrm{IH}}(Z) = f_*(c(T^*Y) \cap [Y]) = (-1)^{\dim Y}\check{c}_*(f_*1\!\!1_Y)$$

and this class is effective by Corollary 6.1.

7. Effective characteristic cycles (II)

In this section we collect several instances of positive characteristic cycles. Some of these were already used in the previous sections, and they will be reproduced here for completeness.

Proposition 7.1. *Let X be a complex nonsingular variety, and let $Z \subseteq X$ be a closed reduced subscheme. If $\varphi \in \mathsf{F}(Z)$ is the constructible function in one of the cases listed below, then $\mathrm{CC}(\varphi)$ is an effective cycle.*

- (a) *$\varphi = (-1)^{\dim Z}\,\mathrm{Eu}_Z$ for Z a closed subvariety of X.*
- (b) *$\varphi = \nu_Z$ (Behrend's constructible function, see §5).*
- (c) *$\varphi = \chi_{stalk}(\mathcal{F})$ for a non-trivial perverse sheaf $\mathcal{F} \in \mathrm{Perv}(Z)$, e.g.:*
  - (c1) *$\varphi = \mathrm{IC}_Z$ for Z a closed subvariety of X, see §6.*
  - (c2) *$\varphi = (-1)^{\dim Z}1\!\!1_Z$ for Z pure-dimensional and smooth, or more generally a rational homology manifold.*
  - (c3) *$\varphi = (-1)^{\dim Z}1\!\!1_Z$ for Z pure-dimensional with only local complete intersection singularities (i.e., $Z \hookrightarrow X$ is a regular embedding).*
  - (c4) *$\varphi = (-1)^{\dim Y}f_*1\!\!1_Y$ for a proper surjective semi-small morphism of varieties $f : Y \to Z$, with Y a rational homology manifold and Z a closed subvariety of X. (See §8.1 for more on semi-small maps.)*
  - (c5) *$\varphi = (-1)^{\dim U}1\!\!1_U$, where $U \subseteq Z$ is a (not necessarily closed) subvariety, such that the inclusion $U \hookrightarrow Z$ is an affine morphism and $\mathbb{C}_U[\dim U]$ is a perverse sheaf on U (e.g., U is smooth, a rational homology manifold, or with only local complete intersection singularities).*

Proof. Part (a) follows from equation (2.2); part (b) from the discussion in §5; part(c) from Proposition 2.4; part (c1) is discussed in §6; part (c5) combines Theorem 1.1 and Remark 2.3. The remaining statements are proved as follows:

(c2): $\mathbb{C}_Z[\dim Z]$ is a perverse sheaf for Z pure-dimensional and smooth, with

$$\chi_{stalk}(\mathbb{C}_Z[\dim Z]) = (-1)^{\dim Z}1\!\!1_Z$$

by definition. The corresponding regular holonomic $\mathcal{D}$-module is just $\mathcal{O}_Z$. Similarly, $\mathbb{C}_Z[\dim Z]$ is a perverse sheaf for Z pure-dimensional and a rational homology manifold, since then $\mathcal{IC}_Z \simeq \mathbb{C}_Z[\dim Z]$; cf. [BM83, p. 34] or [HTT08, Proposition 8.2.21].

(c3): $\mathbb{C}_Z[\dim Z]$ is a perverse sheaf for Z pure-dimensional with only local complete intersection singularities [Sch03, Example 6.0.11, p. 404].

(c4): In the given hypotheses, the push-forward $Rf_*\mathbb{C}_Y[\dim Y]$ is a perverse sheaf; cf. [Sch03, Example 6.0.9, p. 400] or [HTT08, Definition 8.2.30]. Of course

$$\chi_{stalk}(Rf_*\mathbb{C}_Y[\dim Y]) = (-1)^{\dim Y} f_*1\!\!1_Y \ ,$$

since f is proper. □

To check that for a constructible function φ, the coefficients a_Y in the expansion $\mathrm{CC}(\varphi) = \sum_Y a_Y[T^*_Y X]$ are nonnegative, one can also use the following description of $\mathrm{CC}(\varphi)$ in terms of 'stratified Morse theory for constructible functions' from [Sch05, ST10] or [Sch03, §5.0.3]:

$$\mathrm{CC}(\varphi) = \sum_S (-1)^{\dim S} \cdot \chi(NMD(S), \varphi) \cdot [\overline{T^*_S X}]$$

if φ is constructible with respect to a complex algebraic Whitney stratification of Z with connected smooth strata S. Here $\chi(NMD(S), \varphi)$ is the Euler characteristic of a corresponding *normal Morse datum* $NMD(S)$ weighted by φ. Then $\mathrm{CC}(\varphi)$ is non-negative (resp., effective) if and only if

$$(-1)^{\dim S} \cdot \chi(NMD(S), \varphi) \geq 0$$

for all S (and, resp., $(-1)^{\dim S'} \cdot \chi(NMD(S'), \varphi) > 0$ for at least one stratum S'). If the complex of sheaves $\mathcal{F}$ is constructible with respect this complex algebraic Whitney stratification of Z, then one gets for $\varphi := \chi_{stalk}(\mathcal{F})$ and $x \in S$ [Sch03, (5.38) on p. 294]:

$$\chi(NMD(\mathcal{F}, x)[-\dim S]) = (-1)^{\dim S} \cdot \chi(NMD(S), \varphi) \ .$$

This leads to a direct proof of Proposition 2.4 in the case of perverse sheaves, without using $\mathcal{D}$-modules: $NMD(\mathcal{F}, x)[-\dim S]$ as above is concentrated in degree zero for all S if and only if $\mathcal{F}$ is a perverse sheaf [Sch03, Remark 6.0.4, p. 389]. This argument also shows that the condition that $\mathrm{CC}(\chi_{stalk}(\mathcal{F}))$ be *effective* is much weaker than the condition that $\mathcal{F}$ be perverse.

We end this section by listing some basic operations of constructible functions which preserve the property of having an effective characteristic cycle. These operations may be used to construct many more examples to which our Theorem 2.2 applies.

Proposition 7.2. *Let Z be a closed reduced subscheme of a nonsingular complex algebraic variety X, and assume that φ is a constructible function on Z with $\mathrm{CC}(\varphi)$ effective.*

(1) *Let Z' be a closed reduced subscheme of a nonsingular variety X', with $\mathrm{CC}(\varphi')$ effective. Then $\mathrm{CC}(\varphi \boxtimes \varphi')$ is also effective for the constructible function $\varphi \boxtimes \varphi'$ on $Z \times Z'$ defined by*
$$(\varphi \boxtimes \varphi')(z, z') := \varphi(z) \cdot \varphi'(z') \ .$$
(2) *Let $f : Z \to Z'$ be a* finite *morphism, i.e., f is proper with finite fibers, with Z' a closed reduced subscheme of a nonsingular complex algebraic variety X'. Then $f_*(\varphi)$ is a constructible function on Z' with $\mathrm{CC}(f_*(\varphi))$ effective. Here*
$$f_*(\varphi)(z') := \sum_{z \in f^{-1}(z')} \varphi(z) \ .$$
(3) *Let $f : X' \to X$ be a morphism of nonsingular complex algebraic varieties such that $f : Z' := f^{-1}(Z) \to Z$ is a* smooth *morphism of relative dimension d. Then $(-1)^d f^*(\varphi) = (-1)^d \varphi \circ f$ is a constructible function on Z' with effective characteristic cycle.*

(4) *Let $f : X \to \mathbb{C}$ be a morphism and let D be the hypersurface $\{f = 0\}$. Denote by $\psi_f : \mathsf{F}(Z) \to \mathsf{F}(Z \cap D)$ the corresponding specialization of constructible functions* [Sch05, §2.4.7]. *Here*

$$\psi_f(\varphi)(x) := \chi(M_{f|Z,x}, \varphi) ,$$

with $M_{f|Z,x}$ a local Milnor fiber *of $f|Z$ at $x \in Z \cap D$. Then $\mathrm{CC}(-\psi_f(\varphi))$ is* non-negative. *It is* effective *in case $\varphi \neq 0$ has a presentation as in* (2.4) *and at least one Y with $a_Y > 0$ is not contained in D.*

(5) *Let $f : X' \to X$ be a morphism of nonsingular complex algebraic varieties such that f is* non-characteristic *with respect to the support* $\mathrm{supp}(\mathrm{CC}(\varphi))$ *of the characteristic cycle of φ (e.g., f is* transversal *to all strata S of a complex algebraic Whitney stratification of Z for which φ is constructible). Let $d := \dim X' - \dim X$. Then $(-1)^d f^*(\varphi) = (-1)^d \varphi \circ f$ is a constructible function on $Z' := f^{-1}(Z)$ with effective characteristic cycle.*

Proof. These results can be deduced from the following facts:

(1) $\mathrm{CC}(\varphi \boxtimes \varphi') = \mathrm{CC}(\varphi) \boxtimes \mathrm{CC}(\varphi')$, which follows from $\mathrm{Eu}_Y \boxtimes \mathrm{Eu}_{Y'} = \mathrm{Eu}_{Y \times Y'}$ [Mac74], or from stratified Morse theory for constructible functions or sheaves [Sch03, (5.6) on p. 277]:

$$\chi(NMD(S), \varphi) \cdot \chi(NMD(S'), \varphi') = \chi(NMD(S \times S'), \varphi \boxtimes \varphi') .$$

(2) Using the graph embedding, we can assume that the finite map $f : Z \to Z'$ is induced from a submersion $f : X \to X'$ of ambient nonsingular varieties. Consider the induced correspondence of cotangent bundles:

$$T^*X \xleftarrow{\;df\;} f^*T^*X' \xrightarrow{\;\tau\;} T^*X' .$$

Here df is a closed embedding (since f is a submersion), and

$$\tau : df^{-1}(\mathrm{supp}(\mathrm{CC}(\varphi))) \to T^*X'|Z'$$

is finite, since $f : Z \to Z'$ is finite. Now $\tau(df^{-1}\,\mathrm{supp}(\mathrm{CC}(\varphi)))$ is known to be contained in a conic Lagrangian subset of $T^*X'|Z'$ (e.g., coming from a stratification of f [Sch03, (4.16) on p. 249]). Therefore its dimension is bounded from above by $\dim X'$. Then also the dimension of $df^{-1}(\mathrm{supp}(\mathrm{CC}(\varphi)))$ is bounded from above by $\dim X'$ by the finiteness of τ, so that

$$df^*(\mathrm{CC}(\varphi)) = \mathrm{CC}(\varphi) \cap [f^*T^*X']$$

is a *proper intersection.* But then $\tau_*(df^*(\mathrm{CC}(\varphi)))$ is an *effective* cycle on $T^*X'|Z'$, and [Sch05, §4.6]:

$$\tau_*(df^*(\mathrm{CC}(\varphi))) = \mathrm{CC}(f_*(\varphi)) .$$

(3) This follows from $f^*\,\mathrm{Eu}_Y = \mathrm{Eu}_{f^{-1}(Y)}$ for Y a closed subvariety in Z. This can be checked locally, e.g., for $f : Z \times Y' \to Z$ the projection along a smooth factor Y', with $f^*\,\mathrm{Eu}_Y = \mathrm{Eu}_Y \boxtimes 1\!\!1_{Y'} = \mathrm{Eu}_Y \boxtimes \mathrm{Eu}_{Y'}$.

(4) Again it is enough to consider $\check{\mathrm{E}}\mathrm{u}_Y := (-1)^{\dim Y}\,\mathrm{Eu}_Y$ for some subvariety Y of Z. If $Y \subseteq \{f = 0\}$, then $\psi_f(\check{\mathrm{E}}\mathrm{u}_Y) = 0$ by definition. So we can assume $Y \not\subseteq \{f = 0\}$. Then $\mathrm{CC}(-\psi_f(\check{\mathrm{E}}\mathrm{u}_Y))$ is by [Sab85, Theorem 4.3] the (Lagrangian) specialization of the relative conormal space $[T^*_{f|Z}X]$ along the hypersurface $\{f = 0\}$, so that it is also effective.

(5) Consider again the induced correspondence of cotangent bundles:

$$T^*X' \xleftarrow{\;df\;} f^*T^*X \xrightarrow{\;\tau\;} T^*X .$$

Then by definition, f is *non-characteristic* with respect to the support $\mathrm{supp}(\mathrm{CC}(\varphi))$ of the characteristic cycle of φ if and only if

$$df : \tau^{-1}(\mathrm{supp}(\mathrm{CC}(\varphi))) \to T^*X'$$

is *proper* and therefore finite, cf. [Sch17, Lemma 3.2] or [Sch03, Lemma 4.3.1, p. 255]. If f is non-characteristic, then

$$\mathrm{CC}((-1)^d f^*(\varphi)) = df_*(\tau^*(\mathrm{CC}(\varphi)))$$

by [Sch17, Theorem 3.3], and this cycle is *effective* if $\mathrm{CC}(\varphi)$ is effective. Indeed the proof of [Sch17, Theorem 3.3] is done in two steps: first for a submersion, where our claim follows from the case (3) above; then the case of a closed embedding of a nonsingular subvariety is (locally) reduced by induction to the case of a hypersurface of codimension one (locally) given by an equation $\{f = 0\}$. Here it is deduced from case (4) above, with $Y \not\subseteq \{f = 0\}$ by the 'non-characteristic' assumption if $[T^*_Y X]$ appears with positive multiplicity in $\mathrm{CC}(\varphi)$. □

8. Further applications

In this final section we explain further applications of Theorem 2.2 and Proposition 2.4 via the theory of perverse sheaves.

8.1. **Semi-small maps.** Recall that a morphism $f : Y \to Z$ is called *semi-small* if for all $i > 0$,

$$\dim\{z \in Z|\ \dim f^{-1}(z) \geq i\} \leq \dim Z - 2i;$$

the morphism f is *small* if in addition all inequalities are strict for $i > 0$. See [BM83, p. 30], [Sch03, Example 6.0.9, p. 400], or [HTT08, Definition 8.2.29].

Proposition 8.1. *Let $f : Y \to Z$ be a proper surjective semi-small morphism of varieties, with Y a rational homology manifold and Z a closed subvariety of a smooth variety X with TX globally generated. Then $(-1)^{\dim Y}\check{s}_*(f_*1\!\!1_Y, X)$ is effective.*

In particular, if TX is trivial then $(-1)^{\dim Y}\check{c}_*(f_*1\!\!1_Y) = (-1)^{\dim Y} f_*\check{c}_{SM}(Y)$ is effective. If moreover X is complete (i.e., an abelian variety) then $(-1)^{\dim Y}\chi(Y) \geq 0$.

The simplest example of a proper semi-small map $f : Y \to Z := f(Y) \subseteq X$ is a closed embedding. A smooth projective variety Y has a proper semi-small morphism (onto its image) into an abelian variety X if and only if its Albanese morphism $alb_X : X \to Alb(X)$ is semi-small (onto its image) [LMW19, Remark 1.3]. The corresponding signed Euler characteristic bound $(-1)^{\dim Y}\chi(Y) \geq 0$ is further refined in [PS13, Corollary 5.2]. As an example, if C is a smooth curve of genus $g \geq 3$ and X is its Jacobian, then the induced Abel-Jacobi map $C^d \to C^{(d)} \to X$ (with $C^{(d)}$ the corresponding symmetric product) is semi-small (onto its image) for $1 \leq d \leq g-1$ [Wei06, Corollary 12].

Proof of Proposition 8.1. By the given hypotheses, the push-forward $Rf_*\mathbb{C}_Y[\dim Y]$ is a perverse sheaf; cf. [Sch03, Example 6.0.9, p. 400] or [HTT08, Definition 8.2.30]. Further

$$\chi_{stalk}(Rf_*\mathbb{C}_Y[\dim Y]) = (-1)^{\dim Y} f_*1\!\!1_Y\ ,$$

since f is proper. The statement follows then from Proposition 2.4. □

8.2. **Regular embeddings and Milnor classes.** For this application, assume that $Z \subseteq X$ is a regular embedding, as in [Ful84, Appendix B.7]. For instance, Z could be a smooth closed subvariety, a hypersurface, or a local complete intersection in X.

Proposition 8.2. *Let X be a complex nonsingular variety such that TX is globally generated, and let $Z \subseteq X$ be a regular embedding. Then $(-1)^{\dim Z}\check{s}_{SM}(Z, X)$ is effective. If TX is trivial, then $(-1)^{\dim Z}\check{c}_{SM}(Z)$ is effective.*

Proof. The hypothesis imply that $\mathbb{C}_Z[\dim Z]$ is a perverse sheaf, by Proposition 7.1. Then the claim follows from Theorem 2.2 and Proposition 2.4. □

In particular for X an abelian variety this implies that $(-1)^{\dim Z}\chi(Z) \geq 0$, recovering [EGM18, Theorem 5.4].

If $Z \subseteq X$ is a closed embedding, with X smooth, then the class

$$c_{\mathrm{F}}(Z) := c(TX) \cap s(Z, X) \in A_*(Z)$$

is the *Chern-Fulton class* of Z; this is another intrinsic Chern class of Z [Ful84, Example 4.2.6]. If Z is a local complete intersection in X, then the normal cone $C_Z X = N_Z X$ is a vector bundle, and we have

$$c_{\mathrm{F}}(Z) = c(TX) \cap \big(c(N_Z X)^{-1} \cap [Z]\big) \in A_*(Z)\,;$$

in this case, this class is also called the *virtual Chern class* of Z. If Z is smooth, $N_Z X$ is the usual normal bundle, so that $c_{\mathrm{F}}(Z) = c(TZ) \cap [Z] = c_{\mathrm{SM}}(Z)$. In general, for singular Z, these classes can be different, and their difference[3]

$$\mathrm{Mi}(Z) := (-1)^{\dim Z}\left(c_{\mathrm{F}}(Z) - c_{\mathrm{SM}}(Z)\right) \in A_*(Z)$$

is called the *Milnor class* of Z. Let $\mathrm{SMi}(Z, X)$ be the corresponding *Segre-Milnor class*

$$\begin{aligned}\mathrm{SMi}(Z, X) &:= c(TX)^{-1} \cap \mathrm{Mi}(Z) = (-1)^{\dim Z}\big(c(N_Z X)^{-1} \cap [Z] - s_{\mathrm{SM}}(Z, X)\big)\\ &= (-1)^{\dim Z}\left(s(Z, X) - s_{\mathrm{SM}}(Z, X)\right).\end{aligned}$$

As before consider the associated signed classes $\check{\mathrm{M}}\mathrm{i}(Z)$ and

$$\mathrm{S}\check{\mathrm{M}}\mathrm{i}(Z, X) := c(T^*X|_Z)^{-1} \cap \check{\mathrm{M}}\mathrm{i}(Z)\,.$$

Assume now that X is projective, with very ample line bundle L, and that

$$Z = \{s_j = 0 \mid j = 1, \ldots, r\}$$

is the global complete intersection of codimension $r > 0$ defined by sections $s_j \in \Gamma(X, L^{\otimes a_j})$ for suitable positive integers a_j. Then [MSS13, Theorem 1 and Corollary 1] implies that $\mathrm{Mi}(Z) = c_*(\varphi)$ for a constructible function φ associated to a perverse sheaf supported on the singular locus Z_{sing} of Z.[4]

Applying Proposition 2.4 we obtain:

Corollary 8.3. *Let X be a smooth projective variety with TX globally generated, and let the subvariety $Z = \{s_j = 0 \mid j = 1, \ldots, r\} \subseteq X$ be a global complete intersection as described above. Then $\mathrm{S}\check{\mathrm{M}}\mathrm{i}(Z, X) \in A_*(Z)$ is non-negative. If X is an Abelian variety, then $\check{\mathrm{M}}\mathrm{i}(Z) \in A_*(Z)$ is non-negative.*

As an illustration, if Z has only isolated singularities, then one can consider the specialization $y = -1$ from [MSS13, Corollary 2] to deduce:

$$\varphi = \sum_{z \in Z_{sing}} \mu_z \cdot 1\!\!1_z \quad \text{so that} \quad \mathrm{Mi}(Z) = c_*(\varphi) = \sum_{z \in Z_{sing}} \mu_z \cdot [z]\,,$$

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[3] There are different sign conventions in the literature. Here we adopt the convention used in the original definition of Milnor classes, [PP01].

[4] More precisely, [MSS13] studies the *Hirzebruch-Milnor class* $M_y(Z)$ of Z, measuring the difference between the *virtual* and the *motivic Hirzebruch class* $T^{vir}_{y*}(Z)$ and $T_{y*}(Z)$ of Z. Specializing to $y = -1$ shows ([MSS13, Corollary 1]) that $\mathrm{Mi}(Z) = c_*(\varphi)$ for φ the constructible function associated with the underlying perverse sheaf of a mixed Hodge module $\mathcal{M}(s'_1, \ldots, s'_r)$ determined by the sections $s_1, \ldots, s_r$.

with $\mu_z > 0$ the corresponding *Milnor number* of the isolated complete interesction singularity $z \in Z_{sing}$. Here the last formula for the Milnor class $\mathrm{Mi}(Z)$ is due to [SS98, Suw97]. Therefore in this case, $\mathrm{SMi}(Z, X) = \check{\mathrm{SMi}}(Z) = \mathrm{Mi}(Z) = \check{\mathrm{Mi}}(Z)$.

8.3. **Semi-abelian varieties.** Recall that a *semi-abelian* variety G is a group scheme given as an extension

$$0 \to \mathbb{T} \to G \to A \to 0$$

of an abelian variety A by a torus $\mathbb{T} \simeq (\mathbb{C}^*)^n$ ($n \geq 0$), so that

$$G \simeq L_1^0 \times_A \cdots \times_A L_n^0 \to A$$

for some degree-zero line bundles L_i over A, with L_i^0 the open complement of the zero-section in the total space $L_i \to A$ (cf. [FK00, (5.5)] or [LMW18b, p. 12]). Then the projection $p : G \to A$ has the following important stability property:

> (stab): The group homomorphism $p_* : \mathsf{F}(X') \to \mathsf{F}(X)$ induced by the morphism $p : X' \to X$ maps the image $\mathrm{im}(\chi_{stalk} : \mathrm{Perv}(X') \to \mathsf{F}(X'))$ to $\mathrm{im}(\chi_{stalk} : \mathrm{Perv}(X) \to \mathsf{F}(X))$.

Note that the constant morphism $p : X' \to pt$ satisfies the property (stab) if and only if X' has the following Euler characteristic property:

$$\chi(X', \mathcal{F}) \geq 0 \quad \text{for all perverse sheaves } \mathcal{F} \in \mathrm{Perv}(X') \,, \tag{8.1}$$

since $\mathbb{Z}_{\geq 0} = \ im(\chi_{stalk} : \mathrm{Perv}(pt) \to \mathsf{F}(pt)) \subseteq \mathsf{F}(pt) = \mathbb{Z}$. In particular an abelian variety satisfies (8.1) by Theorem 2.2 and Proposition 2.4.

Proposition 8.4. *The class of morphisms satisfying property (stab) is closed under composition. Further, the following morphisms satisfy property (stab):*

(a) *$p : X' \to X$ is an affine morphism with all fibers zero-dimensional (e.g., a finite morphism or an affine inclusion).*
(b) *$p : X' \to X$ is an affine smooth morphism of relative dimension one, with all fibers non-empty, connected and of the same non-positive Euler characteristic $\chi_p \leq 0$.*

Example 8.5. The following morphisms $p : X' \to X$ are affine smooth morphisms of relative dimension one, with all fibers non-empty, connected and constant non-positive Euler characteristic $\chi_p \leq 0$:

(1) $p : L^0 \to X$ is the open complement of the zero-section in the total space of a line bundle $L \to X$.
(2) p is the projection $p : X \times C \to X$ of a product with a smooth non-empty, connected affine curve C with non-positive Euler characteristic $\chi(C) \leq 0$.
(3) More generally, let $p : X' \to X$ be an *elementary fibration* in the sense of M. Artin (cf. [AB01, Definition 1.1, p. 105]), i.e., such that it can be factorized as an open inclusion $j : X' \to \overline{X'}$ followed by a projective smooth morphism of relative dimension one $\bar{p} : \overline{X'} \to X$ with irreducible (or connected) fibers, such that the induced map of the reduced complement $\bar{p} : Z := \overline{X'} \setminus X' \to X$ is a surjective étale covering. Then $p : X' \to X$ is an affine morphism [AB01, Lemma 1.1.2, p. 106]. If X is connected, then the genus $g \geq 0$ of the fibers of $\bar{p}$ and the degree $n \geq 1$ of the covering $\bar{p} : Z \to X$ are constant, so that all fibers of p have the same Euler characteristic $\chi_p = 2 - 2g - n$. The final assumption $\chi_p = 2 - 2g - n \leq 0$ just means $(g, n) \neq (0, 1)$, i.e., only the affine line $\mathbb{A}^1(\mathbb{C})$ (with $\chi(\mathbb{A}^1(\mathbb{C})) = 1$) is not allowed as a fiber of p.

The stabilization property (stab) is preserved by compositions, so that the projection

$$G \simeq L_1^0 \times_A \cdots \times_A L_n^0 \to L_2^0 \times_A \cdots \times_A L_n^0 \to \cdots \to L_n^0 \to A$$

for a semi-abelian variety has the property (stab) by the first example above. Similarly for the composition of 'elementary fibrations'

$$X'_n \xrightarrow{p_n} X'_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} X'_1 \xrightarrow{p_1} X$$

over a connected base X, with all fiber Euler characteristics $\chi_{p_i} \leq 0$.

Corollary 8.6. *Assume that the morphism $p : X' \to X$ has the property (stab), with X a smooth variety such that TX is globally generated. Let $\mathcal{F} \in \mathrm{Perv}(X')$ be a perverse sheaf on X', with $\varphi := \chi_{stalk}(\mathcal{F})$. Then $\check{s}_*(p_*(\varphi))$ is non-negative. If TX is trivial, then $\check{c}_*(p_*(\varphi))$ is non-negative. In particular $\chi(X', \mathcal{F}) = \chi(X', \varphi) = \chi(X, p_*(\varphi)) \geq 0$ if X is an abelian variety.*

The Euler characteristic property (8.1) for a semi-abelian variety G is due to [FK00, Corollary 1.4] (but their proof uses results about characteristic cycles on suitable compactifications and does not extend to the more general context considered above). The Euler characteristic property (8.1) for an algebraic torus $\mathbb{T} \simeq (\mathbb{C}^*)^n$ is due to [GL96, Corollary 3.4.4] in the ℓ-adic context as an application of the *generic vanishing theorem*:

$$H^i(\mathbb{T}, \mathcal{F} \otimes L) = 0 \quad \text{for } i \neq 0$$

for a given perverse sheaf $\mathcal{F} \in \mathrm{Perv}(\mathbb{T})$ and a *generic* rank one local system L on $\mathbb{T}$. See also [LMW18a, Theorem 1.2] resp., [LMW19, Theorem 1.1] and [LMW18b, Theorem 1.2] for the *generic vanishing theorem* for complex tori, resp., semi-abelian varieties and algebraically constructible perverse sheaves in the classical topology (as used in this paper). The following proof of Proposition 8.4(b) is an adaption to the language of constructible functions of techniques used in these references for their proofs of the generic vanishing theorem. In this way we can prove the Euler characteristic property (8.1) in a much more general context, e.g., for any product of connected smooth affine curves different from the affine line $\mathbb{A}^1(\mathbb{C})$ (instead of complex tori).

We close the paper with the proof of Proposition 8.4.

Proof of Proposition 8.4. Note that $\chi_{stalk} : \mathrm{Perv}(X) \to \mathsf{F}(X)$ is *additive* in the sense that $\chi_{stalk}(\mathcal{F}) = \chi_{stalk}(\mathcal{F}') + \chi_{stalk}(\mathcal{F}'')$ for any short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

in the abelian category $\mathrm{Perv}(X)$. In particular $\chi_{stalk}(\mathcal{F}' \oplus \mathcal{F}'') = \chi_{stalk}(\mathcal{F}') + \chi_{stalk}(\mathcal{F}')$ and the zero-sheaf is mapped to the zero-function. Therefore χ_{stalk} induces a map from the corresponding Grothendieck group $\chi_{stalk} : K_0(\mathrm{Perv}(X)) \to \mathsf{F}(X)$, and

$$\mathrm{im}(\chi_{stalk} : \mathrm{Perv}(X) \to \mathsf{F}(X))$$

is a *submonoid* of the abelian group $\mathsf{F}(X)$. Moreover, χ_{stalk} commutes with both pushforwards $Rf_!, Rf_*$ for a morphism $f : X' \to X$ [Sch03, §2.3], with

$$f_! = f_* : K_0(\mathrm{Perv}(X')) \to K_0(\mathrm{Perv}(X)) \quad \text{and} \quad f_! = f_* : \mathsf{F}(X') \to \mathsf{F}(X)$$

in this complex algebraic context [Sch03, (6.41), (6.42), p. 413]. In particular, the pushforward for constructible functions is functorial with $\chi(X', \mathcal{F}) = \chi(X', \varphi) = \chi(X, f_*(\varphi))$ for $\varphi = \chi_{stalk}(\mathcal{F})$ and $\mathcal{F} \in \mathrm{Perv}(X')$. Also, this shows that the property (stab) is preserved by compositions, as claimed in Proposition 8.4. The other parts of Proposition 8.4 are proved as follows.

(a) An affine morphism $p : X' \to X$ with zero-dimensional fibers induces *exact* functors $Rp_!, Rp_* : \mathrm{Perv}(X') \to \mathrm{Perv}(X)$ [Sch03, Corollary 6.0.5, p. 397 and Theorem 6.0.4, p. 409].

(b) Let $p : X' \to X$ be an affine smooth morphism of relative dimension one, with all fibers non-empty, connected and with the same non-positive Euler characteristic $\chi_p \leq 0$. Then the shifted pullback $p^*[1] : \mathrm{Perv}(X) \to \mathrm{Perv}(X')$ is *exact*, since p is smooth of relative dimension one [Sch03, Lemma 6.0.3, p. 386]. Note that Rp_* is not necessarily exact for the perverse t-structure. Nevertheless, since p is affine of relative dimension one, the perverse cohomology sheaves ${}^m\mathcal{H}^i(Rp_*\mathcal{F})$ vanish for $i \neq -1, 0$ for every perverse sheaf $\mathcal{F} \in \mathrm{Perv}(X')$ [Sch03, Corollary 6.0.5, p. 397 and Theorem 6.0.4, p. 409]. Moreover, the abelian category $\mathrm{Perv}(X')$ is a *length category,* i.e., it is noetherian and artinian, so that $\mathcal{F} \in \mathrm{Perv}(X')$ is a finite iterated extension of *simple* perverse sheaves on X' [BBD82, Theorem 4.3.1, p. 112]. By the additivity of χ_{stalk}, it is enough to consider a *simple* perverse sheaf $\mathcal{F}$ on X'. If ${}^m\mathcal{H}^{-1}(Rp_*\mathcal{F}) = 0$, then Rp_*F is also perverse, with

$$p_*(\chi_{stalk}(\mathcal{F})) = \chi_{stalk}(Rp_*\mathcal{F}) \in \mathrm{im}(\chi_{stalk} : \mathrm{Perv}(X) \to \mathsf{F}(X)) \, .$$

Assume now that ${}^m\mathcal{H}^{-1}(Rp_*\mathcal{F}) \neq 0 \in \mathrm{Perv}(X)$. Then also

$$p^* \left({}^m\mathcal{H}^{-1}(Rp_*\mathcal{F})\right)[1] \neq 0 \in \mathrm{Perv}(X')$$

by the surjectivity of p. Since the fibers of p are non-empty and connected, one gets by [BBD82, Corollary 4.2.6.2, p. 111] a *monomorphism*

$$0 \to p^* \left({}^m\mathcal{H}^{-1}(Rp_*\mathcal{F})\right)[1] \to \mathcal{F} \, .$$

This has to be an isomorphism $p^* \left({}^m\mathcal{H}^{-1}(Rp_*\mathcal{F})\right)[1] \simeq \mathcal{F}$, since $\mathcal{F}$ is simple. As mentioned before, $Rp_!$ and Rp_* induce the same constructible function under χ_{stalk}, and the stalk of $Rp_!$ calculates the compactly supported cohomology in the corresponding fiber. But

$$\chi_{stalk}(\mathcal{F}) = \chi_{stalk}\left(p^* \left({}^m\mathcal{H}^{-1}(Rp_*\mathcal{F})\right)[1]\right) = -p^*(\varphi')$$

is constant along the fibers of p, with

$$\varphi' := \chi_{stalk}({}^m\mathcal{H}^{-1}(Rp_*\mathcal{F})) \in \mathrm{im}(\chi_{stalk} : \mathrm{Perv}(X) \to \mathsf{F}(X)) \, .$$

Finally, all fibers of p have by assumption the same non-positive Euler characteristic $\chi_p \leq 0$, so that

$$p_*(\chi_{stalk}(\mathcal{F})) = -p_*(p^*(\varphi')) = -\chi_p \cdot \varphi' \in \mathrm{im}(\chi_{stalk} : \mathrm{Perv}(X) \to \mathsf{F}(X)) \, ,$$

since $\mathrm{im}(\chi_{stalk} : \mathrm{Perv}(X) \to \mathsf{F}(X))$ is a submonoid of the abelian group $\mathsf{F}(X)$. □

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Mathematics Department, Florida State University, Tallahassee FL 32306, USA
E-mail address: `aluffi@math.fsu.edu`

Department of Mathematics, Virginia Tech University, Blacksburg VA 24061, USA
E-mail address: `lmihalce@vt.edu`

Mathematisches Institut, Universität Münster, Germany
E-mail address: `jschuerm@uni-muenster.de`

Department of Mathematics, University of Toronto, Toronto ON M5S 1A1, Canada
E-mail address: `changjiansu@gmail.com`