MOTIVIC CHERN CLASSES OF SCHUBERT CELLS, HECKE ALGEBRAS, AND APPLICATIONS TO CASSELMAN'S PROBLEM

PAOLO ALUFFI, LEONARDO C. MIHALCEA, JÖRG SCHÜRMANN, AND CHANGJIAN SU

Abstract. Motivic Chern classes are elements in the K theory of an algebraic variety X depending on an extra parameter y. They are determined by functoriality and a normalization property. In this paper we calculate the motivic Chern classes of Schubert cells in the (equivariant) K-theory flag manifolds G/B. The calculation is recursive starting from the class of a point, and using the Demazure-Lusztig operators in the Hecke algebra. The resulting classes are conjectured to satisfy a positivity property. We use the recursions to give a new proof that they are equivalent to certain K-theoretic stable envelopes recently defined by Okounkov and collaborators, thus recovering results of Fehér, Rimányi and Weber. The Hecke algebra action on the K-theory of the dual flag manifold matches the Hecke action on the Iwahori invariants of the principal series representation associated to an unramified character for a group over a nonarchimedean local field. This gives a correspondence identifying the Poincaré dual version of the motivic Chern class to the standard basis in the Iwahori invariants, and the fixed point basis to Casselman's basis. We apply this to prove two conjectures of Bump, Nakasuji and Naruse concerning factorizations, and holomorphic properties, of the coefficients in the transition matrix between the standard and the Casselman's basis.

1. Introduction

Let X be a complex algebraic variety, and let $G_0(var/X)$ be the (relative) Grothendieck group of varieties over X. It consists of classes of morphisms $[f : Z \to X]$ modulo the scissor relations; cf. [Loo02, Bit04] and §4 below. Brasselet, Schürmann and Yokura [BSY10] defined the *motivic Chern transformation* $MC_y : G_0(var/X) \to K(X)[y]$ with values in the K-theory group of coherent sheaves in X to which one adjoins a formal variable y. The transformation MC_y is a group homomorphism, it is functorial with respect to proper push-forwards, and if X is smooth it satisfies the normalization condition

$$MC_y[id_X : X \to X] = \sum [\wedge^j T^*(X)] y^j.$$

Here $[\wedge^j T^*(X)]$ is the K-theory class of the bundle of degree j differential forms on X. If $Z \subset X$ is a constructible subset, we denote by $MC_y(Z) := MC_y(Z \hookrightarrow X) \in K(X)[y]$ the motivic Chern class of Z. Because MC_y is a group homomorphism, it follows that if $X = \bigcup Z_i$ is a disjoint union of constructible subsets, then $MC_y(X) = \sum MC_y(Z_i)$. As explained in [BSY10], the motivic Chern class $MC_y(Z)$ is related by a Hirzebruch-Riemann-Roch type statement to the Chern-Schwartz-MacPherson class $c_{\mathrm{SM}}(Z)$ in the homology of X, a class whose existence was conjectured by Deligne and Grothedieck and proved by MacPherson [Mac74]. Both the motivic and the CSM classes give a functorial way to attach

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K-theory, respectively cohomology classes, to locally closed subvarieties, and both satisfy the usual motivic relations. There is also an equivariant version of the motivic Chern class transformation, which uses equivariant varieties and morphisms, and has values in the suitable equivariant K-theory group. Its definition was given in [FRW18], based on [BSY10].

Let G be a complex, simple, Lie group, and B a Borel subgroup. The main purpose of this paper is to build the computational foundations for the study of the motivic Chern classes of Schubert cells in a generalized flag manifold G/B. By functoriality, this determines the motivic classes of Schubert cells in any flag manifold G/P, where P is a parabolic subgroup. Based on the previous behavior of the CSM classes of Schubert cells [AM16, RV, AMSS17] it was expected that the motivic classes will be closely related to objects which appear in geometric representation theory. We will show in this note that the motivic Chern classes of Schubert cells are recursively determined by the Demazure-Lusztig operators which appear in early works of Lusztig on Hecke algebras [Lus85]. Further, the motivic classes of Schubert cells are equivalent (in a precise sense) to the K-theoretic stable envelopes defined by Okounkov and collaborators in [Oko15, AO, OS16]. This coincidence was proved recently by Fehér, Rimányi and Weber [FRW18]; our approach, based on comparing the Demazure-Lusztig recursions to the recursions for the stable envelopes found by Su, Zhao and Zhong in [SZZ17] gives another proof of their result. Via this equivalence, the motivic Chern classes can be considered as natural analogues of the Schubert classes in the K theory of the cotangent bundle of G/B.

As in the authors' previous work [AMSS17], combining the connections to Hecke algebras and to K-theoretic stable envelopes yields certain remarkable identities among (Poincaré duals of) motivic Chern classes. We use these identities to prove two conjectures of Bump, Nakasuji and Naruse [BN11, BN17, NN15] about the coefficients in the transition matrix between the Casselman's basis and the standard basis in the Iwahori-invariant space of the principal series representation for an unramified character for a group over a non archimedean local field.

We present next a more precise description of our results.

1.1. **Statement of results.** Recall that G is a complex, simple, Lie group, and fix B, B^- a pair opposite Borel subgroups of G. Denote by $T := B \cap B^-$ the maximal torus, by $W := N_G(T)/T$ the Weyl group, and by $X := G/B$ the (generalized) flag variety. For each Weyl group element $w \in W$ consider the Schubert cell $X(w)^\circ := BwB/B$, a subvariety of (complex) dimension $\ell(w)$. The opposite Schubert cell $Y(w)^\circ := B^-wB/B$ has complex codimension $\ell(w)$. The closures $X(w)$ and $Y(w)$ of these cells are the Schubert varieties. Let $\mathcal{O}_w$, respectively $\mathcal{O}^w$ be the (K-theoretic) Schubert classes associated to the structure sheaves of $X(w)$, respectively $Y(w)$. The equivariant K-theory ring of X, denoted by $K_T(X)$, is an algebra over $K_T(pt) = R(T)$ - the representation ring of T - and it has a $R(T)$-basis given by the Schubert classes $\mathcal{O}_w$ (or $\mathcal{O}^w$), where w varies in the Weyl group W.

If E is an equivariant vector bundle over X, we denote by $[E]$ its class in $K_T(X)$, and by $\lambda_y(E)$ the class

$$\lambda_y(E) = \sum [\wedge^i E] y^i \in K_T(X)[y].$$

For a T-stable subvariety $\Omega \subset X$ recall the notation

$$MC_y(\Omega) := MC_y[\Omega \hookrightarrow X] \in K_T(X)[y].$$

Our first main result is a recursive formula to calculate $MC_y(X(w)^\circ)$, the (equivariant) motivic Chern class of the Schubert cell. For each simple positive root α_i, consider the

Demazure operator $\partial_i : K_T(X) \to K_T(X)$; this is a $K_T(pt)$-linear endomorphism. The Demazure-Lusztig (DL) operators are defined by

$$\mathcal{T}_i := \lambda_y(\mathcal{L}_{\alpha_i})\partial_i - id; \quad \mathcal{T}_i^\vee := \partial_i \lambda_y(\mathcal{L}_{\alpha_i}) - id,$$

where $\mathcal{L}_{\alpha_i} = G \times^B \mathbb{C}_{\alpha_i}$ is the line bundle whose fibre over $1.B$ has weight α_i. The operator $\mathcal{T}_i^\vee$ appeared classically in Lusztig's study of Hecke algebras [Lus85], and $\mathcal{T}_i$ appeared recently in related works [LLL17, BBL15]. The two operators are adjoint to each other via the K-theoretic Poincaré pairing; see section 3.2 below. Our first main result is the following (cf. Theorem 4.5).

Theorem 1.1. *Let $w \in W$ and let s_i be a simple reflection such that $\ell(ws_i) > \ell(w)$. Then*

$$MC_y(X(ws_i)^\circ) = \mathcal{T}_i(MC_y(X(w)^\circ)).$$

Using the (equivariant) K-theoretic Chevalley formula [FL94, PR99, LP07] to multiply by classes of line bundles, the DL operators give a recursive formula to calculate the motivic Chern classes, starting from the class of a point.

Theorem 1.1 generalizes the similar result from [AM16] where it was proved that the CSM classes of Schubert cells are recursively determined by operators in the degenerate Hecke algebra; see [Gin98]. In fact, the proof of Theorem 1.1 generalizes from cohomology to K-theory the proof from [AM16]. It relies on the calculations of push-forwards of classes from the Bott-Samelson desingularizations of Schubert varieties.

To illustrate the results, we list below the non-equivariant motivic Chern classes of Schubert cells in $\mathrm{Fl}(3) := \mathrm{SL}_3(\mathbb{C})/B$, the manifold parametrizing flags in $\mathbb{C}^3$. In this case, the Weyl group is the symmetric group S_3, generated by simple reflections s_1 and s_2, and $w_0 = s_1 s_2 s_1$ is the longest element.

$$\begin{aligned}
MC_y(X(id)) =& \mathcal{O}_{id};\\
MC_y(X(s_1)^\circ) =& (1+y)\mathcal{O}_{s_1} - (1+2y)\mathcal{O}_{id};\\
MC_y(X(s_2)^\circ) =& (1+y)\mathcal{O}_{s_2} - (1+2y)\mathcal{O}_{id};\\
MC_y(X(s_1s_2)^\circ) =& (1+y)^2\mathcal{O}_{s_1s_2} - (1+y)(1+2y)\mathcal{O}_{s_1} - (1+y)(1+3y)\mathcal{O}_{s_2} + (5y^2+5y+1)\mathcal{O}_{id};\\
MC_y(X(s_2s_1)^\circ) =& (1+y)^2\mathcal{O}_{s_2s_1} - (1+y)(1+2y)\mathcal{O}_{s_2} - (1+y)(1+3y)\mathcal{O}_{s_1} + (5y^2+5y+1)\mathcal{O}_{id};\\
MC_y(X(w_0)^\circ) =& (1+y)^3\mathcal{O}_{w_0} - (1+y)^2(1+2y)(\mathcal{O}_{s_1s_2} + \mathcal{O}_{s_2s_1}) +\\
& (1+y)(5y^2+4y+1)(\mathcal{O}_{s_1} + \mathcal{O}_{s_2}) - (8y^3+11y^2+5y+1)\mathcal{O}_{id}.
\end{aligned}$$

One observes in these examples, and one can also prove it in general, that the specialization $y \mapsto 0$ in $MC_y(X(w)^\circ)$ yields the ideal sheaf of the boundary of the Schubert variety $X(w)$. The Schubert class $\mathcal{O}_w$ is obtained if one takes $y \mapsto 0$ in a recursion given by a renormalization of the dual operator $\mathcal{T}_i^\vee$. In fact, Theorem 1.1 generalizes the well-known fact from Schubert Calculus that the Schubert classes $\mathcal{O}_w$ are obtained recursively by the Demazure operators ∂_i. These and other combinatorial properties of the motivic Chern classes will be studied in a continuation to this paper.

A remarkable feature in the examples above is a positivity property. Based on substantial computer evidence we make the following positivity conjecture:

Conjecture 1 (Positivity Conjecture)**.** *Consider the Schubert expansion*

$$MC_y(X(w)^\circ) = \sum c(w;u)\mathcal{O}_u \in K_T(X)[y].$$

Then the coefficients $c(w;u) \in K_T(pt)$ satisfy $(-1)^{\ell(w)-\ell(u)}c(w,u) \in \mathbb{Z}_{\geq 0}[y][e^{-\alpha_1}, \dots, e^{-\alpha_r}]$, where α_i are the positive simple roots. In particular, in the non-equivariant case,

$$(-1)^{\ell(w)-\ell(u)}c(w;u) \geq 0.$$

The analogous cohomological version, involving CSM classes, was conjectured by Aluffi and Mihalcea [AM09, AM16]. For Grassmannians, the positivity property of CSM classes was established by J. Huh [Huh16]; few special cases were settled earlier in [AM09, Mih15, Jon10, Str11]. In full G/P generality, and in the non-equivariant case, the conjecture was recently proved in [AMSS17], using the theory of characteristic cycles in the cotangent bundle of G/B, and Aluffi's notion of *shadows* [Alu04]. There is also a stronger version of this conjecture, which claims that in addition $c(w;u) \neq 0$ whenever $u \leq w$. Huh's proof shows this, and also establishes (implicitly) the equivariant version for Grassmannians. The statement of the conjecture is reminiscent of the positivity in (equivariant) K theory proved by Buch [Buc02], Brion [Bri02], and Anderson, Griffeth and Miller [AGM11].

Let $MC_y^\vee(Y(v)^\circ)$ be the classes obtained by applying the dual operators $\mathcal{T}_i^\vee$ instead of $\mathcal{T}_i$. Then we prove in Theorem 5.2 that for any $u, v \in W$,

$$\langle MC_y(X(u)^\circ), MC_y^\vee(Y(v)^\circ)\rangle = \delta_{u,v}(-y)^{\ell(u)-\dim G/B}\prod_{\alpha>0}(1+ye^{-\alpha}), \tag{1}$$

where $\langle\cdot,\cdot\rangle$ is the K-theoretic Poincaré pairing. This orthogonality property, which we call 'Hecke duality', mirrors the similar orthogonality of Chern-Schwartz-MacPherson (CSM) classes proved by the authors in [AMSS17].

Our main application of the calculations of motivic Chern classes is to relate the equivariant K theory $K_{T^\vee}(G^\vee/B^\vee)$ of the Langlands dual flag manifold $G^\vee/B^\vee$ to the principal series representation $I(\tau)$ associated to an unramified character τ for a split reductive Chevalley group $G(F)$ over a nonarchimedean local field F. The representation $I(\tau)$ consists of functions $f : G(F) \to \mathbb{C}$ such that $f(bg) = (\tau\delta)^{1/2}(b)f(g)$ for any b in the standard Borel $B(F)$ in $G(F)$, and where δ is the modulus function; see §9 for more details. We consider the submodule $I(\tau)^I$ of Iwahori invariants of $I(\tau)$. Both $I(\tau)^I$ and $K_{T^\vee}(G^\vee/B^\vee)$ are Hecke modules, with additive bases indexed by the Weyl group W. On the representation theory side, the standard basis of $I(\tau)^I$ is given by the characteristic functions φ_w; the *Casselman basis* $\{f_w\}$ is another basis defined using certain intertwiners; see (30) below.

Consider the expansion

$$\varphi_u = \sum a_{u,w} f_w. \tag{2}$$

A question of Casselman [Cas80] is to find the transition matrix between the two bases φ_w and f_w. As observed by Bump and Nakasuji [BN11], it is better to consider the basis $\psi_u := \sum_{w \geq u}\varphi_w$ and the expansion

$$\psi_u = \sum \tilde{m}_{u,w} f_w.$$

By Möbius inversion, the problems of finding either of the transition matrices are equivalent. Recent solutions to the Casselman's problem were obtained by Naruse and Nakasuji [NN15], using the Yang-Baxter basis in the Hecke algebra introduced by Lascoux, Leclerc and Thibon [LLT97], and by Su, Zhao and Zhong [SZZ17], using the theory of stable envelopes developed in [Oko15, OS16, AO]. The K-theoretic stable envelopes are certain classes in the equivariant K theory of the cotangent bundle $T^*(G/B)$, indexed by the Weyl group elements; see §7 below. Su, Zhao and Zhong proved that the Hecke algebra action on the basis of stable envelopes coincides with the Hecke algebra action on the standard basis φ_w. Under their correspondence, the Hecke action on the Casselman's basis fits with the Hecke action on the fixed point basis in equivariant K theory.

It was observed by Fehér, Rimányi and Weber [FRW18, FRW] that motivic Chern classes and K-theoretic stable envelopes are closely related; see also §7 below. Therefore it is not a surprise that one can recover the Hecke correspondence from [SZZ17] using motivic Chern classes. The advantage of this point of view is that motivic Chern classes satisfy strong functoriality properties, and this will allow us to obtain additional properties of the coefficients $\tilde{m}_{u,v}$, including proofs for conjectures of Bump, Nakasuji and Naruse [BN11, NN15, BN17]. We explain all of this next.

An unramified character τ can be identified with an element of the dual torus $T^\vee$; cf. §9. Consider the $K_{T^\vee}(pt)$-module $\mathbb{C}_\tau$ obtained by evaluation at τ. Let ι_w be the fixed point basis in $K_{T^\vee}(G^\vee/B^\vee)$, and let b_w be the multiple of ι_w determined by the localization $b_w|_w = MC_y^\vee(Y(w)^\circ)|_w$ at the fixed point e_w. The formula for $MC_y^\vee(Y(w)^\circ)|_w \in K_{T^\vee}(pt)[y^{-1}]$ is explicit; see Prop. 6.3. We show that for a sufficiently general τ there is an isomorphism of Hecke modules $\Psi : K_{T^\vee}(G^\vee/B^\vee) \otimes_{K_{T^\vee}(pt)} \mathbb{C}_\tau \to I(\tau)^I$, such that

$$\Psi(MC_y^\vee(Y(w)^\circ) \otimes 1) = \varphi_w; \quad \Psi(b_w \otimes 1) = f_w; \quad \Psi(y) = -q.$$

Using this result, we prove that $\tilde{m}_{u,w} = m_{u,w}(\tau)$, where $m_{u,w}$ are the coefficients in the expansion

$$MC_y^\vee(Y(u)) = \sum m_{u,w} b_w \quad \in K_{T^\vee}(G^\vee/B^\vee)[y^{\pm 1}].$$

Implicit in this is that the coefficients $m_{u,w}$ may be regarded as complex valued functions defined on a certain Zariski open subset of the dual torus $T^\vee$. Therefore we can translate all statements about $\tilde{m}_{u,w}$ into 'geometric' statements about $m_{u,w}$. We will do this next, and from now on we return to the usual flag manifold G/B instead of the Langlands dual $G^\vee/B^\vee$ needed on the representation theoretic side. The key result for the representation theoretic applications is that the coefficients $m_{u,w}$ are given by localization (cf. Proposition 8.8 below):

Theorem 1.2. *(a) For any $w \geq u \in W$, the coefficient $m_{u,w} \in \mathrm{Frac}(K_T(pt))[y^{-1}]$ equals*

$$m_{u,w} = \Big(\frac{MC_y(Y(u))|_w}{MC_y(Y(w)^\circ)|_w}\Big)^\vee,$$

where $\vee$ is the operator sending $e^\lambda \mapsto e^{-\lambda}$ for $e^\lambda \in K_T(pt)$, and $y \mapsto y^{-1}$.

(b) Assume that $Y(u)$ is smooth at the fixed point e_w, and denote by $N_{w,Y(w)}Y(u)$ the normal space at e_w inside $Y(u)$, regarded as a trivial (but not equivariantly trivial) vector bundle. Then

$$m_{u,w} = \frac{\lambda_{y^{-1}}(N_{w,Y(w)}Y(u))}{\lambda_{-1}(N_{w,Y(w)}Y(u))}. \tag{3}$$

In particular, the entries $m_{id,w}$ are obtained from the motivic Chern class of the full flag variety $MC_y(Y(id)) = MC_y(G/B) = \lambda_y(T^*(G/B))$, and one recovers the (geometric version of the) classical Gindikin-Karpelevich formula, proved by Langlands [Lan71]:

$$m_{id,w} = \prod \frac{1 + y^{-1}e^\alpha}{1 - e^\alpha},$$

where the product of over positive roots α such that $w^{-1}(\alpha) < 0$. Let

$$S(u, w) := \{\alpha \in R^+ | u \leq s_\alpha w < w\}.$$

Our main application is the following factorization formula for $m_{u,v}$, see Theorem 8.5 below:

Theorem 1.3 (Geometric Bump-Nakasuji-Naruse Conjecture)**.** *For any $u \leq w \in W$,*

$$m_{u,w} = \prod_{\alpha \in S(u,w)} \frac{1 + y^{-1} e^{\alpha}}{1 - e^{\alpha}},$$

if and only if the Schubert variety $Y(u)$ is smooth at the torus fixed point e_w.

This is the geometric analogue of a conjecture of Bump and Nakasuji [BN11, BN17] for simply laced types, generalized to all types by Naruse [Nar14], and further analyzed by Nakasuji and Naruse [NN15]. While this paper was in preparation, Naruse informed us that he also obtained an (unpublished) proof of the implication assuming factorization. Both his and our proofs are based on Kumar's cohomological criterion for smoothness of Schubert varieties [Kum96], although Naruse did not use motivic Chern classes. The original conjecture of Bump and Nakasuji from [BN11] was stated in terms of when certain Kazhdan-Lusztig polynomials $P_{w_0 w^{-1}. w_0 u^{-1}} = 1$; we explain the equivalence to the statement above (in simply laced types) and make further remarks about this conjecture in sections §8.2 and 9.2.

We return for the moment to the principal series representation: in relation to Kazhdan-Lusztig theory, Bump and Nakasuji [BN17] defined the coefficients

$$\tilde{r}_{u,w} := \sum_{u \leq x \leq w} (-1)^{\ell(x) - \ell(u)} \overline{\tilde{m}}_{x,w},$$

where the bar operator replaces $q \mapsto q^{-1}$. Using the Möbius inversion it follows that the coefficients $a_{u,w}$ from (2) satisfy $\bar{a}_{u,w} = \tilde{r}_{u,w}$. Geometrically (and again changing from $G^\vee/B^\vee$ to G/B), these correspond to the coefficients $r_{u,w}$ obtained from the expansion

$$MC_y^\vee(Y(u)^\circ) = \sum \overline{r}_{u,w} b_w \quad \in K_T(G/B)[y],$$

where $\bar{f}(y) := f(y^{-1})$. Another application of Theorem 1.2 is a proof of Conjecture 1 from [BN17], see Theorem 9.4 below:

Theorem 1.4. *Let $u \leq w$ be two Weyl group elements. Then the functions*

$$\prod_{\alpha \in S(u,w)} (1 - e^{\alpha}) r_{u,w} \quad , \quad \prod_{\alpha \in S(u,w)} (1 - e^{\alpha}) m_{u,w}$$

are holomorphic on the torus T.

The proof of Theorem 1.2 requires a second orthogonality property between motivic Chern classes and their duals, coming from the theory of K-theoretic stable envelopes. From this orthogonality we deduce the following key formula, proved in Theorem 7.10:

$$MC_y^\vee(Y(u)^\circ) = \prod_{\alpha > 0} (1 + y e^{-\alpha}) \frac{\mathcal{D}(MC_y(Y(u)^\circ))}{\lambda_y(T^*(G/B))},$$

where $\mathcal{D}[E] = (-1)^{\dim X} [E^\vee] \otimes [\wedge^{\dim X} T^*(X)]$ is the Grothedieck-Serre duality operator. The proof requires a precise relationship between the motivic Chern classes and stable envelopes. If $\iota : X \to T^*_X$ is the zero section, then our statement is that (roughly)

$$\mathcal{D}(\iota^*(\mathrm{stab}_+(w))) = N(q) MC_{-q^{-1}}(X(w)^\circ), \tag{4}$$

where $\mathrm{stab}_+(w)$ is a stable envelope, $N(q)$ is a normalization parameter, and q is determined from the dilation action of $\mathbb{C}^*$ on the fibers of the cotangent bundle. The precise statement is given in Theorem 7.5.

The formula (4) is part of a more general paradigm, stemming from the classical works of Sabbah [Sab85] and Ginzburg [Gin86], relating intersection theory on the cotangent bundle to that of characteristic classes of singular varieties. For instance, the cohomological analogues of the motivic Chern classes of Schubert cells - the CSM classes - are equivalent to Maulik and Okounkov's *cohomological stable envelopes* [MO]. This statement, observed by Rimányi and Varchenko [RV], and by the authors in [AMSS17], is a consequence of the fact that both the stable envelopes and the CSM classes are determined by certain interpolation conditions obtained from equivariant localization; cf. Weber's work [Web12]. The relation to stable envelopes was recently extended to K-theory by Fehér, Rimányi and Weber [FRW18] (see also [FRW]). They showed that the motivic Chern classes of the Schubert cells satisfy the same localization conditions as the K-theoretic stable envelopes appearing in papers by Okounkov and Smirnov [Oko15, OS16] for a particular choice of parameters. (The result from [FRW18] is more general, involving the motivic Chern classes for orbits in a space with finitely many orbits under a group action.) We reprove this result by comparing the Demazure-Lusztig type recursions for motivic Chern classes to the recursions for the stable envelopes found by Su, Zhao and Zhong in [SZZ17]. We also discuss the relation between the motivic Chern class and various choices of parameters for the K-theoretic stable envelopes, which might be of independent interest; see section 7 below.

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2. Schubert varieties and their Bott-Samelson resolution

In this section we recall the basic definitions and facts about the Bott-Samelson resolution of Schubert varieties. These will be used in the next section to perform the calculation of the motivic Chern class of a Schubert cell. Our main references are [AM16] and [BK05].

Let G be a complex semisimple Lie group, and fix B a Borel subgroup and $T := B \cap B^-$ its maximal torus, where B^- denotes the opposite Borel subgroup. Let $W := N_G(T)/T$ be the Weyl group, and $\ell : W \to \mathbb{N}$ the associated length function. Denote by w_0 the longest element in W; then $B^- = w_0 B w_0$. Let also $\Delta := \{\alpha_1, \dots, \alpha_r\} \subset R^+$ denote the set of simple roots included in the set of positive roots for (G, B). The simple reflection for the root $\alpha_i \in \Delta$ is denoted by s_i and the *minimal* parabolic subgroup is denoted by P_i.

Let $X := G/B$ be the flag variety. It has a stratification by Schubert cells $X(w)^\circ := BwB/B$ and opposite Schubert cells $Y(w)^\circ := B^- wB/B$. The closures $X(w) := \overline{X(w)^\circ}$ and $Y(w) := \overline{Y(w)^\circ}$ are the Schubert varieties. With these definitions, $\dim_{\mathbb{C}} X(w) = \mathrm{codim}_{\mathbb{C}} Y(w) = \ell(w)$. The Weyl group W admits a partial ordering, called the Bruhat ordering, defined by $u \le v$ if and only if $X(u) \subset X(v)$.

We recall next the definition of the Bott-Samelson resolution of a Schubert variety, following [AM16, §2.3] and [BK05, §2.2]. Fix $w \in W$ and a reduced decomposition of w, i.e. a sequence $(i_1, \dots, i_k)$ such that $w = s_{i_1} \cdot \ldots \cdot s_{i_k}$ and $\ell(w) = k$. Associated to this data one constructs a tower Z of $\mathbb{P}^1$-bundles and a birational map $\theta : Z \to X(w)$ as follows.

If the word is empty, then define $Z := pt$. In general assume we have constructed $Z' := Z_{i_1,\dots,i_{k-1}}$ and the map $\theta' : Z' \to X(w')$, for $w' = s_{i_1} \cdots s_{i_{k-1}}$. Define $Z = Z_{i_1,\dots,i_k}$ so that the left square in the diagram

$$
\begin{tikzcd}
Z \arrow[r, "\theta_1"] \arrow[d, "\pi"'] & G/B \times_{G/P_{i_k}} G/B \arrow[r, "pr_1"] \arrow[d, "pr_2"] & G/B \arrow[d, "p_{i_k}"] \\
Z' \arrow[r, "\theta'"] & G/B \arrow[r, "p_{i_k}"] & G/P_{i_k}
\end{tikzcd}
\tag{5}
$$

is a fiber square; the morphisms pr_1, pr_2, p_{i_k} are the natural projections. From this construction it follows that $Z_{i_1,\dots,i_k}$ is a smooth, projective variety of dimension k.

The Bott-Samelson variety Z is equipped with a simple normal crossing (SNC) divisor D_Z. We recall next an explicit inductive construction of this divisor, which will be needed later. If $Z = pt$, then $D_Z = \emptyset$. In general, G/B is the projectivization $\mathbb{P}(E)$ of a homogeneous rank-2 vector bundle $E \to G/P_k$, defined up to tensoring with a line bundle. Define $\mathcal{E} := E \otimes \mathcal{O}_E(1)$, a vector bundle over $G/B = \mathbb{P}(E)$. Then we have the Euler sequence of the projective bundle $\mathbb{P}(E)$

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{6}
$$

where $\mathcal{Q}$ is the relative tangent bundle $T_{p_{i_k}}$. Note that $\mathcal{E}$ is independent of the specific choice of E, and $pr_2 : G/B \times_{G/P_{i_k}} G/B \to G/B$, that is, the pull-back of $\mathbb{P}(E)$ via p_{i_k}, may be identified with $\mathbb{P}(\mathcal{E})$. Let $\mathcal{E}' := (\theta')^*\mathcal{E}$ and $\mathcal{Q}' := (\theta')^*\mathcal{Q}$, and pull-back the previous sequence via θ' to get an exact sequence

$$
0 \longrightarrow \mathcal{O}_{Z'} \longrightarrow \mathcal{E}' \longrightarrow \mathcal{Q}' \longrightarrow 0\,. \tag{7}
$$

The inclusion $\mathcal{O}_{Z'} \hookrightarrow \mathcal{E}'$ gives a section $\sigma : Z' \to Z$ of π and therefore a divisor $D_k := \sigma(Z') = \mathbb{P}(\mathcal{O}_{Z'})$ in $\mathbb{P}(\mathcal{E}') = Z$. The SNC divisor on Z is defined by

$$
D_Z = \pi^{-1}(D_{Z'}) \cup D_k
$$

where $D_{Z'}$ is the inductively constructed SNC divisor on Z'. The following result is well known, see e.g., [BK05, §2.2]:

Proposition 2.1. *If $s_{i_1} \dots s_{i_k}$ is a reduced word for w, then the image of the composition $\theta = pr_1 \circ \theta_1 : Z_{i_1,\dots,i_k} \to G/B$ is the Schubert variety $X(w)$. Moreover, $\theta^{-1}(X(w) \smallsetminus X(w)^\circ) = D_{Z_{i_1,\dots,i_k}}$ and the restriction map*

$$
\theta : Z_{i_1,\dots,i_k} \smallsetminus D_{Z_{i_1,\dots,i_k}} \to X(w)^\circ
$$

is an isomorphism.

The group G acts diagonally on the fibre product $G/B \times_{G/P_{i_k}} G/B$ with two orbits (in fact the fibre product itself is the closure of the G-orbit of $(id.B, s_{i_k}.B)$ under the diagonal action of G on $G/B \times G/B$.) The closed orbit is the diagonal $\mathcal{D} \subset G/B \times_{G/P_{i_k}} G/B$, which is naturally isomorphic to G/B. We will need the following result:

Lemma 2.2. *There is a natural isomorphism of equivariant vector bundles between the normal bundle $N_{\mathcal{D}}(G/B \times_{G/P_{i_k}} G/B)$ and the restriction $(T_{pr_2})_{|\mathcal{D}}$.*

Proof. Denote by $\mathcal{O}(\mathcal{D})$ the line bundle $\mathcal{O}_{G/B \times_{G/P_{i_k}} G/B}(\mathcal{D})$; it is naturally isomorphic to the normal bundle in the statement, as equivariant bundles. By [AM16, Proposition 6.2(a)] there is an isomorphism of equivariant bundles $\mathcal{O}(\mathcal{D}) \simeq T_{pr_2} \otimes \mathcal{O}_{\mathcal{E}}(1)$. The part (a) of the same lemma shows that the restriction $(\mathcal{O}_{\mathcal{E}}(1))_{|\mathcal{D}}$ is equivariantly trivial. The lemma follows from this. □

3. Equivariant K theory of flag manifolds and Demazure-Lusztig operators

In this section we recall the definition and basic properties of equivariant K theory of flag manifolds, and of certain Demazure-Lusztig operators acting on equivariant K theory. This setup is well known from the theory of Hecke algebras, see e.g. works of Lusztig [Lus85] and Ginzburg [Gin98].

3.1. **Equivariant K theory.** Let X be a smooth, quasi-projective algebraic variety with a T-action. The (algebraic) equivariant K theory ring $K_T(X)$ is the ring generated by symbols $[E]$, where $E \to X$ is an equivariant vector bundle modulo the relations $[E] = [E_1] + [E_2]$ for any short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of equivariant vector bundles. The ring addition is given by direct sums, and multiplication by tensor products. Since X is smooth, any (equivariant) coherent sheaf has a finite resolution by (equivariant) vector bundles [CG09, Prop.5.1.28], and $K_T(X)$ coincides with the Grothendieck group of (equivariant) coherent sheaves on X. The ring $K_T(X)$ is an algebra over the Laurent polynomial ring $K_T(pt) = \mathbb{Z}[e^{\pm t_1}, \dots, e^{\pm t_r}]$ where e^{t_i} are characters corresponding to a basis of the Lie algebra of T; alternatively $K_T(pt) = R(T)$, the representation ring of T.

In our situation $X = G/B$ and T acts on X by left multplication. Since X is smooth, the ring $K_T(X)$ coincides with the Grothendieck group of T-linearized coherent sheaves on X. Indeed, any coherent sheaf has a resolution by vector bundles. There is a pairing

$$\langle -, - \rangle : K_T(X) \otimes K_T(X) \to K_T(pt) = R(T); \quad \langle [E], [F] \rangle := \int_X E \otimes F = \chi(X; E \otimes F),$$

where $\chi(X; E)$ is the (equivariant) Euler characteristic, i.e. the virtual character

$$\chi(X; E) = \sum (-1)^i ch_T H^i(X; E).$$

Let $\mathcal{O}_w := [\mathcal{O}_{X(w)}]$ be the Grothedieck class determined by the structure sheaf of $X(w)$ (a coherent sheaf), and similarly $\mathcal{O}^w := [\mathcal{O}_{Y(w)}]$. The equivariant K-theory ring has $K_T(pt)$-bases $\{\mathcal{O}_w\}_{w \in W}$ and $\mathcal{O}^w$ for $w \in W$. Let $\partial X(w) := X(w) \setminus X(w)^\circ$ be the boundary of the Schubert variety $X(w)$, and similarly $\partial Y(w)$ the boundary of $Y(w)$. It is known that the dual bases of $\{\mathcal{O}_w\}$ and $\{\mathcal{O}^w\}$ are given by the classes of the ideal sheaves $\mathcal{I}^w := [\mathcal{O}_{Y(w)}(-\partial Y(w))]$ respectively $\mathcal{I}_w := [\mathcal{O}_{X(w)}(-\partial X(w))]$. I.e.

$$\langle \mathcal{O}_u, \mathcal{I}^v \rangle = \langle \mathcal{O}^u, \mathcal{I}_v \rangle = \delta_{u,v}.$$

See e.g. [Bri05, Prop. 4.3.2] for the non-equivariant case - same proof works equivariantly; see also[GK08, AGM11]. In fact, the same result in [Bri05] shows that

$$\mathcal{O}_w = \sum_{v \le w} \mathcal{I}_v \text{ and } \quad \mathcal{I}_w = \sum_{v \le w} (-1)^{\ell(w) - \ell(v)} \mathcal{O}_v. \tag{8}$$

3.2. **Demazure-Lusztig operators.** Fix a simple root $\alpha_i \in \Delta$ and $P_i \subset G$ the corresponding minimal parabolic group. Consider the right hand side of the diagram (5):

$$\begin{tikzcd} FP := G/B \times_{G/P_i} G/B \arrow[r, "pr_1"] \arrow[d, "pr_2"] & G/B \arrow[d, "p_i"] \\ G/B \arrow[r, "p_i"] & G/P_i \end{tikzcd} \tag{9}$$

The Demazure operator [Dem74] $\partial_i : K_T(X) \to K_T(X)$ is defined by $\partial_i := (p_i)^*(p_i)_*$. It satisfies

$$\partial_i(\mathcal{O}_w) = \begin{cases} \mathcal{O}_{ws_i} & \text{if } ws_i > w; \\ \mathcal{O}_w & \text{otherwise .} \end{cases} \tag{10}$$

From this, one deduces that $\partial_i^2 = \partial_i$ and the operators ∂_i satisfy the same braid relations as the elements in the Weyl group W.

Fix y an indeterminate. Recall that for a vector bundle E the λ_y-class of E is the class

$$\lambda_y(E) := \sum_k [\wedge^k E] y^k \in K_T(X)[y].$$

The λ_y-class is multiplicative, i.e. for any short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of equivariant vector bundles there is an equality $\lambda_y(E) = \lambda_y(E_1)\lambda_y(E_2)$ as elements in $K_T(X)[y]$. We refer to the book [Hir95] for details.

We define next the main operators used in this paper. First, consider the projection $p_i : G/B \to G/P_i$ determined by the minimal parabolic subgroup P_i, and let $T^*_{p_i}$ be the relative cotangent bundle. It is given by

$$T^*_{p_i} = \mathcal{L}_{\alpha_i} := G \times^B \mathbb{C}_{\alpha_i},$$

i.e. it is the equivariant line bundle on G/B with character α_i in the fibre over $1.B$.

Definition 3.1. Let $\alpha_i \in \Delta$ be a simple root. Define the operators

$$\mathcal{T}_i := \lambda_y(T^*_{p_i})\partial_i - id; \quad \mathcal{T}_i^\vee := \partial_i \lambda_y(T^*_{p_i}) - id.$$

The operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are $K_T(pt)$-module endomorphisms of $K_T(X)$.

Remark 3.2. The operator $\mathcal{T}_i^\vee$ was defined by Lusztig [Lus85, Eq. (4.2)] in relation to affine Hecke algebras and equivariant K theory of flag varieties. As we shall see below, the 'dual' operator $\mathcal{T}_i$ arises naturally in the study of motivic Chern classes of Schubert cells. In an algebraic form, $\mathcal{T}_i$ appeared recently in [LLL17], in relation to Whittaker functions. In future work, both operators will be realized as convolution operators in the context of the motivic Hecke algebra formalism of Tanisaki [Tan87].

Lemma 3.3. *The operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are adjoint to each other, i.e. for any $a, b \in K_T(X)$,*

$$\langle \mathcal{T}_i(a), b \rangle = \langle a, \mathcal{T}_i^\vee(b) \rangle.$$

Proof. Clearly, the identity is self adjoint and ∂_i is also self adjoint. Indeed, by projection formula

$$\langle \partial_i(a), b \rangle = \int_{G/B} p_i^*(p_i)_*(a)) \cdot b = \int_{G/P_i} (p_i)_*(a) \cdot (p_i)_*(b),$$

and the last expression is symmetric in a, b. It remains to show that coefficient of y in both sides is the same, i.e. $\langle T^*_{p_i} \partial_i(a), b \rangle = \langle a, \partial_i T^*_{p_i}(b) \rangle$. We calculate

$$\begin{aligned} \langle T^*_{p_i} \partial_i(a), b \rangle &= \int_{G/B} T^*_{p_i} p_i^*((p_i)_* a) \cdot b = \int_{G/P_i} (p_i)_*(a) \cdot (p_i)_*(T^*_{p_i} \cdot b) \\ &= \int_{G/B} a \cdot p_i^*(p_i)_*(T^*_{p_i} \cdot b) = \langle a, \partial_i(T^*_{p_i} \cdot b) \rangle \end{aligned}.$$

□

According to Lusztig's main result in [Lus85], the operators $\mathcal{T}_i^\vee$ satisfy the braid relations and the quadratic relations defining the (affine) Hecke algebra. The operators $\mathcal{T}_i$ also show up in the papers [BBL15, LLL17]. (In the language of this paper, the variable q from [Lus85] satisfies $y = -q$.) Since the K-theoretic Poincaré pairing is non-degenerate, Lemma 3.3 implies that both sets of operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ satisfy the same algebraic formulas. We record this next.

Proposition 3.4 (Lusztig)**.** *The operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ satisfy the braid relations for the group G. For each simple root $\alpha_i \in \Delta$ the following quadratic formula holds:*

$$(\mathcal{T}_i + id)(\mathcal{T}_i + y) = (\mathcal{T}_i^\vee + id)(\mathcal{T}_i^\vee + y) = 0.$$

An immediate corollary of the quadratic formula is that for $y \neq 0$, the operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are invertible. In fact,

$$\mathcal{T}_i^{-1} = -\frac{1}{y}\mathcal{T}_i - \frac{1+y}{y} id. \tag{11}$$

Same formula holds when $\mathcal{T}_i$ is exchanged with $\mathcal{T}_i^\vee$.

The cohomological versions of the Demazure-Lusztig polynomials, which appear in the study of degenerate Hecke algebras [Gin98, AM16] are self inverse (i.e. $(\mathcal{T}_i^{coh})^2 = id$). Obviously, this is not true in K-theory, due to the presence of quadratic relations. However, the multiplication of Demazure-Lusztig operators behaves rather nicely, as shown in the following proposition.

Proposition 3.5. *Let $u, v \in W$ be two Weyl group elements. Then*

$$\mathcal{T}_u \cdot \mathcal{T}_v^{-1} = c_{uv^{-1}}(y)\mathcal{T}_{uv^{-1}} + \sum_{w < uv^{-1}} c_w(y)\mathcal{T}_w,$$

where $c_w(y)$ is a rational function in y. Further, if $\ell(u) + \ell(v^{-1}) = \ell(uv^{-1})$ then $c_{uv^{-1}}(y) = (-y)^{-\ell(v)}$. Same statements hold for the multiplication of the dual operators $\mathcal{T}_u^\vee \cdot (\mathcal{T}_v^\vee)^{-1}$.

To prove the proposition we need the following lemma.

Lemma 3.6. *Let $u, v \in W$ be two Weyl group elements, and let s be a reflection such that $us > u$ and $sv < v$. Then $usv < uv$.*

Proof. Let $s = s_\alpha$ for some positive root α. The identity $s_{v^{-1}(\alpha)} = v^{-1}s_\alpha v$ implies that $s_\alpha v = v s_{v^{-1}(\alpha)}$. Then $us_\alpha v = uvv^{-1}s_\alpha v = uvs_{v^{-1}(\alpha)}$. Note that the hypothesis implies that $v^{-1}s < v^{-1}$, thus $v^{-1}(\alpha) < 0$. To finish the proof, observe that $uv(-v^{-1}(\alpha)) = -u(\alpha) < 0$, thus $uvs_{v^{-1}(\alpha)} < uv$. ☐

Proof of Proposition 3.5. We use ascending induction on $\ell(v) \geq 0$. The statement is clear if $\ell(v) = 0$. For $\ell(v) > 0$ write $v = v's_k$ where $\ell(v) > \ell(v')$. By definition, $\mathcal{T}_u \cdot \mathcal{T}_v^{-1} = \mathcal{T}_u \cdot \mathcal{T}_k^{-1} \cdot \mathcal{T}_{v'}^{-1}$. We have two cases: $us_k > u$ and $us_k < u$. Consider first the situation $us_k < u$. Then u has a reduced decomposition ending in s_k, i.e. $u = u's_k$ and $\ell(u') < \ell(u)$. Then

$$\mathcal{T}_u \cdot \mathcal{T}_k^{-1} \cdot \mathcal{T}_{v'}^{-1} = \mathcal{T}_{u'} \cdot \mathcal{T}_k \cdot \mathcal{T}_k^{-1} \cdot \mathcal{T}_{v'}^{-1} = \mathcal{T}_{u'} \cdot \mathcal{T}_{v'}^{-1},$$

and since $v' < v$ the result is known by induction. Assume now that $us_k > u$. Using equation (11) we obtain

$$\mathcal{T}_u \cdot \mathcal{T}_v^{-1} = \mathcal{T}_u \cdot \mathcal{T}_k^{-1} \cdot \mathcal{T}_{v'}^{-1} = -\frac{1}{y}\mathcal{T}_u \cdot (\mathcal{T}_k + y + 1) \cdot \mathcal{T}_{v'}^{-1} = -\frac{1}{y}\mathcal{T}_{us_k} \cdot \mathcal{T}_{v'}^{-1} - \frac{1+y}{y}\mathcal{T}_u \cdot \mathcal{T}_{v'}^{-1}.$$

By induction, the leading term of $\mathcal{T}_{us_k} \cdot \mathcal{T}_{v'}^{-1}$ is $\mathcal{T}_{us_k v'^{-1}}$, and the leading term of $\mathcal{T}_u \cdot \mathcal{T}_{v'}^{-1}$ is $\mathcal{T}_{uv'^{-1}}$. From lemma 3.6 we obtain that $us_k v'^{-1} = uv^{-1} > us_k v^{-1} = uv'^{-1}$ and we are done. The statement for the dual operators is proved in the same way. $\square$

3.3. **Actions on Schubert and fixed point bases.** We will need several formulas about the action of the Demazure-Lusztig operators on the Schubert classes, and on the classes determined by the torus fixed points. For instance, the definition and a standard localization argument imply that for any $w \in W$,

$$\mathcal{T}_i(\mathcal{O}_w) = \begin{cases} (1 + ye^{-w(\alpha_i)})\mathcal{O}_{ws_i} + l.o.t. & \text{if } ws_i > w; \\ ye^{w(\alpha_i)}\mathcal{O}_w + l.o.t. & \text{if } ws_i < w. \end{cases} \tag{12}$$

where l.o.t. (lower order terms) encodes terms $P(y, e^t)\mathcal{O}_u$ where $u < ws_i$ on the first branch, and $u < w$ on the second branch. A similar formula holds for the dual operator:

$$\mathcal{T}_i^\vee(\mathcal{O}^w) = (1 + ye^{w_0 w(\alpha_i)})\mathcal{O}^{ws_i} + l.o.t. \text{ if } ws_i < w, \tag{13}$$

where now the l.o.t. encodes terms containing $\mathcal{O}^v$ such that $v > ws_i$.

Consider next the localized equivariant K theory ring $K_T(G/B)_{loc} := K_T(G/B) \otimes_{K_T(pt)} \mathrm{Frac}(K_T(pt))$ where Frac denotes the fraction field. The Weyl group elements $w \in W$ are in bijection with the torus fixed points $e_w \in G/B$. Let $\iota_w := [\mathcal{O}_{e_w}] \in K_T(G/B)_{loc}$ be the class of the structure sheaf of e_w. By the localization theorem, the classes ι_w form a basis for the localized equivariant K theory ring; we call this the *fixed point basis.* For a weight λ consider the G-equivariant line bundle $\mathcal{L}_\lambda := G \times^B \mathbb{C}_\lambda$. We need the following lemma.

Lemma 3.7. *The following formulas hold in $K_T(G/B)_{loc}$:*

(a) For any weight λ, $\mathcal{L}_\lambda \cdot \iota_w = e^{w\lambda}\iota_w$;

(b) For any simple root α_i,

$$\partial_i(\iota_w) = \frac{1}{1 - e^{w\alpha_i}}\iota_w + \frac{1}{1 - e^{-w\alpha_i}}\iota_{ws_i};$$

(c) The action of the operator $\mathcal{T}_i$ on the fixed point basis is given by the following formula

$$\mathcal{T}_i(\iota_w) = -\frac{1 + y}{1 - e^{-w\alpha_i}}\iota_w + \frac{1 + ye^{-w\alpha_i}}{1 - e^{-w\alpha_i}}\iota_{ws_i}.$$

(d) The action of the adjoint operator $\mathcal{T}_i^\vee$ is given by

$$\mathcal{T}_i^\vee(\iota_w) = -\frac{1 + y}{1 - e^{-w\alpha_i}}\iota_w + \frac{1 + ye^{w\alpha_i}}{1 - e^{-w\alpha_i}}\iota_{ws_i}.$$

(e) The action of the inverse operator $(\mathcal{T}_i^\vee)^{-1}$is given by

$$(\mathcal{T}_i^\vee)^{-1}(\iota_w) = -\frac{1 + y^{-1}}{1 - e^{w\alpha_i}}\iota_w - \frac{y^{-1} + e^{w\alpha_i}}{1 - e^{-w\alpha_i}}\iota_{ws_i}.$$

Proof. Part (a) is a standard calculation. For part (b), notice that $\partial_i(\iota_w) = [\mathcal{O}_{p_i^{-1}(e_w)}]$, where $p_i : G/B \to G/P_i$ is the projection, and (abusing notation) e_w denotes the fixed point in G/P. The fibre $p_i^{-1}(e_w)$ equals $w.[X(s_i)]$, the w-translate of the Schubert curve, thus isomorphic to $\mathbb{P}^1$. It contains only two T-fixed points: e_w and e_{ws_i}. Let $[\mathcal{O}_{p_i^{-1}(e_w)}] = a\iota_w + b\iota_{ws_i}$. By projection formula we can regard this as an expansion in the localized equivariant K theory of the fibre itself. Then $(\iota_w)_{|w} = 1 - e^{w\alpha_i}$ (the Euler class of the tangent space at w), and $(\iota_w)_{|ws_i} = 0$. Similarly $(\iota_{ws_i})_{|ws_i} = 1 - e^{-w\alpha_i}$ and the localization at e_w equals 0. (For instance, this can be seen to hold if $w = id$, then one translates.) Since $[\mathcal{O}_{p_i^{-1}(e_w)}]$ is the identity in the K-theory ring, both localizations equal 1. Then the statement in (b) follows after localizing both sides at e_w and e_{ws_i}. To prove (c), we use that ∂_i is a homomorphism of $K_T(pt)$-modules to calculate

$$\mathcal{T}_i(\iota_w) = (1 + y\mathcal{L}_{\alpha_i})\partial_i(\iota_w) - \iota_w = -\frac{1+y}{1-e^{-w\alpha_i}}\iota_w + \frac{1+ye^{-w\alpha_i}}{1-e^{-w\alpha_i}}\iota_{ws_i}$$

The formula for the adjoint operator follows similarly. Finally, part (e) follows because $(\mathcal{T}_i^\vee)^{-1} = -y^{-1}(\mathcal{T}_i^\vee + y + 1)$ (cf. equation (11)). $\square$

We also record the action of several specializations of the Demazure-Lusztig operators.

Lemma 3.8. *(a) The specializations*

$$(\mathcal{T}_i)_{y=0} = \partial_i - id; \quad (\mathcal{T}_i^\vee)_{y=0} = \partial_i - id;$$

Further, for any $w \in W$, the following hold:

$$(\partial_i - id)(\mathcal{I}_w) = \begin{cases} \mathcal{I}_{ws_i} & \text{if } ws_i > w; \\ -\mathcal{I}_w & \text{if } ws_i < w. \end{cases} \quad \partial_i(\mathcal{O}_w) = \begin{cases} \mathcal{O}_{ws_i} & \text{if } ws_i > w; \\ \mathcal{O}_w & \text{if } ws_i < w. \end{cases}$$

(b) Let $w \in W$. Then the specializations at $y = -1$ satisfy

$$(\mathcal{T}_i)_{y=-1}(\iota_w) = \iota_{ws_i}; \quad (\mathcal{T}_i^\vee)_{y=-1}(\iota_w) = \frac{1-e^{w\alpha_i}}{1-e^{-w\alpha_i}}\iota_{ws_i} = -e^{w\alpha_i}\iota_{ws_i}.$$

In other words, this specialization is compatible with the right Weyl group multiplication.

Proof. Part (b) is immediate from Lemma 3.7. Part (a) is an easy exercise using equation (10) and duality between ideal and structure sheaf bases. $\square$

4. Motivic Chern classes

We recall the definition of the motivic Chern classes, following [BSY10] and [AMSS]. For now let X be a quasi-projective, complex algebraic variety, with an action of T. We recall the definition of the (relative) motivic Grothendieck group $G_0^T(var/X)$ of varieties over X, mostly following Looijenga's notes [Loo02]; see also Bittner [Bit04]. The group $G_0^T(var/X)$ is the free abelian group generated by symbols $[f : Z \to X]$ where Z is a T-variety and $f : Z \to X$ is a G-equivariant morphism modulo the usual additivity relations $[f : Z \to X] = [f : U \to X] + [f : Z \setminus U \to X]$ for $U \subset Z$ an open invariant subvariety. If $X = pt$ then $G_0^T(var/pt)$ is a ring with the product given by the external product of morphisms, and the groups $G_0^T(var/X)$ acquire a module structure over $G_0^T(var/pt)$.

For any equivariant morphism $f : X \to Y$ there are well defined push-forwards $f_! : G_0^T(var/X) \to G_0^T(var/Y)$ (given by composition) and $f^! : G_0^T(var/Y) \to G_0^T(var/X)$ (given by fibre product); see [Bit04, §6]. (There are also maps $f_* : G_0^T(var/X) \to G_0^T(var/Y)$ and $f^* : G_0^T(var/Y) \to G_0^T(var/X)$ defined using a certain duality, but we do not need those.)

Remark 4.1. For any variety X, similar functors can be defined on the ring of constructible functions $\mathcal{F}(X)$, and the Grothendieck ring $G_0(var/X)$ may be regarded as a motivic version $\mathcal{F}(X)$. In fact, there is a map $e : G_0(var/X) \to \mathcal{F}(X)$ sending $[f : Y \to X] \mapsto f_!(1\!\!1_Y)$, where $f_!(1\!\!1_Y)$ is defined using compactly supported Euler characteristic. The map e is a group homomorphism, and if $X = pt$ then e is a ring homomorphism. The constructions extend equivariantly.

The following theorem was proved by Brasselet, Schürmann and Yokura [BSY10, Thm. 2.1] in the non-equivariant case. Minor changes in arguments are needed in the equivariant case - see [FRW18]. In the upcoming paper [AMSS] we will reprove the theorem below, and relate equivariant motivic Chern classes to certain classes in the equivariant K theory of the cotangent bundle.

Theorem 4.2. *Let X be a quasi-projective, non-singular, complex algebraic variety with an action of the torus T. There exists a unique natural transformation $MC_y : G_0^T(var/X) \to K_T(X)[y]$ satisfying the following properties:*

(1) *It is functorial with respect to projective morphisms of non-singular, irreducible, varieties.*
(2) *It satisfies the normalization condition*

$$MC_y[id_X : X \to X] = \lambda_y(T_X^*) = \sum y^i[\wedge^i T_X^*]_T \in K_T(X)[y].$$

The transformation MC_y satisfies the following properties:

(3) *It is determined by its image on classes $[f : Z \to X]$ where Z is a non-singular, irreducible and quasi-projective algebraic variety and f is a projective morphism.*
(4) *It satisfies a Verdier-Riemann-Roch (VRR) formula: for any smooth, equivariant morphism $\pi : X \to Y$ of quasi-projective, irreducible and non-singular algebraic varieties, and any $[f : Z \to Y] \in G_0^T(var/Y)$, the following holds:*

$$\lambda_y(T_\pi^*) \cap \pi^* MC_y[f : Z \to Y] = MC_y[\pi^! f : Z \times_Y X \to X].$$

Proof. As remarked above, in the non-equivariant case the statements (1), (2) and (4) were proved in [BSY10]. Fehér, Rimányi an Weber used this to extend the motivic Chern classes to the equivariant situation in [FRW18] and in the process proved (1) and (2).

We now prove part (3). Consider an equivariant morphism $f : Z \to X$. By additivity we can assume Z is connected, quasi-projective and non-singular. Using normalization property from part (2) and additivity again it suffices to show that there exists a non-singular algebraic variety $\overline{Z}$ containing Z such that the following are satisfied:

- $\overline{Z}$ admits an action of T and the inclusion $i : Z \to \overline{Z}$ is an open, T-equivariant embedding;
- the boundary $\overline{Z} \setminus Z$ is a T-equivariant simple normal crossing divisor;
- there exists a T-equivariant projective morphism $\overline{f} : \overline{Z} \to X$ such that $f = \overline{f} \circ i$.

In other words, we need to show that the Grothendieck group $G_0^T(var/X)$ is generated by morphisms $\overline{f} : \overline{Z} \to X$ as before. This is known in the non-equivariant case, using resolution of singularities; see e.g. results of Bittner [Bit04, Thm. 5.1(bl)]. One can extend this equivariantly using the results of Sumihiro [Sum74] as follows. By general arguments using equivariant resolution of singularities one can construct a quasi-projective variety $\overline{Z}$ satisfying the first two properties above, and a *proper* T-equivariant morphism $\overline{f} : \overline{Z} \to X$ as in the third property; see e.g. [FRW18, Def. 2.1]. We obtained a T-equivariant, proper morphism $\overline{f} : \overline{Z} \to X$ of quasi-projective varieties. Such morphisms must be projective, e.g. by [Sta, Lemma 28.41.13].

We turn to the Verdier-Riemann-Roch formula in (4). By part (3) we can assume that Z is non-singular and quasi-projective and the morphism $f : Z \to Y$ is projective. Consider the fibre diagram

$$\begin{tikzcd} Z \times_Y X \arrow[r, "f'"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Z \arrow[r, "f"] & Y \end{tikzcd}$$

Then f' is projective, π' is smooth, $Z \times_Y X$ is non-singular, and $(f')^*(\lambda_y(T^*_\pi)) = \lambda_y(T^*_{\pi'})$. Using the functoriality of the motivic Chern class transformation with respect to projective morphisms of non-singular varieties, we calculate:

$$MC_y[f' : Z \times_Y X \to X] = (f')_* MC_y(id : Z \times_Y X \to Z \times_Y X] = (f')_* \lambda_y(T^*_{Z \times_X Y}).$$

Since π' is smooth, we have that

$$\lambda_y(T^*_{Z \times_X Y}) = (\pi')^* \lambda_y(T^*_Z) \cdot \lambda_y(T^*_{\pi'}),$$

and by projection formula and base-change [Har77, Prop. 9.3],

$$(f')_*((\pi')^* \lambda_y(T^*_Z) \cdot \lambda_y(T^*_{\pi'})) = (f')_*((\pi')^* \lambda_y(T^*_Z) \cdot (f')^*(\lambda_y(T^*_\pi))) = \pi^* f_*(\lambda_y(T^*_Z)) \cdot \lambda_y(T^*_\pi).$$

The last quantity equals $\pi^* MC[f : Z \to Y] \cdot \lambda_y(T^*_\pi)$ as claimed. $\square$

Most of the time the variety X will be understood from the context. If $Y \subset X$, is a subvariety, not necessarily closed, denote by

$$MC_y(Y) := MC_y[Y \hookrightarrow X].$$

If $i : Y \subset X$ is closed and $Y' \subset Y$ then by functoriality $MC_y[Y' \hookrightarrow X] = i_* MC_y[Y' \hookrightarrow Y]$ (K theoretic push-forward). For instance if $Y' = Y$ is smooth then $MC_y[i : Y \to X] = i_*(\lambda_y(T^*_Y) \otimes [\mathcal{O}_Y])$, an element in $K_T(X)$. We will often abuse notation and suppress the push-forward notation.

We need the following general lemma.

Lemma 4.3. *Let $X_1, X_2 \subset X$ be three T-equivariant smooth proper varieties such that the intersection $X_1 \cap X_2$ is smooth and transversal in X. Then the following equalities hold in $K_T(X)[y, y^{-1}]$:*

(a) The inclusion exclusion formula:

$$MC_y[X_1 \cup X_2 \hookrightarrow X] = MC_y[X_1 \hookrightarrow X] + MC_y[X_2 \hookrightarrow X] - MC_y[X_1 \cap X_2 \hookrightarrow X].$$

(b) The motivic class of the intersection is given by

$$MC_y[X_1 \cap X_2 \hookrightarrow X] = \frac{MC_y(X_1) MC_y(X_2)}{MC_y(X)}.$$

(c) Same formulas apply in the case when X_1 is a union of smooth irreducible varieties, intersecting mutually transversally.

Proof. The statement in (a) is immediate from the additivity property in the Grothendieck group. Part (b) follows from standard exact sequences, using that the normal bundle $N_{X_1 \cap X_2} X$ is the restriction to $X_1 \cap X_2$ of $N_{X_1} X \oplus N_{X_2} X$. Finally, part (c) follows from repeated application of (a) and (b), using inclusion-exclusion and induction on the number of components of X_1. $\square$

Remark 4.4. In the cohomological case, i.e. after replacing the motivic Chern classes by CSM classes, this lemma is well known, and it was extended by Schürmann [Sch17] to the singular case.

4.1. **Motivic Chern classes of Schubert cells via Demazure-Lusztig operators.** We return to the situation when $X = G/B$. In this section we will calculate the motivic Chern classes of Schubert cells using Demazure-Lusztig operators. Our main theorem is:

Theorem 4.5. *Let $w \in W$ and s_i a simple reflection such that $ws_i > w$. Then*

$$\mathcal{T}_i MC_y[X(w)^\circ \hookrightarrow G/B] = MC_y[X(ws_i)^\circ \hookrightarrow G/B],$$

as elements in $K_T(G/B)[y]$.

Proof. Fix a reduced decomposition for w, and consider the Bott-Samelson $\theta : Z \to X(ws_i)$ for the corresponding reduced word of ws_i. We use the notation from section 2 above; in particular $\sigma : Z' \to Z$ denotes the section of the projection $\pi : Z \to Z'$. For notational simplicity let $D := \sigma(Z')$ be the image. Then the boundary $\partial Z = \pi^{-1}(\partial Z') \cup D$ is a SNC divisor with smooth components. Recall the diagram (5) from §2 above.

$$\begin{tikzcd} Z \arrow[r, "\theta_1"] \arrow[d, "\pi"'] & G/B \times_{G/P_i} G/B \arrow[r, "pr_1"] \arrow[d, "pr_2"] & G/B \arrow[d, "p_i"] \\ Z' \arrow[r, "\theta'"] & G/B \arrow[r, "p_i"] & G/P_i \end{tikzcd}$$

By functoriality $MC_y(X(ws_i)^\circ) = \theta_* MC_y(Z \setminus \partial Z)$ and by additivity

$$MC_y(Z \setminus \partial Z) = MC_y(Z) - MC_y(\partial Z) = MC_y(Z) - MC_y(\pi^{-1}(\partial Z') \cup D).$$

Inclusion-exclusion and Lemma 4.3(c) imply that the last quantity equals

$$MC_y(Z) - MC_y(\pi^{-1}(\partial Z')) - MC_y(D) + \frac{MC_y(\pi^{-1}(\partial Z'))MC_y(D)}{MC_y(Z)}.$$

By Verdier-Riemann-Roch formula from Theorem 4.2 and by additivity $MC_y(\pi^{-1}(\partial Z')) = \lambda_y(T^*_\pi)\pi^* MC_y(\partial Z')$ and again by additivity $MC_y(\partial Z') = MC_y(Z') - MC_y((Z')^\circ)$, where $(Z')^\circ = Z' \setminus \partial Z'$. Combining everything we obtain

$$\begin{aligned} MC_y(Z \setminus \partial Z) = MC_y(Z) - MC_y(D) + \lambda_y(T^*_\pi)\pi^*(MC_y((Z')^\circ))(1 - \frac{MC_y(D)}{MC_y(Z)}) \\ -\lambda_y(T^*_\pi)\pi^* MC_y(Z')(1 - \frac{MC_y(D)}{MC_y(Z)}). \end{aligned}$$

Note that $\lambda_y(T^*_\pi)\pi^* MC_y(Z') = MC_y(Z)$ because of the short exact sequence defining the relative cotangent bundle. Therefore the last term equals $-(MC_y(Z) - MC_y(D))$, and

$$MC_y(Z \setminus \partial Z) = \lambda_y(T^*_\pi)\pi^*(MC_y((Z')^\circ))(1 - \frac{MC_y(D)}{MC_y(Z)}).$$

The relative cotangent bundle $T^*_\pi = \theta_1^* T^*_{pr_2}$, while D is the pull-back of the diagonal $\mathcal{D} \subset G/B \times_{G/P_i} G/B$. Further, the quotient

$$\frac{MC_y(D)}{MC_y(Z)} \otimes [\mathcal{O}_D] = \frac{1}{\lambda_y(N^*_D Z)} \otimes [\mathcal{O}_D],$$

and the conormal bundle $N^*_D Z = \theta_1^*(N^*_{\mathcal{D}}(FP))$, where $FP = G/B \times_{G/P_i} G/B$ is the fibre product in the diagram above. We deduce that

$$MC_y(Z \setminus \partial Z) = \lambda_y(T^*_\pi)\pi^*(MC_y((Z')^\circ)) - \pi^*(MC_y((Z')^\circ))\theta_1^*\big(\frac{\lambda_y(T^*_{pr_2})}{\lambda_y(N^*_{\mathcal{D}}(FP))} \otimes \mathcal{O}_D\big).$$

Now lemma 2.2 implies that

$$\frac{\lambda_y(T^*_{pr_2})}{\lambda_y((N_{\mathcal{D}}(FP))^*)} \otimes [\mathcal{O}_{\mathcal{D}}] = [\mathcal{O}_{\mathcal{D}}],$$

therefore

$$MC_y(Z \setminus \partial Z) = \lambda_y(T^*_\pi)\pi^*(MC_y((Z')^\circ)) - \pi^*(MC_y((Z')^\circ)) \otimes \mathcal{O}_D. \tag{14}$$

A standard diagram chase, using projection formula, the base-change formula $(\theta_1)_*\pi^* = pr_2^*\theta'_*$, and the fact that $\theta'_*(MC_y((Z')^\circ)) = MC_y(X(w)^\circ)$ (by functoriality), gives that

$$(\theta_1)_*MC_y(Z \setminus \partial Z) = \lambda_y(T^*_{pr_2})pr_2^*MC_y(X(w)^\circ) - pr_2^*MC_y(X(w)^\circ) \otimes \mathcal{O}_{\mathcal{D}}.$$

Applying $(pr_1)_*$ to both sides, and observing that $T^*_{pr_2} = pr_1^*T^*_{p_i}$ gives

$$\theta_*MC_y(Z \setminus \partial Z) = \lambda_y(T^*_{p_i})(pr_1)_*pr_2^*MC_y(X(w)^\circ) - (pr_1)_*(pr_2^*MC_y(X(w)^\circ) \otimes \mathcal{O}_{\mathcal{D}}).$$

But $(pr_1)_*pr_2^* = \partial_i$ and the restriction $(pr_2)_{|\mathcal{D}}$ to the diagonal $\mathcal{D} \simeq G/B$ is the identity $G/B \to G/B$, therefore the last expression is

$$(\lambda_y(T^*_{p_i})\partial_i - id)MC_y(X(w)^\circ) = \mathcal{T}_iMC_y(X(w)^\circ),$$

as claimed. □

We record the following corollary.

Corollary 4.6. *Let $w \in W$ and s_i a simple reflection. Then*

$$\mathcal{T}_i(MC_y(X(w)^\circ)) = \begin{cases} MC_y(X(ws_i)^\circ) & \text{if } ws_i > w; \\ -(y+1)MC_y(X(w)^\circ) - yMC_y(X(ws_i)^\circ) & \text{if } ws_i < w. \end{cases}$$

Proof. The identity from the $ws_i > w$ branch was proved in Theorem 4.5. Assume that $ws_i < w$. Then same theorem shows that $MC_y(X(w)^\circ) = \mathcal{T}_i(MC_y(X(ws_i)^\circ))$, thus

$$\mathcal{T}_i(MC_y(X(w)^\circ)) = \mathcal{T}_i^2(MC_y(X(ws_i)^\circ)).$$

By the quadratic relations from Proposition 3.4, $\mathcal{T}_i^2 = -(y+1)\mathcal{T}_i - y \cdot id$. Now we apply the right hand side to $MC_y(X(ws_i)^\circ)$, using again Theorem 4.5 and that $(ws_i)s_i > ws_i$. □

4.2. **Motivic Chern classes in G/P.** Let $P \supset B$ be a parabolic subgroup containing B and let $W^P \subset W$ be the subset of minimal length representatives for W/W_P, the quotient of W by the subgroup W_P generated by the reflections in P. For $wW_P \in W/W_P$, $\ell(wW_P)$ denotes the length of the (unique) representative of wW_P in W^P. The Schubert cells in G/P are $X(wW_P)^\circ := BwP/P \subset G/P$; then $X(wW_P)^\circ \simeq \mathbb{A}^{\ell(wW_P)}$. The natural projection $\pi : G/B \to G/P$ sends $X(w)^\circ$ to $X(wW_P)^\circ$ and it is an isomorphism if $w \in W^P$. From this and functoriality of motivic Chern classes it follows that

$$\pi_*MC_y[X(w)^\circ \hookrightarrow G/B] = MC_y[X(wW_P)^\circ \hookrightarrow G/P], \forall w \in W^P.$$

Remark 4.7. In fact, one can prove more: the restriction $\pi|_{X(w)^\circ} : X(w)^\circ \to X(wW_P)^\circ$ is a trivial fibration with fibre a Schubert cell of dimension $\ell(w)-\ell(wW_P)$ in $\pi^{-1}(e_{wW_P}) \simeq P/B$, regarded as a homogeneous space for the Levi subgroup of P. It is not difficult to show that this implies that for all $w \in W$,

$$\pi_*MC_y[X(w)^\circ \hookrightarrow G/B] = (-y)^{\ell(w)-\ell(wW_P)}MC_y[X(wW_P)^\circ \hookrightarrow G/P].$$

Details of the proof and applications to point counting in characteristic p will be included in a continuation to this paper.

5. The Hecke duality for motivic Chern classes

It was proved in [AMSS17, §5] that the Poincaré duals of the CSM classes of Schubert cells are given by the operators which are adjoint to the Hecke-type operators which determine the CSM classes. Same phenomenon holds in K-theory, with the same idea of proof. The difference in K-theory comes from the twist arising from the quadratic relations, resulting in calculations which are somewhat more involved. In analogy with the dual CSM class from [AMSS17, Def. 5.3] we make the following definition.

Definition 5.1. Let $w \in W$. The *dual motivic Chern class* is defined by

$$MC_y^\vee(Y(w)^\circ) := (\mathcal{T}_{w_0w}^\vee)^{-1}(MC_y(Y(w_0))) = (\mathcal{T}_{w_0w}^\vee)^{-1}(\mathcal{O}^{w_0}) \in K_T(G/B)[y, y^{-1}].$$

The name of this class is explained by the following theorem, which is the K-theoretic analogue of [AMSS17, Theorem 5.7].

Theorem 5.2. *For any $u, v \in W$,*

$$\langle MC_y(X(u)^\circ), MC_y^\vee(Y(v)^\circ)\rangle = \delta_{u,v}(-y)^{\ell(u)-\dim G/B}\prod_{\alpha>0}(1+ye^{-\alpha}).$$

Remark 5.3. Geometrically, the quantity $\prod_{\alpha>0}(1+ye^{-\alpha})$ equals $\lambda_y(T^*_{w_0}(G/B))$, i.e. it is the λ_y class of the fibre of the cotangent bundle at w_0.

Proof of Theorem 5.2. Using the definition of both flavors of motivic classes, and the fact that $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are adjoint to each other, we obtain that

$$\langle MC_y(X(u)^\circ), MC_y^\vee(Y(v)^\circ)\rangle = \langle \mathcal{T}_{u^{-1}}(\mathcal{O}_{id}), (\mathcal{T}_{w_0v}^\vee)^{-1}(\mathcal{O}^{w_0})\rangle = \langle \mathcal{O}_{id}, \mathcal{T}_u^\vee \cdot (\mathcal{T}_{w_0v}^\vee)^{-1}(\mathcal{O}^{w_0})\rangle.$$

By proposition 3.5,

$$\mathcal{T}_u^\vee \cdot (\mathcal{T}_{w_0v}^\vee)^{-1} = c_{uv^{-1}w_0}(y)\mathcal{T}_{uv^{-1}w_0}^\vee + \sum_{w<uv^{-1}w_0} c_w(y)\mathcal{T}_w^\vee.$$

Since $\mathcal{T}_w^\vee(\mathcal{O}^{w_0})$ contains Schubert classes $\mathcal{O}^{w'}$ such that $w \leq w'$ and $\langle \mathcal{O}_{id}, \mathcal{O}^w\rangle = \delta_{w,id}$, $\langle \mathcal{O}_{id}, \mathcal{T}_u^\vee \cdot (\mathcal{T}_{w_0v}^\vee)^{-1}(\mathcal{O}^{w_0})\rangle$ is 0 unless $uv^{-1}w_0 = w_0$, i.e. $u = v$. In this case, by Equation (13), the coefficient of $\mathcal{O}^{id}$ in $\mathcal{T}_{w_0}^\vee(\mathcal{O}^{w_0})$

$$\prod_{\alpha>0}(1+ye^{-\alpha}).$$

By Proposition 3.5, the coefficient of $\mathcal{T}_{w_0}^\vee$ in $\mathcal{T}_u^\vee \cdot (\mathcal{T}_{w_0u}^\vee)^{-1}$ is $(-y)^{-\ell(w_0u)}$. Therefore,

$$\begin{aligned}\langle MC_y(X(u)^\circ), MC_y^\vee(Y(u)^\circ)\rangle &= \langle \mathcal{O}_{id}, \mathcal{T}_u^\vee \cdot (\mathcal{T}_{w_0u}^\vee)^{-1}(\mathcal{O}^{w_0})\rangle \\ &= \langle \mathcal{O}_{id}, (-y)^{-\ell(w_0u)}T_{w_0}^\vee(\mathcal{O}^{w_0})\rangle \\ &= \langle \mathcal{O}_{id}, (-y)^{-\ell(w_0u)}\prod_{\alpha>0}(1+ye^{-\alpha})\mathcal{O}^{id}\rangle \\ &= (-y)^{\ell(u)-\dim G/B}\prod_{\alpha>0}(1+ye^{-\alpha}).\end{aligned}$$

□

Remark 5.4. It is natural to consider the normalized class

$$\widetilde{MC}_y(Y(w)^\circ) := (-y)^{\dim G/B-\ell(w)}MC_y^\vee(Y(w)^\circ).$$

The classes $\widetilde{MC}_y(Y(w)^\circ)$ are given by the normalized operator $\mathcal{L}_i := \mathcal{T}_i^\vee + (1+y)id$; cf. equation (11). The Schubert expansion of this class has coefficients which are polynomial in y.

Example 5.5. The motivic Chern class for Schubert cells in Fl(3) were listed in the introduction. The normalized dual motivic classes $\widetilde{MC}_y(Y(w)^\circ)$ for the Schubert cells in Fl(3) are:

$$\begin{aligned}
\widetilde{MC}_y(Y(w_0)) =&\mathcal{O}^{w_0};\\
\widetilde{MC}_y(Y(s_1s_2)^\circ) =&(1+y)\mathcal{O}^{s_1s_2}+y\mathcal{O}^{w_0};\\
\widetilde{MC}_y(Y(s_2s_1)^\circ) =&(1+y)\mathcal{O}^{s_2s_1}+y\mathcal{O}^{w_0};\\
\widetilde{MC}_y(Y(s_1)^\circ) =&(1+y)^2\mathcal{O}^{s_1}+y(1+y)\mathcal{O}^{s_1s_2}+2y(1+y)\mathcal{O}^{s_2s_1}+y^2\mathcal{O}^{w_0};\\
\widetilde{MC}_y(Y(s_2)^\circ) =&(1+y)^2\mathcal{O}^{s_2}+2y(1+y)\mathcal{O}^{s_1s_2}+y(1+y)\mathcal{O}^{s_2s_1}+y^2\mathcal{O}^{w_0};\\
\widetilde{MC}_y(Y(id)^\circ) =&(1+y)^3\mathcal{O}^{id}+y(1+y)^2(\mathcal{O}^{s_1}+\mathcal{O}^{s_2})+\\
&2y^2(1+y)(\mathcal{O}^{s_1s_2}+\mathcal{O}^{s_2s_1})+y^3\mathcal{O}^{w_0}.
\end{aligned}$$

An algebra check together with fact that $\langle \mathcal{O}_u, \mathcal{O}^w\rangle = 1$ if and only if $u \geq w$, shows that the pairing

$$\langle MC_y(X(u)^\circ), \widetilde{MC}_y(Y(w)^\circ)\rangle = (1+y)^{\dim \mathrm{Fl}(3)}\delta_{u,w},$$

as predicted by Theorem 5.2. (In this situation, the equivariant variables $e^\alpha \mapsto 1$.) At this time we note that the analogue of the positivity Conjecture 1 is false for the dual classes. For instance the coefficient of $\mathcal{O}^{s_3s_1s_2}$ in the expansion of $\widetilde{MC}_y(Y(id)^\circ) \in K(\mathrm{Fl}(4))$ equals $y^2(4y-1)(1+y)^3$.

In the next result we determine the action of the operators $\mathcal{T}_i^\vee$ on the dual motivic classes.

Proposition 5.6. *Let $w \in W$ be a Weyl group element and s_i a simple reflection. Then the following equalities hold:*

(a)

$$\mathcal{T}_i^\vee(MC_y^\vee(Y(w)^\circ)) = \begin{cases} MC_y^\vee(Y(ws_i)^\circ) & \textit{if } ws_i > w;\\ -(y+1)MC_y^\vee(Y(w)^\circ) - yMC_y^\vee(Y(ws_i)^\circ) & \textit{if } ws_i < w.\end{cases}$$

(b)

$$(\mathcal{T}_i^\vee)^{-1}(MC_y^\vee(Y(ws_i)^\circ)) = \begin{cases} MC_y^\vee(Y(w)^\circ) & \textit{if } ws_i > w;\\ -\frac{1}{y}MC_y^\vee(Y(w)^\circ) - \frac{y+1}{y}MC_y^\vee(Y(ws_i)^\circ) & \textit{if } ws_i < w.\end{cases}$$

Proof. To prove the equality (a), consider first the case when $ws_i > w$. Then $w_0ws_i < w_0w$, thus $\mathcal{T}_{w_0w}^\vee = \mathcal{T}_{w_0ws_i}^\vee\mathcal{T}_i^\vee$. By definition 5.1 we obtain:

$$\mathcal{T}_i^\vee(MC_y^\vee(Y(w)^\circ)) = \mathcal{T}_i^\vee(\mathcal{T}_{w_0w}^\vee)^{-1}(\mathcal{O}^{w_0}) = (\mathcal{T}_{w_0ws_i}^\vee)^{-1}(\mathcal{O}^{w_0}) = MC_y^\vee(Y(ws_i)^\circ)).$$

The situation when $ws_i < w$ is treated as in the proof of Corollary 4.6, using that $\mathcal{T}_i^\vee$ satisfies the quadratic relations from Proposition 3.4. Part (b) follows from (a) by applying $(\mathcal{T}_i^\vee)^{-1}$ to both sides. $\square$

6. Recursions for localizations of motivic Chern classes

In this section, we use the Demazure Lusztig operators $\mathcal{T}_i$ to obtain recursive relations for the ordinary and dual motivic Chern classes of Schubert cells. These recursions will be used to compare the motivic Chern classes both with the stable envelopes, and with the Casselman's basis. We also record a divisibility property for localizations of motivic classes, to be used later in the proof of Theorem 1.4.

6.1. **Recursions.** Consider the localized equivariant K theory ring defined by $K_T(G/B)_{loc} := K_T(G/B)\otimes_{K_T(pt)}\mathrm{Frac}(K_T(pt))$. The K-theoretic analogue of Bott localization theorem (see e.g. [CG09, §5.10]) gives the expansion of the motivic Chern classes in terms of the fixed point classes ι_w, for any $w \in W$:

$$\begin{aligned} MC_y(X(w)^\circ) &= \sum_{u\leq w} MC_y(X(w)^\circ)|_u \frac{\iota_u}{\lambda_{-1}(T^*_u(G/B))} \\ &= \sum_{u\leq w} MC_y(X(w)^\circ)|_u \frac{\iota_u}{\prod_{\alpha>0}(1-e^{u\alpha})}. \end{aligned} \tag{15}$$

The following three propositions give recursions formulas for various flavors of motivic Chern classes. These will be used later to make the connection with the Hecke algebra action on the principal series representation. The recursions are very similar, and this similarity can be explained by the fact that they are related either by an automorphism of G/B, or by the involution exchanging the Demazure-Lusztig operators.

Proposition 6.1. *The localizations $MC_y(X(w)^\circ)|_u$ are uniquely determined by the following conditions:*

(a) $MC_y(X(w)^\circ)|_u = 0$, *unless* $u \leq w$.

(b)
$$MC_y(X(w)^\circ)|_w = \prod_{\alpha>0, w\alpha<0}(1+ye^{w\alpha})\prod_{\alpha>0,w\alpha>0}(1-e^{w\alpha}).$$

(c) *If* $ws_i > w$, *then*
$$MC_y(X(ws_i)^\circ)|_u = -\frac{1+y}{1-e^{-u\alpha_i}}MC_y(X(w)^\circ)|_u + \frac{1+ye^{u\alpha_i}}{1-e^{-u\alpha_i}}MC_y(X(w)^\circ)|_{us_i}.$$

Proof. Part (a) follows because the motivic class is supported on the Schubert variety $X(w)$. To prove part (b), observe that $MC_y(X(w)^\circ)|_w = MC_y(X(w))|_w$, by additivity and because $MC_y(X(v)^\circ)|_w = 0$ for $v < w$, as the latter class is supported on $X(v)$. Then

$$MC_y(X(w)^\circ)|_w = MC_y(X(w))|_w = \lambda_y(T^*_w X(w))\lambda_{-1}(N^\vee_w),$$

where $T^*_w X(w)$ and $N^\vee_w$ are the fibers at the fixed point w of the dual of the cotangent, respectively the conormal bundle for $X(w)$. (A more general result is proved in Theorem 8.1 below.) The last part follows by applying the operator $\mathcal{T}_i$ to Equation (15) and taking the coefficients of ι_u; this requires the action of $\mathcal{T}_i$ on the fixed point basis described in Lemma 3.7. Finally, the uniqueness follows by induction on the length of w. □

For further use, we also record the similar result for the motivic Chern class of the opposite Schubert cells.

Proposition 6.2. *The localizations $MC_y(Y(w)^\circ)|_u$ are uniquely determined by the following conditions:*

(1) $MC_y(Y(w)^\circ)|_u = 0$, *unless* $u \geq w$.

(2)
$$MC_y(Y(w)^\circ)|_w = \prod_{\alpha>0, w(\alpha)>0}(1+ye^{w\alpha})\prod_{\alpha>0,w(\alpha)<0}(1-e^{w\alpha}).$$

(3) *If* $ws_i > w$, *then*
$$MC_y(Y(w)^\circ)|_u = -\frac{1+y}{1-e^{-u\alpha_i}}MC_y(Y(ws_i)^\circ)|_u + \frac{1+ye^{u\alpha_i}}{1-e^{-u\alpha_i}}MC_y(Y(ws_i)^\circ)|_{us_i}.$$

Proof. The left multiplication by w_0 induces an automorphism of G/B sending $X(w)$ to $Y(w_0w)$. This is not T-equivariant, but it is equivariant with respect to the map $T \to T$ sending $t \mapsto w_0tw_0$. This induces an automorphism of $K_T(G/B)$ and its localized version, twisting the coefficients by w_0. Then the proposition follows from Proposition 6.1 above, by applying w_0. □

Proposition 6.3. *The localizations $MC_y^\vee(Y(w)^\circ)|_u$ are uniquely determined by the following conditions:*

(1) $MC_y^\vee(Y(w)^\circ)|_u = 0$, *unless* $u \geq w$.

(2)
$$MC_y^\vee(Y(w)^\circ)|_w = (-1)^{\dim G/B-\ell(w)} \prod_{\alpha>0,w\alpha>0}(y^{-1}+e^{-w\alpha}) \prod_{\alpha>0,w\alpha<0}(1-e^{w\alpha}).$$

(3) *If $ws_i > w$, then*
$$MC_y^\vee(Y(w)^\circ)|_u = \frac{1+y^{-1}}{e^{u\alpha_i}-1} MC_y^\vee(Y(ws_i)^\circ)|_u + \frac{y^{-1}+e^{-u\alpha_i}}{e^{-u\alpha_i}-1} MC_y^\vee(Y(ws_{\alpha_i})^\circ)|_{us_{\alpha_i}}.$$

Proof. The uniqueness follows directly from induction. So we only need to show $MC_y^\vee(Y(w)^\circ)|_u$ satisfies these properties. The support property follows because $(\mathcal{T}_i^\vee)^{-1}$ sends a Schubert class $\mathcal{O}^u$ into classes supported on $Y(u) \cup Y(us_i)$; then one applies Proposition 5.6. To calculate the localization at w, we use the duality from theorem 5.2 and Bott localization to obtain

$$\begin{aligned}(-y)^{\ell(w)-\dim G/B}\prod_{\alpha>0}(1+ye^{-\alpha}) &= \langle MC_y(X(w)^\circ), MC_y^\vee(Y(w)^\circ)\rangle \\ &= \sum_{u\in W} \frac{(MC_y(X(w)^\circ)\cdot MC_y^\vee(Y(w)^\circ))|_u}{\prod_{\alpha>0}(1-e^{u(\alpha)})} \cdot \int_{G/B} \iota_u \ .\end{aligned}$$

The only non-zero contribution is for $u = w$, and the integral equals 1, thus

$$MC_y^\vee(Y(w)^\circ)|_w = \frac{(-y)^{\ell(w)-\dim G/B}\prod_{\alpha>0}(1+ye^{-\alpha})(1-e^{w(\alpha)})}{MC_y(X(w)^\circ)|_w}.$$

Part (b) follows from this and the localization from Proposition 6.1. Part (c) follows as in Proposition 6.1, using now Proposition 5.6 and part (e) of Lemma 3.7. □

6.2. **A divisibility property for localization coefficients.** We record the following property which will be used in our applications to p-adic groups.

Theorem 6.4. *For any $w \leq u \in W$, the Laurent polynomial $MC_y(Y(w)^\circ)|_u$ is divisible by*
$$\prod_{\alpha>0,u\alpha>0}(1+ye^{u\alpha}) \prod_{\alpha>0,w\nleq us_\alpha<u}(1-e^{u\alpha}).$$

Proof. [1] By a general property of motivic classes proved in [FRW18, Theorem 5.3(ii)], the localization coefficient $MC_y(Y(w)^\circ)|_u$ is divisible by $\lambda_y(T_u^*Y(u)^\circ) = \prod_{\alpha>0,u\alpha>0}(1+ye^{u\alpha})$. As α varies in the set of positive roots, the factors $1-e^{u\alpha}$ and $1+ye^{u\alpha}$ are relative prime to each other. Then it remains to show that for any $\alpha > 0$ such that $w \nleq us_\alpha < u$, the

[1] We thank A. Okounkov for comments leading to this proof.

localization coefficient $MC_y(Y(w)^\circ)|_u$ is divisible by $1-e^{u\alpha}$. Let $C \simeq \mathbb{P}^1$ denote the T-stable curve connecting the fixed points e_u and e_{us_α}. The T-weight of the tangent space T_uC is $-u\alpha$. By localization, the equivariant Euler characteristic

$$\chi(C, MC_y(Y(w)^\circ)) = \frac{MC_y(Y(w)^\circ)|_u}{1-e^{u\alpha}} + \frac{MC_y(Y(w)^\circ)|_{us_\alpha}}{1-e^{-u\alpha}} = \frac{MC_y(Y(w)^\circ)|_u}{1-e^{u\alpha}},$$

where the last equality holds because $MC_y(Y(w)^\circ)|_{us_\alpha} = 0$, as the hypothesis $w \not\leq us_\alpha$ implies that $e_{us_\alpha} \notin Y(w)$. Since the Euler characteristic $\chi(C, MC_y(Y(w)^\circ))$ is a Laurent polynomial, $MC_y(Y(w)^\circ)|_u$ must be divisible by $1-e^{u\alpha}$. This finishes the proof. ☐

7. Motivic Chern classes and K-theoretic stable envelopes

In this section, we recall some basic properties of the K-theoretic stable basis of $T^*(G/B)$, including a recursive relation. Our main references are [SZZ17, Oko15, OS16]). We compare the recursive relation obtained in [SZZ17] to the one for motivic Chern classes, and we deduce that the two objects are closely related. This was also found by Féher and Rimányi and Weber in [FRW18] (see also [FRW]), using interpolation techniques for motivic Chern classes. In cohomology, the relation between stable envelopes and Chern-Schwartz-MacPherson classes was noticed in [RV, AMSS17] and it was used in [AMSS17] to obtain a second 'stable basis duality'. In theorem 7.10 we generalize this duality to K theory.

7.1. **K-theoretic stable envelopes.** The cotangent bundle of G/B is the homogeneous bundle $T^*(G/B) := G \times^B T^*_{1.B}(G/B)$ which is given by equivalence classes

$$\{[g,v] : (g,v) \in G \times T^*_{1.B}(G/B) \text{ and } (gb,v) \sim (g, b.v), \forall g \in G, b \in B\};$$

here $T^*_{1.B}(G/B)$ is the cotangent space at the identity with its natural B-module structure. As before, let T be the maximal torus in B, and consider the $\mathbb{C}^*$-module $\mathbb{C}$ with character $q^{1/2}$. We let $\mathbb{C}^*$ act trivially on G/B and we consider the $T\times\mathbb{C}^*$ action on the cotangent bundle defined by $(t,z).[g,v] = [tg, z^{-2}v]$. In other words, T acts via its natural left action; while $\mathbb{C}^*$ acts such that the cotangent fibres get a weight q^{-1} and $K_{T\times\mathbb{C}^*}(pt) = K_T(pt)[q^{\pm 1/2}]$.

The stable basis is a certain basis for the localized equivariant K theory

$$K_{T\times\mathbb{C}^*}(T^*(G/B))_{loc} := K_{T\times\mathbb{C}^*}(T^*(G/B)) \otimes_{K_{T\times\mathbb{C}^*}(pt)} \mathrm{Frac}(K_{T\times\mathbb{C}^*}(pt)),$$

where Frac means taking the fraction field. The basis elements are called the *stable envelopes* $\{\mathrm{stab}_{\mathcal{C},T^{\frac{1}{2}},\mathcal{L}}(w) | w \in W\}$ and were defined by Maulik and Okounkov in the cohomological case. We recall their definition in K theory below, following mainly Okounkov's lectures [Oko15] and [SZZ17].

For a fixed Weyl group element, the definition of the stable envelope $\mathrm{stab}_{\mathcal{C},T^{\frac{1}{2}},\mathcal{L}}(w)$ depends on three parameters:

- a chamber $\mathcal{C}$ in the Lie algebra of the maximal torus T, or equivalently, a Borel subgroup of G.
- a polarization $T^{\frac{1}{2}} \in K_{T\times\mathbb{C}^*}(T^*(G/B))$ of the tangent bundle $T(T^*(G/B))$, i.e., a Lagrangian subbundle $T^{\frac{1}{2}}$ of the tangent bundle $T(T^*(G/B)) \in K_{T\times\mathbb{C}^*}(T^*(G/B))$ which is a solution of the equation
$$T^{\frac{1}{2}} + q^{-1}(T^{\frac{1}{2}})^\vee = T(T^*(G/B))$$
in the ring $K_{T\times\mathbb{C}^*}(T^*(G/B))_{loc}$. The most important solutions are $T(G/B)$ and $T^*(G/B)$. For any polarization $T^{\frac{1}{2}}$, there is an opposite polarization defined as $T^{\frac{1}{2}}_{\mathrm{opp}} = q^{-1}(T^{\frac{1}{2}})^\vee$.

- A sufficiently general fractional equivariant line bundle on G/B, i.e. a general element $\mathcal{L} \in \mathrm{Pic}_T(T^*(G/B)) \otimes_{\mathbb{Z}} \mathbb{Q}$, called the *slope* of the stable envelope. The dependence on the slope parameter is locally constant, in the following sense.

  The choice of a maximal torus $T \subset G$ determines a decomposition of $(\mathrm{Lie}\, T)^* \otimes \mathbb{R}$ into *alcoves*; these are the complement of the affine hyperplanes $H_{\alpha^\vee,n} = \{\lambda \in (\mathrm{Lie}\, T)^* \otimes \mathbb{R} : \langle \lambda, \alpha^\vee \rangle = n\}$ as $\alpha^\vee$ varies in the set of positive coroots, and n over the integers. The alcove structure is independent the choice of a chamber (and hence of the Borel subgroup B), and the stable envelopes are constant for fractional multiples of (pull-backs of) line bundles $\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda$ for weights λ in a given alcove.

The torus fixed point set $(T^*(G/B))^T$ is in one-to-one correspondence with the Weyl group W. For any $w \in W$, we still use w to denote the corresponding fixed point. For a chosen Weyl chamber $\mathfrak{C}$ in $\mathrm{Lie}\, T$, pick any cocharacter $\sigma \in \mathfrak{C}$. Then the attracting set of $w \in W$ is defined as

$$\mathrm{Attr}_{\mathfrak{C}}(w) = \left\{ x \in T^*(G/B) \mid \lim_{z\to 0} \sigma(z)\cdot x = w \right\}.$$

It is not difficult to show that $\mathrm{Attr}_{\mathfrak{C}}(w)$ is the conormal space over the attracting variety in G/B for w; the latter attracting variety is a Schubert cell in G/B. Define a partial order on the fixed point set W as follows:

$$w \preceq_{\mathfrak{C}} v \quad \text{if} \quad \overline{\mathrm{Attr}_{\mathfrak{C}}(v)} \cap w \neq \emptyset.$$

Then the order determined by the positive (resp. negative) chamber is the same as the Bruhat order (resp. the opposite Bruhat order).

Any chamber $\mathfrak{C}$ determines a decomposition of the tangent space $N_w := T_w(T^*(G/B))$ as $N_w = N_{w,+} \oplus N_{w,-}$ into T-weight spaces which are positive and negative with respect to $\mathfrak{C}$ respectively. For any polarization $T^{\frac{1}{2}}$, denote $N_w^{\frac{1}{2}}$ by $N_w \cap T^{1/2}|_w$. Similarly, we have $N_{w,+}^{\frac{1}{2}}$ and $N_{w,-}^{\frac{1}{2}}$. In particular, $N_{w,-} = N_{w,-}^{\frac{1}{2}} \oplus q^{-1}(N_{w,+}^{\frac{1}{2}})^\vee$. Consequently, we have

$$N_{w,-} - N_w^{\frac{1}{2}} = q^{-1}(N_{w,+}^{\frac{1}{2}})^\vee - N_{w,+}^{\frac{1}{2}}$$

as virtual vector bundles. The determinant bundle of the virtual bundle $N_{w,-} - N_w^{\frac{1}{2}}$ is a complete square and its square root will be denoted by $\left(\frac{\det N_{w,-}}{\det N_w^{\frac{1}{2}}}\right)^{\frac{1}{2}}$; [Oko15, §9.1.5]. For instance, if we choose the polarization $T^{1/2} = T(G/B)$, the positive chamber, and $w = id$ then both $N_{id}^{\frac{1}{2}}$ and $N_{id,-}$ have weights $-\alpha$, where α varies in the set of positive roots; in this case the virtual bundle $N_{id,-} - N_{id}^{\frac{1}{2}}$ is trivial. To calculate the localizations at other fixed points w, one may use the equivariance; see e.g. [SZZ17, Lemma 2.2] for such calculations.

For a Laurent polynomial $f := \sum_\mu f_\mu e^\mu \in K_{T\times\mathbb{C}^*}(\mathrm{pt})$, where $e^\mu \in K_T(\mathrm{pt})$ and $f_\mu \in \mathbb{Q}[q^{1/2}, q^{-1/2}]$, we define its *Newton Polygon*, denoted by $\deg_T f$ to be

$$\deg_T f = \textit{Convex hull}(\{\mu | f_\mu \neq 0\}) \subset X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The following theorem defines the K-theoretic stable envelopes.

Theorem 7.1. [Oko15] *For any chamber $\mathfrak{C}$, a sufficiently general $\mathcal{L}$, and a polarization $T^{1/2}$, there exists a unique map of $K_{T\times\mathbb{C}^*}(pt)$-modules*

$$\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}} : K_{T\times\mathbb{C}^*}((T^*(G/B))^T) \to K_{T\times\mathbb{C}^*}(T^*(G/B))$$

such that for any $w \in W$, $\Gamma = \mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(w)$ satisfies:

(1) *(support)* $\operatorname{Supp}\Gamma \subset \cup_{z \preceq_{\mathfrak{C}} w} \overline{\operatorname{Attr}_{\mathfrak{C}}(z)}$;

(2) *(normalization)* $\Gamma|_w = (-1)^{\operatorname{rank} N^{\frac{1}{2}}_{w,+}} \left(\frac{\det N_{w,-}}{\det N_w^{\frac{1}{2}}}\right)^{\frac{1}{2}} \mathcal{O}_{\operatorname{Attr}_{\mathfrak{C}}(w)}|_w$;

(3) *(degree)* $\deg_T \Gamma|_v \subseteq \deg_T \operatorname{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, \mathcal{L}}(v)|_v + \mathcal{L}|_v - \mathcal{L}|_w$, *for any* $v \prec_{\mathfrak{C}} w$.

The difference $\mathcal{L}|_v - \mathcal{L}|_w$ in the degree condition implies that the stable basis does not depend on the choice of the linearization of $\mathcal{L}$.

Let $+$ denote the chamber such that all the roots in B are positive on it, and $-$ denote the opposite chamber. From now on we fix the 'fundamental slope' given by $\mathcal{L} := \mathcal{L}_\rho \otimes 1/N$, where ρ is the sum of fundamental weights and N is a large enough positive integer. Recall that $\omega_{G/B} := \mathcal{L}_{2\rho}$ is the canonical bundle of G/B, therefore the slope $\mathcal{L}$ can also be thought as a (fractional version of a) square root of the canonical line bundle. We will use the following notations:

$$\operatorname{stab}_+(w) := \operatorname{stab}_{+, T(G/B), \mathcal{L}^{-1}}(w), \text{ and } \quad \operatorname{stab}_-(w) := \operatorname{stab}_{-, T^*(G/B), \mathcal{L}}(w).$$

The positive chamber and negative chamber stable basis are dual bases in the localized equivariant ring, i.e.,

$$\langle \operatorname{stab}_+(w), \operatorname{stab}_-(u) \rangle_{T^*(G/B)} = \delta_{w,u}, \tag{16}$$

where $\langle \cdot, \cdot \rangle_{T^*(G/B)}$ is the equivariant K theory pairing on $T^*(G/B)$ defined via localization; see [Oko15, Ex. 9.1.17],[OS16, §2.2.1, Prop. 1] or [SZZ17, Remark 1.3]). We will study the pairing on $K_{T\times\mathbb{C}^*}(T^*(G/B))$ in more detail in section 7.5 below.

7.2. **Automorphisms.** The stable envelopes for various triples of parameters can be related to each other by automorphisms of the equivariant K theory ring $K_{T\times\mathbb{C}^*}(T^*(G/B))$. We will use the following types of automorphisms:

a. the automorphism induced by the left Weyl group multiplication. Recall that this induces an automorphism of $K_{T\times\mathbb{C}^*}(T^*(G/B))$ which twists the coefficients in $K_{T\times\mathbb{C}^*}(pt)$ by w. In terms of localization, for any $\mathcal{F} \in K_{T\times\mathbb{C}^*}(T^*(G/B))$, we have

$$w(\mathcal{F})|_u = w(\mathcal{F}|_{w^{-1}u}). \tag{17}$$

b. The duality automorphism, taking $[E] \mapsto [E^\vee]$, i.e. the class of a vector bundle to its dual; this automorphism also acts on $K_{T\times\mathbb{C}^*}(pt)$ by taking $e^\lambda \mapsto e^{-\lambda}$ and $q \mapsto q^{-1}$.

c. The multiplication by the class of a line bundle $\mathcal{L}$. Often we will fix an integral weight $\lambda \in X^*(T)$ and a Borel subgroup B, and associate the equivariant line bundle $\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda$. We will abuse notation and will denote with the same symbol a line bundle on G/B and on its cotangent bundle.

d. For the ring $K_T(G/B)[q, q^{-1}]$, a composition of the previous two automorphisms gives the Grothendieck-Serre duality. This is a $K_T(pt)$-module automorphism $\mathcal{D}$ on $K_T(G/B)[q, q^{-1}]$ defined as follows. For any $\mathcal{F} \in K_T(G/B)$, define

$$\mathcal{D}(\mathcal{F}) := RHom(\mathcal{F}, \omega^\bullet_{G/B}) = \omega^\bullet_{G/B} \otimes \mathcal{F}^\vee \in K_T(G/B),$$

where $\omega^\bullet_{G/B} \simeq \omega_{G/B}[\dim G/B]$ is the dualizing complex of the flag variety (the canonical bundle $\omega_{G/B} = \mathcal{L}_{2\rho}$ shifted by dimension). The class $\mathcal{F}^\vee$ is obtained by taking an equivariant resolution of $\mathcal{F}$ by vector bundles, and then taking duals. Observe that

$$(\mathcal{F}^\vee)^\vee = \mathcal{F}; \quad \mathcal{D}(\mathcal{F} \otimes \omega^\bullet_{G/B}) = \mathcal{F}^\vee.$$

Extend the operations $\mathcal{D}$ and $(-)^\vee$ to $K_T(G/B)[q, q^{-1}]$ by sending q to q^{-1}.

The following lemma, proved in the Appendix, records the effect of these automorphisms.

Lemma 7.2. *(a) Let $u, w \in W$. Under the left Weyl group multplication,*

$$w.stab_{\mathcal{C},T^{1/2},\mathcal{L}}(u) = stab_{w\mathcal{C},wT^{1/2},w.\mathcal{L}}(wu).$$

In particular, if both the polarization $T^{1/2}$ and the line bundle $\mathcal{L}$ are G-equivariant, then

$$w.stab_{\mathcal{C},T^{1/2},\mathcal{L}}(u) = stab_{w\mathcal{C},T^{1/2},\mathcal{L}}(wu).$$

(b) The duality automorphism acts by sending $q \mapsto q^{-1}$ and

$$(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(w))^\vee = q^{-\frac{\dim G/B}{2}} \mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}}_{opp},\mathcal{L}^{-1}}(w), \tag{18}$$

where $T^{\frac{1}{2}}_{opp} := q^{-1}(T^{\frac{1}{2}})^\vee$ is the opposite polarization; see [OS16, Equation (15)]*. I.e., this duality changes the polarization and slope parameters to the opposite ones, while keeping the chamber parameter invariant.*

(c) Let $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}_T(T^*(G/B))$ be any equivariant line bundles, let $w \in W$ and $a \in \mathbb{Q}$ a rational number. Then

$$\mathrm{stab}_{\mathcal{C},T^{1/2},a\mathcal{L}\otimes\mathcal{L}'}(w) = \frac{1}{\mathcal{L}'|_w}\mathcal{L}' \otimes \mathrm{stab}_{\mathcal{C},T^{1/2},a\mathcal{L}}(w),$$

as elements in $K_{T\times\mathbb{C}^*}(T^*(G/B))_{loc}$.

7.3. **Recursions for stable envelopes.** Because the $\mathbb{C}^*$-fixed locus of the cotangent bundle is the zero section (i.e. G/B), it follows that the torus fixed point locus $(T^*(G/B))^{T\times\mathbb{C}^*}$ coincides with the fixed locus $(G/B)^T$, a discrete set indexed by the Weyl group W. Therefore the equivariant K theory classes associated to fixed points form a basis in the localized ring $K_{T\times\mathbb{C}^*}(T^*(G/B))_{loc}$. In order to compare motivic Chern classes to stable envelopes, we need the following result proved in [SZZ17, Prop. 3.6].

Proposition 7.3. *The restriction coefficients $\mathrm{stab}_-(w)|_u$ are uniquely characterized by*

(1) $\mathrm{stab}_-(w)|_u = 0$*, unless $u \geq w$.*
(2) $\mathrm{stab}_-(w)|_w = q^{\frac{\ell(w)}{2}} \prod_{\alpha>0,w\alpha<0}(1-e^{-w\alpha}) \prod_{\alpha>0,w\alpha>0}(1-qe^{-w\alpha})$.
(3) *If $ws_i > w$, then*

$$q^{\frac{1}{2}} \mathrm{stab}_-(w)|_u = \frac{(1-q)}{1-e^{-u\alpha_i}} \mathrm{stab}_-(ws_i)|_u + \frac{1-qe^{-u\alpha_i}}{1-e^{u\alpha_i}} \mathrm{stab}_-(ws_i)|_{us_i}.$$

Applying parts (a) and (b) from Lemma 7.2, and from the definitions of the stable envelopes, we obtain that for any $u \in W$,

$$w_0.(\mathrm{stab}_-(u))^\vee = q^{-\frac{\dim G/B}{2}} \mathrm{stab}_+(w_0u). \tag{19}$$

Then we immediately obtain the following analogue of Proposition 7.3:

Proposition 7.4 ([SZZ17])**.** *The localizations $\mathrm{stab}_+(w)|_u$ are uniquely characterized by the following properties:*

(1) $\mathrm{stab}_+(w)|_u = 0$*, unless $u \leq w$.*
(2) $\mathrm{stab}_+(w)|_w = q^{\frac{\ell(w)}{2}} \prod_{\alpha>0,w\alpha<0}(1-q^{-1}e^{w\alpha}) \prod_{\alpha>0,w\alpha>0}(1-e^{w\alpha})$.
(3) *If $ws_i > w$, then*

$$q^{\frac{1}{2}} \mathrm{stab}_+(ws_i)|_u = \frac{q-1}{1-e^{u\alpha_i}} \mathrm{stab}_+(w)|_u - \frac{e^{u\alpha_i}-q}{1-e^{-u\alpha_i}} \mathrm{stab}_+(w)|_{us_i}.$$

7.4. **Motivic classes are pull-backs of stable envelopes.** One of the key formulas in [AMSS17] shows that the dual CSM class equals the Segre-Schwartz-MacPherson (SSM) class, up to a normalization coefficient. The proof of that identity is based on a transversality argument, which can be expressed either in terms of (cohomological) stable basis elements, or in terms of transversality of characteristic cycles. Same phenomenon happens in K-theory. Let $i : G/B \hookrightarrow T^*(G/B)$ is the inclusion of the zero section into the cotangent bundle. Define

$$\mathrm{stab}'_+(w) := \mathcal{D}(i^* \mathrm{stab}_+(w)) = (-1)^{\dim G/B}(i^* \mathrm{stab}_+(w))^\vee \otimes \mathcal{L}_{2\rho} \in K_T(G/B)[q, q^{-1}].$$

The following relates motivic Chern classes and stable envelopes and it is the K-theoretic analogue of the cohomological results from [AMSS17, Cor. 6.6] and [RV]. It is also equivalent to results from [FRW18], where it is shown that motivic Chern classes satisfy the same localization properties as the stable envelopes for a certain triple of parameters; cf. Remark 7.6 below.

Theorem 7.5. *For any $w \in W$, we have*

$$q^{-\frac{\ell(w)}{2}} \mathrm{stab}'_+(w) = MC_{-q^{-1}}(X(w)^\circ) \in K_T(G/B)[q, q^{-1}].$$

Proof. We compare localization properties of the motivic Chern classes with those for the Grothedieck-Serre dual of $\mathrm{stab}_+(w)$. We have that

$$\mathrm{stab}'_+(w)|_u = (-1)^{\dim G/B} e^{2u\rho}(\mathrm{stab}_+(w)|_u)|_{e^\lambda \to e^{-\lambda}, q \to q^{-1}}. \tag{20}$$

Then the corresponding result from Proposition 7.4 for $\mathrm{stab}'_+(w)$ is that the localizations $\mathrm{stab}'_+(w)|_u$ are uniquely characterized by the following properties

(1) $\mathrm{stab}'_+(w)|_u = 0$, unless $u \leq w$.
(2) $\mathrm{stab}'_+(w)|_w = q^{\frac{\ell(w)}{2}} \prod_{\alpha>0, w\alpha<0}(1 - q^{-1}e^{w\alpha}) \prod_{\alpha>0, w\alpha>0}(1 - e^{w\alpha})$.
(3) If $ws_{\alpha_i} > w$, then

$$q^{-\frac{1}{2}} \mathrm{stab}'_+(ws_{\alpha_i})|_u = \frac{q^{-1} - 1}{1 - e^{-u\alpha_i}} \mathrm{stab}'_+(w)|_u + \frac{1 - q^{-1}e^{u\alpha_i}}{1 - e^{-u\alpha_i}} \mathrm{stab}'_+(w)|_{us_{\alpha_i}}.$$

Comparing this with localizations of motivic Chern classes from Proposition 6.1 finishes the proof. ☐

Remark 7.6. For any $w \in W$, we have that

$$q^{\frac{-\ell(w)}{2}} i^* \mathrm{stab}_{+, T(G/B), \mathcal{L}}(w) = MC_{-q^{-1}}(X(w)^\circ), \tag{21}$$

where (recall) $\mathcal{L}$ is the fundamental slope. Indeed, from Theorem 7.5 and Lemma 7.2, it suffices to show that

$$q^{-(\dim G/B)/2}(-1)^{\dim G/B} \mathrm{stab}_{+, T^*(G/B), \mathcal{L}}(w) \otimes \mathcal{L}_{2\rho} = \mathrm{stab}_{+, T(G/B), \mathcal{L}}(w).$$

By the uniqueness of the stable basis, we only need to check the left hand side satisfies the defining properties of the stable basis on the right hand side. Since the two sides have the same chamber and slope, the support and degree conditions follow directly. The normalization condition is checked as follows:

$$
\begin{aligned}
&\left(q^{-(\dim G/B)/2}(-1)^{\dim G/B}\operatorname{stab}_{+,T^*(G/B),\mathcal{L}}(w)\otimes\mathcal{L}_{2\rho}\right)|_w\\
=&\left((-1)^{\dim G/B}(\operatorname{stab}_{+,T(G/B),\mathcal{L}^{-1}}(w))^\vee\otimes\mathcal{L}_{2\rho}\right)|_w\\
=&(-1)^{\dim G/B}q^{-\frac{\ell(w)}{2}}\prod_{\alpha>0,w\alpha<0}(1-qe^{-w\alpha})\prod_{\alpha>0,w\alpha>0}(1-e^{-w\alpha})e^{2w\rho}\\
=&q^{\frac{\ell(w)}{2}}\prod_{\alpha>0,w\alpha<0}(1-q^{-1}e^{w\alpha})\prod_{\alpha>0,w\alpha>0}(1-e^{w\alpha})\\
=&\operatorname{stab}_{+,T(G/B),\mathcal{L}}(w)|_w,
\end{aligned}
$$

where the first equality follows from Lemma 7.2, and the second and the last equalities follow from Proposition 7.4(2). While not explicitly stated, the left hand side of equation (21) corresponds to the parameters used in [FRW18] to index stable envelopes. In particular, this shows our results are compatible with those from *loc.cit.*

A similar statement relates $\operatorname{stab}_-(w)$ to the motivic Chern class of the opposite Schubert cells. We record the statement next; the proof is essentially the same, and details are left to the reader. Define

$$\operatorname{stab}'_-(w) := q^{-\dim G/B}i^*(\operatorname{stab}_-(w))\otimes\omega^\bullet_{G/B}\in K_T(G/B)[q,q^{-1}].$$

Theorem 7.7. *For any $w\in W$,*

$$q^{\frac{\ell(w)}{2}}\operatorname{stab}'_-(w) = MC_{-q^{-1}}(Y(w)^\circ)\in K_T(G/B)[q,q^{-1}].$$

Remark 7.8. We prove in the upcoming paper [AMSS] that

$$(-q)^{-\dim G/B}i^*(gr^H(i_{w!}\mathbb{Q}^H_{Y(w)^\circ}))\otimes\omega^\bullet_{G/B} = MC_{-q^{-1}}(Y(w)^\circ),$$

where $i_w : Y(w)^\circ\to G/B$ is the inclusion, and $gr^H(i_{w!}\mathbb{Q}^H_{Y(w)^\circ})$ is the associated graded sheaf on the cotangent bundle of G/B determined by the shifted mixed Hodge module $\mathbb{Q}^H_{Y(w)^\circ}$; see [Tan87, AMSS]. Since i^* is an isomorphism, we deduce from Theorem 7.7 that:

$$\operatorname{stab}_-(w) = (-1)^{\dim G/B}gr^H(i_{w!}\mathbb{Q}^H_{Y(w)^\circ}). \tag{22}$$

Since the support of the sheaf $gr^H((i_w)_!\mathbb{Q}^H_{Y(w)^\circ})$ is the characteristic cycle of constructible function $\mathbb{1}_{Y(w)^\circ}$, this equation can be seen as the K-theoretic generalization of the coincidence between (cohomological) stable envelopes and characteristic cycles, indicated by Maulik and Okounkov [MO, Remark 3.5.3]; see also [AMSS17, Lemma 6.5] for a proof.

7.5. **Stable basis duality.** As for the CSM classes, there are two sources for Poincaré type dualities of the motivic Chern classes. The first was a consequence of the existence of two adjoint Demazure-Lusztig operators. The second, which has a geometric origin, uses the duality for the stable envelopes (cf eq. (16)) on the cotangent bundle. Given that the localization pairing on the cotangent bundle can also be expressed in terms of a twisted Poincaré pairing on the zero section, this leads to some remarkable identities among motivic Chern classes. Recall from the equation (16) that the 'opposite' stable envelopes are dual to each other with respect to the K-theoretic pairing on $T^*(G/B)$, defined as follows: for any $\mathcal{F},\mathcal{G}\in K_{T\times\mathbb{C}^*}(T^*(G/B))$,

$$\langle\mathcal{F},\mathcal{G}\rangle_{T^*(G/B)} := \sum_{w\in W}\frac{\mathcal{F}|_w\cdot\mathcal{G}|_w}{\prod_{\alpha>0}(1-e^{w\alpha})(1-qe^{-w\alpha})}.$$

Recall that $i : G/B \hookrightarrow T^*(G/B)$ is the inclusion of the zero section. By localization, the pairing in $T^*(G/B)$ is related to the ordinary Poincaré pairing in equivariant K theory of G/B:

$$\langle \mathcal{F}, \mathcal{G} \rangle_{T^*(G/B)} = \langle i^*\mathcal{F}, \frac{i^*\mathcal{G}}{\lambda_{-q}(T(G/B))} \rangle. \tag{23}$$

We need the following lemma.

Lemma 7.9. *Let $\mathcal{F}, \mathcal{G} \in K_T(G/B)[q, q^{-1}]$ such that*

$$\langle \mathcal{F}, \mathcal{G} \rangle = f(e^t, q) \in K_T(pt)_{loc}[q, q^{-1}].$$

Then

$$\langle \mathcal{D}(\mathcal{F}), \mathcal{G}^\vee \rangle = \langle \mathcal{F}^\vee, \mathcal{D}(\mathcal{G}) \rangle = f(e^{-t}, q^{-1}),$$

i.e. all weights are inverted.

Proof. By the definition of Grothendieck-Serre dual, it suffices to prove the last equality. Applying the K-theoretic Bott localization [CG09, §5.10] we obtain

$$\langle \mathcal{F}, \mathcal{G} \rangle = \sum_w \frac{\mathcal{F}|_w \cdot \mathcal{G}|_w}{\prod_{\alpha>0}(1 - e^{w\alpha})} = f(e^t, q). \tag{24}$$

Recall that $\omega^\bullet_{G/B} = (-1)^{\dim G/B} \mathcal{L}_{2\rho}$. Then

$$\begin{aligned}
\langle (\mathcal{F})^\vee, (\mathcal{G})^\vee \otimes \mathcal{L}_{2\rho} \rangle =& \sum_w \frac{(\mathcal{F}|_w)^\vee (\mathcal{G}|_w)^\vee}{\prod_{\alpha>0}(1 - e^{w\alpha})} e^{2w\rho} \\
=&(-1)^{\dim G/B} \sum_w \frac{(\mathcal{F}|_w)^\vee (\mathcal{G}|_w)^\vee}{\prod_{\alpha>0}(1 - e^{-w\alpha})} \\
=&(-1)^{\dim G/B} f(e^{-t}, q^{-1}).
\end{aligned}$$

The first equality holds because $2\rho = \sum_\alpha \alpha$, thus $e^{2w(\rho)} = \prod_{\alpha>0} e^{w(\alpha)}$, and the last equality follows by inverting every term in the middle of Equation (24). The T and $\mathbb{C}^*$-weights are inverted because this is the effect of taking $(-)^\vee$. □

Theorem 7.10. *Let $u, w \in W$ and $y = -q^{-1}$. Then the following orthogonality relation holds:*

$$\langle MC_y(X(w)^\circ), \frac{\mathcal{D}(MC_y(Y(u)^\circ))}{\lambda_y(T^*(G/B))} (-y)^{\dim G/B - \ell(u)} \rangle = \delta_{w,u}.$$

In particular,

$$MC_y^\vee(Y(u)^\circ) = \prod_{\alpha>0} (1 + ye^{-\alpha}) \frac{\mathcal{D}(MC_y(Y(u)^\circ))}{\lambda_y(T^*(G/B))}.$$

Proof. The idea is to use Theorem 7.5 to express $MC_y(X(u)^\circ)$ in terms of the Grothendieck-Serre dual, then use Lemma 7.9 to relate the pairing in the statement of the theorem to the pairing between orthogonal stable envelopes. We start by observing that $\lambda_y(T^*(G/B)) = (\lambda_{y^{-1}} T(G/B))^\vee \in K_T(G/B)[y, y^{-1}]$, and that by Theorem 7.7

$$\mathcal{D}(MC_y(Y(u)^\circ)) = \mathcal{D}((-y)^{-\frac{\ell(u)}{2}} (-y)^{\dim G/B} \iota^*(\mathrm{stab}_-(u)) \otimes \omega^\bullet_{G/B}) = (-y)^{\frac{\ell(u)}{2} - \dim G/B} (\iota^*(\mathrm{stab}_-(u)))^\vee.$$

From this, the second term of the pairing equals

$$\frac{\mathcal{D}(MC_y(Y(u)^\circ))}{\lambda_y(T^*(G/B))} (-y)^{\dim G/B - \ell(u)} = \Big(\frac{\iota^*(\mathrm{stab}_-(u))}{\lambda_{y^{-1}}(T(G/B))} (-y)^{\frac{\ell(u)}{2}} \Big)^\vee.$$

Then Theorem 7.5, Lemma 7.9, and orthogonality of stable envelopes (Eq. (16)) imply that

$$\begin{aligned}
\langle MC_y(X(w)^\circ), \frac{\mathcal{D}(MC_y(Y(u)^\circ))}{\lambda_y(T^*(G/B))}(-y)^{\dim G/B-\ell(u)}\rangle &= \\
\langle \mathcal{D}((-y)^{-\frac{\ell(w)}{2}}\iota^*\operatorname{stab}_+(w)), \Big(\frac{\iota^*(\operatorname{stab}_-(u))}{\lambda_{y^{-1}}(T(G/B))}(-y)^{\frac{\ell(u)}{2}}\Big)^\vee\rangle &= \\
\langle (-y)^{\frac{\ell(u)-\ell(w)}{2}}\iota^*\operatorname{stab}_+(w)), \frac{\iota^*\operatorname{stab}_-(u)}{\lambda_{y^{-1}}(T(G/B))}\rangle_{y\mapsto y^{-1}, e^t\mapsto e^{-t}} &= \\
(-y)^{\frac{\ell(u)-\ell(w)}{2}}\langle \operatorname{stab}_+(w), \operatorname{stab}_-(u)\rangle_{T^*(G/B)}|_{y\mapsto y^{-1}, e^t\mapsto e^{-t}} &= \quad \delta_{w,u},
\end{aligned}$$

where the third equality follows from Equation (23) and the last one follows from Equation (16). This proves the first assertion. The second assertion follows from the 'Hecke orthgonality' of motivic Chern classes, proved in Theorem 5.2. □

The theorem justifies the definition of a motivic Chern class of a Schubert variety:

Definition 7.11. Let $w \in W$. Define the dual motivic Chern class of a Schubert *variety* by

$$MC_y^\vee(Y(w)) := \sum_{u\geq w} MC_y^\vee(Y(u)^\circ).$$

Notice that Theorem 7.10 implies that

$$MC_y^\vee(Y(w)) = \frac{\prod_{\alpha>0}(1+ye^{-\alpha})}{\lambda_y(T^*(G/B))}\mathcal{D}(MC_y(Y(w))),$$

where in the last term we use the ordinary motivic Chern class. The class

$$\frac{\mathcal{D}(MC_y(Y(w)))}{\lambda_y(T^*(G/B))}$$

can be thought as a ($\mathcal{D}$-twisted) *motivic Segre class*, i.e. a K-theoretic analogue of the Segre-Schwartz-MacPherson class discussed in [Ohm06, FR, AMSS17].

Remark 7.12. A K-theoretic analogue of [AMSS17, Lemma 8.1] gives a formula to divide by $\lambda_y(T^*(G/B))$:

$$\lambda_y(T^*(G/B))\lambda_y(T(G/B)) = \prod_{\alpha>0}(1+ye^{\alpha})(1+ye^{-\alpha}).$$

As in *loc. cit.* this is proved again by an easy localization argument.

8. Smoothness of Schubert varieties and localizations of motivic Chern classes

Among the main applications of this paper are properties about the transition matrix between the standard and the Casselman's basis for Chevalley groups over nonarchimedean local fields. The matrix coefficients are rational functions, and of particular interest to us are certain factorization and polynomial properties of these coefficients conjectured by Bump, Naruse and Nakasuji; see section §9 below and [BN11, NN15, BN17]. We will prove in §9 that the transition matrix from the 'Casselman setting' corresponds to transition matrix between (dual) motivic Chern classes of Schubert varieties and an appropriate normalization of the fixed point basis. This motivates the study in this section of the underlying 'geometric' transition matrix between the motivic classes and fixed point basis. The main result of this section is Theorem 8.5 (Theorem 1.3 from the introduction) which is the geometric analogue of the Theorem 9.1 about the principal series representation.

8.1. **A smoothness criterion.** In this section we prove a criterion for the smoothness of Schubert varieties in terms of the motivic Chern classes. The main theorem is the following:

Theorem 8.1. *Let $u, w \in W$ such that $u \le w$. The opposite Schubert variety $Y(u)$ is smooth at the torus fixed point e_w if and only if*

$$MC_y(Y(u))|_w = \prod_{\alpha>0, ws_\alpha \ge u} (1 + ye^{w\alpha}) \prod_{\alpha>0, u \not\le ws_\alpha} (1 - e^{w\alpha}).$$

Further, if $Y(u)$ is smooth at e_w then

$$MC_y(Y(u))|_w = \lambda_y(T_w^* Y(u)) \cdot \lambda_{-1}(N^\vee_{w,Y(u)}(G/B)).$$

Let $S'(u,w) := \{\alpha > 0 | u \le ws_\alpha < w\}$. An immediate consequence of the theorem is that if $Y(u)$ is smooth at e_w then the weights of the normal space $N_{w,Y(w)}Y(u) = T_w(Y(u))/T_w(Y(w))$ of $Y(w)$ at e_w in $Y(u)$ are

$$S(u,w) := \{\beta > 0 : u \le s_\beta w < w\} = \{-w\alpha | \alpha \in S'(u,w)\}. \tag{25}$$

To prove the theorem we first prove several basic results about localization of motivic Chern classes. (These are also implicit in the work of [FRW18].)

Lemma 8.2. *Let $i : X \subset M$ be a proper morphism of G-equivariant, non-singular, quasi-projective, algebraic varieties. Then $i^* MC_y[X \to M] = \lambda_y(T_X^*) \otimes \lambda_{-1}(N_X^\vee M)$, where $N_X^\vee M$ is the conormal bundle of X inside M.*

Proof. By self-intersection formula in K theory [CG09, Prop. 5.4.10] we have

$$i^* MC_y[X \to M] = i^* i_* MC_y[id_X] = MC_y[id_X] \otimes \lambda_{-1}(N_X^\vee M) = \lambda_y(T_X^*) \otimes \lambda_{-1}(N_X^\vee M).$$

□

Lemma 8.3. *Let $X \subset M$ be a proper inclusion of T-equivariant, algebraic varieties, and assume that M is smooth. Let $p \in X$ be a smooth point, and $j : V \subset M$ any T-invariant open set such that $p \in V$ and $X' := V \cap X$ is smooth. Let $\iota_p : \{p\} \to M$ be the inclusion. Then*

$$\iota_p^* MC_y[X \to M] = \lambda_y(T_p^* X) \cdot \lambda_{-1}(N_{p,X}^\vee M).$$

Proof. Let $\iota_p' : p \to V$ denote the embedding. Note that

$$\iota_p^* MC_y[X \to M] = (\iota_p')^* j^* MC_y[X \to M] = (\iota_p')^* MC_y(j^*[X \to M]) = (\iota_p')^* MC_y[X' \to V],$$

where the second equality follows from the Verdier-Riemann-Roch formula from Theorem 4.2, as j is an open embedding (thus a smooth morphism), with relative tangent bundle equal to 1. Observe now that the embedding $X' \to V$ is a proper morphism, because is is obtained by base-change. Therefore Lemma 8.2 applies, and we get

$$MC_y[X' \to V] = \lambda_y(T_{X'}^*) \otimes \lambda_{-1}(N_{X'}^\vee V).$$

Then the claim follows by pulling back via $(\iota_p')^*$, using that $(\iota_p')^*$ is a ring homomorphism in (equivariant) K-theory. □

We will apply this lemma in the case when $Y(u) \subset G/B$ is a Schubert variety which is smooth at a torus fixed point e_w. In this case we need to exhibit the open set V from the lemma. To do this, we consider the shifted Schubert cell $ww_0 Y^\circ$, where $Y^\circ = Y(id)^\circ$ is the opposite Schubert cell in G/B. This is a T-invariant, open neighborhood of e_w in G/B, and the hypothesis on e_w implies that the intersection $V' := ww_0 Y^\circ \cap Y(u)$ is irreducible and non-singular at e_w. The singular locus of the intersection V' is $ww_0 Y^\circ \cap Sing(Y(u))$, which is T-invariant and closed in V' (possibly empty). Then the set $V := ww_0Y^\circ \setminus Sing(Y(u))$ satisfies the required properties.

We also need a variant of Kumar's cohomological criterion for smoothness of Schubert varieties.

Theorem 8.4 ([Kum96])**.** *Let u, w be two Weyl group elements such that $u \le w$. Then the Schubert variety $Y(u)$ is smooth at e_w if and only if the localization*

$$[Y(u)]|_w = \prod_{\beta>0, u\not\leq s_\beta w} \beta.$$

If $Y(u)$ is smooth at e_w then the torus weights of $T_wY(u)$ are $\{-w\alpha | \alpha > 0, ws_\alpha \ge u\}$.

Proof. Consider the automorphism of the set R^+ of positive roots given by $\alpha \mapsto -w_0(\alpha)$. One checks that this is actually an automorphism of the Dynkin diagram. It induces the automorphism $w \mapsto w_0ww_0$ of the Weyl group W, preserving the length and the Bruhat order. It also induces an automorphism of G/B sending the Schubert cell $Y(w)^\circ$ to $Y(w_0ww_0)^\circ$. In particular, $Y(u)$ is smooth at e_w if and only if $Y(w_0uw_0)$ is smooth at w_0ww_0. Then

$$\begin{aligned} & Y(w_0uw_0) \textit{ is smooth at } e_{w_0ww_0} \\ \Longleftrightarrow & X(uw_0) \textit{ is smooth at } e_{ww_0} \\ \Longleftrightarrow & [Y(u)]|_w = \prod_{\beta>0, u\not\leq s_\beta w} \beta, \end{aligned}$$

where the last equivalence follows from the original version of Kumar's criterion, as stated in [BL00, Corollary 7.2.8][2], together with [BL00, Thm.7.2.11] and Billey's formula for localization of equivariant Schubert classes [Bil99].

To show that the weights on the tangent space are those claimed, observe first that $T_{id}(G/B) = \mathrm{Lie}(G)/\mathrm{Lie}(B)$ has weights $\{-\alpha : \alpha > 0\}$, thus after translation by w, $T_w(G/B)$ has weights $\{-w(\alpha) : \alpha > 0\}$. The tangent space $T_wY(u)$ is a T-submodule of $T_w(G/B)$, and its weights correspond to edges $w \mapsto ws_\alpha$ in the Bruhat graph of G/B, where $ws_\alpha \in Y(u)$, i.e. $ws_\alpha \ge u$. □

We are now ready to prove theTheorem 8.1.

Proof of Theorem 8.1. We only need to show the smoothness criterion. Assume first $Y(u)$ is smooth at e_w. We apply Lemma 8.3 and we use the weights of the tangent bundle from Theorem 8.4 to obtain:

$$\begin{aligned} MC_y(Y(u))|_w &= \lambda_y(T_w^*Y(u))\lambda_{-1}(N^\vee_{w,Y(u)}G/B) \\ &= \lambda_y(T_w^*Y(u))\frac{T_w^*(G/B)}{\lambda_{-1}(T_w^*Y(u))} \\ &= \prod_{\alpha>0, ws_\alpha\ge u}(1+ye^{w\alpha})\frac{\prod_{\alpha>0}(1-e^{w\alpha})}{\prod_{\alpha>0,u\le ws_\alpha}(1-e^{w\alpha})} \\ &= \prod_{\alpha>0, ws_\alpha\ge u}(1+ye^{w\alpha})\prod_{\alpha>0,u\not\leq ws_\alpha}(1-e^{w\alpha}). \end{aligned}$$

The converse follows immediately from Kumar criterion, after making $y = 0$, and taking leading terms, in order to the reduce to the cohomological case. More precisely, Lemma 3.8 implies that the specialization $MC_y(Y(u))|_{y=0}$ is the sum of the ideal sheaves

[2] In *loc. cit.*, $d_{w,u} = [Y(u)]|_w$, see [BL00, Theorem 7.2.11].

$\sum_{w\geq u}\mathcal{I}^w=\mathcal{O}^u$, by Möbius inversion; see (8). Consider now the equivariant Chern character $ch_T: K_T(G/B)\to \widehat{A}^T_*(G/B)$, where the hat denotes an appropriate completion of the equivariant Chow group; see [EG00]. From the definition of ch_T and by [Ful84, Thm. 18.3] it follows that the top degree term of the equivariant Chern character $ch_T(\mathcal{O}^u)$ is the equivariant fundamental class $[Y(u)]_T$. Together with the fact that $ch_T(e^\lambda)=1+\lambda+$ higher degree *cohomological* terms, this implies that the top degree term of $ch_T(MC_y(Y(u))|_{y=0})$ is $[Y(u)]_T$ as claimed. □

8.2. **The geometric Bump-Nakasuji-Naruse conjecture.** Motivated by the applications to representation theory from section §9, we study the following problem. Define the element $b_w\in K_T(G/B)_{loc}[y,y^{-1}]$ by the formula

$$b_w:=(-y)^{\ell(w)-\dim G/B}\prod_{\alpha>0,w\alpha>0}\frac{1+ye^{-w\alpha}}{1-e^{w\alpha}}\iota_w. \tag{26}$$

Equivalently, b_w this is the multiple of the fixed point basis element ι_w which satisfies $(b_w)|_w=MC_y^\vee(Y(w)^\circ)|_w$. Recall from Definition 7.11 that $MC_y^\vee(Y(u)):=\sum_{w\geq u}MC_y^\vee(Y(w)^\circ)$. Consider the expansion

$$MC_y^\vee(Y(u))=\sum m_{u,w}b_w. \tag{27}$$

It is easy to see that $m_{u,w}=0$ unless $u\leq w$. For pairs $u\leq w\in W$, recall that $S(u,w):=\{\beta\in R^+|u\leq s_\beta w<w\}$. The main result of this section is the following geometric analogue of Bump, Naruse and Nakasuji conjecture [BN11, NN15, BN17].

Theorem 8.5 (Geometric Bump-Nakasuji-Naruse Conjecture)**.** *For any $u\leq w\in W$,*

$$m_{u,w}=\prod_{\alpha\in S(u,w)}\frac{1+y^{-1}e^\alpha}{1-e^\alpha}, \tag{28}$$

if and only if the Schubert variety $Y(u)$ is smooth at the torus fixed point e_w.

This is the geometric version of Theorem 9.1 in p-adic representation theory; the coefficients $m_{u,w}$ calculate the transition matrix between the 'standard basis' and 'Casselman's basis' for the Iwahori invariants of the principal series representation. All of this will be discussed in §9.2 below. The statement is a generalization of the original Bump-Nakasuji conjecture, communicated to us by H. Naruse; see [Nar14]. In fact, Naruse informed us that he obtained the implication of this theorem which assumes the factorization. Naruse's proof of this implication, and ours, are both based on Kumar's cohomological criterion for smoothness (Theorem 8.4 above, see [Kum96]), but Naruse's proof is based on Hecke algebra calculations and does not use motivic Chern classes.

After harmonizing conventions between this note and [BN11, BN17], and passing to the 'geometric' version, the original conjecture states the following (see [BN11, Conj. 1.2] and [BN17, p. 3]):

Corollary 8.6. *Let G be a complex simply laced reductive Lie group. Then the coefficient $m_{u,v}$ satisfies the factorization in* (28) *if and only if the Kazhdan-Lusztig polynomial*

$$P_{w_0w^{-1},w_0u^{-1}}=1.$$

We prove this statement assuming Theorem 8.5.

Proof. Since the group G is simply laced, a theorem of Carrell and Peterson (see e.g. [CK03] or [BL00, Thm. 6.0.4]) shows that the condition that $Y(u)$ is smooth at e_w is equivalent to $Y(u)$ being rationally smooth at e_w. For arbitrary G, rational smoothness is equivalent to the fact that the Kazhdan-Lusztig polynomial $P_{w_0u,w_0w} = 1$, by a theorem Kazhdan and Lusztig [KL79, Thm. A2]. Then by Theorem 8.5, it remains to shows that $P_{w_0w^{-1},w_0u^{-1}} = 1$ if and only if $P_{w_0w,w_0u} = 1$. In turn, this is equivalent to

$$P_{u,w} = 1 \Longleftrightarrow P_{w_0u^{-1}w_0,w_0w^{-1}w_0} = 1.$$

This is proved in the next lemma below. □

Lemma 8.7. *Let G be a complex reductive Lie group of arbitrary Lie type. Then $P_{u,w} = 1$ if and only if the polynomial $P_{w_0u^{-1}w_0,w_0w^{-1}w_0} = 1$.*

Proof. We use a characterization of the condition that the Kazhdan-Lusztig polynomials equal to 1, proved in various generality by Deodhar, Carrell and Peterson; see [BL00, Thm. 6.2.10]. Let $\mathcal{R}$ be the set of (not necessarily simple) reflections in W. Then $P_{u,w} = 1$ if and only if

$$\#\{r \in \mathcal{R} : y < ry \leq w\} = \ell(w) - \ell(y), \quad \forall u \leq y \leq w.$$

It is well known that taking inverses, and conjugating by w_0 are bijections of W which preserve both the length, and the Bruhat order of elements. Thus $y < w$ if and only if $w_0y^{-1}w_0 < w_0w^{-1}w_0$ and $\ell(w) - \ell(y) = \ell(w_0w^{-1}w_0) - \ell(w_0y^{-1}w_0)$. This finishes the proof. □

We note that in general rational smoothness is different from smoothness, therefore the statement from Cor. 8.6 does not generalize to non-simply laced case. The statement of [BN11, Conj. 1.2] is slightly different from the final version stated in Cor. 8.6 and in [BN17]; the initial statement was analyzed by Lee, Lenart and Liu in [LLL17]. They found that under certain conditions on the reduced words of w and z the factorization holds, but in general there are counterexamples. We refer further to [Nar14, NN15, BN17] for work closely related to [BN11].

We now return to the proof of Theorem 8.5. The key part is the following result, which may be of interest in its own right.

Proposition 8.8. *(a) For any $w \geq u \in W$, the coefficient $m_{u,w}$ equals*

$$m_{u,w} = \Big(\frac{MC_y(Y(u))|_w}{MC_y(Y(w)^\circ)|_w}\Big)^\vee.$$

(b) Assume that $Y(u)$ is smooth at e_w. Then

$$m_{u,w} = \frac{\lambda_{y^{-1}}(N_{w,Y(w)}Y(u))}{\lambda_{-1}(N_{w,Y(w)}Y(u))}.$$

In particular, we obtain a geometric analogue of the Langlands-Gindikin-Karpelevich formula [Lan71]*:*

$$m_{1,w} = \prod_{\alpha<0,w^{-1}(\alpha)>0} \frac{1 + y^{-1}e^\alpha}{1 - e^\alpha}.$$

Proof. To calculate $m_{u,w}$ we localize both sides at the fixed point e_w. Using Theorem 7.10 and Definition 7.11, we calculate:

$$m_{u,w} = \frac{MC_y^\vee(Y(u))|_w}{MC_y^\vee(Y(w)^\circ)|_w} = \frac{\mathcal{D}(MC_y(Y(u))|_w)}{\mathcal{D}(MC_y(Y(w)^\circ)|_w)} = \Big(\frac{MC_y(Y(u))|_w}{MC_y(Y(w)^\circ)|_w}\Big)^\vee.$$

This finishes the proof of part (a). For part (b), we use part (a), and Lemma 8.3 (or Theorem 8.1) to obtain

$$\begin{aligned} m_{u,w} &= \Big(\frac{MC_y(Y(u))|_w}{MC_y(Y(w)^\circ)|_w}\Big)^\vee \\ &= \Big(\frac{\lambda_y(T^*_wY(u))\cdot\lambda_{-1}(N^\vee_{w,Y(u)}(G/B))}{\lambda_y(T^*_wY(w))\cdot\lambda_{-1}(N^\vee_{w,Y(w)}G/B)}\Big)^\vee \\ &= \frac{\lambda_{y^{-1}}(T_wY(u))\cdot\lambda_{-1}(N_{w,Y(u)}G/B)}{\lambda_{y^{-1}}(T_wY(w))\cdot\lambda_{-1}(N_{w,Y(w)}G/B)} \\ &= \frac{\lambda_{y^{-1}}(N_{w,Y(w)}Y(u))}{\lambda_{-1}(N_{w,Y(w)}Y(u))}. \end{aligned}$$

The last equality follows from the multiplicativity of the λ_y class, and the short exact sequences $0 \to T_wY(w) \to T_wY(u) \to N_{w,Y(w)}(Y(u)) \to 0$ and $0 \to N_{w,Y(u)}(G/B) \to N_{w,Y(w)}(G/B) \to N_{w,Y(w)}(Y(u)) \to 0$. The case when $u = 1$ follows from the description of the weights from Theorem 8.1. □

Proof of Theorem 8.5. If $Y(u)$ is smooth at e_w the claim follows from Proposition 8.8(b), using the description of appropriate weights from Theorem 8.1. Conversely, assume that

$$m_{u,w} = \prod_{u\le s_\alpha w<w}\frac{1+y^{-1}e^\alpha}{1-e^\alpha}.$$

Part (a) of Proposition 8.8, together with the localization result from Proposition 6.2 imply that

$$MC_y(Y(u))|_w = \prod_{\alpha>0, ws_\alpha\ge u}(1+ye^{w\alpha})\prod_{\alpha>0, u\not\le ws_\alpha}(1-e^{w\alpha}).$$

Therefore, Theorem 8.5 follows from Theorem 8.1. □

9. Motivic Chern classes and the principal series representation

The goal of this section is to establish a relation between the Iwahori invariants of the unramified principal series representations of a group over a non archimedean local field and the equivariant K theory of the flag variety for the complex Langlands dual group. A similar relation was established recently in [SZZ17], using the equivariant K theory of the cotangent bundle, and the stable basis. The advantage of using motivic Chern classes is that their functoriality properties will help get additional properties of this correspondence. For instance, we use functoriality to relate localization coefficients of the motivic Chern classes to coefficients in the transition matrix between the standard basis and *Casselman's basis* (defined below). This will be applied to solve a refined conjecture of Bump–Nakasuji [BN11] about Casselman basis, which was communicated to the authors by H. Naruse.

9.1. **Iwahori invariants of the principal series representations.** In this section, we recall the definition and properties of the two bases in the Iwahori invariants of the principal series representations. The literature in this subject uses several normalization conventions. We will be consistent with the conventions used in the paper of Reeder [Ree92] and [SZZ17], because they fit with our previous geometric calculations in this paper; these conventions differ from those in [BN11, BN17] or [BBL15], and when necessary we will explain the differences. Let G be a split, reductive, Chevalley group defined over $\mathbb{Z}$; see e.g. [Ste16]. We will also consider its Langlands dual, denoted $G^\vee$; by definition this will be a *complex* reductive Lie group, of type dual to the Lie type of G. See e.g. [Bor79, §2.1].[3] Let $B = TN \leq G$ be a standard Borel subgroup containing a maximal torus T and its unipotent radical N. Let $W := N_G(T)/T$ be the Weyl group.

Let F be a non archimedean local field, with ring of integers $\mathcal{O}$, uniformizer $\varpi \in \mathcal{O}$, and residue field $\mathbb{F}_q$. Examples are finite extensions of the field of p-adic numbers, or of the field of Laurent series over $\mathbb{F}_p$. Since G is defined over $\mathbb{Z}$, we may consider $G(F)$, the group of the F-points of G, with maximal torus $T(F)$ and Borel subgroup $B(F) = T(F)N(F)$. Let I be an Iwahori subgroup, i.e., the inverse image of $B(\mathbb{F}_q)$ under the evaluation map $G(\mathcal{O}) \to G(\mathbb{F}_q)$. To simplify formulas, we let α, β denote the coroots of G. Let R^+ and $R^{+\vee}$ denote the positive roots and coroots, respectively.

Let $\mathbb{H} = \mathbb{C}_c[I\backslash G(F)/I]$ be the Iwahori Hecke algebra, consisting of compactly supported functions on $G(F)$ which are bi-invariant under I. As a vector space, $\mathbb{H} = \Theta \otimes_{\mathbb{C}} H_W$, where Θ is a commutative subalgebra isomorphic to the coordinate ring $\mathbb{C}[T^\vee]$ of the complex dual torus $T^\vee = \mathbb{C}^* \otimes X^*(T)$, and where H_W is the finite Hecke (sub)algebra associated to the (finite) Weyl group W. The finite Hecke algebra H_W, is also a subalgebra of $\mathbb{H}$, and it is generated by elements T_w such that the following relations hold: $T_uT_v = T_{uv}$ if $\ell(uv) = \ell(u) + \ell(v)$, and $(T_{s_i} + 1)(T_{s_i} - q) = 0$ for a simple reflection s_i.

For any character τ of T, and α a coroot define e^α by $e^\alpha(\tau) = \tau(h_\alpha(\varpi))$, where $h_\alpha : F^\times \to T(F)$ is the one parameter subgroup. There is a pairing

$$\langle \cdot, \cdot \rangle : T(F)/T(\mathcal{O}) \times T^\vee \to \mathbb{C}^*$$

given by $\langle a, z \otimes \lambda \rangle = z^{val(\lambda(a))}$. This induces an isomorphism between $T(F)/T(\mathcal{O})$ and the group $X^*(T^\vee)$ of rational characters of $T^\vee$. It also induces an identification between $T^\vee$ and unramified characters of $T(F)$, i.e., characters which are trivial on $T(\mathcal{O})$.

For simplicity, from now on we take τ to be an unramified character of $T(F)$ such that $e^\alpha(\tau) \neq 1$ for all coroots α, and for which the stabilizer $W_\tau = 1$. The *principal series representation* is the induced representation $I(\tau) := \mathrm{Ind}_{B(F)}^{G(F)}(\tau)$. As a $\mathbb{C}$-vector space, $I(\tau)$ consists of locally constant functions f on $G(F)$ such that $f(bg) = \tau(b)\delta^{\frac{1}{2}}(b)f(g)$ for any $b \in B(F)$, where $\delta(b) := \prod_{\alpha>0} |\alpha^\vee(b)|_F$ is the modulus function on the Borel subgroup. The Hecke algebra $\mathbb{H}$ acts through convolution from the right on the Iwahori invariant subspace $I(\tau)^I$, so that the restriction of this action to H_W is a regular representation. One can pass back and forth between left and right $\mathbb{H}$-modules by using the standard anti-involution ι on $\mathbb{H}$ given by $\iota(h)(x) = h(x^{-1})$. If T_w denote the standard generators of the Hecke algebra H_W, then $\iota(T_w) = T_{w^{-1}}$ and $\iota(q) = q$, see [HKP10, Section 3.2]. This is of course consistent with the left $\mathbb{H}$-action on $I(\tau)$ described by Reeder on [Ree92, p. 325].

We are interested in the Iwahori invariants $I(\tau)^I$ of the principal series representation for an unramified character. One reason to study the invariants is that as a $G(F)$-module, the principal series representation $I(\tau)$ is generated by $I(\tau)^I$; cf. [Cas80, Prop. 2.7]. As a vector space, $\dim_{\mathbb{C}} I(\tau)^I = |W|$, the order of the Weyl group W. We will study the transition between two bases of $I(\tau)^I$. From the decomposition $G(F) = \bigsqcup_{w\in W} B(F)wI$ one obtains the basis of the *characteristic functions* on the orbits, denoted by $\{\varphi_w \mid w \in W\}$[4]. For $w \in W$, the element φ_w is characterized by the two conditions: [Ree92, pg. 319]:

(1) φ_w is supported on $B(F)wI$;

[3]In the previous section we used flag varieties associated to complex simple Lie groups. However, all results extend without change to the semisimple case. Further, the flag varieties for the semisimple Lie groups are the same as those for reductive groups. We will tacitly use these facts in this section.

[4]Our φ_w is equal to $\phi_{w^{-1}}$ in [BN11, BN17]

(2) $\varphi_w(bwg) = \tau(b)\delta^{\frac{1}{2}}(b)$ for any $b \in B(F)$ and $g \in I$.

The (left) action of $\mathbb{H}$ on $I(\tau)^I$, denoted by π, was calculated e.g. by Casselman in [Cas80, Thm. 3.4]. With the conventions from Reeder [Ree92, pg. 325], for any simple coroot α_i:

$$\pi(T_{s_i})(\varphi_w) = \begin{cases} q\varphi_{ws_i} + (q-1)\varphi_w & \text{if } ws_i < w; \\ \varphi_{ws_i} & \text{if } ws_i > w. \end{cases} \tag{29}$$

The second basis, called *Casselman's basis*, and denoted by $\{f_w \mid w \in W\}$, was defined by Casselman [Cas80, §3] by duality using certain intertwiner operators. We recall the relevant definitions, following again [Ree92]. For any character τ and $x \in W$, define $x\tau \in X^*(T)$ by the formula $x\tau(a) := \tau(x^{-1}ax)$ for any $a \in T$. Since τ that is unramified and it has trivial stabilizer under the Weyl group action, the space $\mathrm{Hom}_{G(F)}(I(\tau), I(x^{-1}\tau))$ is known to be one dimensional, spanned by an operator $\mathcal{A}_x = \mathcal{A}_x^\tau$ [5] defined by

$$\mathcal{A}_x(\varphi)(g) := \int_{N_x} \varphi(\dot{x}ng)dn,$$

where $\dot{x}$ is a representative of $x \in W$, $N_x = N(F) \cap \dot{x}^{-1}N^-(F)\dot{x}$ where N^- is the unipotent radical of the opposite Borel subgroup B^-; the measure on N_x is normalized by the condition that $vol(N_x \cap G(\mathcal{O})) = 1$ [Ree92]. If $x, y \in W$ satisfy $\ell(x) + \ell(y) = \ell(xy)$, then $\mathcal{A}_y^{x^{-1}\tau}\mathcal{A}_x^\tau = \mathcal{A}_{xy}^\tau$. Then there exist unique functions $f_w \in I(\tau)^I$ such that

$$\mathcal{A}_x^\tau(f_w)(1) = \delta_{x,w}. \tag{30}$$

(Again under our conventions f_w equals the element denoted $f_{w^{-1}}$ in [BN11].) For the longest element w_0 in the Weyl group, Casselman showed in [Cas80, Prop. 3.7] that

$$\varphi_{w_0} = f_{w_0}.$$

Reeder [Ree92] calculated the action of $\mathbb{H}$ on the functions f_w: he showed in [Ree92, Lemma 4.1] that the functions f_w are Θ-eigenvectors, and he calculated in [Ree92, Prop. 4.9] the action of H_W. To describe the latter, let

$$c_\alpha = \frac{1 - q^{-1}e^\alpha(\tau)}{1 - e^\alpha(\tau)}. \tag{31}$$

For any simple coroot α_i and $w \in W$, write

$$J_{i,w} = \begin{cases} c_{w(\alpha_i)}c_{-w(\alpha_i)} & \text{if } ws_i > w; \\ 1 & \text{if } ws_i < w. \end{cases}$$

Then, we have

$$\pi(T_{s_i})(f_w) = q(1 - c_{w(\alpha_i)})f_w + qJ_{i,w}f_{ws_i}. \tag{32}$$

9.2. **A conjecture of Bump, Nakasuji and Naruse.** In this section we state a conjecture of Bump, Nakasuji and Naruse, regarding a factorization of certain coefficients of the transition matrix between the bases $\{\varphi_w\}$ and $\{f_w\}$. We follow mainly [BN11], and we recall that we use opposite notations from those in *loc.cit*: our φ_w and f_w are $\phi_{w^{-1}}$ and $f_{w^{-1}}$ respectively in [BN11]. Let

$$\phi_u := \sum_{u \leq w} \varphi_w \in I(\tau)^I,$$

[5] The intertwiner $\mathcal{A}_x$ is related to M_x from [Cas80, BN11] by the formula $\mathcal{A}_x = M_{x^{-1}}$.

and consider the expansion in terms of the Casselman's basis:

$$\phi_u = \sum_w \tilde{m}_{u,w} f_w.$$

Then by the definition of f_w, $\tilde{m}_{u,w} = \mathcal{A}_w(\phi_u)(1)$. It is also easy to see that $\tilde{m}_{u,w} = 0$ unless $u \leq w$, see [BN11, Theorem 3.5]. We shall see below that $\tilde{m}_{u,v}$ equals the evaluation at τ of the coefficient $m_{u,v}$ from (27) above, defined for the Langlands dual flag variety. For any $u \leq w \in W$, recall the definition $S(u,w) := \{\beta \in R^{+,\vee} | u \leq s_\beta w < w\}$. Recall that $G^\vee$ is the complex Langlands dual group, with the corresponding Borel subgroup $B^\vee$ and the maximal torus $T^\vee$. Then the goal is to prove the following statement.

Theorem 9.1 (Bump-Nakasuji-Naruse Conjecture)**.** *For any $u \leq w \in W$,*

$$\tilde{m}_{u,w} = \prod_{\alpha \in S(u,w)} \frac{1 - q^{-1} e^{\alpha}(\tau)}{1 - e^{\alpha}(\tau)}, \tag{33}$$

if and only if the opposite Schubert variety $Y(u) := \overline{B^{\vee -} u B^\vee / B^\vee}$ in the (dual, complex) flag manifold $G^\vee / B^\vee$ is smooth at the torus fixed point e_w.

This is the representation theoretic counterpart of Theorem 8.5; its proof will be given in the next section.

We provide further historical context. It is a question of Casselman [Cas80] to write the basis f_w as a linear combination of the standard basis φ_w. It was Bump and Nakasuji who found that the basis ϕ_w is better behaved to this problem. Of course the original Casselman's basis can be obtained from the Möbius inversion

$$\varphi_u = \sum_{w \geq u} (-1)^{\ell(u) - \ell(w)} \phi_w.$$

The case $u = 1$ is well known. In this case ϕ_1 is the spherical vector in $I(\tau)$, i.e. the vector fixed by the maximal compact subgroup $G(\mathcal{O})$, and

$$\mathcal{A}_w(\phi_1)(1) = \tilde{m}_{1,w} = \prod_{\alpha \in S(1,w)} \frac{1 - q^{-1} e^{\alpha}(\tau)}{1 - e^{\alpha}(\tau)}. \tag{34}$$

This is the *Gindikin–Karpelevich formula*, which in the non-archimedean setting was actually proved by Langlands [Lan71] after Gindikin and Karpelevich proved a similar statement for real groups. Casselman gave another proof using his basis f_w, and this plays an crucial role in his computation of Macdonald formula and the spherical Whittaker functions, see [Cas80, CS80]. See also [SZZ17] for an approach using the stable basis and the equivariant K theory of the cotangent bundle $T^*(G^\vee / B^\vee)$. Other special cases of the conjecture follow from the work of Reeder in *loc. cit.*

9.3. **Casselman's problem and motivic Chern classes.** In this section, we construct the promised isomorphism between the H_W-module of the Iwahori invariants of the principal series representation of G and the equivariant K group of the flag variety for the dual group $G^\vee$, regarded as an H_W-module via the action of the operators $\mathcal{T}_i^\vee$. This construction, together with the cohomological properties of the motivic Chern classes from the section §8 will be used to prove the Theorem 9.1.

For now we assume that the unramified character τ is in the open set in $T^\vee$ such that $1 - q e^{\alpha}(\tau) \neq 0$, for any (positive or negative) coroot α. Regard the representation ring

$K_{T^\vee}(pt)$ as a subring of $\mathbb{C}[T^\vee]$ and let $\mathbb{C}_\tau$ denote the one dimensional $K_{T^\vee}(pt)$-module induced by evaluation at τ. Recall the operators $\mathcal{T}_i^\vee$ in Definition 3.1 satisfy

$$(\mathcal{T}_i^\vee + 1)(\mathcal{T}_i^\vee + y) = 0,$$

and the braid relations. Hence, they induce an action of the Hecke algebra H_W on the K-theory ring $K_{T^\vee}(G^\vee/B^\vee)[y, y^{-1}]$ by sending T_w to $\mathcal{T}_w^\vee$, and q to $-y$, see [Lus85]. We use the symbol π to denote this action.

We are now ready state the main result which connects the equivariant K theory of $G^\vee(\mathbb{C})/B^\vee(\mathbb{C})$ to the Iwahori invariants in $I(\tau)$. Define a $K_{T^\vee}(pt)$-module homomorphism

$$\Psi : K_{T^\vee}(G^\vee/B^\vee)[y, y^{-1}] \otimes_{K_{T^\vee}(pt)[y,y^{-1}]} \mathbb{C}_\tau \xrightarrow{\sim} I(\tau)^I,$$

by $\Psi(MC_y^\vee(Y(w)^\circ) \otimes 1) = \varphi_w$ and sending $y \mapsto -q$. Since both the motivic Chern classes and the elements φ_w form bases in the appropriate modules, Ψ is clearly a module isomorphism. As in (26), define the element $\tilde{b}_w \in K_{T^\vee}(G^\vee/B^\vee)_{loc}[y, y^{-1}]$ by the formula

$$\tilde{b}_w := (-y)^{\ell(w)-\dim G^\vee/B^\vee} \prod_{\alpha>0, w\alpha>0} \frac{1+ye^{-w\alpha}}{1-e^{w\alpha}} \iota_w \otimes 1.$$

Equivalently, $\tilde{b}_w = b_w \otimes 1$, where b_w is the multiple of the fixed point basis element ι_w which satisfies $(b_w)|_w = MC_y^\vee(Y(w)^\circ)|_w$. We now state the main comparison theorem.

Theorem 9.2. *The following hold:*

(a) Ψ is an isomorphism of left H_W-modules;

(b) There is an equality $\Psi(\tilde{b}_w) = f_w$.

(c) The coefficients $\tilde{m}_{u,w}$ are represented by the meromorphic functions $m_{u,w}$ on $T^\vee$, defined in equation (27)*, for the Langlands dual complex flag variety $G^\vee/B^\vee$. More precisely, let $\tau \in T^\vee$ be any regular unramified character, i.e. the stabilizer $W_\tau = 1$. Then*

$$\tilde{m}_{u,w} = m_{u,w}(\tau).$$

Remark 9.3. For the cotangent bundle case, there is an analogue of this theorem in [SZZ17]. This is also studied by Lusztig and Braverman–Kazhdan in [Lus98, BK99] from different points of view.

Proof. The fact that Ψ is a map of H_W modules follows from comparing Proposition 5.6 to equation (29); these describe the Hecke actions on the basis of dual motivic Chern classes $MC_y^\vee(Y(w)^\circ)$, and the basis of characteristic functions φ_w. To prove part (b), start by recalling that $f_{w_0} = \varphi_{w_0}$. Further, since $b_{w_0} = \iota_{w_0} = MC_y^\vee(Y(w_0))$ (from the definition 5.1 of dual motivic classes) we deduce that $\Psi(\tilde{b}_{w_0}) = f_{w_0}$. Using that Ψ is a homomorphism of Hecke modules, we compare the formula for $\Psi(\mathcal{T}_i^\vee(\tilde{b}_w)) = \pi(T_i)(\Psi(\tilde{b}_w))$ from Lemma 3.7 to the Hecke action on the Casselman's basis $\pi(T_i)(f_w)$ from (32). Induction on Bruhat ordering, and a standard calculation using that

$$\{\alpha > 0 : ws_i(\alpha) > 0\} = \{\alpha > 0 : w(\alpha) > 0\} \cup \{\alpha_i\},$$

whenever $ws_i < w$, shows that for any $w \in W$, $\pi(T_i)(\Psi(\tilde{b}_w)) = \pi(T_i)(f_w)$. Since T_i is invertible in the Hecke algebra, the result follows.

For part (c), observe that the previous two parts, together with the definitions of $\tilde{m}_{u,w}$ and $m_{u,w}$ imply the equality $\tilde{m}_{u,w} = m_{u,w}(\tau)$ for all regular unramified characters τ satisfying $1 - qe^\alpha(\tau) \neq 0$, for any coroot α. However, it is known that the intertwiners $\mathcal{A}_x$ depend holomorphically on regular characters $\tau \in T^\vee$; see e.g. [Cas80, §3] or [Cas, §6.4]. Then one can extend meromorphically the equality $\tilde{m}_{u,w} = m_{u,w}(\tau)$ to any regular unramified character τ. $\square$

Combining Theorem 9.2 with Proposition 8.8 above gives a formula for $\tilde{m}_{u,w}$ in terms of localizations of motivic Chern classes, and in particular it recovers the Langlands-Gindikin-Karpelevich formula from (34). We are now ready to prove the main Theorem 9.1:

Proof of Theorem 9.1. This follows from Theorem 8.5 above together with the equality $\tilde{m}_{u,w} = m_{u,w}(\tau)$ for all regular unramified characters τ. □

9.4. **Analytic properties of transition coefficients.** The goal of this section is to prove a conjecture of Bump and Nakasuji [BN17, Conj. 1] about analytic properties for the transition coefficients $\tilde{m}_{u,w}$ and the set of coefficients $\tilde{r}_{u,w}$ defined as follows (cf. [BN17, Thm.3]). If $f = f(q)$ is a function, let $\bar{f}(q) := f(q^{-1})$. Then

$$\tilde{r}_{u,w} := \sum_{u \le x \le w} (-1)^{\ell(x)-\ell(u)} \overline{\tilde{m}}_{x,w}. \tag{35}$$

Since we are interested only in analytic properties, by the comparison Theorem 9.2(c), we can replace the coefficients $\tilde{m}_{u,w}$ by the 'geometric' ones $m_{u,w}$, and we abuse notation to denote by $r_{u,w} := \sum_{u \le x \le w} (-1)^{\ell(x)-\ell(u)} \overline{m}_{x,w}$ the resulting coefficients. With these notations, the main result in this section is to prove the following statement, see Conjecture 1 in [BN17]:

Theorem 9.4. *Let $u \le w$ be two Weyl group elements. Then the functions*

$$\prod_{\alpha \in S(u,w)} (1 - e^{\alpha}) r_{u,w} \quad , \quad \prod_{\alpha \in S(u,w)} (1 - e^{\alpha}) m_{u,w}$$

are holomorphic on the full dual torus $T^\vee$.

Proof. As observed by Bump and Nakasuji in *loc. cit.*, the conjecture for $r_{u,w}$ implies the conjecture for $m_{u,w}$. Further, by the formula from Proposition 8.8 of $m_{u,v}$,

$$\begin{aligned}(\overline{r}_{u,w})^\vee &= \sum_{u \le x \le w} (-1)^{\ell(x)-\ell(u)} m^\vee_{x,w} = \frac{1}{MC_{-q}(Y(w)^\circ)|_w} \sum_{u \le x \le w} (-1)^{\ell(x)-\ell(u)} MC_{-q}(Y(x))|_w \\ &= \frac{1}{MC_{-q}(Y(w)^\circ)|_w} \sum_{u \le x} (-1)^{\ell(x)-\ell(u)} MC_{-q}(Y(x))|_w = \frac{MC_{-q}(Y(u)^\circ)|_w}{MC_{-q}(Y(w)^\circ)|_w};\end{aligned}$$

here the second equality holds because $MC_{-q}(Y(x))|_w = 0$ for $x \not\le w$, as $e_w \notin Y(x)$, and the third equality follows by Möbius inversion on the Bruhat poset W. It follows that

$$\Big(\prod_{\alpha \in S(u,w)} (1 - e^{\alpha}) r_{u,w} \Big)^\vee = \prod_{\alpha \in S(u,w)} (1 - e^{-\alpha}) \frac{MC_{-q^{-1}}(Y(u)^\circ)|_w}{MC_{-q^{-1}}(Y(w)^\circ)|_w},$$

therefore it suffices to show that the right hand side is holomorphic. Using Proposition 6.2, equation (25), and the description of $MC_y(Y(w)^\circ)|_w$ from Theorem 8.1, we obtain

$$\begin{aligned}\prod_{\alpha \in S(u,w)} (1 - e^{-\alpha}) \frac{MC_{-q^{-1}}(Y(u)^\circ)|_w}{MC_{-q^{-1}}(Y(w)^\circ)|_w} &= \frac{MC_{-q^{-1}}(Y(u)^\circ)|_w}{\lambda_{-q^{-1}}(T^*_w Y(w))} \times \frac{\prod_{\alpha \in S(u,w)} (1 - e^{-\alpha})}{\prod_{\alpha > 0, w s_\alpha < w} (1 - e^{w\alpha})} \\ &= \frac{MC_{-q^{-1}}(Y(u)^\circ)|_w}{\lambda_{-q^{-1}}(T^*_w Y(w))} \times \frac{\prod_{\alpha > 0, u \le w s_\alpha < w} (1 - e^{w\alpha})}{\prod_{\alpha > 0, w s_\alpha < w} (1 - e^{w\alpha})} \\ &= \frac{MC_{-q^{-1}}(Y(u)^\circ)|_w}{\lambda_{-q^{-1}}(T^*_w Y(w)) \cdot \prod_{\alpha > 0, u \not\le w s_\alpha < w} (1 - e^{w\alpha})}.\end{aligned}$$

The last expression is a holomorphic function by Theorem 6.4 and we are done. □

We also record the following corollary of this proof:

Corollary 9.5. *The coefficients $r_{u,w}$ are obtained from the expansion:*

$$MC^{\vee}_{-q}(Y(u)^\circ) = \sum \overline{r}_{u,w} b_w$$

or, equivalently

$$\overline{r}^{\vee}_{u,w} = \frac{MC_{-q}(Y(u)^\circ)|_w}{MC_{-q}(Y(w)^\circ)|_w} \quad .$$

Remark 9.6. The coefficients $r_{u,w}$ satisfy other remarkable properties, such as a certain duality and orthogonality; see [NN15, BN17]. We will study these properties using motivic Chern classes in an upcoming note.

10. Appendix: proof of Lemma 7.2

Recall Lemma 7.2.

Lemma 10.1. *(a) Let $u, w \in W$. Under the left Weyl group multiplication,*

$$w.stab_{\mathcal{C},T^{1/2},\mathcal{L}}(u) = stab_{w\mathcal{C},wT^{1/2},w.\mathcal{L}}(wu).$$

In particular, if both the polarization $T^{1/2}$ and the line bundle $\mathcal{L}$ are G-equivariant, then

$$w.stab_{\mathcal{C},T^{1/2},\mathcal{L}}(u) = stab_{w\mathcal{C},T^{1/2},\mathcal{L}}(wu).$$

(b) The duality automorphism acts by sending $q \mapsto q^{-1}$ and

$$(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(w))^{\vee} = q^{-\frac{\dim G/B}{2}} \mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}}_{opp},\mathcal{L}^{-1}}(w), \tag{36}$$

where $T^{\frac{1}{2}}_{opp} := q^{-1}(T^{\frac{1}{2}})^{\vee}$ is the opposite polarization; see [OS16, Equation (15)]. *I.e., this duality changes the polarization and slope parameters to the opposite ones, while keeping the chamber parameter invariant.*

(c) Fix integral weights $\lambda, \mu \in X^*(T)$ and the equivariant line bundles $\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda$ and $\mathcal{L}_\mu$. Let $a \in \mathbb{Q}$ be a rational number. Then

$$\mathrm{stab}_{\mathcal{C},T^{1/2},a\mathcal{L}_\lambda \otimes \mathcal{L}_\mu}(w) = e^{-w(\mu)}\mathcal{L}_\mu \otimes \mathrm{stab}_{\mathcal{C},T^{1/2},a\mathcal{L}_\lambda}(w),$$

as elements in $K_{T\times\mathbb{C}^*}(T^*(G/B))$

Proof. We sketch the proof of part (a). By the uniqueness of the stable basis, we only need to check the expression on the left hand side satisfies the defining properties of the stable basis $\mathrm{stab}_{w\mathfrak{C},w(T^{\frac{1}{2}}),w\mathcal{L}}(wu)$ from Theorem 7.1. The support condition follows from the fact that

$$u \preceq_{\mathfrak{C}} v \Longleftrightarrow wu \preceq_{w\mathfrak{C}} wv. \tag{37}$$

The normalization condition follows from an explicit computation. We now check the degree condition.We need to show that

$$\deg_T \left(w(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(v))|_{wu}\right) \subset \deg_T \left(w(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(u))|_{wu}\right) + (w\mathcal{L})|_{wu} - (w\mathcal{L})|_{wv} \quad ,$$

for any $wu \prec_{w\mathfrak{C}} wv$. By Equations (17) and (37), this is equivalent to

$$\deg_T w(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(v)|_u) \subset \deg_T w(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(u)|_u) + w(\mathcal{L}|_u) - w(\mathcal{L}|_v) \quad ,$$

for any $u \prec_{\mathfrak{C}} v$, which is the defining property for the stable basis $\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(v)$. Hence, the degree condition is satisfied.

Part (b) is [OS16, Equation (15)] with $\hbar^{-\frac{\dim X}{2}}$ replaced by $q^{-\frac{\dim G/B}{2}}$.[6] Since this equation is not proved in *loc. cit.*, we include a proof in the case when the fixed point set X^T satisfies $\dim X^T = 0$ (e.g. $X = T^*(G/B)$).[7] By the uniqueness of stable envelopes, we need to show $q^{\frac{\dim X}{4}}(\mathrm{stab}_{\mathfrak{C},T^{\frac12},\mathcal{L}})^\vee$ satisfies the defining properties of $\mathrm{stab}_{\mathfrak{C},T^{\frac12}_{opp},\mathcal{L}^{-1}}$. The support condition is obvious, and the degree condition follows because $\deg_T$ remains unchanged after multiplication by powers of $q^{\pm\frac12}$. We turn to the normalization condition. Denote by F a component of X^T (a point, in our case), and we use the notation $N_+, N_-, N^{\frac12}$ etc. for the appropriate normal subspaces to F, as before Theorem 7.1. Since $N_- - T^{\frac12} = q^{-1}(T^{\frac12}_{>0})^\vee - T^{\frac12}_{>0}$ (see [OS16, p.13]), the normalization is (see [OS16, Equation (10)])

$$\mathrm{stab}_{\mathfrak{C},T^{\frac12},\mathcal{L}}|_F = (-1)^{\mathrm{rk}\,T^{\frac12}_{>0}}\left(\frac{\det N_-}{\det T^{\frac12}}\right)^{\frac12}\mathcal{O}_{\mathrm{Attr}}|_F = (-1)^{\mathrm{rk}\,T^{\frac12}_{>0}} q^{-\frac{\mathrm{rk}\,T^{\frac12}_{>0}}{2}}(\det T^{\frac12}_{>0})^\vee\mathcal{O}_{N_+}|_F.$$

The last equality follows because the normal bundle of Attr at F is spanned by the *non-attracting* weights at F; this is the same as the normal bundle of N_+ inside N, therefore $\mathcal{O}_{\mathrm{Attr}}|_F = \mathcal{O}_{N_+}|_F = \lambda_{-1}^{T\times\mathbb{C}^*}(N_-^\vee)$.

Assume the torus weights of $T^{\frac12}_{>0}|_F$ are $\{\gamma_j\}$, and the torus weights of $T^{\frac12}_{<0}|_F$ are $\{\beta_i\}$. Since $T(X) = T^{\frac12} + q^{-1}(T^{\frac12})^\vee$, the torus weights of $N_-|_F$ are $\{\beta_i\}$ and $\{q^{-1}\gamma_j^{-1}\}$. We abuse notation and write λ for $e^\lambda \in R(T)$. Then,

$$\begin{aligned}\mathrm{stab}_{\mathfrak{C},T^{\frac12},\mathcal{L}}|_F &= (-1)^{\mathrm{rk}\,T^{\frac12}_{>0}} q^{-\frac{\mathrm{rk}\,T^{\frac12}_{>0}}{2}}\prod_j\gamma_j^{-1}\prod_i(1-\beta_i^{-1})\prod_j(1-q\gamma_j)\\ &= q^{\frac{\mathrm{rk}\,T^{\frac12}_{>0}}{2}}\prod_i(1-\beta_i^{-1})\prod_j(1-q^{-1}\gamma_j^{-1}).\end{aligned}\tag{38}$$

Since $T^{\frac12}_{opp} = q^{-1}(T^{\frac12})^\vee$, the torus weights of $T^{\frac12}_{opp,>0}|_F$ are $\{q^{-1}\beta_i^{-1}\}$, and the torus weights of $T^{\frac12}_{opp,<0}|_F$ are $\{q^{-1}\gamma_j^{-1}\}$. A similar calculation shows

$$\mathrm{stab}_{\mathfrak{C},T^{\frac12}_{opp},\mathcal{L}^{-1}}|_F = q^{\frac{\mathrm{rk}\,T^{\frac12}_{<0}}{2}}\prod_j(1-q\gamma_j)\prod_i(1-\beta_i).$$

Taking the dual of (38), we get

$$\left(\mathrm{stab}_{\mathfrak{C},T^{\frac12},\mathcal{L}}\right)^\vee|_F = q^{-\frac{\mathrm{rk}\,T^{\frac12}_{>0}}{2}}\prod_i(1-\beta_i)\prod_j(1-q\gamma_j).$$

Therefore,

$$\left(\mathrm{stab}_{\mathfrak{C},T^{\frac12},\mathcal{L}}\right)^\vee|_F = q^{-\frac{\mathrm{rk}\,T^{\frac12}}{2}}\mathrm{stab}_{\mathfrak{C},T^{\frac12}_{opp},\mathcal{L}^{-1}}|_F = q^{-\frac{\dim X}{4}}\mathrm{stab}_{\mathfrak{C},T^{\frac12}_{opp},\mathcal{L}^{-1}}|_F.$$

This proves the normalization condition, whence part (b). Part (c) follows directly from the uniqueness of the stable envelope. $\square$

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[6]In [OS16] the variety X is the symplectic resolution, and it corresponds to our $T^*(G/B)$, so $\hbar^{-\frac{\dim X}{2}}$ should be $q^{-\dim G/B}$; the missing factor of 2 is a typo.

[7]Our T is denoted by A in *loc. cit.* and this is the torus preserving the symplectic form of X.

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Mathematics Department, Florida State University, Tallahassee FL 32306
E-mail address: aluffi@math.fsu.edu

Department of Mathematics, Virginia Tech University, Blacksburg, VA 24061
E-mail address: lmihalce@vt.edu

Mathematisches Institut, Universität Münster, Germany
E-mail address: jschuerm@uni-muenster.de

Department of Mathematics, University of Toronto, Toronto, ON, M5S, 2E4, Canada
E-mail address: changjiansu@gmail.com