PARTIALLY HYPERBOLIC DIFFEOMORPHISMS HOMOTOPIC TO THE IDENTITY IN DIMENSION 3

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Abstract. We study partially hyperbolic diffeomorphisms homotopic to the identity in 3-manifolds. Under a general minimality condition, we show a dichotomy for the dynamics of the (branching) foliations in the universal cover. This allows us to give a full classification in certain settings: dynamically coherent partially hyperbolic diffeomorphisms on hyperbolic 3-manifolds (proving a classification conjecture of Hertz-Hertz-Ures in this setting), and partially hyperbolic diffeomorphisms homotopic to the identity on Seifert fibered manifolds (proving a conjecture of Pujals in this setting). In both cases, up to iterates we prove that the diffeomorphism is leaf conjugate to the time one map of an Anosov flow. Several other results of independent interest are obtained along the way.

Keywords: Partial hyperbolicity, 3-manifold topology, foliations, classification.


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1. Introduction

A diffeomorphism $f$ of a 3-manifold $M$ is partially hyperbolic if it preserves a splitting of the tangent bundle $TM$ into three 1-dimensional sub-bundles

$$TM = E^s \oplus E^c \oplus E^u,$$

where the stable bundle $E^s$ is eventually contracted, the unstable bundle $E^u$ is eventually expanded, and the center bundle $E^c$ is distorted less than the stable and unstable bundles at each point.

From a dynamical perspective, the interest in partial hyperbolicity stems from its appearance as a generic consequence of certain dynamical conditions, such
as stable ergodicity and robust transitivity. For example, a diffeomorphism is *transitive* if it has a dense orbit, and *robustly transitive* if this behavior persists under $C^1$-small deformations. Every robustly transitive diffeomorphism on a 3-manifold is either Anosov or “weakly” partially hyperbolic [DPU99]. Analogous results hold for stable ergodicity and in higher dimensions [BDV05].

From a geometric perspective, one can think of partial hyperbolicity as a generalization of the discrete behavior of Anosov flows, which feature prominently in the theory of 3-manifolds. Recall that a flow $\Phi$ on a 3-manifold $M$ is *Anosov* if it preserves a splitting of the unit tangent bundle $TM$ into three 1-dimensional sub-bundles

$$TM = E^s \oplus T\Phi \oplus E^u,$$

where $T\Phi$ is the tangent direction to the flow, $E^s$ is eventually exponentially contracted, and $E^u$ is eventually exponentially expanded. After flowing for a fixed time, an Anosov flow generates a partially hyperbolic diffeomorphism of a particularly simple type, where the stable and unstable bundles are contracted uniformly, and the center direction, which corresponds to $T\Phi$, is left undistorted. More generally, one can construct partially hyperbolic diffeomorphisms of the form $f(x) = \Phi_{\tau(x)}(x)$ where $\Phi$ is a (topological) Anosov flow and $\tau : M \to \mathbb{R}_{>0}$ is a positive continuous function; the partially hyperbolic diffeomorphisms obtained in this way are called *discretized Anosov flows*.

In this article we show that large classes of partially hyperbolic diffeomorphisms can be identified with discretized Anosov flows. This confirms a large part of the well-known conjecture by Pujals [BW05], which attempts to classify 3-dimensional partially hyperbolic diffeomorphisms by asserting that they are all either discretized Anosov flows or deformations of certain kinds of algebraic examples.

1.1. **Homotopy, integrability, and conjugacy.** There are two important obstructions to identifying a partially hyperbolic diffeomorphism with a discretized Anosov flow. The first comes from the fact that the latter are homotopic to the identity, while the former may be homotopically nontrivial. Homotopically nontrivial examples include Anosov diffeomorphisms on the 3-torus with distinct eigenvalues, “skew products,” and the counterexamples to Pujals’ conjecture constructed in [BPP16, BGP16, BZ17, BGHP17].

The second major obstruction comes from the integrability of the bundles in a partially hyperbolic splitting. In the context of an Anosov flow $\Phi$, the stable and unstable bundles $E^s$ and $E^u$ integrate uniquely into a pair of 1-dimensional foliations, the stable foliation $W^s$ and unstable foliation $W^u$. In fact, even the weak stable and weak unstable bundles $E^{s}_w \oplus T\Phi$ and $E^{u}_w \oplus T\Phi$ integrate uniquely into a transverse pair of $\Phi$-invariant 2-dimensional foliations, the weak stable foliation $W^{sw}$ and weak unstable foliation $W^{wu}$.

In the context of a partially hyperbolic diffeomorphism $f$, the stable and unstable bundles still integrate uniquely into stable and unstable foliations, $W^s$ and $W^u$ [HP18]. However, the center stable and center unstable bundles $E^c \oplus E^s$ and $E^c \oplus E^u$ may fail to be uniquely integrable. In fact, there are examples where it is impossible to find any $f$-invariant 2-dimensional foliation tangent to the center stable or center unstable bundle [RHRHU16, BGHP17].

\footnote{Note that a discretized Anosov flow is not in general the time-1 map of a reparametrization of the Anosov flow, see Appendix G}
If one can find a pair of $f$-invariant foliations tangent to the center stable and center unstable bundles then $f$ is said to be dynamically coherent. This condition must certainly be satisfied for $f$ to be a discretized Anosov flow (cf. Appendix G).

1.2. Major results. Most of the existing progress towards classifying partially hyperbolic diffeomorphisms takes an outside-in approach, restricting attention to particular classes of manifolds, and comparing to an a priori known model partially hyperbolic (see [CRRU15, HP18] for recent surveys). In particular, partially hyperbolic diffeomorphisms have been completely classified in manifolds with solvable or virtually solvable fundamental group [HP14, HP15].

Ours is an inside-out approach, using the theory of foliations to understand the way the local structure that defines partial hyperbolicity is pieced together into a global picture. We then relate the dynamics of these foliations to the large-scale structure of the ambient manifold. An advantage of this method is that, since it does not rely on a model partially hyperbolic to compare to, we can consider any manifold, not just one where an Anosov flow is known to exist.

The following two theorems are the main consequences of our work, applied to two of the major classes of 3-manifolds. Note that the classification of partially hyperbolic diffeomorphisms is always considered up to finite lifts and iterates, since one can easily build infinitely many different but uninteresting examples by taking finite covers.

Theorem A (Seifert manifolds). Let $f : M \to M$ be a partially hyperbolic diffeomorphism on a closed Seifert fibered 3-manifold. If $f$ is homotopic to the identity, then it is dynamically coherent, and some iterate is a discretized Anosov flow.

This resolves the Pujals’ Conjecture for Seifert fibered manifolds$^2$.

Note that the preceding theorem does not assume dynamical coherence, nor does it use the classification of Anosov flows on Seifert fibered 3-manifolds [Gly84, Bar96]. A weaker version of this theorem was recently announced by Ures [Ure], with the additional assumption that $f$ is isotopic, through partially hyperbolic diffeomorphisms, to the time-1 map of an Anosov flow.

Theorem B (Hyperbolic manifolds). Let $f : M \to M$ be a partially hyperbolic diffeomorphism on a closed hyperbolic 3-manifold. If $f$ is dynamically coherent, then some iterate is a discretized Anosov flow.

This resolves a classification conjecture in [CRRU15] for hyperbolic 3-manifolds.

Note that this theorem does not assume that $f$ is homotopic to the identity, since any homeomorphism on a closed hyperbolic 3-manifold has a finite power that is homotopic to the identity. It does, however, assume dynamical coherence.

1.3. Results. Theorems A and B are the consequences of some stronger and more general statements, which require some knowledge of taut foliations. See Appendix B for the relevant definitions.

Also, as stated previously, the complete classification of partially hyperbolic diffeomorphisms is known when the 3-manifold has virtually solvable fundamental group [HP14, HP15, HP18], in particular Theorem A holds in this setting (cf. Theorem F.8). Thus we always assume our manifolds to have non virtually solvable fundamental group.

$^2$The conjecture is true for Seifert manifolds with fundamental group with polynomial growth [HP14] and false in Seifert fibered manifolds when the isotopy class is not the identity as the examples in [BGP16, BGHP17] are not homotopic to identity and so cannot be discretized Anosov flows.
1.3.1. Dynamically coherent case. Let $f: M \to M$ be a partially hyperbolic diffeomorphism on a closed 3-manifold $M$. When $f$ is homotopic to the identity, we denote by $\tilde{f}$ the specific lift to the universal cover $\tilde{M}$ that is obtained by lifting such a homotopy. We begin with the dynamically coherent case, where we denote the center stable and center unstable foliations by $W^{cs}$ and $W^{cu}$, and their lifts to $\tilde{M}$ by $\tilde{W}^{cs}$ and $\tilde{W}^{cu}$.

**Theorem 1.1.** Let $f: M \to M$ be a partially hyperbolic diffeomorphism on a closed 3-manifold $M$ that is dynamically coherent and homotopic to the identity. If $W^{cs}$ and $W^{cu}$ are $f$-minimal, then either

(i) $f$ is a discretized Anosov flow, or

(ii) $W^{cs}$ and $W^{cu}$ are $\mathbb{R}$-covered and uniform, and $\tilde{f}$ acts as a translation on the leaf spaces of $\tilde{W}^{cs}$ and $\tilde{W}^{cu}$.

Here, $f$-minimal means that the only closed sets that are both $f$-invariant and saturated by the foliation are the empty set and the whole manifold $M$. If $f$ is transitive or volume preserving, then it is already known that $W^{cs}$ and $W^{cu}$ are $f$-minimal [BW05]. We will show that this holds as well when $M$ is hyperbolic or Seifert and the lift $\tilde{f}$ fixes a leaf in the universal cover (see Proposition 3.15). We show that (ii) cannot occur in a hyperbolic manifold, and Theorem B follows.

It is likely that Theorem B could be shown in the setting of 3-manifolds with atoroidal pieces in their JSJ decomposition, but we have not pursued this here as it would require proving results similar to [Thu, Cal00, Fen02] in this setting.

The technique to eliminate the possibility (ii) in Theorem 1.1 is more widely applicable: In a companion article [BFFP], we use the same ideas to show that a partially hyperbolic diffeomorphism on a Seifert manifold which acts as a pseudo-Anosov on (part of) the base is not dynamically coherent.

For Seifert manifolds, it is possible to show that, after taking an iterate, there is another lift $\tilde{f}$ that is still a bounded distance from the identity and which fixes a leaf of $\tilde{W}^{cs}$ (see Proposition 13.2), and $f$-minimality follows. We show that (ii) implies leaf conjugacy of (possibly an iterate of) $f$ to a time one map of an Anosov flow on a Seifert fibered manifold, and Theorem A follows under the additional assumption of dynamical coherence. We also completely classify the partially hyperbolic diffeomorphisms for which it is necessary to take an iterate, as opposed to $f$ itself, to get a discretized Anosov flow (see Remark 7.4).

1.3.2. Non-dynamically coherent case. In the non-dynamically coherent case, we work with the center-stable and center-unstable “branching” foliations $W^{cs}_{\text{bran}}$ and $W^{cu}_{\text{bran}}$ introduced in [BI08]. These behave like foliations, but leaves are allowed to merge together (see Definition 10.2). Their lifts to $\tilde{M}$ are denoted by $\tilde{W}^{cs}_{\text{bran}}$ and $\tilde{W}^{cu}_{\text{bran}}$.

**Theorem 1.2.** Let $f: M \to M$ be a partially hyperbolic diffeomorphism on a closed 3-manifold $M$ that is homotopic to the identity. If $f$ preserves two branching foliations $W^{cs}_{\text{bran}}$ and $W^{cu}_{\text{bran}}$ that are $f$-minimal, then either

(i) $f$ is a discretized Anosov flow (and in particular dynamically coherent),

(ii) $\tilde{f}$ fixes the leaves of one of the lifted branching foliations in $\tilde{M}$, and the other branching foliation is $\mathbb{R}$-covered, uniform, and $\tilde{f}$ acts as a translation on its leaf space in the universal cover, or

(iii) $W^{cs}_{\text{bran}}$ and $W^{cu}_{\text{bran}}$ are $\mathbb{R}$-covered and uniform, and $\tilde{f}$ acts as a translation on the leaf spaces of $\tilde{W}^{cs}_{\text{bran}}$ and $\tilde{W}^{cu}_{\text{bran}}$. 
As in the dynamically coherent case, we already know that $f$-minimality holds when $f$ is either transitive or volume preserving.

We will show that case (ii) of Theorem 1.2 cannot occur when $M$ is hyperbolic or Seifert fibered (in §15 and §13 respectively), in which case we can also eliminate the hypothesis of $f$-minimality, obtaining the following:

**Theorem 1.3.** Let $f: M \to M$ be a partially hyperbolic diffeomorphism on a closed hyperbolic or Seifert fibered 3-manifold that is homotopic to the identity. Then either

(i) $f$ is a discretized Anosov flow, or

(ii) both $W_{\text{cs}}^{\text{bran}}$ and $W_{\text{cu}}^{\text{bran}}$ are $\mathbb{R}$-covered and uniform, and $\tilde{f}$ acts as a translation on both leaf spaces in $\tilde{M}$.

Again, it is likely that this result may be proven under the assumption of $f$-minimality and the existence of an atoroidal piece in the JSJ decomposition of $M$.

**Remark 1.4.** Case (ii) of Theorem 1.2 may also be ruled out under the assumption of absolute partial hyperbolicity (cf. §16).

When $M$ is Seifert fibered, we know that the branching foliations must be horizontal [HPS18], i.e., that they can be isotoped to be transverse to the Seifert fibers. Using this, we can eliminate possibility (ii) of Theorem 1.2 after taking a finite iterate. The following proposition, together with Theorem 1.3, yields Theorem A.

**Proposition 1.5.** When $M$ is Seifert and $f$ is homotopic to the identity, there always exists a lift $\tilde{f}$ of an iterate of $f$ which fixes at least one leaf of $\tilde{W}_{\text{cs}}^{\text{bran}}$.

When $M$ is hyperbolic, it is not known whether the second possibility in Theorem 1.3 (which we refer to as a double translation) may occur, but it follows from Theorem B that an example with such behavior could not be dynamically coherent.

We end the introduction by stating a dynamical consequence of our results and analysis.

**Theorem 1.6.** Let $f: M \to M$ be a partially hyperbolic diffeomorphism of a closed 3-manifold $M$ homotopic to the identity and assume that one of the following conditions is verified:

- $M$ is hyperbolic or Seifert fibered, or,
- the (branching) center stable foliation is $f$-minimal,

then $f$ has no contractible periodic points.

This result will be proven as Corollary 11.11.

1.4. **Remarks and references.** The definition of a partially hyperbolic diffeomorphism traces back to [HPS77] and [BP73].

The classification problem for 3-dimensional partially hyperbolic diffeomorphisms has attracted significant attention since the pioneering work of [BW05, BBI04], which was partially motivated by Pujals’ conjecture (see also [PS04, §20] or [HP06, §2.2.6]). For example, there is interesting recent work done under some restrictions on the center dynamics [Zha17, BZ19] or assuming some conditions on the derivatives [CPR19]. See also the surveys [CRRU15, HP18, Pot18].

Besides its intrinsic interest, the classification problem for partially hyperbolic diffeomorphisms has dynamical implications. For example, several finer dynamical and ergodic properties have been studied under the assumption of having
a discretized Anosov flow (while not using that terminology), for instance in [AVW15, BFT19] (see also [Pot18, Wil10]). Our result makes that condition checkable. Several of the techniques presented here also yield information about the dynamics along the center direction, which is so far poorly understood (see, e.g. [FP18]).

In addition, this article contains several new results of independent interest. Indeed, important steps in our study do not use partial hyperbolicity, but more general foliation preserving diffeomorphisms. Thus some results (see §3 and §8 for instance) are much more widely applicable. In particular, in §8 we use regulating pseudo-Anosov flows to understand the dynamics of a diffeomorphism that translates the leaves of an $\mathbb{R}$-covered foliation, showing that any such diffeomorphism has “invariant cores” that shadow the closed orbits of the corresponding flow.

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2. Outline and discussion

In this section we will set some basic terminology, outline our major arguments, and detail the organization of this paper.

Definition 2.1. A $C^1$-diffeomorphism $f: M \to M$ on a 3-manifold $M$ is partially hyperbolic if there is a $Df$-invariant splitting of the tangent bundle $TM$ into three 1-dimensional bundles $TM = E^s \oplus E^c \oplus E^u$ such that for some $n > 0$, one has

$$\|Df^n|_{E^s(x)}\|<1,$$

$$\|Df^n|_{E^c(x)}\|>1,$$

and

$$\|Df^n|_{E^c(x)}\|<\|Df^n|_{E^u(x)}\|<\|Df^n|_{E^c(x)}\|,$$

for all $x \in M$.

See Appendix F for more details. Our major goal is to show that large classes of partially hyperbolic diffeomorphisms are discretized Anosov flows:

Definition 2.2. A discretized Anosov flow is a partially hyperbolic diffeomorphism $g: M \to M$ on a 3-manifold $M$ that is of the form $g(p) = \Phi_t(p)$ for a topological Anosov flow $\Phi$ and a map $t: M \to (0, \infty)$.

The precise definition of a topological Anosov flow is given in Appendix G, where we also explain the relationship between discretized Anosov flows and the more common notion of partially hyperbolic diffeomorphisms that are leaf conjugate to time-1 maps of Anosov flows.

Consider a discretized Anosov flow $g: M \to M$ on a closed 3-manifold $M$. We will see (Proposition G.2) that $g$ is dynamically coherent, and that the center
leaves of \( g \) are exactly the orbits of the underlying flow. This means that \( g \) fixes each leaf of the center foliation. Moreover, it has a natural lift \( \tilde{g} : \tilde{M} \to \tilde{M} \) to the universal cover that fixes the lift of each center leaf, but fixes no point in \( \tilde{M} \). Indeed, such a lift may be obtained by flowing points along lifted orbits. That is, \( \tilde{g}(p) = \tilde{\Phi}_{t(\pi(p))}(p) \), where \( \tilde{\Phi} \) is the lifted flow and \( \pi : \tilde{M} \to M \) is the covering map.

In fact, to show that a partially hyperbolic diffeomorphism \( f : M \to M \) is a discretized Anosov flow, it will suffice to find a lift \( \tilde{f} : \tilde{M} \to \tilde{M} \) with this property, i.e., that fixes the leaves of the lifted center foliation, but fixes no point in \( \tilde{M} \). This argument is essentially given in [BW05, Section 3.5] — see Section 6.2.

2.1. Organization. This article is organized in two parts: §3–9 work under the additional assumption of dynamical coherence, and §10–15 handle the general case. This is an expository choice, not a logical one, as many of our arguments can be made to handle the dynamically coherent and incoherent cases simultaneously.

In addition to reducing the initial difficulty of certain arguments, this split illuminates some of the specific guises in which dynamical incoherence can appear. For instance, we will find some phenomena stemming from dynamical incoherence that are compatible with absolute partial hyperbolicity, and some that are not (see Section 16).

In addition to the thousand words that follow, the picture below summarizes the interdependence of the sections of this article.

\[
\begin{array}{cccc}
\text{§3} & \text{§10} \\
\text{§4} & \text{§11} & \text{Thm 1.6} \\
\text{Thm 1.1} & \text{§5} & \text{Rmk 1.4} & \text{§16} & \text{§12} & \text{Thm 1.2} \\
\text{§6} & \text{Thm A} & \text{§13} \\
\text{Thm A} & \text{DC case} & \text{§7} & \text{§8} & \text{§14} \\
\text{Thm B} & \text{§9} & \text{§15} & \text{Thm 1.3} \\
\end{array}
\]

2.2. Setup. We will now set some basic definitions and outline our major arguments. We will assume some familiarity with 3-manifold topology, taut foliations, and leaf spaces; see Appendices A and B for an outline of the necessary background.

In this paper, \( M \) will be a closed 3-manifold, and \( f : M \to M \) will be a partially hyperbolic diffeomorphism that is homotopic to the identity.

**Convention:** Throughout this paper we will assume that \( \pi_1(M) \) is not virtually solvable.

This assumption implies that there is no closed surface tangent to either \( E^{cs} \) or \( E^{cu} \) (Theorem F.1), a fact that we will use often.
The classification of partially hyperbolic diffeomorphisms on manifolds with virtually solvable fundamental group is complete [HP14, HP15], and our assumption does not affect our main results (see Theorem F.8).

2.2.1. Good lifts. Since \( f \) is homotopic to the identity, we can lift such a homotopy to \( \tilde{M} \), and obtain a lift \( \tilde{f} : \tilde{M} \to \tilde{M} \) that is good:

**Definition 2.3.** A lift \( \tilde{f} : \tilde{M} \to \tilde{M} \) of a homeomorphism \( f : M \to M \) is called a good lift if

(i) it moves each point a uniformly bounded distance (i.e., there exists \( K > 0 \) such that \( d_{\tilde{M}}(x, \tilde{f}(x)) < K \) for all \( x \in \tilde{M} \)), and

(ii) it commutes with every deck transformation.

In the sequel, we will always take \( \tilde{f} \) to be a good lift of \( f \).

**Remark 2.4.** In fact, it is easy to show that (i) follows from (ii) on a closed manifold.

A homeomorphism may have more than one good lift. Indeed, composing a good lift with a deck transformation in the center of the fundamental group yields another good lift. Conversely, the existence of more than one good lift implies that the fundamental group has non-trivial center. By the Seifert fibered space conjecture [CJ94, Gab92], this implies that the manifold is Seifert-fibered with orientable Seifert fibration.

2.3. Part 1: The dynamically coherent case. We will begin by outlining our arguments with the assumption that \( f \) is dynamically coherent. The center stable, center unstable, stable, unstable, and center foliations on \( M \) are denoted by \( W_{cs}, W_{cu}, W_s, W_u, \) and \( W_c \). These lift by the universal covering map \( \pi : \tilde{M} \to M \) to foliations on \( \tilde{M} \) which we denote by \( \tilde{W}_{cs}, \tilde{W}_{cu}, \tilde{W}_s, \tilde{W}_u, \) and \( \tilde{W}_c \).

Recall that there are no closed surfaces tangent to the center stable or center unstable bundles. In particular, \( W_{cs} \) and \( W_{cu} \) have no closed leaves, which implies that they are taut. Furthermore, \( \tilde{M} \) is homeomorphic to \( \mathbb{R}^3 \), and each leaf of \( \tilde{W}_{cs} \) or \( \tilde{W}_{cu} \) is a properly embedded plane that separates \( \tilde{M} \) into two open balls (cf. Theorem B.1).

2.3.1. Dichotomies for foliations. In §3 we study the basic structure of the center stable and center unstable foliations, and the way that \( \tilde{f} \) permutes their lifted leaves. Much of this section applies more generally to a homeomorphism that is homotopic to the identity and preserves a foliation.

The basic tool is Lemma 3.3, which says that the complementary components of a lifted center stable or center unstable leaf are “large” in the sense that they contains balls of arbitrary radius. Since \( \tilde{f} \) moves points a uniformly bounded distance, this has immediate consequences for the way that it acts on the leaf spaces of \( \tilde{W}_{cs} \) and \( \tilde{W}_{cu} \).

In particular, we deduce that the set of center stable leaves that are fixed by \( \tilde{f} \) is closed in the leaf space \( \mathcal{L}^{cs} \) of \( \tilde{W}_{cs} \), each complementary component of this set is an open interval that is acted on by \( \tilde{f} \) as a translation, and any two leaves in one of these “translation regions” are a finite Hausdorff distance apart (Proposition 3.7). The same holds for the center unstable foliation.
When $\mathcal{W}^{cs}$ is $f$-minimal, or $M$ is hyperbolic or Seifert-fibered, we use this to show that either:

- $\tilde{f}$ fixes every leaf of $\tilde{\mathcal{W}}^{cs}$, or
- $\mathcal{W}^{cs}$ is $\mathbb{R}$-covered and uniform, and $\tilde{f}$ acts as a translation on the leaf space of $\tilde{\mathcal{W}}^{cs}$.

Recall that $\mathbb{R}$-covered means that the leaf space in the universal cover is $\simeq \mathbb{R}$, and uniform means that any two leaves in the universal cover are a finite Hausdorff distance apart.

This dichotomy is easy to show under the assumption of $f$-minimality, where it does not use partial hyperbolicity (Corollary 3.10). It takes significantly more work under the assumption that $M$ is hyperbolic or Seifert-fibered (Proposition 3.15).

If $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ are $f$-minimal, or $M$ is hyperbolic or Seifert-fibered, we are left with three possibilities:

1. **Double invariance:** $\tilde{f}$ fixes every leaf of both $\tilde{\mathcal{W}}^{cs}$ and $\tilde{\mathcal{W}}^{cu}$;
2. **Mixed behavior:** $f$ fixes every leaf of either $\mathcal{W}^{cs}$ or $\mathcal{W}^{cu}$, and acts as a translation on the leaf space of the other, which is $\mathbb{R}$-covered and uniform;
   or
3. **Double translation:** $\tilde{f}$ acts as a translation on both $\tilde{\mathcal{W}}^{cs}$ and $\tilde{\mathcal{W}}^{cu}$, which are $\mathbb{R}$-covered and uniform.

The remainder of the argument is arranged around these three possibilities. We will see in §5 that mixed behavior cannot happen. In §6 we show that double invariance implies that $f$ is a discretized Anosov flow. The double translation case is ruled out for Seifert-fibered manifolds in §7, and for hyperbolic manifolds in §8–§9.

### 2.3.2. Center dynamics in fixed leaves.

In §4, we work under the assumption that $\tilde{f}$ fixes every leaf of $\tilde{\mathcal{W}}^{cs}$, and study the dynamics within each center stable leaf. In particular, we show (Proposition 4.4):

- If $\tilde{f}$ fixes every leaf of $\tilde{\mathcal{W}}^{cs}$, then any leaf of $\tilde{\mathcal{W}}^{cs}$ that is fixed by a nontrivial deck transformation contains a center leaf that is fixed by $\tilde{f}$.

This immediately eliminates the possibility of mixed behavior (see §5). It will also be used in §6 to show that double invariance implies that $f$ is a discretized Anosov flow.

Consider a center stable leaf $L$ that is fixed by a deck transformation $\gamma$. The proof of (**) comes down to understanding the topology of the stable foliation within $L$ “in the direction of” $\gamma$. The formal meaning of this is the axis for the action of $\tilde{f}$ on the stable leaf space in $L$ (see Appendix E), but it can be understood intuitively as the set of all stable leaves that cross the core of the cylinder $\tilde{M}/\langle \gamma \rangle$ essentially.

Suppose that there is an line’s worth of stable leaves in this direction, which corresponds to circle’s worth in $\tilde{M}/\langle \gamma \rangle$ as depicted (roughly) in the left half of Figure 1. Then one can find a curve representing $\gamma$ that is transverse to the stable foliation, and a “graph transform argument” finds a corresponding center leaf preserved by both $\gamma$ and $\tilde{f}$ (Lemma 4.5).
The other possibility is that one finds gaps, which look roughly like Reeb components as in the right half of Figure 1. We eliminate the possibility of such gaps by combining the dynamics coming from partial hyperbolicity with two conflicting forces:

(i) On one hand, the topology of the stable and center foliations within $L$ forces the existence of a center ray within this gap that is expanded by $f$ (Lemma 4.7).

(ii) On the other hand, we find from the geometry of $L$ that the entire gap, and any center leaf within it, must be coarsely contracted (Lemma 4.9).

These conclusions are contradictory, so there can be no gaps, completing the proof of (⋆⋆).

The existence of the expanded center ray (i) is delicate, and may disappear in the dynamically incoherent case when center leaves are allowed to merge (see Remark 4.8 and Figure 6). The coarse contraction of gaps (ii) is more robust, and will be used again in the dynamically incoherent case.

2.3.3. Double invariance. In the doubly invariant case (1), one would like to show that $\tilde{f}$ fixes each center leaf. Since by assumption it fixes each center stable and center unstable leaf, it fixes the intersection between any two such leaves. Each component of this intersection is a collection of center leaves, but there is no a priori reason for it to have a single component. In §6.1, we show that $\tilde{f}$ fixes either every center leaf or no center leaf, and so by (⋆⋆) it fixes every center leaf.

Once we know that $\tilde{f}$ fixes every center leaf, we can use the arguments of Bonatti–Wilkinson [BW05] to show that the center foliation is the orbit foliation of a topological Anosov flow, and hence that $f$ is a discretized Anosov flow. This is done in §6.2, completing the proof of Theorem 1.1.

2.3.4. Double translation in Seifert-fibered manifolds. The double translation case (3) turns out to be the trickiest. The preceding results work with either topological conditions ($M$ being Seifert fibered or hyperbolic) or dynamical conditions (minimality or $f$-minimality). To handle double translations we will need topological restrictions.

Part of the difficulty is that double translation do in fact exist (Remark 7.4)! However, these examples live in Seifert-fibered manifolds, and have iterates that are discretized Anosov flows. Using a similar idea (see section 6.2 of [BFFP18]), one can build homeomorphisms that act as a double translation on any manifold,
but our techniques do show that they cannot be partially hyperbolic (even in a topological sense) when the ambient manifold is hyperbolic.

Eliminating double translations when $M$ is a Seifert manifold relies on a trick: Since there are many good lifts, we show in §7 that some good lifts (of a power of $f$) must fix the leaf of at least one foliation. This completes the proof of Theorem A under the additional assumption of dynamical coherence (see Theorem 7.3).

2.3.5. Double translation in hyperbolic manifolds. We are left to treat the case of double translations in hyperbolic manifolds, which we do in §8 and §9.

In §8, we prove a result about $\mathbb{R}$-covered foliations that is of general interest. In a hyperbolic 3-manifold, an $\mathbb{R}$-covered foliation admits a transverse regulating pseudo-Anosov flow (see Appendix D). We will use this flow to understand the dynamics of any homeomorphism that acts as a translation on its leaf space (Proposition 8.1):

Let $f: M \to M$ be a homeomorphism on a closed hyperbolic 3-manifold that is homotopic to the identity and preserves a taut, $\mathbb{R}$-covered foliation $\mathcal{T}$. Suppose that a good lift acts as a translation on $\mathcal{T}$.

\[\ast\ast\ast\]

Then for each periodic orbit $\gamma$ of the regulating pseudo-Anosov flow $\Phi$, there is a corresponding invariant “core” $T_\gamma$ for $f$. Moreover, the dynamics of $f$ at $T_\gamma$ is coarsely identical to the dynamics of $\Phi$ at $\gamma$ (in the sense that they have the same Lefschetz index).

There is a little lie in this description, as the core $T_\gamma$ is in fact in the cover $\widetilde{M}/\langle \gamma \rangle$ and is invariant under the appropriate lift of $f$ to that cover.

In fact, having a hyperbolic manifold is not essential — we use similar techniques on Seifert-fibered manifolds in [BFFP].

The result (\(*\ast\ast\ast\)) is the main ingredient in §9, where we show that double translations cannot occur in hyperbolic manifolds and complete the proof of Theorem B.

The rough idea is that (\(*\ast\ast\ast\)) gives a circle invariant by (a lift of) $f$ and with at least one fixed point, but partial hyperbolicity implies that any fixed point must be, say, repulsive. But the devil is in the details, and while one can make this rough idea precise in the case of a foliation, it does not lead to a contradiction for branching foliation. This is why Theorem B requires dynamical coherence.

2.4. Part 2: The general case. The second part of this article does away with the assumption of dynamical coherence, and uses the “branching foliations” of Burago–Ivanov [BI08] to stand in for the true foliations of Part 1.

Many of the results in Part 1 adapt directly, though some take much more care or a different route. There are two places where the general case diverges significantly from the dynamically coherent case: The first is in the proof of (\(*\ast\ast\ast\)), which we manage to work around and prove a weaker statement that serves the same purpose. The second and more important difference is that we cannot deduce the impossibility of double translations from the general version of (\(*\ast\ast\ast\)).

This leads to what we feel is one of the most important question left open by our work:

**Question.** Does there exists a (necessarily dynamically incoherent) partially hyperbolic diffeomorphism of a hyperbolic 3-manifold which acts as a double translation?
We do obtain (in §14) some dynamical properties that would have to be satisfied by such an example. This behavior is akin to what is seen in the examples of [BGHP17], so we refrain from giving a conjectural answer to our question.

Let us now take $f: M \to M$ to be a partially hyperbolic diffeomorphism, not necessarily dynamically coherent. In §10, we review Burago–Ivanov’s [BI08] construction of branching center stable and center unstable foliations. We also show that these branching foliations have leaf spaces that behave like the leaf spaces of true foliations.

2.4.1. Dichotomies for branching foliations. In §11, we recover all that we can from Part 1 and adapt it to the branching foliation case.

In particular, in §11.1–11.5 we show that the dichotomy result (*) holds without assuming dynamical coherence, so we can arrange our arguments around the same trichotomy: If $\tilde{W}_{\text{cs}}^{\text{bran}}$ and $\tilde{W}_{\text{cu}}^{\text{bran}}$ are $f$-minimal, or $M$ is hyperbolic or Seifert-fibered, then one of the following holds:

1. **double invariance**: $\tilde{f}$ fixes every leaf of both $\tilde{W}_{\text{cs}}^{\text{bran}}$ and $\tilde{W}_{\text{cu}}^{\text{bran}}$;
2. **mixed behavior**: $\tilde{f}$ fixes every leaf of either $\tilde{W}_{\text{cs}}^{\text{bran}}$ or $\tilde{W}_{\text{cu}}^{\text{bran}}$, and acts as a translation on the leaf space of the other, which is $\mathbb{R}$-covered and uniform; or
3. **double translation**: $\tilde{f}$ acts as a translation on both $\tilde{W}_{\text{cs}}^{\text{bran}}$ and $\tilde{W}_{\text{cu}}^{\text{bran}}$, which are $\mathbb{R}$-covered and uniform.

Along the way, we also prove Theorem 1.6 (see §11.3 and Corollary 11.11).

2.4.2. Center dynamics in fixed leaves. In §11.7, we work under the assumption that $\tilde{f}$ fixes every leaf of $\tilde{W}_{\text{cs}}^{\text{bran}}$, and study the dynamics within each center stable leaf. Although (***) fails without dynamical coherence, we still find that gaps are contracted, and obtain the following (Proposition 11.27):

Suppose that $W_{\text{bran}}^{\text{cs}}$ is $f$-minimal, that all the leaves of $\tilde{W}_{\text{bran}}^{\text{cs}}$ are fixed by $\tilde{f}$, and that $\tilde{f}$ does not fix any center leaf in $\tilde{M}$.

If $c$ is a periodic center leaf of $f$ in $M$, then $c$ is coarsely contracted by $h$. In particular, $c$ contains a periodic point of $f$.

This fact, together with the fact that periodic center leaves exist on any leaf with non-trivial fundamental group (see Proposition 11.32) gives us the tool to pursue the proof program further.

At this point, the reader interested in absolutely partially hyperbolic diffeomorphisms can fast forward to §16 to see how one can recover (***') under that stronger dynamical assumption (see Proposition 16.3).

2.4.3. Double invariance implies dynamical coherence. With (**') in hand, we show in §12 that the existence of a good lift $\tilde{f}$ with doubly invariant behavior implies that $f$ is dynamically coherent. By the work of §6.2 in Part 1, we get that $f$ is a discretized Anosov flow.

There is one unsavory issue that we have to address in this section: The theorem of Burago–Ivanov gives the existence of branching foliations under some orientability conditions (see Theorem 10.4), which can always be achieved by taking an appropriate lift and power. However, in order not to have these conditions appear in Theorem A, we need to show that if a lift and power of a partially hyperbolic diffeomorphism is dynamically coherent, then so is the original one. We do not know if this statement is true in general, but we prove it (in §12.3) when the lift is further assumed to be doubly invariant.

The work up to this section implies Theorem 1.2.
2.4.4. General version of Theorem A. In §13 we finish the proof of Theorem A by ruling out both mixed behavior and double translations when $M$ is a Seifert-fibered manifold.

This uses a combination of the good lift trick, which allows to take one good lift that fixes one of the foliations, and Proposition 11.27. If a good lift (of a power) does not fix both branching foliations, then we obtain periodic center leaves that must be both coarsely expanding and contracting, a contradiction.

2.4.5. No mixed behavior in hyperbolic manifolds. Sections 14 and 15 deal with the last property we want to show in order to obtain Theorem 1.3 in the hyperbolic case. That is, we want to eliminate mixed behavior.

To reach this goal, we first get, in §14, a better understanding of homeomorphisms that act as a translation on a branching foliation. Recall that, in §8 (and later extended to the branching case in §11.8), we proved that the dynamics of such a homeomorphism resembles the one of the regulating pseudo-Anosov. We push the understanding of that resemblance further and show (see Proposition 14.1) that, on periodic center stable leaves, at least some center rays that are fixed must be expanding, i.e., act in a similar way as the strong unstable foliation of the pseudo-Anosov regulating flow.

This property is then used in §15 to rule out mixed behavior, but it does not rule out double translations.

Part 1. The dynamically coherent case

3. Foliations and good lifts

In this section we will study the way that a good lift $\tilde{f}$ of a dynamically coherent partially hyperbolic diffeomorphism $f: M \rightarrow M$ that is homotopic to the identity permutes the leaves of the lifted center stable and center unstable foliations.

Most of the arguments in this section apply to any homeomorphism of a 3-manifold that preserves an appropriate foliation and is homotopic to the identity, so we will work for a while in this more general setting. At the end, we obtain the following results for our setting.

**Proposition 3.1.** Let $f: M \rightarrow M$ be a partially hyperbolic diffeomorphism on a closed 3-manifold that is dynamically coherent and homotopic to the identity, and let $\tilde{f}$ be a good lift of $f$. If $W^{cs}$ is $f$-minimal, or $M$ is hyperbolic or Seifert fibered, then either

1. $W^{cs}$ is $\mathbb{R}$-covered and uniform, and $\tilde{f}$ acts on the leaf space of the lifted foliations as a translation, or
2. $\tilde{f}$ fixes each leaf of the lifted foliation.

The same holds for the center unstable foliation $W^{cu}$.

3.1. General homeomorphisms. Let $\mathcal{T}$ be a taut foliation on a closed 3-manifold $M$ that is not finitely covered by $S^2 \times S^1$. Then the universal cover $\tilde{M}$ is homeomorphic to $\mathbb{R}^3$, and each leaf of the lifted foliation $\tilde{T}$ is a properly embedded plane (see Theorem B.1).

Fix a homeomorphism $f: M \rightarrow M$ that preserves $\mathcal{T}$ and is homotopic to the identity, and a good lift $\tilde{f}: \tilde{M} \rightarrow \tilde{M}$ (Definition 2.3).

3.1.1. Complementary regions. Being a properly embedded plane, each leaf $K \in \tilde{T}$ separates $\tilde{M}$ into two open balls. We will call these two components of $\tilde{M} \setminus K$ the complementary regions of $K$. The closure of such a complementary region $U$ is called a side of $K$ and is simply $\overline{U} = U \cup K$. 
If \( K, L \in \tilde{T} \) are distinct leaves, then \( K \cup L \) separates \( \tilde{M} \) into three open complementary regions, which can be built from the complementary regions of \( K \) and \( L \): Let \( U, U' \) be the complementary regions of \( K \), labeled so that \( L \subset U' \), and let \( W, W' \) be the complementary regions of \( L \), labeled so that \( K \subset W' \). Then the complementary regions of \( K \cup L \) are \( U, V = U' \cap W', \) and \( W \). See Figure 2. We call \( V \) the (open) region between \( K \) and \( L \). Its closure, which is simply \( \overline{V} = K \cup V \cup L \), is called the closed region between \( K \) and \( L \).

\[ \begin{array}{c}
K \\
\begin{array}{c}
\left\{ \begin{array}{c}
U \\
U'
\end{array} \right. \\
\left\{ \begin{array}{c}
W \\
W'
\end{array} \right.
\end{array}
\end{array} \]

**Figure 2.** The region between two leaves

Since \( \tilde{M} \) is simply connected, the lifted foliation \( \tilde{T} \) is coorientable. A coorientation determines a labeling of the complementary regions of each leaf \( L \in \tilde{T} \) as a positive complementary region denoted \( L^+ \) and a negative complementary region denoted \( L^- \). The corresponding positive and negative sides are denoted by \( L^+ = L^+ \cup L \) and \( L^- = L^- \cup L \).

**Remark 3.2.** We stress that a priori, some deck transformations or lifts of \( T \)-preserving homeomorphisms may exchange the coorientations of \( \tilde{T} \).

3.1.2. *The big half-space lemma.* The following lemmas will be used to understand the way that \( \tilde{f} \) can act on the leaf space of \( \tilde{T} \).

**Lemma 3.3.** For every leaf \( L \in \tilde{T} \), and every \( R > 0 \), there is a ball of radius \( R \) contained in each of the complementary regions of \( L \).

**Proof.** If necessary, we pass to a double cover of \( M \) for which \( T \) is coorientable, and choose such a coorientation. Then every deck transformation preserves the corresponding coorientation on \( \tilde{T} \) and orientation on the leaf space \( L = L_{\tilde{T}} \).

Fix a ball \( B \subset \tilde{M} \) of arbitrary radius, and a leaf \( L \in \tilde{T} \). We will find a deck transformation \( g \) that takes \( B \) into \( L^+ \); a similar argument would find a deck transformation that takes \( B \) into \( L^- \), completing the proof.

Since \( B \) is compact, we can find a leaf \( F \in \tilde{M} \) such that \( B \subset F^\oplus \). Indeed, the quotient map \( \nu : \tilde{M} \to L \) takes \( B \) to a compact subset \( \nu(B) \) of the leaf space, which can be covered by a finite collection of open intervals \( I_1, I_2, \ldots, I_n \); We may assume that \( \nu(B) \) intersects every one of these intervals. At least one of these intervals has an initial point (with respect to the orientation on \( L \)) that is not contained in any other interval, and is therefore disjoint from \( \nu(B) \). Then \( B \) is contained in the positive complementary region of the leaf \( F \) corresponding to this initial point.

Let us now find a deck transformation \( g \) that takes \( F^\oplus \) into \( L^\oplus \). Since \( T \) is taut, we can find a positively oriented closed transversal \( \gamma : [0, 1] \to M \), based at a point in \( \pi(F) \), that passes through \( \pi(L) \). Let \( \tilde{\gamma} \) be the lift of \( \gamma \) based at a point in \( F \), which passes through some lift \( L' \) of \( \pi(L) \). Then we can take \( g = h' \circ h \), where \( h \) is the deck transformation that takes \( \tilde{\gamma}(0) \) to
\( \gamma(1) \), and \( h' \) takes \( L' \) to \( L \). The oriented transversal \( \gamma \) certifies that \( h(F^\oplus) \subset L^\oplus \), and \( h'(L'^\oplus) \subset L^\oplus \) because our deck transformations preserve coorientation. \( \square \)

It follows that \( \tilde{f} \) can never take a complementary region of a leaf off of itself: This would mean that it takes an arbitrarily large ball off of itself, which contradicts the fact that \( \tilde{f} \) moves points a uniformly bounded distance. This has important consequences for the way that \( \tilde{f} \) behaves with respect to each leaf.

In particular, if \( \tilde{f} \) fixes a leaf, then it cannot interchange its complementary components, and we have:

**Corollary 3.4.** If \( L \in \tilde{T} \) is fixed by \( \tilde{f} \), then \( \tilde{f} \) preserves coorientations at \( L \).

\[ \begin{array}{c}
\includegraphics[width=\textwidth]{translation-like-behavior.png}
\end{array} \]

**Figure 3. Translation-like behavior**

### 3.1.3. Translated leaves.
Similarly, if \( \tilde{f} \) moves some leaf, then it does so in a “translation-like” manner as is illustrated in Figure 3. In fact, something a bit stronger is true:

**Proposition 3.5.** Let \( L \in \tilde{T} \) be a leaf that is not fixed by \( \tilde{f} \), then

1. the closed region between \( L \) and \( \tilde{f}(L) \) is foliated as a product,
2. \( \tilde{f} \) takes each coorientation at \( L \) to the corresponding coorientation at \( \tilde{f}(L) \), and
3. the closed region between \( L \) and \( \tilde{f}(L) \) is contained in the closed \( 2R \)-neighborhood of \( L \), where \( R = \max_{y \in \widetilde{M}} d(y, \tilde{f}(y)) \).

**Proof.** As in Figure 3, let \( U, U' \) be the complementary components of \( L \), labeled so that \( \tilde{f}(L) \subset U' \), and let \( W, W' \) be the complementary components of \( \tilde{f}(L) \), labeled so that \( L \subset W' \). Then \( V = U' \cap W' \) is the open region between \( L \) and \( f(L) \).

Note that \( \tilde{f} \) must take \( U \) to either \( W \) or \( W' \). But \( W \) is disjoint from \( U \), so we cannot have \( \tilde{f}(U) = W \) by Lemma 3.3. Thus \( \tilde{f} \) takes \( U \) to \( W' \), and \( U' \) to \( W \). This is what is meant formally by the aforementioned “translation-like” behavior.
(1) It follows, in particular, that \( \tilde{f} \) takes \( V \) off of itself and into \( W \). To see that \( V = K \cup V \cup L \) is foliated as a product, it suffices to show that every leaf that lies in \( V \) separates \( K \) from \( L \). Suppose that some leaf \( F \subset V \) does not separate \( K \) from \( L \). Then \( K \) and \( L \) are contained in the same complementary region of \( F \), so the other complementary region is contained entirely in the open region \( V \) between \( K \) and \( L \). But this means that \( V \) contains balls of arbitrary radius, which contradicts the fact that \( \tilde{f} \) takes \( V \) off of itself. Thus every leaf that lies in \( V \) separates \( K \) from \( L \), and \( V \) is foliated as a product.

(2) Since \( V = K \cup V \cup L \) is foliated as a product, it follows that a coorientation taking \( L^\circ = U' \) will take \( f(L)^\circ = W \). We have already seen that \( \tilde{f}(U') = W \), so (2) follows.

(3) Suppose for a contradiction that there is a point \( p \in V \) with \( d(p, L) = 2R + \epsilon \) for some \( \epsilon > 0 \). Then since \( d(L, \tilde{f}(L)) \leq R \), it follows from the triangle inequality that \( d(p, \tilde{f}(L)) \geq R + \epsilon \). This means that the open ball \( B_{R+\epsilon}(p) \) at \( p \) of radius \( R + \epsilon \) is contained in \( V \). But we have already seen that \( \tilde{f} \) takes \( V \) off of itself, so this implies that \( d(p, \tilde{f}(p)) \geq R + \epsilon > R \), a contradiction.

\[ \square \]

It follows that if \( L \in \tilde{T} \) is not fixed by \( \tilde{f} \), then one can string together the \( \tilde{f} \)-translates of the closed region \( V \) between \( L \) and \( \tilde{f}(L) \) to see that their union

\[ U = \cdots \cup \tilde{f}^{-2}(V) \cup \tilde{f}^{-1}(V) \cup V \cup \tilde{f}^1(V) \cup \tilde{f}^2(V) \cup \cdots \]

is an open, product-foliated set that is preserved by \( \tilde{f} \). This corresponds to an open interval in the leaf space on which \( \tilde{f} \) acts as a translation.

Let \( X \subset M \) be the union of all leaves of \( \tilde{T} \) that are fixed by \( \tilde{f} \). Then \( U \) is contained in a connected component of \( M \setminus X \). In fact, the following lemma says that \( U \) is exactly a connected component of \( M \setminus X \).

**Lemma 3.6.** Let \( L \) be a leaf of \( \tilde{T} \) that is not fixed by \( \tilde{f} \), and let \( U = \bigcup_{i=-\infty}^\infty f^i(V) \), where \( V \) is the closed region between \( L \) and \( f(L) \). Then each leaf in \( \partial U = \overline{U} \setminus U \) is fixed by \( \tilde{f} \).

**Proof.** The frontier \( \partial U \) can be broken into “forwards” and “backwards” frontiers

\[ \partial_\omega U = \lim_{i \to \infty} f^i(L) \] and \( \partial_\alpha U = \lim_{i \to -\infty} f^i(L) \),

each of which is preserved by \( f \).

Let \( K \) be a leaf in \( \partial_\omega U \), and suppose that \( \tilde{f}(K) \neq K \). Then the closed region between \( K \) and \( \tilde{f}(K) \) would be product foliated, and it follows that either \( K \) separates \( U \) from \( \tilde{f}(K) \) or \( \tilde{f}(K) \) separates \( U \) from \( K \). This contradicts the fact that \( K, \tilde{f}(K) \subset \partial_\omega U \), so we must have \( \tilde{f}(K) = K \). A similar argument shows that every leaf in \( \partial_\alpha U \) is fixed by \( \tilde{f} \).\[ \square \]

3.1.4. **The dichotomy.** We summarize the preceding discussion in terms of the leaf space:

**Proposition 3.7.** Let \( M \) be a closed 3-manifold that is not finitely covered by \( S^2 \times S^1 \), \( f: M \to M \) a homeomorphism homotopic to the identity that preserves a taut foliation \( T \), and \( \tilde{f} \) a good lift.
The set \( \Lambda \subset L \) of leaves that are fixed by \( \widetilde{f} \) is closed and \( \pi_1(M) \)-invariant. Moreover, each connected component \( I \) of \( L \setminus \Lambda \) is an open interval that \( \widetilde{f} \) preserves and acts on as a translation, and every pair of leaves in \( I \) are a finite Hausdorff distance apart.

**Proof.** The only detail that needs to be pointed out is that \( \Lambda \) is \( \pi_1(M) \)-invariant, which follows from the fact that \( \widetilde{f} \) commutes with every deck transformation. \( \square \)

In particular, one may have \( \Lambda = \emptyset \):

**Corollary 3.8.** Let \( M \) be a closed 3-manifold that is not finitely covered by \( S^2 \times S^1 \), \( f: M \to M \) a homeomorphism homotopic to the identity that preserves a taut foliation \( T \), and \( \widetilde{f} \) a good lift.

If \( \widetilde{f} \) fixes no leaf of \( \widetilde{T} \), then \( T \) is \( \mathbb{R} \)-covered and uniform, and \( \widetilde{f} \) acts on \( L \widetilde{T} \simeq \mathbb{R} \) as a translation.

This leads to a simple dichotomy when the foliation is \( f \)-minimal. Recall:

**Definition 3.9.** A foliation \( T \) that is preserved by a map \( f: M \to M \) is said to be \( f \)-minimal if the only closed sets that are both \( f \)-invariant and saturated are \( M \) and \( \emptyset \).

**Corollary 3.10.** Let \( M \) be a closed 3-manifold that is not finitely covered by \( S^2 \times S^1 \), \( f: M \to M \) a homeomorphism homotopic to the identity that preserves a taut foliation \( T \), and \( \widetilde{f} \) a good lift.

If \( T \) is \( f \)-minimal, then either

1. \( \widetilde{f} \) fixes every leaf of \( \widetilde{T} \), or
2. \( T \) is \( \mathbb{R} \)-covered and uniform, and \( \widetilde{f} \) acts as a translation on the leaf space of \( \widetilde{T} \).

**Proof.** Since \( \widetilde{f} \) commutes with each deck transformation, each deck transformation preserves the set \( \Lambda \subset L \) of fixed leaves. In particular, if \( I \) is a component of \( L \setminus \Lambda \) and \( g \in \pi_1(M) \) then one has either \( g(I) = I \) or \( g(I) \cap I = \emptyset \).

If \( \Lambda = \emptyset \) then we are in case (2) by the preceding corollary.

Suppose that \( \Lambda \neq \emptyset \). If \( \Lambda \neq L \), then it corresponds to a closed, \( T \)-saturated subset of \( M \) that is preserved by \( f \). Furthermore, this subset is not all of \( M \) since it cannot accumulate on a leaf lying in the interior of a complementary interval to \( \Lambda \). This contradicts \( f \)-minimality, so we have \( \Lambda = L \) and are in case (1). \( \square \)

**3.1.5. Bounded movement inside leaves.** We end this section by showing that a good lift that fixes every leaf will be within a bounded distance of the identity not only in \( \widetilde{M} \) but also in each leaf.

**Lemma 3.11.** Let \( M \) be a closed 3-manifold that is not finitely covered by \( S^2 \times S^1 \), \( f: M \to M \) a homeomorphism homotopic to the identity that preserves a taut foliation \( T \), and \( \widetilde{f} \) a good lift.

If \( \widetilde{f} \) fixes every leaf of \( \widetilde{T} \), then there is a uniform bound \( K > 0 \) such that for any leaf \( L \in \widetilde{T} \) one has

\[
    d_L(x, \widetilde{f}(x)) < K \quad \text{for all } x \in L,
\]

where \( d_L \) is the path metric on \( L \).

**Proof.** Suppose for a contradiction that there is a sequence of points \( x_i \in \widetilde{M} \) for which \( d_{L_i}(x_i, \widetilde{f}(x_i)) \) tends to infinity, where \( d_{L_i} \) is the path metric on the leaf \( L_i \in \widetilde{T} \) containing \( x_i \).
Since $M$ is compact, we can pass to a subsequence and find a sequence of deck transformations $\gamma_i$ such that $\gamma_i(x_i)$ converges to a point $x_{\infty}$. Since $\tilde{f}$ commutes with $\gamma_i$, we have that $\gamma_i(\tilde{f}(x_i)) = \tilde{f}(\gamma_i(x_i))$ converges to $\tilde{f}(x_{\infty})$.

Now, since $d_{L_i}(\gamma_i(\tilde{f}(x_i)), (\gamma_i(x_i)) = d_{L_i}(\tilde{f}(x_i), x_i)$ goes to infinity, the points $\tilde{f}(x_{\infty})$ and $x_{\infty}$ must be in different leaves of $\mathcal{T}$. This contradicts the fact that $\tilde{f}$ fixes each leaf of $\mathcal{T}$.

**Remark 3.12.** This lemma applies as well to a leaf of a closed sublamination of $\mathcal{T}$ whose lift is leafwise fixed by $\tilde{f}$. In fact, it also works for a closed sublamination of a branching foliation — see Definition 10.2 and Section 11.5.

3.2. **Consequences for partially hyperbolic systems.** Let us now specialize, and fix a closed 3-manifold $M$ whose fundamental group is not virtually solvable, a dynamically coherent partially hyperbolic diffeomorphism $f : M \to M$ that is homotopic to the identity, and a good lift $\tilde{f} : \tilde{M} \to \tilde{M}$.

We denote by $\mathcal{W}^{cs}, \mathcal{W}^{cu}, \mathcal{W}^{s}, \mathcal{W}^{u}$, and $\mathcal{W}^{c}$ the center stable, center unstable, stable, unstable, and center foliations.

#### 3.2.1. Fixed points and the topology of leaves.

**Lemma 3.13.** Let $L \in \tilde{\mathcal{W}}^{cs}$ be a leaf that is fixed by $\tilde{f}$. If there is a sequence of leaves $L_i \in \tilde{\mathcal{W}}^{cs}$ that are fixed by $\tilde{f}$ and accumulate on $L$, then there are no points in $L$ fixed by $f$ of non-trivial power of $\tilde{f}$.

**Proof.** Suppose that $\tilde{f}^n$, $n > 0$, fixes some point $x \in L$. Then it fixes the unstable leaf $\tilde{W}^{u}(x)$ through that point. When $n$ is sufficiently large, $\tilde{W}^{u}(x)$ intersects $L_i$ at a single point $x_i$, which must therefore also be fixed by $\tilde{f}^n$. This contradicts the fact that $\tilde{f}^n$ contracts unstable leaves.

**Proposition 3.14.** Let $L \in \tilde{\mathcal{W}}^{cs}$ be a leaf that is fixed by $\tilde{f}$. If $\tilde{f}$ fixes no point in $L$, then $A = \pi(L)$ has cyclic fundamental group (and is therefore a plane, cylinder, or Möbius band).

**Proof.** Let $L$ be the leaf space of the stable foliation within $L$.

Since $\tilde{f}$ fixes no point in $L$, it cannot fix any stable leaf in $L$, since a stable leaf that is fixed by $\tilde{f}$ would contain a fixed point. In other words, $f$ acts freely on $L$, and hence has an axis $A_f$ by Proposition E.2.

Consider two elements $\gamma_1, \gamma_2 \in \pi_1(M)$ that fix $L$. Since the stable foliation can have no circular leaves, neither of these elements may fix a stable leaf. Hence each $\gamma_i$ acts freely on $L$ with an axis $A_i$.

As $\tilde{f}$ commutes with both $\gamma_1$ and $\gamma_2$ it follows from Proposition E.2 that in fact these axes are the same, i.e., $A_1 = A_f = A_2$. Proposition E.2 further implies that the subgroup generated by $\gamma_1$ and $\gamma_2$ is abelian. Since there are no compact leaves in $\tilde{\mathcal{W}}^{cs}$, it follows that this subgroup is cyclic, and hence $\gamma_1^n = \gamma_2^m$ for some $n, m$.

#### 3.2.2. Minimality in hyperbolic or Seifert manifolds.** The following proposition implies that the dichotomy in Corollary 3.10 holds, without the assumption of $f$-minimality, when $M$ is hyperbolic or Seifert-fibered.

**Proposition 3.15.** Let $M$ be a closed 3-manifold that is hyperbolic or Seifert-fibered, $f : M \to M$ a dynamically coherent partially hyperbolic diffeomorphism that is homotopic to the identity, and $\tilde{f}$ a good lift.

If $\tilde{f}$ fixes one leaf of $\tilde{\mathcal{W}}^{cs}$, then $\tilde{\mathcal{W}}^{cs}$ is a minimal foliation, and $\tilde{f}$ fixes every leaf of $\tilde{\mathcal{W}}^{cs}$.
The same statement holds for $\mathcal{W}^{cu}$.

Proof. Without loss of generality, we may assume, by passing to a finite cover of $M$ and power of $f$, that $\mathcal{W}^{cs}$ is orientable and coorientable, and $f$ preserves all orientations and coorientations.

Let $X \subset \tilde{M}$ be the union of all leaves of $\tilde{\mathcal{W}}^{cs}$ that are fixed by $\tilde{f}$. This set is non-empty by hypothesis, $\pi_1(M)$-invariant as $\tilde{f}$ is a good lift and closed by Proposition 3.7. It follows that $\pi(X) \subset M$ is compact and non-empty. By Zorn's lemma, we can find a minimal compact, non-empty, $\mathcal{W}^{cs}$-saturated subset $\Lambda \subset \pi(X)$. We will show that $\Lambda = M$, which implies both that $\mathcal{W}^{cs}$ is minimal and that $\tilde{f}$ fixes every leaf.

Note that $\Lambda$ cannot contain any isolated leaves. Indeed, it cannot consist solely of isolated leaves since then these leaves would be compact, and $\mathcal{W}^{cs}$ has no compact leaves. Deleting an isolated leaf from $\Lambda$ still leaves a closed, saturated subset, so the existence of an isolated leaf would contradict our minimality assumption.

Let $\tilde{\Lambda}$ be the preimage of $\Lambda$ in $\tilde{M}$. Since no leaf in $\Lambda$ is isolated, every leaf in $\tilde{\Lambda}$ is accumulated on by a sequence of leaves in $\tilde{\Lambda}$. Since these leaves are all fixed by $\tilde{f}$, Lemma 3.13 implies that $\tilde{f}$ has no fixed points in $\tilde{\Lambda}$. It follows from Proposition 3.14 that each leaf of $\Lambda$ is either a cylinder or a plane.

Assume for a contradiction that $\Lambda \neq M$, and hence $\tilde{\Lambda} \neq \tilde{M}$. Then we can choose a nontrivial connected component $V$ of $\tilde{M} \setminus \tilde{\Lambda}$.

**Claim 3.16.** The projection $\pi(\partial V)$ to $M$ consists of finitely many leaves.

This is a standard fact in the theory of foliations [CC00, Lemma 5.2.5].

For each $x \in \partial V$, let $u_x$ be the maximal connected unstable segment that starts at $x$ and is contained in $\tilde{V}$, which is either a closed interval or a ray. That is, $u_x$ is the component of $\tilde{\mathcal{W}}^u(x) \cap \tilde{V}$ that contains $x$. Given $r > 0$, and a leaf $L \subset \partial V$ define

$$A^r_L = \{ x \in L \mid \ell(J_x) \geq r \}.$$ 

**Claim 3.17.** For any leaf $L \subset \partial V$, and any $r > 0$, the set $\pi(A^r_L)$ is compact as a subset of $\pi(L)$.

Proof. It is straightforward to see that $\pi(A^r_L)$ is closed as a subset of $\pi(L)$. If it is not compact, then one can find sequence of points $x_i \subset \pi(A^r_L)$ that escapes every compact subset of $\pi(L)$. After taking a subsequence we can assume that the $x_i$ converges in $M$ to some point $x$. Take a chart around $x$ of the form $D^2 \times (0,1)$ where each $D^2 \times \{y\}$ is a plaque of $\mathcal{W}^{cs}$, and each $\{p\} \times (0,1)$ is an oriented plaque of $\mathcal{W}^u$. Since the $x_i$ escape every compact subset of $\pi(L)$, we can pass to a subsequence such that each $x_i$ is contained in a different plaque. Then it is easy to see that the lengths of the unstable segments at $x_i$ that stay in $\pi(\tilde{V})$ must go to $0$, a contradiction. \qed

**Claim 3.18.** Each leaf $L \subset \partial V$ corresponds to an annulus $\pi(L)$ in $M$.

Proof. Fix a leaf $L \subset \partial V$ and an $r > 0$ for which $A^r_L := A^r_L$ is non-empty. As $\pi(L)$ is either a plane or an annulus, we assume for a contradiction that it is a plane. Then the covering map $\pi$ restricts to a homeomorphism on $L$, so the fact that $\pi(A)$ is compact means that $A$ is compact. Let $D$ be a disk in $L$ containing $A^r_L$ in its interior.

Since the leaves of $\partial V$ are fixed by $\tilde{f}$, and a positive iterate of $\tilde{f}$ expands the lengths of unstable arcs, we can find an $n \geq 1$ for which $\tilde{f}^n(D) \subset A^r_L \subset D$. Then Brouwer's fixed point theorem implies that $\tilde{f}^n$ has a fixed point in $L$, which contradicts Lemma 3.13. So $L$ must be an annulus. \qed
Now we can complete the proof of Proposition 3.15. Let $L_1, \ldots, L_k$ be a finite collection of leaves in $\partial V$ that cover $\pi(\partial V)$, and fix $r > 0$ such that each $A_i := A_r^f$ is nonempty. Choose a compact annulus $C_i$ in each $\pi(L_i)$ that contains $\pi(A_i)$. Since $f$ preserves orientations and coorientations, we can join each $C_i$ to an adjoining $C_j$ with an annulus built out of unstable segments $u_x$ for points $x \in \partial C_i$. Iterating this procedure, we obtain a torus $T$ that consists of alternating annuli contained in leaves of $\mathcal{W}^{cs}$ and annuli transverse to $\mathcal{W}^{cs}$ inside $W = \pi(V)$.

We will now (for the first time) use the assumption that $M$ is hyperbolic or Seifert-fibered to see that $T$ bounds a solid torus.

If $M$ is hyperbolic, then $T$ either bounds a solid torus or is contained in a 3-ball (Lemma A.2). If $T$ is contained in a 3-ball, then the annuli $C_i$ are contained in that ball, so the $\mathcal{W}^{cs}$ leaf containing $C_i$ is compressible. This contradicts the fact that $\mathcal{W}^{cs}$ is a taut foliation (see Theorem B.1), so $T$ bounds a closed solid torus $U$.

If $M$ is Seifert-fibered, then $\mathcal{W}^{cs}$ is a horizontal foliation. That is, one can isotope $\mathcal{W}^{cs}$ so that all leaves are transverse to the Seifert fibers of $M$ (Theorem F.3). It follows that the complementary regions of the lamination $\Lambda$ are horizontal. In particular, the region $\pi(V \cup \partial V)$ is a product, which means that the torus $T$ is made up of two horizontal $C_i$ and two transverse annuli, and hence bounds a closed solid torus $U$.

We will now use a “volume vs. length” argument to get a contradiction. We refer to [HPS18, Proposition 5.2] for a detailed proof and give only a sketch: Consider an unstable arc inside $U$ from a point in some $\pi(A_i)$ to some $C_j$. Fix some $\epsilon > 0$, and call $u$ the non-empty part of that unstable segment that is at distance $\geq \epsilon$ from both $C_i$ and $C_j$. Up to taking $\epsilon > 0$ smaller if necessary, we can then assume that $u$ is at distance at least $\epsilon > 0$ from $T$. Consider a lift $\tilde{u} \subset V$ of $u$, and note that for any positive $n$, $\tilde{f}^n(\tilde{u})$ stays a bounded distance away from the corresponding lift $\tilde{T}$ of $T$. The length of $\tilde{f}^n(\tilde{u})$ will grow exponentially in $n$, while the volume of its maximal tubular neighborhood can only grow linearly, as $\tilde{T}$ is at bounded distance from the identity and the fundamental group of $T$ is $\mathbb{Z}$. This means that $\tilde{f}^n(\tilde{u})$ must auto-accumulate in $\tilde{M}$, contradicting the fact that it is transverse to $\mathcal{W}^{cs}$. Thus $\Lambda = M$ as desired. \hfill $\Box$

Remark 3.19. We point out here that the hypothesis of $M$ being hyperbolic or Seifert fibered is used in a single place, but it is crucial. To see this, it is enough to consider the time-one map of Franks-Williams intransitive Anosov flow [FW80] (or any other non-transitive Anosov flow), for which neither the center stable nor the center unstable foliations are minimal.

3.3. Gromov hyperbolicity of leaves. In this section we show that Candel’s Theorem (Theorem C.1) applies under the assumptions that $f$ is partially hyperbolic and that $\tilde{f}$ fixes the leaves of the center stable foliation. It is known that the assumption for Candel’s Theorem is always satisfied for hyperbolic 3-manifolds (see e.g., [Cal07]), as well as for horizontal foliations in Seifert fibered manifolds with exponential growth of fundamental group (which is automatic in our case thanks to Theorem F.3). However, in order to deal with other 3-manifolds, we need a more general version.

Lemma 3.20. Let $M$ be a closed 3-manifold, $f : M \to M$ a dynamically coherent partially hyperbolic diffeomorphism homotopic to the identity, and $\tilde{f}$ a good lift.

Suppose $\mathcal{W}^{cs}$ has no compact leaves, and $\tilde{f}$ fixes every leaf of $\mathcal{W}^{cs}$. Then every leaf of $\mathcal{W}^{cs}$ is Gromov-hyperbolic. Moreover, there is a metric on $M$ which restricts to a metric of constant negative curvature on each leaf of $\mathcal{W}^{cs}$.
Proof. Thanks to Candel’s Theorem (Theorem C.1), all we have to show is that \( W^{cs} \) does not admit a holonomy invariant transverse measure.

So we suppose that there is an invariant transverse measure \( \mu \) to \( W^{cs} \). Let \( S \) be its support. First notice that, as there are no compact leaves in \( W^{cs} \), \( \mu \) has no atoms, so there are no isolated leaves in \( S \). Let \( \tilde{\mu} \) be the lift of \( \mu \) to \( \tilde{M} \).

The fact that \( \tilde{f} \) fixes every leaf of \( \tilde{W}^{cs} \) implies that the measure \( \mu \) is \( \tilde{f} \)-invariant. To see this, consider \( \tau \) a small transversal to \( W^{cs} \), and \( \tilde{\tau} \) its lift to \( \tilde{M} \). Then, since \( \tilde{f} \) fixes every leaf of \( \tilde{W}^{cs} \), the transversals \( \tilde{\tau} \) and \( \tilde{f}(\tilde{\tau}) \) intersect the same set of leaves of \( W^{cs} \). Hence, \( \tilde{\tau} \) and \( \tilde{f}(\tilde{\tau}) \) have the same \( \tilde{\mu} \)-measure (because \( \mu \) is an invariant transverse measure), thus \( \mu(\tau) = \mu(f(\tau)) \) as desired.

Now let \( \tau \) be a closed segment on an unstable leaf and call \( x \) one of its endpoints. Note that \( \tau \) is a transversal to \( W^{cs} \), and, up to taking a different unstable segment, we assume that \( \tau \) is chosen so that \( \mu(\tau) > 0 \).

We can choose a sequence \( (n_i) \) of negative integers converging to \(-\infty\) such that \( (f^{n_i}(x)) \) converges to some \( y \in M \).

Then, as the \( n_i \) are negative integers, \( f^{n_i} \) contracts the unstable length, so the sequence of segments \( (f^{n_i}(\tau)) \) also converges to \( y \). Now, since \( \mu \) is \( f \)-invariant, it implies that \( \mu(f^{n_i}(\tau)) = \mu(\tau) > 0 \), for all \( n_i \). By taking the limit, we get that the \( W^{cs} \) leaf containing \( y \) must be an atom of \( \mu \), in contradiction with the fact recalled earlier that \( \mu \) has no atoms.

Thus \( W^{cs} \) does not admit an invariant transverse measure and Candel’s Theorem yields the conclusion of our lemma.

We will use the metric given by this lemma on \( M \) in the specific situations where a hyperbolic metric makes the proof less technical. But all such results only need a Gromov-hyperbolic metric in the center stable or center unstable leaves.

3.4. Summary. For convenience, we summarize the results obtained in Section 3.

Corollary 3.21. Let \( f : M \to M \) be a partially hyperbolic, dynamically coherent, diffeomorphism of a 3-manifold \( M \) that is homotopic to the identity. Suppose that \( W^{cs} \) is \( f \)-minimal, or that \( M \) is hyperbolic or Seifert fibered. Let \( f : M \to M \) be any good lift of \( f \).

Then, \( \tilde{f} \) has no fixed points and either

1. the foliation \( W^{cs} \) is \( \mathbb{R} \)-covered and uniform, and \( \tilde{f} \) acts as a translation on the leaf space of \( \tilde{W}^{cs} \); or,
2. the map \( \tilde{f} \) leaves every leaf of \( \tilde{W}^{cs} \) fixed and every leaf of \( W^{cs} \) is a plane, an annulus or a Möbius band. Moreover, there is a metric on \( M \) that restricted to each leaf has constant negative curvature \(-1\).

Proof. By Proposition 3.7, either the foliation \( W^{cs} \) is \( \mathbb{R} \)-covered and uniform, and \( \tilde{f} \) acts as a translation on the leaf space of \( \tilde{W}^{cs} \), or, if \( \tilde{f} \) does not act as a translation, then it must fix at least one leaf.

Thus, if \( M \) is hyperbolic or Seifert fibered, we can apply Proposition 3.15 and deduce that \( W^{cs} \) is \( f \)-minimal.

Hence, if \( \tilde{f} \) does not act as a translation, then we can apply Corollary 3.10 to get that \( \tilde{f} \) must fix every leaf of \( \tilde{W}^{cs} \).

Now, if no leaf of \( \tilde{W}^{cs} \) is fixed by \( \tilde{f} \) then \( \tilde{f} \) cannot have fixed points. On the other hand, if all leaves of \( \tilde{W}^{cs} \) are fixed, then we can apply Lemma 3.13 to deduce that \( \tilde{f} \) still does not fix points.
Finally, Proposition 3.14 implies that when all leaves of $\widetilde{W}^{cs}$ are fixed then every leaf is a plane, an annulus, or a Möbius band. The existence of the claimed metric follows from Lemma 3.20.

The same statement holds for the foliation $W^{cu}$. Notice that, in principle, the behavior of each foliation is independent. The goal of the next few sections is to show that the behavior of one of the foliations forces the same behavior in the other foliation.

4. Center dynamics in fixed leaves

In this section we will study the dynamics within center stable leaves. The main result is Proposition 4.4, which will be used to understand the doubly invariant and mixed cases (see §2.3.1).

4.1. Perfect fits. Much of this section will be concerned with transverse pairs of foliations of a plane — in particular, the stable and center foliations within a center stable leaf. We begin by introducing some basic tools, in particular the idea of “perfect fits” first used by Barbot and the second author [Fen94, Bar95].

Let $L$ be a complete plane equipped with a transverse pair of one-dimensional foliations $S$ and $C$. We denote by $L^s := L/S$ and $L^c := L/C$ their respective leaf spaces. These are simply-connected, separable 1-manifolds which may not be Hausdorff (see e.g., [Bar98, Cal07, CLN85]).

**Definition 4.1.** A leaf $c \in C$ and leaf $s \in S$ are said to make a $CS$-perfect fit, if they do not intersect, but there is a local transversal $\tau$ to $C$ through $c$, such that every leaf $c' \in C$ that intersects $\tau$ on one side of $c$ must intersect $s$.

On the other hand, if there exists $\tau'$ a local transversal to $s \in S$, such that every leaf $s' \in S$ that intersect $\tau'$ on one side of $s$ has to intersect $c$, we will say that $s$ and $c$ make a $SC$-perfect fit.

If $c$ and $s$ make both a $CS$-perfect fit and a $SC$-perfect fit, we say that they make a perfect fit.

![Figure 4](image-url)

**Figure 4.** The leaves $c$ and $s$ make a $CS$-perfect fit, but not a $SC$-perfect fit. The leaves $c$ and $s'$ make a perfect fit.
Lemma 4.2. If two leaves $c \in C$ and $s \in S$ make a $CS$-perfect fit, then there exists $s' \in S$, possibly distinct from $s$ such that $c$ and $s'$ make a perfect fit. The symmetric statement holds for $SC$-perfect fits.

Proof. Fix a small transversal $\tau$ to $c$. Let $c'$ near enough $c$ which also intersects $s$. Let $p = c' \cap \tau$ and $q = c' \cap s$. For any $x$ in $c'$ between $p$ and $q$ and near enough $p$, the stable leaf of $x$ intersects $c$. Let $y$ in $c'$ between $p$ and $q$ be the first point such that the stable leaf of $y$ does not intersect $c$. Let $s'$ be this stable leaf. Then $c, s', s$ form a perfect fit. □

A straightforward argument shows the following — see, e.g., [Fen98, Claim in Theorem 3.5].

Lemma 4.3. If two leaves $s, s' \in S$ are nonseparated in the leaf space $L^s$, then there is a unique leaf $c \in C$ that separates $s$ from $s'$ and makes a perfect fit with $s$.

4.2. Finding fixed center leaves. The following proposition is the main result of this section.

Proposition 4.4. Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism homotopic to the identity, and $\bar{f}$ a good lift.

Suppose that $\bar{f}$ fixes every leaf of $W^{cs}$. Then any leaf of $\bar{W}^{cs}$ that is fixed by a nontrivial element of $\pi_1(M)$ contains a center leaf that is fixed by $\bar{f}$.

The proof of this will span the rest of this section. Let us fix $M, f$, and $\bar{f}$ as above, along with a leaf $L \in W^{cs}$ and a nontrivial element $\gamma \in \pi_1(M)$ that fixes $L$. Our goal is to find a center leaf $c \subset L$ that is fixed by $\bar{f}$.

Let $L^s$ and $L^c$ be the leaf spaces of the foliations $\bar{W}^s$ and $\bar{W}^c$ restricted to $L$. These are simply connected, separable, 1-manifolds that may not be Hausdorff.

Since $\bar{f}$ fixes every center stable leaf, Lemma 3.13 implies that $\bar{f}$ has no fixed points. This means that $\bar{f}$ cannot fix any stable leaf, since such a leaf would be contracted and hence contain a fixed point, so $\bar{f}$ acts freely on $L^s$.

Since there are no circular stable leaves downstairs, $\gamma$ must also act freely on $L^s$. By Proposition 3.14, the stabilizer of $L$ is cyclic, so we may take $\gamma$ to be a generator.

Since $\bar{f}$ and $\gamma$ commute and act freely on $L^s$, they preserve an axis $A^s \subset L^s$, which is either a line or a $\mathbb{Z}$-union of intervals (see Proposition E.2 and Remark E.3).

The following lemma completes the proof of Proposition 4.4 when $A^s$ is a line.

Lemma 4.5. If $A^s \simeq \mathbb{R}$, then there exists a center leaf $c \subset L$ that is fixed by both $\bar{f}$ and $\gamma$.

Proof. We will use the graph transform argument (Lemma H.1).

Since $A^s$ is homeomorphic to $\mathbb{R}$, one can find a bi-infinite curve $\eta$ in $L$ that is transverse to the stable foliation and invariant under $\gamma$. For instance, pick a point $x$ in $L$ and an arc $a$ from $x$ to $\gamma x$ transverse to the stable foliation. Concatenating the positive and negative iterates of $a$ by $\gamma$ gives such a curve $\eta$ (that can be smoothed if required).

In particular, $\eta$ represents the axis $A^s$ of $\gamma$, in the sense that a stable leaf is in $A^s$ if and only if it intersects $\eta$.

Since $A^s$ is also the axis for the action of $\bar{f}$ on $L^s$, every $\bar{f}$-iterate of $\eta$ also represents $A^s$. In particular, $\bar{f}(\eta)$ and $\eta$ intersect the exact same set of stable leaves.
So the curve $\eta$ satisfies the two hypothesis of Lemma H.1, and we obtain a curve $\beta$ in $L$ that is tangent to $E^c$ and invariant under both $\tilde{f}$ and $\gamma$. It remains to show that $\beta$ is in fact a center leaf.

Choose a point $x \in \beta$, and let $\beta'$ be the compact subsegment of $\beta$ running from $x$ to $\gamma x$. This is a fundamental domain for the action of $\gamma$ on $\beta$.

At each point $y \in \beta$, one can find a compact center segment $c_y$ through $y$ that intersects the same set of stable leaves as some compact subsegment $\beta_y$ of $\beta$, where the interior of $\beta_y$ contains $y$. By compactness of $\beta'$, one can find a finite collection $c_1, c_2, \cdots, c_k$ of these center segments such that the corresponding subsegments of $\beta$ covers $\beta'$.

Projecting to $M$, we have a finite union of center segments $\bigcup \pi(c_i)$ that intersects the same set of stable leaves as the closed curve $\pi(\beta)$. Since $f$ contracts stable segments, $f^n(\bigcup \pi(c_i))$ converges to $\pi(\beta)$ as $n \to \infty$. Since a sequence of center segments can only converge to a center segment, it follows that $\pi(\beta)$ is a center leaf, and so is $\beta$.

\[ \square \]

4.2.1. Gaps. To complete the proof of Proposition 4.4 we will show that $A^s$ is indeed a line.

Suppose that $A^s$ is not a line. Then it is a $\mathbb{Z}$-union of closed intervals

\[ A^s = \bigcup_{i \in \mathbb{Z}} [s_i^-, s_i^+]. \]

We will call each of the pairs $s_i^+, s_{i+1}^-$ a gap in this axis.

The following lemma says that some positive power of $f$ fixes the image of each gap in $M$.

**Lemma 4.6.** There are $m \neq 0$ and $n > 0$ such that $h = \gamma^m \circ \tilde{f}^n$ fixes every $s_i^\pm$.

**Proof.** Since $\tilde{f}$ and $\gamma$ act freely on $A^s$, they act freely on the index set of the collection of intervals, which is $\mathbb{Z}$. It follows that some nontrivial element of the group generated by $\tilde{f}$ and $\gamma$ acts trivially on $\mathbb{Z}$. This element is of the form $h := \gamma^m \circ f^n$. Since both $\gamma$ and $\tilde{f}$ act freely on $A^s$, neither $n$ nor $m$ can be equal to zero, and we can take $n > 0$. Since $h$ fixes each interval, it fixes the endpoints of each interval as desired. \[ \square \]

For the remainder of this section, we fix $h$ as in this lemma, and look at a single gap, setting $s^+ = s_i^+$ and $s^- = s_{i+1}^-$ for some fixed $i$.

Let us name some features of this gap — refer to Figure 5.

Proposition E.2 says that $s^+$ is non-separated from $s^-$ in the leaf space $L^s$, so Lemma 4.3 provides a center leaf $c$ that makes a perfect fit with $s^+$ and separates $s^+$ from $s^-$. Since there is a unique such leaf, it follows that $h$ fixes $c$.

Note that $h$ eventually contracts stable leaves; this is because $\gamma$ is an isometry, $\tilde{f}$ eventually contracts stable leaves, and $n > 0$. Up to an iterate we can assume that this contraction is immediate. It follows that $h$ fixes a single point $x$ within $s^+$. Let $c'$ be the center ray that starts at $x$ on the side of $c$.

We will show that $h$ expands $c'$ in Lemma 4.7, and that it contracts $c'$ in Corollary 4.13. This is contradictory, so there are no gaps in $A^s$, i.e., it is a line, and the proof of Proposition 4.4 is complete.

4.2.2. Perfect fits and expanded center rays. In the following lemma we find that the topology of the stable and center foliations in $L$ forces a stable ray in our gap to expand.

**Lemma 4.7.** $h$ acts as an expansion on $c'$ with unique fixed point $x$. 
Proof. Refer to Figure 5.

Note that the stable leaf $s_y = \tilde{\mathcal{W}}^s(y)$ through any point $y \in c'$ that is sufficiently close to $x$ will intersect $c$. This is because $s^+$ and $c$ make a perfect fit, and $c'$ is a transversal to $\mathcal{S}$ on the side of $c$. Given such a point, let $s'_y$ be the compact segment of $s_y$ that runs from $c'$ to $c$. Since the lengths of $h$-iterates of this segment go to zero, i.e., $\lim_{n \to \infty} \ell(h^n(s')) = 0$, it follows that the $h$-iterates of $y$ eventually escape every compact set. Indeed, otherwise one would find that $c$ and $c'$ intersect at some accumulation point of $h^n(y)$.

The lemma follows since we can take $y \in c'$ arbitrarily close to $x$. □

Remark 4.8. The proof of Lemma 4.7 uses the structure of the transverse pair of foliations in an essential way. It does not hold when the center leaves are allowed to merge — see Figure 6. This is exactly the type of behavior that arises in the examples of [RHRHU16].

4.2.3. Coarse contraction in stable gaps. In the following lemma we find that the geometry of the gap forces it to contract laterally. This contradicts the expansion found in Lemma 4.7 — see Corollary 4.13.
Lemma 4.9. There is a rectangle $R$ bounded by segments of $s^+$ and $s^-$ that contain the fixed points, together with two arcs $\tau_1, \tau_2$, such that $h(R)$ is contained in the interior of $R$.

See Figure 7.

We will need two lemmas. The first is that the gap is "uniformly thin":

Lemma 4.10. The leaves $s^+$ and $s^-$ are a bounded Hausdorff distance apart with respect to the path metric on $L$.

Proof. Since this gap is part of the axis $A^s = \bigcup_{k \in \mathbb{Z}} [s^-_k, s^+_k]$, it follows that $s^-$ separates $s^+$ from either $\tilde{f}(s^+)$ or $\tilde{f}^{-1}(s^+)$. Then Lemma 3.11 implies that the Hausdorff distance between $s^+$ and $\tilde{f}(s^+)$ is uniformly bounded above, and the same holds for the Hausdorff distance between $s^+$ and $s^-$. \hfill \Box

Recall that, since $\tilde{f}$ fixes all leaves of $\tilde{W}^{cs}$, Candel’s theorem (Theorem C.1) implies that there is a metric $g$ on $M$ such that $W^{cs}$ is leafwise hyperbolic. Let $d$ be the associated path metric on the leaf $L$.

Lemma 4.11. For any $K_0 > 0$, and any ray $r \subset s^+$, there exists $y \in r$ such that $d(y, h(y)) > K_0$.

Proof. Let $r$ be a ray of $s^+$. Suppose for a contradiction that there exists $K_0$ such that for all $y$ in $r$ one has $d(y, h(y)) < K_0$.

Recall that $h = \gamma^n \tilde{f}^m$, where $m$ and $n$ are fixed. By Lemma 3.11, there exists a constant $K_1$ such that, for any $z$ in $L$, $d(z, \tilde{f}^m(z)) < K_1$.

Thus, by assumption, for any $y \in r$,

$$d(y, \gamma^m y) \leq d(y, \tilde{f}^m(\gamma^m y)) + d(\gamma^m y, \tilde{f}^m(\gamma^m y)) < K_0 + K_1.$$ 

Now $\gamma$ is an hyperbolic isometry for $d$ (since $\gamma$ acts without fixed points on $L$, and $\gamma$ is not parabolic because $M$ is compact). Hence, since $d(y, \gamma^m y)$ stays bounded for all $y$ in $r$, it implies that $r$ has to stay a bounded distance from the (or any when $d$ is only supposed Gromov-hyperbolic) geodesic in $L$ that is the axis for the action of $\gamma$ on $L$.

So $\pi(r)$ stays a bounded distance away from the (or all) geodesic in $A = \pi(L)$ that lifts to the axis of $\gamma$. Thus, Poincaré–Bendixon Theorem implies that $\pi(r)$ must accumulate onto a closed stable leaf in $M$, which is impossible (see Figure 8). \hfill \Box

Remark 4.12. Notice that this is the only place in the proof of Proposition 4.4 that Theorem C.1 is used.

In fact, the proof does not actually need $d$ to come from a Riemannian hyperbolic metric — only that it is Gromov-hyperbolic — so Lemma 4.11 will hold as
long as we know Gromov-hyperbolicity of the leafwise metric. We will need this in the proof of Proposition 11.27.

\[ A = \pi(L) \]

Figure 8. If a stable ray in \( L \) stays close to the axis of the deck transformation \( \gamma \) which is a hyperbolic isometry, then its projection in \( M \) has to accumulate on a circle stable leaf.

**Proof of Lemma 4.9.** Let \( y_1 \) and \( y_2 \) be points in \( s^+ \) that lie on either side of, and far away from, the fixed point \( x^+ \), and let \( \tau_1 \) and \( \tau_2 \) be geodesic arcs from \( y_1 \) and \( y_2 \) to \( s^- \). See Figure 7. Note that the lengths of \( \tau_i \), \( i = 1, 2 \), are uniformly bounded by Lemma 4.10, and since \( f \) has bounded derivatives, the length of \( h(\tau_i) \) is bounded as well. By Lemma 4.11, we can ensure that \( h \) moves \( y_i \) far enough to ensure that \( h(\tau_i) \) is disjoint from \( \tau_i \), and the lemma follows. \( \Box \)

**Corollary 4.13.** Some subsegment of \( c' \) is contracted by \( h \).

**Proof.** This follows from Lemma 4.9, noting that \( c' \) must intersect either \( \tau_1 \) or \( \tau_2 \). \( \Box \)

This completes the proof of Proposition 4.4.

**Remark 4.14.** Note that Lemma 4.9 also implies that the center leaf \( c \) that separates \( s^+ \) from \( s^- \) is “coarsely contracted” by \( h \), in the sense that sufficiently large subsegments of \( c \) are taken properly into themselves.

This generalizes as follows:

**Lemma 4.15.** Let \( c \) be a center leaf in a center stable leaf \( L \subset \tilde{M} \). Suppose that \( L \) is Gromov-hyperbolic, and fixed by \( \tilde{f} \) and some nontrivial \( \gamma \in \pi_1(M) \). Moreover, assume that there exist two stable leaves \( s_1, s_2 \) on \( L \) such that:

1. The center leaf \( c \) is in the region between \( s_1 \) and \( s_2 \);
2. The leaves \( s_1 \) and \( s_2 \) are a bounded Hausdorff distance apart;
3. The leaves \( c, s_1 \) and \( s_2 \) are all fixed by \( h = \gamma^n \circ \tilde{f}^m, m \neq 0 \).

Then, there exists a compact segment \( I \subset c \), such that \( h \) (if \( m > 0 \)) or \( h^{-1} \) (if \( m < 0 \)) acts as a contraction on \( c \setminus I \).
This remains true without assuming dynamical coherence — we will use it in Propositions 11.27 and 11.30. The proof of this lemma is very similar to that of Lemma 4.9. Note that we do not need $c$ to make a perfect fit with $s_1$ or $s_2$, nor do we need that $c$ necessarily goes to both ends of the band determined by $s_1$ and $s_2$ as in Figure 7. All we need is that $c$ is between $s_1$ and $s_2$, and that both rays of $c$ escape every compact set in $L$. That last fact is true of any center leaf in $M$.

5. Mixed behavior

We can now eliminate mixed behavior in our cases of interest.

**Theorem 5.1.** Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism homotopic to the identity. Assume that $W^{cu}$ is $f$-minimal or that $M$ is hyperbolic or Seifert.

If a good lift $\tilde{f}$ fixes all the leaves of $\tilde{W}^{cs}$, then it also fixes all the leaves of $\tilde{W}^{cu}$.

**Proof.** Since $M$ is not $T^3$ (recall that $\pi_1(M)$ is not virtually solvable), Proposition B.2 says that we can find a leaf in $W^{cs}$ with nontrivial fundamental group. Let $L \in \tilde{W}^{cs}$ be a lift of such a leaf, which is invariant by some nontrivial $\gamma \in \pi_1(M)$.

By Proposition 4.4, $\tilde{f}$ fixes some center leaf $c$ in $L$, so it must fix the center unstable leaf $K \in \tilde{W}^{cu}$ that contains $L$. From the dichotomy in Corollary 3.21, it follows that $\tilde{f}$ fixes every leaf of $\tilde{W}^{cu}$ as desired. $\square$

In particular, under the assumptions of this theorem, one rules out the mixed case (see item (2) of section 2.3.1).

6. Double invariance

In this section we show that, under the appropriate conditions, the **doubly invariant case** (see item (1) of section 2.3.1) leads to a discretized Anosov flow.

**Theorem 6.1.** Let $f$ be a dynamically coherent partially hyperbolic diffeomorphism. Assume that $M$ is hyperbolic or Seifert or that $W^{cs}$ and $W^{cu}$ are $f$-minimal. If there exists a good lift $\tilde{f}$ which fixes a leaf of $\tilde{W}^{cs}$, then $f$ is a discretized Anosov flow.

Thanks to Proposition 3.15, under any of the hypothesis of the Theorem, both $W^{cs}$ and $W^{cu}$ are $f$-minimal. Notice also that Theorem 5.1 implies that $\tilde{f}$ must fix every leaf of both $\tilde{W}^{cs}$ and $\tilde{W}^{cu}$.

Notice that Theorem 6.1 together with the dichotomy of Corollary 3.10 proves Theorem 1.1 from the introduction.

We will first show that connected components of the intersections of center stable and center unstable leaves are fixed by $\tilde{f}$ (i.e., that $\tilde{f}$ fixes leaves of the center foliation). Proving that $f$ is a discretized Anosov flow will then follow rather easily.

We will prove that the set of connected components of intersections fixed by $\tilde{f}$ is both open and closed and then that it is non-empty, thus proving that all center leaves are fixed.

6.1. **Fixing center leaves.** The main step in the proof of Theorem 6.1 is the following proposition. Recall that the lift $\tilde{W}$ of the center foliation $W^c$ consists of the connected components of the intersections between leaves of $\tilde{W}^{cs}$ and $\tilde{W}^{cu}$.
Proposition 6.2. Let $f$ be a dynamically coherent partially hyperbolic diffeomorphism homotopic to the identity. Let $\tilde{f}$ be a good lift of $f$ which fixes every leaf of $\tilde{W}^{cs}$ and $\tilde{W}^{cu}$. Suppose that $W^{cs}$ and $W^{cu}$ are $f$-minimal in $M$. Then $\tilde{f}$ fixes every leaf of $\tilde{W}^{c}$.

The key point in the proof of this proposition is to show that either all leaves of $\tilde{W}^{c}$ are fixed by $\tilde{f}$, or no leaf of $\tilde{W}^{c}$ is fixed by $\tilde{f}$. In the latter case we will use an argument similar to that of the analysis of the mixed behavior case, reaching a contradiction from the results of section 4.

Lemma 6.3. The set

$$\text{Fix}_{\tilde{f}}^{c} := \left\{ x \in \tilde{M} \mid \text{the center leaf through } x \text{ is fixed by } \tilde{f} \right\}$$

is open in $\tilde{M}$. In addition $\text{Fix}_{\tilde{f}}^{c}$ is invariant under deck transformations. 

Proof. Let $c \in \tilde{W}^{c}$ be such that $\tilde{f}(c) = c$. Let $L = \tilde{W}^{cs}(c)$ be the center-stable leaf containing $c$.

Let $\epsilon > 0$ be small enough so that the center and stable foliations restricted to any ball of radius $\epsilon$ in $L$ is product (i.e., every stable and central leaf in the ball intersect each other).

Let $x \in c$. By continuity of $f$, pick $\delta > 0$ such that if $d(x, y) < \delta$ then $d(\tilde{f}(x), \tilde{f}(y)) < \epsilon$. Up to taking $\delta$ smaller, and since $\tilde{f}(x) \in c$, we can assume that for any $y \in L$ such that $d(x, y) < \delta$, we have that $c(y)$, the central leaf through $y$, intersects $s(\tilde{f}(x))$, the stable leaf through $\tilde{f}(x)$. This $\delta$ a priori depends on $x$.

Let $y \in L$ such that $d(x, y) < \delta$, then $c(y) \cap s(\tilde{f}(x)) \neq \emptyset$. Moreover, since $d(\tilde{f}(x), \tilde{f}(y)) < \epsilon$, we have that $c(f(y)) \cap s(\tilde{f}(x)) \neq \emptyset$. So the stable leaf $s(\tilde{f}(x))$ intersects both $c(y)$ and $c(\tilde{f}(y))$.

Now, as $\tilde{f}$ fixes the leaves of the central unstable foliations, we have that $\tilde{W}^{cu}(c(y)) = \tilde{W}^{cu}(c(\tilde{f}(y)))$. But $s(\tilde{f}(x))$ is transverse to $\tilde{W}^{cu}$, so it cannot intersect the same leaf of $\tilde{W}^{cu}$ more than once (see Theorem B.1). Hence $c(\tilde{f}(y)) = c(y)$.

Thus, the set of center leaves fixed by $\tilde{f}$ in a center stable leaf is open in that center stable leaf. As the same argument applies to center unstable leaves, we obtain that the union of points in center leaves in $\text{Fix}_{\tilde{f}}^{c}$ is open in $\tilde{M}$.

Finally, since $\tilde{f}$ commutes with every deck transformation, $\text{Fix}_{\tilde{f}}^{c}$ is $\pi_{1}(M)$-invariant. 

We will think of $\text{Fix}_{\tilde{f}}^{c}$ as both a subset of $\tilde{M}$ and a collection of center leaves in $\tilde{W}^{c}$.

Let $D := \pi(\text{Fix}_{\tilde{f}}^{c})$. By Lemma 6.3, $D$ is open in $M$, and, obviously, $f$-invariant.

Lemma 6.4. Either $\text{Fix}_{\tilde{f}}^{c} = \tilde{M}$ or $\text{Fix}_{\tilde{f}}^{c} = \emptyset$. 

Proof. Assume that $\text{Fix}_{\tilde{f}}^{c} \neq \emptyset$, and thus, $D \neq \emptyset$.

We start by showing that every leaf of $\tilde{W}^{cs}$ has at least some fixed center leaves: Let $E$ be the $\tilde{W}^{cs}$-saturation of $\text{Fix}_{\tilde{f}}^{c}$, and suppose for a contradiction that there exists $L$ a leaf of $\tilde{W}^{cs}$ such that $L \cap E = \emptyset$. 

Since $\text{Fix}^c_\gamma$ is $\pi_1(M)$-invariant, so is $E$. Hence, for any $\gamma \in \pi_1(M)$, we have $\gamma L \cap \text{Fix}^c_\gamma = \emptyset$. Therefore, in $M$, we have

$$\pi(L) \cap \pi(E) = \emptyset.$$ 

So $\pi(L)$ is contained in the set $M \setminus \pi(E)$, which is thus non empty. But $\pi(E)$ is the $\mathcal{W}^{cs}$-saturation of $D$, hence open since $D$ is open. The set $\pi(E)$ is also $f$-invariant since $D$ is. Therefore, $M \setminus \pi(E)$ is a non empty, closed, $f$-invariant subset of $M$ saturated by $\mathcal{W}^{cs}$. The $f$-minimality of $\mathcal{W}^{cs}$ implies that $M \setminus \pi(E) = M$, which is in contradiction with the fact that $\pi(E)$ is non empty.

It follows that, for any center-stable leaf $L$, we have $L \cap \text{Fix}^c_\gamma \neq \emptyset$.

Our next step is to prove that any center stable leaf that has a non-trivial stabilizer in $\pi_1(M)$ is contained in $\text{Fix}^c_\gamma$. Let $L$ be a leaf of $\mathcal{W}^{cs}$ such that its projection $A = \pi(L)$ is not simply connected (in which case it must be an annulus or a Möbius band according to Corollary 3.21). As we proved above, we know that $L \cap \text{Fix}^c_\gamma \neq \emptyset$. We now want to show that $L \subset \text{Fix}^c_\gamma$. Let us assume for a contradiction that $\text{Fix}^c_\gamma \cap L \neq L$.

Recall that $\text{Fix}^c_\gamma$ is open (by Lemma 6.3), thus so is $B = \text{Fix}^c_\gamma \cap L$ (for the relative topology on $L$). Notice that, since both $\text{Fix}^c_\gamma$ and $L$ are invariant by $\tilde{f}$, so is $B$, and in turn, so is its boundary $\partial B$.

Let $c_1$ be a center leaf in $\partial B$. Then $\tilde{f}(c_1) \neq c_1$, but arbitrarily near $c_1$ there are some fixed center leaves.

Since $c_1$ and $\tilde{f}(c_1)$ are both in $\partial B$, they are non separated from each other in the leaf space of the center foliation in $L$. Indeed, if one takes a sequence $(c_n)$ of central leaves in $B$ that accumulates on $c_1$, then, since $\tilde{f}(c_n) = c_n$, the sequence also accumulates on $\tilde{f}(c_1)$.

As $c_1$ and $\tilde{f}(c_1)$ are not separated in the center leaf space of $L$, it follows that there exists a stable leaf $s_1$ making a perfect fit with $c_1$, such that $s_1$ separates $c_1$ from $\tilde{f}(c_1)$.

If some power of $\tilde{f}$ fixes $s_1$, then that power of $\tilde{f}$ has a fixed point in $s_1$, contradicting Lemma 3.13 (since $\tilde{f}$ fixes every leaves of $\mathcal{W}^{cs}$).

It follows that the sequence $\left(\tilde{f}^n(s_1)\right)$ is infinite. Moreover, there exists $c \in \text{Fix}^c_\gamma$ that intersects all the leaves $\left(\tilde{f}^n(s_1)\right)$. Indeed, taking $c \in \text{Fix}^c_\gamma$ to be a central leaf close enough to $c_1$ so that $c \cap s_1 \neq \emptyset$, then $c$ intersects every $\tilde{f}^n(s_1)$, because $\tilde{f}(c) = c$.

Furthermore, for all $i$, $\tilde{f}^i(s_1)$ separates $\tilde{f}^{i-1}(s_1)$ from $\tilde{f}^{i+1}(s_1)$, because $\tilde{f}$ acts as a translation on $c$ (because $\tilde{f}$ cannot have a fixed point in $L$ by Lemma 3.13).

As $\tilde{f}$ acts freely on the stable leaf space in $L$ (again thanks to Lemma 3.13), then $\tilde{f}$ has an axis $A^\gamma(\tilde{f})$ for this action. By definition, all the leaves $\tilde{f}^i(s_1)$ are in this axis. Since all the leaves $\tilde{f}^i(s_1)$ also intersect a common transversal $c$, we deduce that $A^\gamma(\tilde{f})$ is a line (see Figure 9).

Now recall that $C = \pi(L)$ is an annulus or a Möbius band, (thanks to Corollary 3.21). Let $\gamma$ be the deck transformation associated with the generator of $\pi_1(C)$, so that $\gamma$ fixes $L$.

Recall that, since there does not exist closed stable leaves in $M$, $\gamma$ must act freely on the stable leaf space in $L$. Thus $\gamma$ admits an axis $A^\gamma(\gamma)$. Since $\tilde{f}$ and
Figure 9. The combination of fixed and non-fixed center leaves allows to construct a center leaf intersecting $s_1$ and $\tilde{f}(s_1)$ in the axis $A^s(\tilde{f}) = A^s(\gamma)$.

$\gamma$ commute, then $A^s(\tilde{f}) = A^s(\gamma)$ (see Proposition E.2). In particular, $A^s(\gamma)$ is a line.

Therefore there exists a $\gamma$-invariant curve in $L$, that we call $\alpha$, such that $\alpha$ is transverse to the stable foliation, and intersects each stable leaf in $A^s(\gamma) = A^s(\tilde{f})$ exactly once. It follows that $\tilde{f}(\alpha)$ and $\alpha$ intersect exactly the same set of stable leaves in $L$. So we can use the Graph Transform argument (Lemma H.1) on $\alpha$ and obtain that there exists a curve $c_0$ in $L$, tangent to the central direction $E^c$, and invariant by both $\tilde{f}$ and $\gamma$.

Since $c_0$ intersects $s_1$, and the leaves $s_1$ and $c_1$ make a perfect fit, we deduce that there exists a leaf $s$, close to $s_1$, that intersects both $c_0$ and $c_1$. Let $x = c_0 \cap s$, $y = c_1 \cap s$ and $z = c_0 \cap s_1$. Up to choosing $s$ closer to $s_1$, we may assume that the distance between $x$ and $z$ is less than some fixed $K > 0$, the length of the closed curve $\pi(c_0)$. Now, since $c_0$ is invariant by $\gamma$, we have that, for all $n$,

$$d(\tilde{f}^n(x), \tilde{f}^n(z)) \leq K.$$  

Moreover, since $\tilde{f}$ contracts stable length, we have that $d(\tilde{f}^n(x), \tilde{f}^n(y))$ converges to 0 as $n$ goes to $+\infty$.

Using the above, together with the invariance of $c_0$ by $\tilde{f}$ and the fact that $c_0$ is tangent to the central direction, we deduce that for some large enough $n$, the leaf $\tilde{f}^n(c_1)$ intersects $\tilde{f}^n(s_1)$, contradicting the fact that $s_1$ and $c_1$ do not intersect.

Hence, we proved thus far that for any $L$ a center-stable leaf with non-trivial stabilizer, we have $L \subset \text{Fix}^c$. We can now finish the proof of Lemma 6.4.

Let $c$ be any center leaf in $\tilde{M}$. Let $x$ be a point in $c$, let $U$ be the center unstable leaf containing $c$, and let $\tau$ be a small unstable segment in $\tilde{W}^{cs}(x)$ that contains $x$ in its interior.

Recall (see Proposition B.2) that there exists leaves in $\tilde{W}^{cs}$ with non-trivial stabilizer. Then $f$-minimality implies that such leaves are dense. Thus, we may

\[ \text{In fact, since there exists a central leaf that is transverse to the axis, an argument used in the proof of Lemma 4.5 shows that $c_0$ is not just tangent to the central direction, but an actual central leaf. However, just having tangency to the central direction is enough to finish the proof.} \]
assume that both endpoints of $\tau$ are on center stable leaves with non-trivial stabilizer. Call $c_1$ and $c_2$ the center leaves through the two endpoints of $\tau$. We proved above that both $c_1$ and $c_2$ are fixed by $\tilde{f}$ (since they are on center stable leaves with non-trivial stabilizer).

Since $c$ intersects $\tau$, an unstable segment from $c_1$ to $c_2$, it follows that $c$ separates $c_1$ from $c_2$ in $U$. As $\tilde{f}$ fixes both $c_1$ and $c_2$ then $\tilde{f}(c)$ also separates $c_1$ from $c_2$ in $U$. This implies that $\tilde{f}(c)$ also intersects $\tau$. As argued before, since $\tilde{f}$ fixes every center stable leaves, $c$ and $\tilde{f}(c)$ must be in the same center stable leaf, and, since they both intersect $\tau$, which is a transversal to the center stable foliation, we deduce that $c = \tilde{f}(c)$.

Therefore, we proved that $\tilde{f}$ fixes every center leaf, i.e., $\operatorname{Fix}_{\tilde{f}}(c) = \tilde{M}$, as desired.

We can now prove Proposition 6.2.

Proof of Proposition 6.2. By assumption, $\tilde{f}$ fixes every leaf of $\tilde{W}^c$, and, by Proposition B.2, there exists some center stable leaf with non-trivial stabilizer. Thus, Proposition 4.4 implies that there exists at least one fixed center leaf, i.e., $\operatorname{Fix}_{\tilde{f}}(c) \neq \emptyset$. Lemma 6.4, then yields that $\operatorname{Fix}_{\tilde{f}}(c) = \tilde{M}$, which is what we wanted to prove.

6.2. Showing that the map is a discretized Anosov flow.

Proposition 6.5 (Leaf conjugacy to a topological Anosov flow). Let $f$ be a partially hyperbolic diffeomorphism on a 3-manifold $M$. Suppose that there exists a lift $\tilde{f}$ to the universal cover $\tilde{M}$ such that $\tilde{f}$ moves points a bounded distance and $\tilde{f}$ fixes every center leaf. Then the center foliation is the orbit foliation of a topological Anosov flow.

The proof is very similar to that given in [BW05, Section 3.5]. We sketch the main points of the proof. We also refer to Appendix G for the precise definition of a topological Anosov flow, and more discussion about discretized Anosov flows.

Proof. Fix a metric on $M$ and consider $X^c$ a unit vector field in $E^c$ which we first assume orientable. In the universal cover, using that $\tilde{f}$ fixes every center leaf, one can show that $\tilde{f}$ does not fix any point in $\tilde{M}$, that there is a uniform estimate for $d_c(x, \tilde{f}(x))$, and it is indeed continuous (see [BW05, Lemma 3.4] for a proof.
with less hypothesis). In particular, we can assume that \([x, f(x)]^c\) is positively oriented with respect to \(X^c\).

Now, let \(c_1, c_2\) be two center leaves in the same center stable leaf such that for some \(x \in c_1\) one has that \(\tilde{W}^s(x) \cap c_2 = \emptyset\). Then, letting \(y\) be the point of intersection, we have that \(d(\tilde{f}^n(x), \tilde{f}^n(y)) \to 0\) as \(n \to \infty\). As the points are moving forward by \(\tilde{f}\) along the orbits of \(X^c\) at bounded speed, this shows that the flow is locally contracted on center stable manifolds. The symmetric arguments give local contraction for the past in center-unstable manifolds. Notice that the fact that \(\tilde{f}\) acts as a translation in all center leaves and that center leaves are fixed by \(\tilde{f}\) implies that no deck transformation can reverse orientation of the center, this implies that our initial assumption is verified.

This shows that the flow generated by \(X^c\) is expansive. Moreover, it preserves the transverse foliations \(W^{cs}\) and \(W^{cu}\), which do not have singularities. Thus, the work of Paternain [Pat93] implies that the flow generated by \(X^c\) is a topological Anosov flow (see also Appendix G).

Putting together Theorem 5.1, Proposition 6.2, Proposition 6.5 and Proposition G.2 one finishes the proof of Theorem 6.1 and of Theorem 1.1.

7. Proof of Theorem A with Dynamical Coherence

We are now ready to finish the proof of Theorem A when the diffeomorphism is assumed to be dynamically coherent.

We start by proving that, in a Seifert manifold, one can always choose a good lift in such a way that it fixes one center-stable leaf.

**Proposition 7.1.** Let \(f: M \to M\) be a dynamically coherent partially hyperbolic diffeomorphism on a Seifert manifold. Suppose that \(f\) is homotopic to the identity and that the Seifert fibration in \(M\) is orientable. Then there exists a good lift of an iterate of \(f\) which fixes a leaf (and therefore every leaf) of \(\tilde{W}^{cs}\).

**Proof.** To prove the result, partial hyperbolicity will only be used to get that \(M\) has non-zero Euler class ([HPS18, Theorem B]).

First up to taking a finite lift we assume that \(M\) is an orientable circle bundle over a higher genus (orientable) surface \(\Sigma\).

Consider the leaf space \(\mathcal{L}^{cs}\) of the center-stable foliation and let \(\delta\) be the deck transformation associated with the center of \(M\). As the foliation is horizontal (see Theorem F.3), it follows that the leaf space \(\mathcal{L}^{cs}\) is homeomorphic to the reals. In addition, \(\mathcal{L}^{cs}/(\delta)\) is a circle that we will call \(S^1_\delta\).

Consider a good lift \(\tilde{f}\) of \(f\). The map \(\tilde{f}\) induces a homeomorphism \(\hat{f}: S^1_\delta \to S^1_\delta\). Moreover, \(\hat{f}\) commutes with the image of the homeomorphisms \(\hat{\rho}(\gamma): S^1_\delta \to S^1_\delta\) which are defined for all \(\gamma \in \pi_1(M)\). These homeomorphisms are well defined up to composition with \(\delta\). So \(\hat{\rho}\) naturally induces a quotient representation \(\rho: \pi_1(\Sigma) \to \text{Homeo}_+(S^1_\delta)\) when using the identification \(\pi_1(M)/(\delta) \cong \pi_1(\Sigma)\). The Euler class of \(M\) coincides with the one of the representation \(\rho\) (see [CC03, Chapter 4]).

We first show that \(\hat{f}\) has rational rotation number. We proceed by contradiction: Assume that \(\hat{f}\) has irrational rotation number.

Suppose first that \(\hat{f}\) is minimal. It directly implies that \(\hat{f}\) is conjugate to an irrational rotation by a homeomorphism \(h: S^1_\delta \to S^1_\delta\). Conjugating the homeomorphisms \(\rho(\gamma)\) (that is, \(h^{-1}\rho(\gamma)h\)), since they commute with an irrational rotation they must commute with every rotation. Therefore the homeomorphisms \(\rho(\gamma)\) are
all conjugate by \( h \) to rigid rotations. This implies that \( \rho: \pi_1(\Sigma) \to \text{Homeo}_+(S^1) \) is conjugate to a representation into \( \text{SO}(2, \mathbb{R}) \). This allows to construct a path to the trivial representation, because one can move freely along \( \text{SO}(2, \mathbb{R}) \) until one gets to the identity without altering the relations. Therefore the representation has zero Euler class. See [Man18, Sections 5.2 and 5.3].

If \( \hat{f} \) is not minimal, it is a Denjoy counterexample, one can see that the representation of \( \pi_1(\Sigma) \) into \( \text{Homeo}(S^1) \) is semi-conjugate to a representation which commutes with a minimal homeomorphism, and so it also has to have zero Euler class (see [Man18, Section 5.2]). This proves that \( \hat{f} \) has rational rotation number.

Now we go back to the original manifold. Since in the finite cover, the corresponding map \( \hat{f} \) had rational rotation number, the same is true for \( \hat{f} \) associated with the original manifold. In particular, \( \hat{f} \) has a periodic point, which means that for some \( i \neq 0 \), \( \delta^i \hat{f} \) has a fixed point. So \( \delta^i \hat{f} \) is the sought good lift (note that it is a good lift because the Seifert fibration is orientable, and thus \( \delta \) is in the center of \( \pi_1(M) \)). This finishes the proof.

Notice that the symmetric statement holds for \( \tilde{W}^{cu} \) but a priori not for both simultaneously.

**Remark 7.2.** In this proof, we did not use dynamical coherence (one only needs to use the leaf space of the branching foliations instead, see section 10). One could give a slightly simpler proof that uses dynamical coherence. However, since we will need this result in the non-dynamically coherent case (see section 13), it is more efficient to give the general proof.

So we can now prove Theorem A in the dynamically coherent case.

**Theorem 7.3.** Let \( f: M \to M \) be a partially hyperbolic diffeomorphism homotopic to identity. Assume that \( M \) is Seifert and that \( f \) is dynamically coherent. Then a power of \( f \) is a discretized Anosov flow.

*Proof.* If the result holds in a finite cover of \( M \), then, by projection, it also holds in \( M \). So, by lifting to a double cover, we may assume that \( M \) has orientable Seifert fibration. Let \( \tilde{f} \) be a good lift of some power \( f^k \) given by Proposition 7.1. Then \( \tilde{f} \) does not act as a translation on both center-stable and center-unstable leaf spaces, so is not in case (ii) of Theorem 1.1. Thus it is in case (i) of Theorem 1.1, i.e., \( f^k \) is a discretized Anosov flow.

**Remark 7.4.** Note that, in Theorem 7.3, we need to take a power of \( f \) to get a discretized Anosov flow, whereas Theorem 1.1 holds for the original \( f \). This condition is necessary, i.e., there are some dynamically coherent partially hyperbolic diffeomorphisms homotopic to the identity on a Seifert manifold that are not discretized Anosov but such that a (non-trivial) iterate is. We will give such an example below and also classify all such examples.

Consider \( \Sigma \) a hyperbolic surface (or orbifold) and \( g^t \) the geodesic flow on \( T^1 \Sigma \). Let \( M \) be a \( k \)-fold cover of \( T^1 \Sigma \) obtained by unwrapping the fiber and \( g^t_M: M \to M \) be a lift of \( g^t \) to \( M \). Call \( s: M \to M \) the map obtained by lifting the “rotation by \( 2\pi \) along the fiber in \( T^1 \Sigma \). Then for any \( i = 1, \ldots, k - 1 \), the diffeomorphism \( f_{k,i} := g_M^i \circ s^i \) is a partially hyperbolic diffeomorphism, dynamically coherent, homotopic to the identity, and it is not a dicretized Anosov flow (but \( f_{k,i}^k \) is a discretized Anosov). Notice that the action of any good lift of \( f_{k,i} \) on the center stable and center unstable leaf spaces is by translations.
Now, suppose that $M$ is a Seifert manifold and $f$ is a dynamically coherent partially hyperbolic diffeomorphisms homotopic to the identity. Then, by Theorem 7.3, there exists $k$ such that $f^k$ is a discretized Anosov flow. Thus, by the classification of Anosov flows on Seifert manifolds (see [Gly84, Bar96]), $M$ is a finite lift of the unit tangent bundle of an orbifold $\Sigma$ and $f^k$ is leaf conjugate to the time-one map of the lift of the geodesic flow. Then the action of (a good lift of) $f$ on both the central stable and central unstable leaf spaces is conjugated to the action of (a good lift of) a diffeomorphism $f_{k,i}$ as above. So $f$ and $f_{k,i}$ are leaf conjugate.

8. Coarse dynamics of translations

In this section, we consider a homeomorphism $f: M \rightarrow M$ of a hyperbolic 3-manifold that preserves a uniform, $\mathbb{R}$-covered foliation $\mathcal{F}$ and acts as a translation on its leaf space. We show that the dynamics of $f$ is comparable to the dynamics of the pseudo-Anosov flow $\Phi$ (given by Theorem D.3) that regulates $\mathcal{F}$. More precisely, for every periodic orbit of $\Phi$, we show that there exists a compact core (in a lift of $M$) invariant by $f$ that plays the role of the periodic orbit of $\Phi$.

So in particular, the result of this section does not require $f$ to be partially hyperbolic and is of independent interest. The description of the dynamics of $f$ in periodic leaves of $\mathcal{F}$ (if any) can be compared to the global shadowing for pseudo-Anosov homeomorphisms done in [Han85]. We will use the results obtained here to complete the proof of Theorem B in the next section.

To make this comparison precise we will introduce some more objects. Let $f: M \rightarrow M$ be a hyperbolic 3-manifold. We assume that $f$ is homotopic to the identity and preserves a foliation $\mathcal{F}$. Furthermore, we suppose that $\mathcal{F}$ is $\mathbb{R}$-covered and uniform and such that a good lift $\tilde{f}$ of $f$ acts as a translation on the leaf space of $\tilde{\mathcal{F}}$.

Since $\tilde{f}$ commutes with any deck transformation and acts as a translation on the leaf space of $\tilde{\mathcal{F}}$, it implies that the foliation $\mathcal{F}$ is actually transversely orientable. Hence Theorem D.3 applies and we call $\Phi$ a transverse regulating pseudo-Anosov flow. We denote by $\tilde{\Phi}$ its lift to the universal cover $\tilde{M}$.

Let $\gamma \in \pi_1(M)$ be an element associated with a periodic orbit of $\Phi$ (i.e., such that there is a flow line of $\tilde{\Phi}$ invariant under $\gamma$). Let $M_\gamma := \tilde{M}/\langle \gamma \rangle$ be the cover of $M$ associated with that deck transformation.

The foliation $\tilde{\mathcal{F}}$, lifted from $\mathcal{F}$ to $M_\gamma$, is a foliation by planes. Indeed, since $\Phi$ is regulating, each orbit of $\tilde{\Phi}$ can represent the leaf space $\mathcal{L}_{\tilde{\mathcal{F}}}$ of $\tilde{\mathcal{F}}$. Thus $\gamma$, and all of its powers, act as a translation on $\mathcal{L}_{\tilde{\mathcal{F}}}$, so no leaf of $\tilde{\mathcal{F}}$ can be fixed by a power of $\gamma$. Therefore, $\tilde{\mathcal{F}}_\gamma$ is a foliation by planes (see, e.g., [Fen02] for more details).

Since $\tilde{f}$ is a good lift of $f$ it induces a lift $\tilde{\gamma}$ of $\gamma$ in $M_\gamma$.

We can now state precisely the main result of this section.

**Proposition 8.1.** Let $M$, $f: M \rightarrow M$, $\mathcal{F}$ and $\Phi$ be as above.

Then, for every $\gamma \in \pi_1(M)$ associated with a periodic orbit of $\Phi$, there is a compact $\tilde{\gamma}$-invariant set $T_\gamma$ in $M_\gamma$ which intersects every leaf of $\tilde{\mathcal{F}}_\gamma$.

Moreover, if an iterate $\tilde{f}_\gamma^k$ of $\tilde{f}_\gamma$ fixes a leaf $L$ of $\tilde{\mathcal{F}}_\gamma$, and $\gamma$ fixes all the prongs of a periodic orbit associated with $\gamma$, then the fixed set of $\tilde{f}_\gamma^k$ in $L$ is contained in $T_\gamma \cap L$ and has negative Lefschetz index.

**Remark 8.2.** In fact, the proof will show that the total Lefschetz index $I_{T_\gamma \cap L}(\tilde{f}_\gamma^k|_L)$ equals $-1$ if the periodic orbit of $\Phi$ is a regular periodic orbit, and equals $1 - p$
if the periodic orbit is a $p$-prong, $p \geq 3$, assuming that $\gamma$ fixes the prongs of the orbit (see Appendix I for the definition of the Lefschetz index).

We also remark that, by construction, the set $T_{\gamma}$ is essential in the sense that any neighborhood of it contains a curve homotopic to (a power of) $\gamma$.

To prove this proposition, we first need to explore some properties of the pseudo-Anosov flow $\Phi$ and its interaction with the foliation $\mathcal{F}$.

Let $\Lambda_s$ and $\Lambda_u$ be the weak stable and weak unstable (singular) foliations of the pseudo-Anosov flow $\Phi$. We denote by $\tilde{\Lambda}_s$ and $\tilde{\Lambda}_u$ their lift to the universal cover.

For any leaf $L$ of $\tilde{\mathcal{F}}$, we denote by $\mathcal{G}^s_L$ and $\mathcal{G}^u_L$ the one dimensional (singular) foliations obtained by intersecting the foliations $\tilde{\Lambda}_s$ and $\tilde{\Lambda}_u$ with $L$.

**Fact 8.3.** The length along foliations $\mathcal{G}^s_L$ and $\mathcal{G}^u_L$ is uniformly efficient up to a multiplicative distortion at measuring distances in the leaves of $\tilde{\mathcal{F}}$. That is, the rays of $\mathcal{G}^s_L$ and $\mathcal{G}^u_L$ are uniform quasi-geodesics for the path metric on $L$.

**Proof.** This fact is a consequence of the construction of the foliations $\tilde{\Lambda}_s$ and $\tilde{\Lambda}_u$. They are obtained by blowing down certain laminations that intersect the leaves of $\tilde{\mathcal{F}}$ along geodesics (with respect to the uniformization metric obtained via Candel’s Theorem C.1). We refer to [Fen02] or [Cal07] for the construction of these laminations.

In particular, there exists a uniform $K_1 > 1$ such that for every $L \in \tilde{\mathcal{F}}$ and $y \in \mathcal{G}^s(x)$ one has

$$\ell([x,y]^{\mathcal{G}^s_L}) \leq K_1 d_L(x,y) + K_1$$

where $\ell([x,y]^{\mathcal{G}^s_L})$ denotes the length of the arc in $\mathcal{G}^s$ joining $x$ and $y$. And similarly for $\mathcal{G}^u(x)$. \[\square\]

The flow $\tilde{\Phi}$ does not preserve the foliation $\tilde{\mathcal{F}}$, but since it is transverse and regulating to the foliation, it makes sense to consider, given $L_1, L_2 \in \tilde{\mathcal{F}}$ two leaves, the map $\tau_{12} : L_1 \rightarrow L_2$ consisting in flowing along $\tilde{\Phi}$ from one leaf to the other. By construction, the map $\tau_{12}$ is a homeomorphism. Notice that since $\mathcal{F}$ is $\mathbb{R}$-covered and uniform, the Hausdorff distance between $L_1$ and $L_2$ is bounded multiplicatively with the flow distance between the leaves – at least for leaves which are sufficiently apart from each other.

By convention, we will always assume that $L_2$ is taken to be above $L_1$, in the sense that one has to follow the orbits of $\tilde{\Phi}$ in the positive direction to go from $L_1$ to $L_2$. Notice that invariance of $\tilde{\Lambda}_s$ and $\tilde{\Lambda}_u$ by $\tilde{\Phi}$ imply that the homeomorphism $\tau_{12}$ maps the foliations $\mathcal{G}^s_{L_1}$ and $\mathcal{G}^u_{L_1}$ into the foliations $\mathcal{G}^s_{L_2}$ and $\mathcal{G}^u_{L_2}$ respectively.

When the leaves $L_1, L_2$ are understood, we will omit them from the notation. It is a standard fact from the dynamics of pseudo-Anosov flows and the bounded comparison between flow distance and leaves\(^5\) that the following holds:

**Fact 8.4.** For any leaves $L_1$ and $L_2$ sufficiently far apart, the map $\tau_{12}$ expands lengths (and, equivalently distances) in $\mathcal{G}^u$ exponentially in terms of the Hausdorff distance between $L_1$ and $L_2$. That is, there exists a $\lambda > 0$, independent of $L_1, L_2$, such that, for any $x \in L_1$ and $y \in \mathcal{G}^u_{L_1}(x)$, we have

$$d_{L_2}(\tau_{12}(x), \tau_{12}(y)) \geq e^{\lambda d_{Haus}(L_1, L_2)},$$

\(^5\)It is worth noting that the pseudo-Anosov property is invariant under reparametrizations of the flows.
as long as \( d_{\text{Haus}}(L_1, L_2) \) is sufficiently big. Similarly, \( \tau_{12}^{-1} \) expands the lengths in \( \mathcal{G}^* \) exponentially in terms of the Hausdorff distance between \( L_1 \) and \( L_2 \).

The following simple result will be extremely useful for us.

**Lemma 8.5.** Suppose that \( \beta \) is a deck transformation that acts freely and decreasingly on the leaf space of \( \tilde{F} \). Let \( L_1 \) be a leaf of \( \tilde{F} \). Let \( \tau_{12} \) be the flow along map from \( L_1 \) to \( L_2 := \beta^{-1}(L_1) \). Define \( g_{\beta,L_1} := \beta \circ \tau_{12} : L_1 \to L_1 \). Then, for every \( K > 0 \), there exists \( R > 0 \) such that if \( d_{L_1}(x, x_1) > R \) then \( d_{L_1}(x, g_{\beta,L_1}(x)) > K \).

**Remark 8.6.** Notice that in this Lemma, we do not ask for \( \beta \) to be associated with a periodic orbit of the pseudo-Anosov regulating flow.

**Proof.** Suppose for a contradiction that there exists \( K > 0 \) and a sequence \( y_n \) escaping to infinity in \( L_1 \) and such that \( d_{L_1}(y_n, g_{\beta,L_1}(y_n)) \leq K \) for all \( n \).

Up to taking a subsequence, there exists \( \gamma_n \in \pi_1(M) \) such that \( \gamma_n(y_n) \) converges to \( y_0 \) in \( M \).

We define a map \( \tau_\beta : \tilde{M} \to \tilde{M} \) as follows: given \( x \) in \( \tilde{M} \), it is in \( L \) a leaf of \( \tilde{F} \), then we let \( \tau_\beta(x) \) be the intersection of the flow line of \( \Phi \) through \( x \) with \( \beta^{-1}(L) \). Notice that if \( x \in L_1 \), then \( \tau_\beta(x) = \tau_{12}(x) \). In particular, for every \( n \), \( \tau_\beta(y_n) = \tau_{12}(y_n) \).

Since \( \gamma_n(y_n) \) converges to \( y_0 \), and \( \tau_\beta \) consists of flowing along \( \tilde{\Phi} \) a uniformly bounded amount, for \( n \) big enough, we have that \( d(\tau_{12}(y_n), \tau_\beta(\gamma_n^{-1}(y_0))) \) is as small as we want. Hence, for \( n \) big enough, we have

\[
d(\beta \circ \tau_{12}(y_n), \beta \circ \tau_\beta(\gamma_n^{-1}(y_0))) < 1.
\]

Now, \( \beta \tau_{12}(y_n) = g_{\beta,L_1}(y_n) \) is at distance less than \( K \) from \( y_n \). Thus, after applying \( \gamma_n \), we get

\[
d(\gamma_n(y_n), \gamma_n \circ \beta \circ \tau_\beta(\gamma_n^{-1}(y_0))) < 1 + K.
\]

Note that the map \( \tau_\beta \) moves every point a bounded distance, the transformations \( \gamma_n, \beta \) are isometries, and, for \( n \) big enough, \( d(\gamma_n(y_n), y_0) \) is very small. Therefore, \( d(y_0, \gamma_n \circ \beta \circ \gamma_n^{-1}(y_0)) < K' \) for all \( n \) big enough and a fixed constant \( K' \).

So we can extract a converging subsequence once more, and get that for any \( n, m \) big enough, the distance between \( \gamma_n \beta \gamma_n^{-1}(y_0) \) and \( \gamma_m \beta \gamma_m^{-1}(y_0) \) is smaller than the injectivity radius of \( M \). It follows that

\[
\gamma_m \beta \gamma_m^{-1} = \gamma_n \beta \gamma_n^{-1},
\]

for all \( n, m \) large enough.

Now we use that \( M \) is hyperbolic. So \( \beta \) is a hyperbolic isometry of \( \mathbb{H}^3 \cong \tilde{M} \). It has an axis with ideal points \( a, b \).

Let \( n_0 \) be fixed large enough. Then, since \( \gamma_n \beta \beta^{-1} = \gamma_{n_0} \beta \beta_{n_0}^{-1} \), we have that \( \gamma_m(a) = \gamma_{n_0}(a) \) and \( \gamma_m(b) = \gamma_{n_0}(b) \).

Let \( c := \gamma_{n_0}(a) \) and \( d := \gamma_{n_0}(b) \). Notice that (for all \( n \) big enough) the axis of the isometry \( \gamma_n \beta \gamma_n^{-1} \) has endpoints \( c \) and \( d \).

Let \( \alpha \) be the generator of the group of deck transformations fixing \( c, d \). Then, for all \( n \) large enough, \( \gamma_n = \alpha^{i_n} \gamma_{n_0} \). In addition since \( y_n \) escapes compact sets but \( \gamma_n(y_n) \) converges to \( y_0 \), it follows that \( |i_n| \) converges to infinity.

Notice that, since \( \gamma_{n_0} \) sends the axis of \( \beta \) to the axis of \( \alpha \), a power of \( \alpha \) is conjugated to a power of \( \beta \) by \( \gamma_{n_0} \). Now, since \( \beta \) acts freely on the leaf space of \( \tilde{F} \), then so does \( \alpha \). But \( \gamma_n(y_n) \) converges to \( y_0 \). So \( \alpha^{i_n}(L_1) = \gamma_n \gamma_{n_0}^{-1}(L_1) \) does not escape in the leaf space, which contradicts the fact that \( \alpha \) acts freely. \( \square \)
We remark that this proof only uses geometry of $M$ and foliations. That is, this proof works for any regulating flow transverse to a transversely oriented, $\mathbb{R}$-covered, uniform foliation in a hyperbolic 3-manifold.

We will also need the following consequence of Lemma 8.5.

**Lemma 8.7.** Let $\gamma \in \pi_1(M)$ be associated with a periodic orbit $\delta_0$ of $\Phi$. Let $\delta$ be the unique lift of $\delta_0$ to $M$. For any leaf $L$ of $\tilde{\mathcal{F}}$, let $x_L$ be the unique intersection point of $L$ and $\delta$.

For any $K > 0$, there exists $R > 0$ such that, for any leaves $L_1, L_2$ in $\tilde{\mathcal{F}}$ with $L_2$ above $\gamma(L_1)$ and for any $x \in L_1$, we have that, if $d_{L_1}(x, x_{L_1}) > R$ then $d_{L_2}(\tau_{12}(x), x_{L_2}) > K$.

**Proof.** Let $K > 0$. First since the Hausdorff distance from $L$ to $\gamma L$ is bounded, then the amount of flowing between them is bounded by a constant $C$, for any $L$ leaf of $\tilde{\mathcal{F}}$. Now, using the fact that the leaves are uniformly properly embedded (see [Cal07, Lemma 4.48]), there is a constant $K_1$ such that if $d_L(u, v) > K_1$ then $d_{\tilde{\mathcal{F}}}(u, v) > K + 2C$.

Let $L_1$ be a leaf in $\tilde{\mathcal{F}}$ and let $L_2$ be such that $\gamma L_1$ separates $L_1$ from $L_2$. Then there exists $n > 0$ such that $L_2$ is in the interval (in the leaf space of $\tilde{\mathcal{F}}$) between $\gamma^n L_1$ and $\gamma^{n+1} L_1$.

By Lemma 8.5 and the fact that deck transformations are isometries in the leafwise metrics, there exists $R$ such that for any $x \in L_1$, if $d_{L_1}(x, x_{L_1}) > R$ then $d_{\gamma^n L_1}(\tau_n(x), x_{\gamma^n L_1}) > K_1$, where $\tau_n: L_1 \to \gamma^n L_1$ is the flow along $\Phi$ map. By the choice of $K_1$ it follows that $d_{\tilde{\mathcal{F}}}(\tau_n(x), x_{\gamma^n L_1}) > K + 2C$.

Now, since $L_2$ is between $\gamma^n L_1$ and $\gamma^{n+1} L_1$, the flow distance from $\gamma^n L_1$ to $L_2$ is bounded above by $C$. It follows that $d_{\tilde{\mathcal{F}}}(\tau_{12}(x), x_{L_2}) > K$. In particular $d_{L_2}(\tau_{12}(x), x_{L_2}) > K$, which proves the lemma. \qed

The reason we will be able to compare the dynamics of $\tilde{f}$ and $\tilde{\Phi}$ is thanks to the fact that they are a uniform bounded distance apart. That is, we have the following.

**Lemma 8.8.** Let $f: M \to M$ be a homeomorphism of a hyperbolic 3-manifold $M$ preserving an $\mathbb{R}$-covered uniform foliation $\mathcal{F}$ and $\tilde{f}$ a good lift to $\tilde{M}$. There exists $R_1 > 0$ so that for every $L_1 \in \tilde{\mathcal{F}}$, if $L_2 = \tilde{f}(L_1)$ and $x \in L_1$ then $d_{L_2}(\tilde{f}(x), \tau_{12}(x)) < R_1$.

**Proof.** Since $\tilde{f}$ is a good lift it follows that one can join $x$ with $\tilde{f}(x)$ by an arc of bounded length. In particular, since the foliation $\mathcal{F}$ is $\mathbb{R}$-covered and uniform, it follows that the Hausdorff distance between $L_1$ and $L_2 = \tilde{f}(L_1)$ is uniformly bounded above and below independently of $L_1 \in \tilde{\mathcal{F}}$. Therefore, as explained before the statement of Fact 8.4 the amount of flowing needed to go from $L_1$ to $L_2$ is also uniformly bounded below and above.

It follows that $d_{\tilde{\mathcal{F}}}(\tilde{f}(x), \tau_{12}(x))$ is uniformly bounded. Again we use the fact that leaves of an $\mathbb{R}$-covered taut foliation are uniformly properly embedded in the universal cover (see [Cal07, Lemma 4.48]). The result follows. \qed

Now we are ready to prove Proposition 8.1.

**Proof of Proposition 8.1.** Let $\gamma \in \pi_1(M)$ be represented by a periodic orbit $\delta_0$ of $\Phi$ and take $\delta$ the unique lift of $\delta_0$ to $M$ fixed by $\gamma$.

We will build the core $T_\gamma$ that we seek by considering a very large tubular neighborhood of $\delta$ and taking the intersection of this tubular neighborhood with
all of its forward and backwards image under $\tilde{f}$ (see figure 11). We will prove that this infinite intersection is non-empty, thus its projection to $M_\gamma$ will have the desired properties.

Note that, if we build the core for a power $\gamma^{k_1}$ and $\tilde{f}^{k_2}$ instead, then taking its intersection with its images by $\gamma, \ldots, \gamma^{k_1-1}$ and $\tilde{f}, \ldots, \tilde{f}^{k_2-1}$ will give a core with the properties we want. So, in this proof we may take any finite power of $\gamma$ or $\tilde{f}$.

Thus, if $\delta_0$ is a $p$-prong, we replace $\gamma$ by a power if necessary, so that $\gamma$ fixes every prongs of $\delta$. Furthermore, we take a power of $f$ so that for any $L$, $\tilde{f}(L)$ is above $\gamma(L)$. For notations sake, we assume this is the original $f$.

As in Lemma 8.7 for any leaf $L$ in $\tilde{F}$, we write $x_L$ to be the (unique) intersection of $\delta$ with $L$.

Let $a_i^L$, with $i = 1, \ldots, p$, be all the ideal points on the boundary at infinity of $L$ of the weak unstable leaf (of $\tilde{\Phi}$) through $\delta$, where $p = 2$ if $\delta$ is a regular orbit and otherwise $\delta$ is a $p$-prong orbit. Equivalently, $a_i^L$ is the ideal point determined by each ray of $G_u(x_L)$.

Similarly, we define $r_i^L$, $i = 1, \ldots, p$, to be the ideal ends of the rays of $G_s(x_L)$.

For every $L \in \tilde{F}$ and for every $i$, we choose $P_i^L$ and $N_i^L$ neighborhoods (in $L \cup \partial_\infty L$) of, respectively $a_i^L$ and $r_i^L$. We also choose these neighborhoods such that their boundary (in $L$) are geodesics for the path metric on $L$. Furthermore, we choose these neighborhoods in such a way that they depend continuously on $L \in \tilde{F}$ and they are $\gamma$-invariant, i.e., $\gamma(P_i^L) = P_{\gamma(L)}^i$ and $\gamma(N_i^L) = N_{\gamma(L)}^i$.

Up to taking the neighborhoods smaller, we assume that for any $L$ and any $i, j$, $P_i^L \cap N_j^L = \emptyset$; and for any $i \neq j$, $P_i^L \cap P_j^L = \emptyset$, $N_i^L \cap N_j^L = \emptyset$.

We define a map $\tau_f: \tilde{M} \to \tilde{M}$ in the following way: For any $L$ in $\tilde{F}$ and any $x \in L$, $\tau_f(x)$ is the intersection of the orbit of $\tilde{\Phi}$ through $x$ with $\tilde{f}(L)$.

Let $R_1$ be the constant given by Lemma 8.8 (i.e., such that $\tilde{f}$ and $\tau_f$ are $R_1$-close). According to Lemma 8.7, we can choose the neighborhoods $P_i^L$ and $N_i^L$ sufficiently small so that:

![Figure 11. The image of a large tubular neighborhood of the lift of the prong by $\tilde{f}$ in a given center stable leaf.](image)
(i) For any $L$ and any $i$,
$$
\tau_f(\tilde{P}_L^i) \subset P_{f(L)}^i \quad \text{and} \quad d_{f(L)}\left(\tau_f(\tilde{P}_L^i), \partial P_{f(L)}^i\right) > 10R_1.
$$
(ii) For any $L$ and any $i$,
$$
\tau_f^{-1}(N_L^i) \subset N_{f^{-1}(L)}^i \quad \text{and} \quad d_{f^{-1}(L)}\left(\tau_f^{-1}(N_L^i), \partial N_{f^{-1}(L)}^i\right) > 10R_1.
$$

To make sense of the distance between sets in $L \cup \partial_\infty L$ above, we decide that ideal points are at infinite distance from any other point. A direct consequence of the conditions above is that

1. For any $L$ and any $i$, $\tilde{f}(P_L^i) \subset P_{\tilde{f}(L)}^i$.
2. For any $L$ and any $i$, $\tilde{f}^{-1}(N_L^i) \subset N_{\tilde{f}^{-1}(L)}^i$.

Lemma 8.8 shows that, for any $L$, the maps $\tau_f|_L$ and $\tilde{f}|_L$ are a finite distance from each other. Thus their extension to the circles at infinity $\partial_\infty L$ is the same. Now, recall that $\tau_f$ corresponds to flowing along the pseudo-Anosov flow $\tilde{\Phi}$. Hence, up to replacing $\tilde{f}$ by a very high power of $\tilde{f}$, we can moreover assume that:

3. For any $L$ and $i$, if $\omega$ is an ideal endpoint of $\partial N_L^i$, then $\tilde{f}(\omega) \in P_{\tilde{f}(L)}^i$, for some $j$ (where $j$ is the unique index such that $a_j^L$ is the first attractor on the side of $\omega$ from $r_k$);
4. For any $L$ and $i$, if $\omega$ is an ideal endpoint of $\partial P_L^i$, then $\tilde{f}^{-1}(\omega) \in N_{\tilde{f}^{-1}(L)}^j$, for some $j$ (where $j$ is uniquely determined as above).

Note that conditions (1) and (2) still are satisfied by our high power of $\tilde{f}$.

Now we choose a constant $R$ large enough so that for every $L$ and every $i$, the ball $D_L := B(x_L, R)$, of radius $R$ around $x_L$, intersects every $P_L^i$. Moreover, we choose it to satisfy:

$$
D_L \supset \tilde{f}\left(\partial N_{\tilde{f}(L)}^i\right) \setminus \bigcup_j P_L^j
$$

$$
D_L \supset \tilde{f}^{-1}\left(\partial P_{\tilde{f}(L)}^i\right) \setminus \bigcup_j N_L^j
$$

This is possible because the ideal points of $A = \tilde{f}(\partial N_{\tilde{f}(L)}^i)$ are contained in the interior of the ideal boundary of the union of the $P_L^i$. It follows that for each $L$ only a compact part of $A \cap L$ is outside the union, and this varies continuously with $L$. By choosing $R$ big enough one satisfies the equations above.

Let

$$
V := \bigcup_{L \in \tilde{F}} D_L.
$$

We will show that the set $\bigcap_{n \in \mathbb{Z}} \tilde{f}^n(V)$ is non-empty and thus its projection to $M_\gamma$ is the core $T_\gamma$ that we seek.

The proof will be done by induction. In order to make that induction work, we need the following

**Claim 8.9.** Let $L$ be a leaf in $\tilde{F}$. Let $C \subset D_L$ be any compact and path-connected set that does not intersect any $N_L^i$. 


If there exists \( i_1, i_2 \) distinct such that \( C \) intersect both \( P_{i_1}^L \) and \( P_{i_2}^L \), then there exists a path-connected component of \( \tilde{f}(C) \cap D_{\tilde{f}(L)} \) that intersects \( P_{i_1}^L \) and \( P_{i_2}^L \), for some \( j_2 \neq i_1 \) (\( j_2 \) is not necessarily \( i_2 \)) and that does not intersect any \( N_{i_1}^L \).

**Proof.** Since \( C \) intersects \( P_{i_1}^L \) and \( P_{i_2}^L \), \( \tilde{f}(C) \) also intersects both \( P_{i_1}^L \) and \( P_{i_2}^L \) (thanks to the condition (1)).

Now, since \( C \) does not intersect any \( N_{i_1}^L \), because of condition (3) and the choice of \( D_L \), the intersections of \( \tilde{f}(C) \) with \( \partial D_{\tilde{f}(L)} \) are contained in the union of the \( P_{i_1}^L \). Thus, as claimed, \( \tilde{f}(C) \cap D_{\tilde{f}(L)} \) contains a connected component that intersects \( P_{i_1}^L \) and \( P_{i_2}^L \), for some \( j_2 \neq i_1 \) and that does not intersect any \( N_{i_1}^L \). \( \square \)

Figure 12 shows a case where \( j_2 \) is not equal to \( i_2 \): It may be that \( \tilde{f}(C) \) stretches well into \( P_{i_2}^L \) and out of \( D_{\tilde{f}(L)} \). Thus, as in the figure, the intersection \( \tilde{f}(C) \cap D_{\tilde{f}(L)} \) can have two components \( C_1 \) and \( C_2 \), neither of which intersects both \( P_{i_1}^L \) and \( P_{i_2}^L \).

![Figure 12](image_url)

**Figure 12.** The intersection \( \tilde{f}(C) \cap D_{\tilde{f}(L)} \) may not have a connected set joining \( P_{i_1}^L \) to \( P_{i_2}^L \).

The same proof as above, using \( \tilde{f}^{-1} \) instead (and the conditions (2) and (4)), gives

**Claim 8.10.** Let \( L \) be a leaf in \( \tilde{F} \). Let \( C \subset D_L \) be any compact and path-connected set that does not intersect any \( P_{i}^L \).

If there exists \( i_1, i_2 \) distinct such that \( C \) intersect both \( N_{i_1}^L \) and \( N_{i_2}^L \), then there exists a path-connected component of \( \tilde{f}^{-1}(C) \cap D_{\tilde{f}^{-1}(L)} \) that intersects \( N_{i_1}^L \) and \( N_{i_2}^L \), for some \( j_2 \neq i_1 \) and that does not intersect any \( P_{i}^L \).
For any leaf \( L \) and any integer \( n \geq 0 \), define
\[
R^n_L = \bigcap_{k=0}^{n} \tilde{f}^k(D_{\tilde{f}^{-k}(L)}) \quad \text{and} \quad Q^n_L = \bigcap_{k=0}^{n} \tilde{f}^{-k}(D_{\tilde{f}^{k}(L)}).
\]

We will show:

**Claim 8.11.** For every \( i \) and every \( n \), \( R^n_L \) contains a subset \( C \), compact and path-connected that does not intersect any \( \mathcal{N}^i_L \) but does intersect \( \mathcal{P}^i_L \) and some \( \mathcal{P}^{i_2} \) (for some \( i_2 \neq i \)).

Similarly, for every \( i \) and every \( n \), \( Q^n_L \) contains a subset \( C \), compact and path-connected that does not intersect any \( \mathcal{P}^j_L \) but does intersect \( \mathcal{N}^i_L \) and some \( \mathcal{N}^{i_2} \) (for some \( i_2 \neq i \)).

**Proof.** We only do the proof for \( R^n_L \), as the claim for \( Q^n_L \) follows similarly.

First, since \( R^0_L = D_L \), the claim is true for \( n = 0 \) and any leaf \( L \) (because \( D_L \) clearly contains such a subset). Let us assume that the claim holds for \( R^{n-1}_L \) and for any \( L \).

Then, Claim 8.9 implies that (for any \( L \)) \( \tilde{f}(R^{n-1}_L) \cap D_{\tilde{f}(L)} \) has a compact and path-connected subset that does not intersect any \( \mathcal{N}^j_L \) but does intersect \( \mathcal{P}^i_L \) and some \( \mathcal{P}^{i_2} \) (for some \( i_2 \neq i \)).

But, by definition, we have
\[
R^n_L = \bigcap_{k=0}^{n} \tilde{f}^k(D_{\tilde{f}^{-k}(L)}) = \tilde{f}(R^{n-1}_L(\mathcal{L})) \cap D_L.
\]

Thus the claim is proved. \( \square \)

Now, since for any \( L \), the ideal points \( a^i_L \) and \( r^i_L \) alternate, the properties of \( R^n_L \) and \( Q^n_L \) given by Claim 8.11 imply that, for all \( n \), \( R^n_L \cap Q^n_L \) is a non-empty compact set.

Since \( R^n_L \) and \( Q^n_L \) are decreasing sets, the set
\[
T_L := \bigcap_{n \geq 0} (R^n_L \cap Q^n_L)
\]

is (for any \( L \)) non-empty and compact. Thus
\[
T := \bigcup_{L \in \tilde{\mathcal{F}}} T_L
\]
is non-empty, and, by construction, \( \tilde{f} \)-invariant (note also that \( T = \cap_{n \in \mathbb{Z}} \tilde{f}^n(V) \) as we claimed above).

Hence, the projection \( T \gamma \) of \( T \) to \( M \gamma \) is non-empty, compact and \( \tilde{f} \)-invariant.

Once \( T \gamma \) is built, the second half of Proposition 8.1 follows directly from Proposition I.2 together with Lemma 8.8. So we finished the proof of Proposition 8.1. \( \square \)

In the proof of Proposition 8.1, we obtained the following result which we state independently for future reference.

**Lemma 8.12.** Let \( f : M \to M \) be a homeomorphism of a hyperbolic 3-manifold, \( f \) homotopic to the identity. Suppose that \( f \) preserves an \( \mathbb{R} \)-covered uniform foliation \( \mathcal{F} \) and that a good lift \( \tilde{f} \) of \( f \) acts as a translation on the leaf space of \( \tilde{\mathcal{F}} \). Let \( \gamma \in \pi_1(M) \) be a deck transformation.
If $h = \gamma \circ \tilde{f}^n$ fixes some leaf $L \in \tilde{\mathcal{F}}$ (with $n \neq 0$) then the set of fixed points of $h$ in $L$ is contained in a compact subset of $L$.

Moreover, given $n > 0$ big enough, then for every $R > 0$ there is a compact set $D \subset L$ such that if $y \notin D$ then $d_L(y, h(y)) > R$.

Finally, let $P$ be the set of ideal points in the boundary at infinity $S^1(L)$ that are attracting and fixed under the map $\gamma \circ \tau_{12}$, where $\tau_{12}: L \to \tilde{f}^n(L)$ is the flow along $\Phi$ map. Then, for any $y \in P$, there exists a neighborhood $U$ of $y$ in $L \cup S^1(L)$ such that

1. $h(U)$ is strictly contained in $U$, and
2. $\bigcap_{i \geq 0} h^i(U) = \{y\}$.

**Remark 8.13.** Notice that the results of this section should be adaptable to the case of a homeomorphism acting as a translation on the leaf space of a manifold with one atoroidal piece. What would be required is some sort of analogue of Theorem D.3. That is, we would need to know that there exists a transverse regulating flow such that any orbit that stays in the atoroidal piece is a hyperbolic $p$-prong ($p \geq 2$). Although that result seems likely to be true, it has not been proven. A similar context is dealt with in a companion paper [BFFP] where we study integrability for partially hyperbolic diffeomorphisms not homotopic to identity in Seifert manifolds.

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9. **Double translations in hyperbolic manifolds**

In this section we prove Theorem B.

Let $f: M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism of a hyperbolic 3-manifold $M$. Recall that we denote by $W^{cs}$ and $W^{cu}$ a pair of $f$-invariant foliations tangent respectively to $E^{cs}$ and $E^{cu}$. Up to taking an iterate, one has that $f$ is homotopic to identity and therefore has a good lift $\tilde{f}$ to $M$. We fix that good lift.

We want to show that $\tilde{f}$ fixes the leaves of both foliations $\tilde{W}^{cs}$ and $\tilde{W}^{cu}$. By Theorem 6.1 this is enough to prove Theorem B. Notice that by Corollary 3.21 and Theorem 5.1 we can assume by contradiction that both foliations are $\mathbb{R}$-covered and uniform and that $\tilde{f}$ acts as translation on both leaf spaces.

We in fact will get a contradiction using just one of the translations thanks to the Proposition 8.1, together with the following result. Notice that we thus obtain an alternative proof, albeit much more complicated, of the fact that there cannot be a mixed behavior in a hyperbolic manifold. For future reference, we remark that the proof which eliminates mixed behavior on hyperbolic manifold in the non-dynamically coherent case (see section 15) will use the same type of ideas as here.

**Proposition 9.1.** Assume that a good lift $\tilde{f}$ of $f$ acts as a translation on the foliation $\tilde{W}^{cs}$ and let $\Phi$ be a transverse regulating pseudo-Anosov flow for $W^{cs}$. Then, for every $\gamma \in \pi_1(M)$ associated to the inverse of a periodic orbit $\gamma$ of $\Phi$ there is $n > 0, m > 0$ such that $h = \gamma^n \circ \tilde{f}^m$ fixes a leaf $L$ of $\tilde{W}^{cs}$.

By symmetry, the same result holds if applied to $W^{cu}$. Notice that once one knows that $h$ fixes a leaf $L$ of $\tilde{W}^{cs}$, the second part of Proposition 8.1 applies to $f$.

**Proof.** Thanks to Proposition 8.1, we can consider the cover $M_\gamma = \tilde{M}/ < \gamma >$ and let $V$ be a compact solid torus in $M_\gamma$ such that $\bigcap_{n \in \mathbb{Z}} \tilde{f}^n(V) = T_\gamma$ is compact and far from $\partial V$. 

---
Let \( z \in T_\gamma \). Let \( y \in T_\gamma \) be an accumulation point of \( \left( \hat{f}_\gamma^n(z) \right) \).

Take \( i, j \) big enough, with \( j \) much bigger than \( i \), such that \( \hat{f}_\gamma^i(z) \) and \( \hat{f}_\gamma^j(z) \) are both very close to \( y \).

Consider a small closed unstable segment containing \( \hat{f}_\gamma^i(z) \) in its interior.

Since \( \hat{f}_\gamma^j \circ \hat{f}_\gamma^{-i} \) increases the unstable length, every leaf of \( \hat{W}_c^u \text{ through } t \) intersects the interior of \( \hat{f}_\gamma^{-i}(t) \). This set of \( \hat{W}_c^u \) leaves is an interval. This produces a fixed \( \hat{W}_c^u \) leaf under \( \hat{f}_\gamma^j \circ \hat{f}_\gamma^{-i} \). Lifting to \( \tilde{M} \) proves the proposition. \( \square \)

We can now finish the proof of Theorem B.

\textit{Proof of Theorem B.} Let \( \tilde{f} \) be a good lift of \( f \) and let \( L_0 \) be a leaf fixed by \( h := \gamma \circ \tilde{f}^k \) for some \( k > 0 \) and \( \gamma \in \pi_1(M) \setminus \{ \text{id} \} \) given by Proposition 9.1.

For any leaf \( L \) fixed by \( h \), the map \( h \) has negative Lefschetz index (according to Proposition 8.1). Thus there exists a point \( x_L \in L \) fixed by \( h \). Now, \( h \) is partially hyperbolic, so any fixed leaf \( L \) is repelling along the unstable manifold through \( x_L \).

But this is impossible, as in the leaf space of \( \tilde{W}_c^u \), the closed interval between \( L_0 \) and \( \gamma(L_0) \) is fixed so cannot contain only repelling fixed points.

This contradiction implies that \( \tilde{f} \) cannot act as a translation on either leaf spaces. It follows that \( \tilde{f} \) has to fix every center stable and center unstable leaf. Therefore by Theorem 6.1, it is conjugate to a discretized Anosov flow. This proves Theorem B. \( \square \)

Part 2. The general case

10. Branching foliations and leaf spaces

Since many non dynamically coherent partially hyperbolic examples have been constructed in recent years, we cannot assume dynamical coherence. The role of the foliations we used in Part 1, will then be replaced by branching foliations, that were constructed by Burago and Ivanov ([BI08], see also [HP18]) for general partially hyperbolic diffeomorphisms under some orientability conditions.

\textbf{Remark 10.1.} Notice that the term \textit{branching} is sometimes used with a different meaning in the study of codimension one foliations (to describe non-separated leaves in the leaf space). Here, branching means that two leaves may merge (and this is irrespective of whether the leaf space in \( \tilde{M} \) is Hausdorff or not).

We start with a proper definition and refer the reader to [HP18] for a detailed explanation on this tool as well as contexts where they are used.

\textbf{Definition 10.2.} A \textit{branching foliation} \( \mathcal{F}_{\text{bran}} \) of a 3-manifold \( M \) is a collection of \( C^1 \)-immersed surfaces complete for the pull-back metric and satisfying:

(i) Every point \( x \in M \) belongs to at least one surface (called leaf) of \( \mathcal{F}_{\text{bran}} \);
(ii) An immersed leaf of \( \mathcal{F}_{\text{bran}} \) does not topologically cross itself;
(iii) Different leaves of \( \mathcal{F}_{\text{bran}} \) do not topologically cross;
(iv) If \( L_n \) are leaves of \( \mathcal{F}_{\text{bran}} \) and \( x_n \in L_n \) is a sequence that converges to \( x \), then, up to taking a subsequence, \( L_n \) converges to a leaf \( L^k \) of \( \mathcal{F}_{\text{bran}} \) with \( x \in L \).

\textsuperscript{6}Here convergence should be understood in the pointed compact-open topology, i.e., given a compact set \( K \) in \( L \) containing \( x \), there is a sequence of compact subsets \( K_n \) of \( L_n \) containing \( x_n \) such that \( K_n \) converges to \( K \) in the Hausdorff topology and \( x_n \) converges to \( x \).
Moreover, we say that a branching foliation is well-approximated by foliations if there exists a family of foliations $\mathcal{F}_\epsilon$, with $C^1$ leaves, and a family of continuous maps $h_\epsilon : M \rightarrow M$, with $\epsilon > 0$, such that, for a fixed Riemannian metric, we have:

(v) The angle between a leaf of $\mathcal{F}_{\text{bran}}$ and $\mathcal{F}_\epsilon$ is less than $\epsilon$;

(vi) The map $h_\epsilon$ is at $C^0$-distance less than $\epsilon$ from the identity;

(vii) The map $h_\epsilon$ maps leaves of $\mathcal{F}_\epsilon$ to leaves of $\mathcal{F}_{\text{bran}}$ by a local diffeomorphism (so in particular, the restriction of $h_\epsilon$ to any leaf is $C^1$);

(viii) For every leaf $L$ of $\mathcal{F}_{\text{bran}}$, there exists a leaf $L_\epsilon$ of $\mathcal{F}_\epsilon$ such that $h_\epsilon(L_\epsilon) = L$.

Notice that, as a branching foliation has $C^1$ leaves and that all possible intersections are not topological crossings, it makes sense to talk about the tangent distribution to a branching foliation.

**Remark 10.3.** When $\mathcal{F}_{\text{bran}}$ is a branching foliation but not a true foliation, then the map $h_\epsilon$ is never a local diffeomorphism, even though it restricts to a local diffeomorphism on each leaf: There are open sets where leaves are collapsed transversely by $h_\epsilon$. In fact, even when restricted to a leaf, it may fail to be a global diffeomorphism as leaves of $\mathcal{F}_{\text{bran}}$ can self intersect, forming branching locus.

As is the case with foliations, there exists a small enough scale at which the branching foliation is “trivially product (branched) foliated”. Let us be more precise: We fix a Riemannian metric. Then there exists $\epsilon_0 > 0$, such that any open set $B$ of diameter less than $\epsilon_0$ satisfies the following. The set $B$ is contained in a smooth chart $\mathbb{D}^2 \times [0, 1]$ such that the local leaves of $\mathcal{F}_{\text{bran}}$ through $B$ intersects the chart in sets transverse to the $[0, 1]$-fibration in $\mathbb{D}^2 \times [0, 1]$, each local leaf intersects every $[0, 1]$-fiber and they are close to being horizontal. This fact readily follows from the fact that the branching foliation are tangent to a continuous distribution.

We call the scale $\epsilon_0 > 0$ above the local product structure size.

The fundational result of Burago and Ivanov states that, under some orientability conditions, a partially hyperbolic diffeomorphism always admits a pair of branching foliations tangent to the center stable and center unstable distributions. We naturally say that a branching foliation is $f$-invariant if the image of any leaf by $f$ is again a leaf.

**Theorem 10.4 (Burago-Ivanov [BI08]).** Let $f$ be a partially hyperbolic diffeomorphism of a 3-manifold $M$. Suppose that the bundles $E^s$, $E^u$ and $E^c$ are orientable and that $Df$ preserves these orientations.

Then there exists two $f$-invariant branching foliations $\mathcal{W}^c_{\text{bran}}$ and $\mathcal{W}^u_{\text{bran}}$ tangent respectively to $E^c_{\text{bran}}$ and $E^u_{\text{bran}}$. Moreover, these branching foliations are well-approximated by foliations $\mathcal{W}^c_{\epsilon}$ and $\mathcal{W}^u_{\epsilon}$, with associated maps denoted by $h^c_{\epsilon}$ and $h^u_{\epsilon}$.

The collections of surfaces $\mathcal{W}^c_{\text{bran}}$ and $\mathcal{W}^u_{\text{bran}}$ are called the center stable and center unstable branching foliations.

There is one property that the center stable and center unstable branching foliations have which will be very useful to us: Since the stable bundle $E^s$ is uniquely integrable, if a point $p$ is in a center stable leaf $L$, then the entire stable leaf $s(p)$ through $p$ is also contained in $L$. As a consequence intersections between distinct center stable leaves are saturated by stable leaves.

**Remark 10.5.** Since the manifolds we consider in this article are not virtually solvable, Theorem F.1 implies that no leaf of the branching foliation, hence no
leaf of the approximating foliation, is compact. Thus the approximating foliations $\mathcal{W}^{cs}_\varepsilon$ and $\mathcal{W}^{cu}_\varepsilon$ are always taut.

Using branching foliations, we can still define center leaves:

**Definition 10.6.** A center leaf $c$ of a partially hyperbolic diffeomorphism is the projection to $M$ of a connected component of the intersection between a leaf of the central stable branching foliation $\tilde{\mathcal{W}}^{cs}_{\text{bran}}$ and a leaf of the central unstable branching foliation $\tilde{\mathcal{W}}^{cu}_{\text{bran}}$ (lifts to $\tilde{M}$).

Even though the collection of center leaves is not a foliation, we will also define a leaf space of center leaves in section 10.1.

**Remark 10.7.** Notice that a center leaf $c$ is automatically tangent to the central direction $E^c$. However, complete curves that are tangent to the central direction may fail to be center leaves for our definition. Indeed, even when the diffeomorphism is dynamically coherent, the central direction may not be uniquely integrable, thus, some complete curves may be tangent to $E^c$, but are not the intersection of a central stable and central unstable (such an example is constructed in [RHRHU16]).

![](image_url)

**Figure 13.** The branching of center and center-stable leaves

10.1. **Leaf Spaces.** When $\mathcal{F}$ is a foliation, the leaf space of $\mathcal{F}$ is the collection of distinct leaves of the lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to $\tilde{M}$. Moreover, it comes naturally equipped with a quotient topology. Indeed, the leaf space of $\mathcal{F}$ can be defined as the set $\tilde{M}$ quotiented by the relation “being on the same leaf of $\tilde{\mathcal{F}}$”.

When $\mathcal{F}$ is a branching foliation, we want to define the leaf space again as the collection of distinct leaves of the lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to $\tilde{M}$. However, this space does not necessarily come from a quotient. Indeed, some points $x \in \tilde{M}$ may belong to more than one (in which case $x$ belongs to uncountably many) distinct leaves, thus one cannot define a quotient projection from $\tilde{M}$.

In the next three sections, we will explain how to put a topology on the leaf spaces of each of the branching foliations. More importantly, we show that these topologies make the leaf spaces of the branching foliations homeomorphic to those of the approximating foliations, for small enough $\varepsilon$.

10.1.1. **Leaf spaces of the center stable and center unstable foliations.** Recall that, by Theorem 10.4, the branching foliations $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ are well-approximated by foliations $\mathcal{W}^{cs}_\varepsilon$ and $\mathcal{W}^{cu}_\varepsilon$. Now property (viii) of Definition 10.2 implies that for $\varepsilon$ sufficiently small (which is assumed from now on), there is a canonical surjection between the leaf spaces of $\mathcal{W}^{cs}_\varepsilon$ and $\mathcal{W}^{cs}_{\text{bran}}$ and the leaf spaces of $\mathcal{W}^{cu}_\varepsilon$ and $\mathcal{W}^{cu}_{\text{bran}}$. 
It is possible to modify the proof of [BI08, Theorem 7.2], where the foliations \(W^{cs}_b\) and the map \(h^{cs}_b\) are constructed, so that the map between leaf spaces given by \(h^{cs}_b\) is also injective. With this result on hand, we could define the topology on the leaf space of \(\widetilde{W}^{cs}_{\text{bran}}\) as the one making that map a homeomorphism. However, proving the injectivity would require to redo the whole proof of [BI08, Theorem 7.2]. So instead, we use a simpler fact which can be easily extracted from the proof of [BI08, Theorem 7.2]: The map \(h^{cs}_b\) is “monotone” meaning that, in local charts, where there is a well defined linear order between leaves, this order is preserved by \(h^{cs}_b\).

**Definition 10.8.** We denote by:

- \(\mathcal{L}^{cs}_b\) the leaf space of the center stable branching foliation \(\widetilde{W}^{cs}_{\text{bran}}\);
- \(\mathcal{L}^{cu}_b\) the leaf space of the center unstable branching foliation \(\widetilde{W}^{cu}_{\text{bran}}\);
- \(\mathcal{L}^{cs}_c\) the leaf space of the approximating center stable foliation \(\widetilde{W}^{cs}_c\);
- \(\mathcal{L}^{cu}_c\) the leaf space of the approximating center unstable foliation \(\widetilde{W}^{cu}_c\).

Furthermore, we denote the surjections between the leaf spaces of the branching foliations and the approximating foliations by

\[
g_{cs}: \mathcal{L}^{cs}_b \to \mathcal{L}^{cs}_c, \quad \text{and} \quad g_{cu}: \mathcal{L}^{cu}_c \to \mathcal{L}^{cu}_b.
\]

Since \(\mathcal{W}^{cs}_c\) is a true foliation, its leaf space \(\mathcal{L}^{cs}_c\) has a natural topology making it a simply connected, but perhaps non Hausdorff, 1-manifold\(^7\).

Each leaf \(L\) of \(\widetilde{W}^{cs}_{\text{bran}}\) is a properly embedded plane in \(\widetilde{M}\). Using this one defines as before \(L^+\) to be the closure of the connected component of \(\widetilde{M} \setminus L\) on the “positive side of \(L^+\), and similarly for \(L^-\). To define positive side pick an orientation to the unstable bundle in \(\widetilde{M}\).

**Topology of \(\mathcal{L}^{cs}_b\).** The topology in \(\mathcal{L}^{cs}_b\) is defined as follows: Consider a finite collection of transversals \(\tau_i\) to \(\mathcal{W}^{cs}_{\text{bran}}\) such that:

(i) Each transversal \(\tau_i\) is open.

(ii) \(\tau_i\) is perpendicular to \(E^{cs}\) everywhere.

(iii) Every leaf of \(\mathcal{W}^{cs}_{\text{bran}}\) intersects at least one of the \(\tau_i\).

Let \(\beta\) be a lift to \(\widetilde{M}\) of some \(\tau_i\). Consider the collection of leaves of \(\widetilde{W}^{cs}_{\text{bran}}\) intersecting \(\beta\). Each such leaf of \(\widetilde{W}^{cs}_{\text{bran}}\) is a properly embedded plane and intersects \(\beta\) only once.

**Claim 10.9.** Let \(x \in \beta\). Let \(I\) be the collection of leaves \(L\) intersecting \(x\). Then \(I\) is a singleton or order isomorphic to a closed interval.

**Proof.** Suppose that \(I\) is not a singleton. Then, given any leaves \(L \neq E\) in \(I\), either \(L \subset E^+\) or \(E \subset L^+\) and only one option occurs (this is thanks to property (iii) of Definition 10.2). We say \(L > E\) in the first case and \(L < E\) in the second case, which gives a total order on \(I\). By property (iv), this order is complete. Moreover, there are no gaps in this order: Let \(L \neq E\) two leaves in \(I\) such that \(L < E\). We want to show that there exists a leaf \(L' \in I\), with \(L < L' < E\). Let \(y\) be a boundary point of the connected component of \(L \cap E\) containing \(x\). Then consider a neighborhood \(B\) of \(y\) of diameter smaller than \(\varepsilon_0\), the local product structure size of the branching foliation \(\mathcal{W}^{cs}_{\text{bran}}\). Since \(\widetilde{W}^{cs}_{\text{bran}}\) has a trivially product foliated structure in \(B\), every leaf that intersects \(B \cap (L^+ \cap E^-)\) must intersect

\(^7\text{This is given by Theorem B.1 since, as explained in Remark 10.5, the approximating foliations are taut.}\)
In particular this shows that contained in of \( \tilde{g} \) transversal \( \beta \) we can assume that \( J \) is the set of branching leaves that intersects a lift \( \beta \) of \( \tau_1 \) makes it homeomorphic to an open interval in \( \mathbb{R} \).

So \( I \) is order isomorphic to a closed interval in \( \mathbb{R} \). 

The claim implies that putting the order topology on the set of leaves of \( \mathcal{W}_{bran}^{cs} \) intersecting a lift \( \beta \) of \( \tau_1 \) makes it homeomorphic to an open interval in \( \mathbb{R} \).

Notice the following: suppose that \( \beta_1, \beta_2 \) are lifts of \( \tau_1, \tau_2 \), and \( L, E \) are leaves of \( \mathcal{W}_{bran}^{cs} \) intersecting both \( \beta_1, \beta_2 \). Then the order induced by \( \beta_1 \) is the same as the order induced by \( \beta_2 \) (in the set of leaves intersecting both transversals). Hence the order topology is well defined when there are intersections.

**Definition 10.10** (topology of \( \mathcal{L}_b^{cs} \)). The topology \( \mathcal{T} \) in \( \mathcal{L}^{cs}_b \) is the one generated by the open intervals defined above. This topology makes \( \mathcal{L}^{cs}_b \) a simply-connected 1-manifold.

**Proposition 10.11.** For \( \epsilon \) small enough (smaller than the local product sizes of \( \mathcal{W}^{cs}_{bran} \) and \( \mathcal{W}^{cu}_{bran} \)), the preimage of a point in \( \mathcal{L}^{cs}_b \) (resp. \( \mathcal{L}^{cu}_b \)) by \( g_{\epsilon,s} \) (resp. \( g_{\epsilon,au} \)) is a closed interval. Moreover, the space \( \mathcal{L}^{cs}_\epsilon \) (resp. \( \mathcal{L}^{cu}_\epsilon \)) is homeomorphic to \( \mathcal{L}^{cs}_b \) (resp. \( \mathcal{L}^{cu}_b \)). The maps \( g_{\epsilon,s} : \mathcal{L}^{cs}_\epsilon \to \mathcal{L}^{cs}_b \) are continuous.

**Proof.** We work with \( \mathcal{L}^{cs}_b \) as the proof for \( \mathcal{L}^{cu}_b \) is identical. The key property is to show that the preimage by \( g_{\epsilon,s} \) of points are closed intervals in the leaf space \( \mathcal{L}^{cs}_\epsilon \), the rest will follow rather easily.

We let \( \mathcal{T}_c \) be the quotient topology induced by \( g_{\epsilon,s} \) on \( \mathcal{L}^{cs}_b \). Our goal is to show that \( \mathcal{T}_c = \mathcal{T} \).

Let \( \epsilon_0 \) be the local product sizes of \( \mathcal{W}^{cs}_{bran} \). Let \( \epsilon < \epsilon_0 / 2 \).

It is in order to prove this proposition that we will use the remark made above that the map \( h^{cs}_\epsilon \) is monotone\(^8\).

Let \( I \) be the preimage of a leaf \( L \in \mathcal{L}^{cs}_b \). Suppose that \( I \) contains two leaves \( \hat{L}_1 \) and \( \hat{L}_2 \), we want to show that every leaf in between \( \hat{L}_1 \) and \( \hat{L}_2 \) is mapped by \( \tilde{h}^{cs}_\epsilon \) to \( L \). From property (vi) of Definition 10.2, we have that the Hausdorff distance between \( \hat{L}_1 \) and \( \hat{L}_2 \) is \( < 2\epsilon \). Now, as \( 2\epsilon \) is chosen smaller than the local product structure size \( \epsilon_0 \), it follows that the region between the leaves \( \hat{L}_1 \) and \( \hat{L}_2 \) has leaf space which is a closed interval (cf. §3.1.1).

Because of the property of monotonicity of \( \tilde{h}^{cs}_\epsilon \) it follows that \( g_{\epsilon,s} \) maps the region between \( \hat{L}_1 \) and \( \hat{L}_2 \) to \( L \). This implies that the preimage of \( L \) is an interval. It remains to show that it is closed, but this is just a consequence of the continuity of \( h^{cs}_\epsilon \).

So the preimage of any point is a closed interval. We now proceed with proving the other needed properties.

Let \( J \) be an open interval \( J \) in \( \mathcal{L}^{cs}_b \) for the topology \( \mathcal{T} \). Up to taking \( J \) smaller, we can assume that \( J \) is the set of branching leaves that intersects a small open transversal \( \beta \). We want to show that \( g^{-1}_{\epsilon,s}(J) \) is open in \( \mathcal{L}^{cs}_\epsilon \). Let \( \hat{L}_1 \) be a leaf in \( g^{-1}_{\epsilon,s}(J) \). Then \( \hat{L}_1 \) intersects \( \beta \) (or a slightly bigger transversal), so all the leaves of \( \mathcal{W}^{cs}_\epsilon \) close enough to \( \hat{L}_1 \cap \beta \) intersect \( \beta \). Thus an open neighborhood of \( \hat{L}_1 \) is contained in \( g^{-1}_{\epsilon,s}(J) \).

Hence the interval \( J \) is also open in the topology \( \mathcal{T}_c \). It follows that \( \mathcal{T} \subset \mathcal{T}_c \). In particular this shows that \( g_{\epsilon,s} \) is continuous.

---

\(^8\)Otherwise, the preimage could be disconnected. One can recover the rest of the statements, but that would need to construct new maps \( \tilde{g}_{\epsilon,s} \) by collapsing closed intervals in both spaces and see that these induce the same topology.
Now for the other inclusion. Suppose $W$ is an open set in $\mathcal{T}$ and $y$ is in $W$. Hence $(g_{\epsilon,s})^{-1}(W)$ is open and contains $(g_{\epsilon,s})^{-1}(y)$, which is an interval $I$ with boundary leaves $L, E$. Since $(g_{\epsilon,s})^{-1}(W)$ is open, it contains and interval of leaves around, say, $L$. Consider the part of this interval made up of $L$ and the side outside $(g_{\epsilon,s})^{-1}(y)$. This projects to an interval in $L_b^c$, which is not just $y$ by definition of $I$. Hence $W$ contains an open interval around $y$, and therefore $W$ is open in $\mathcal{T}$. This shows that $\mathcal{T} = \mathcal{T}_\epsilon$.

We already proved that the preimage of a point in $L_b^c$ is a a closed interval in $L^c$. This implies that $L_b^c$, $L^c$ are homeomorphic. This is because the only collapsing from $L^c$ to $L_b^c$ is done along closed intervals $I$. If $L, E$ are the endpoints of $I$, then there is no other leaf in the region between $L$ and $E$ besides those leaves that are in $I$.

This finishes the proof of the proposition. □

Notice that the leaf spaces $L_b^c, L^c$ are homeomorphic, however the natural map $g_{\epsilon,s} : L^c \to L_b^c$ is not necessarily a homeomorphism, as it may collapse points. In the sequel, we fix some $\epsilon$ small enough so that the previous proposition applies.

10.1.2. Leaf spaces of the center foliation in a center stable or center unstable leaf. We now define a topology on the leaf space of the center branching foliation, restricted to a particular center stable or center unstable leaf.

Remark 10.12. Recall from Definition 10.6 that a center leaf in $\tilde{M}$ is defined as a connected component of the intersection between a leaf of $\tilde{W}_{cu}^{cs} bran$ and a leaf of $\tilde{W}_{bran}^{cu}$. Now, the following situation may arise (see Figure 14): Two leaves $U_1, U_2$ of $\tilde{W}_{bran}^{cu}$ and a leaf $L$ of $\tilde{W}_{bran}^{cs}$ such that the triple intersection $U_1 \cap L \cap U_2$ contains a connected component of $c_1$ of $U_1 \cap L$ as well as a connected component $c_2$ of $U_2 \cap L$. That is, the center leaves $c_1$ and $c_2$ represents the same set in $\tilde{M}$. In this case, we also consider $c_1$ and $c_2$ as the same leaf of the center foliation in $L$.

![Figure 14](image_url)

**Figure 14.** Different center unstable leaves may intersect a given center stable leaf in the same center leaf.

We will describe the topology of the center leaf space $L^c_L$ on a given leaf $L$ of $\tilde{W}_{bran}^{cs}$. The center leaf space $L^c_U$ on a leaf $U$ of $\tilde{W}_{bran}^{cu}$ is defined in the same manner, so we do not explicit it.
Definition 10.13 (topology \( \mathcal{A} \) in \( \mathcal{L}^c_{\epsilon} \)). Consider a countable set of open transversals \( \tau_i \) which are perpendicular to the center bundle in \( L \), and so that the union intersects every center leaf in \( L \). Put the order topology in the set \( I_i \) of center leaves intersecting \( \tau_i \). This induces the topology \( \mathcal{A} \) in \( \mathcal{L}^c_{\epsilon} \).

Let \( L \) be a fixed leaf of \( \mathcal{W}^{cs}_{\text{bran}} \). We again fix an \( \epsilon > 0 \) and consider the approximating foliation \( \mathcal{W}^{cu}_{\epsilon} \). Since \( \mathcal{W}^{cu}_{\epsilon} \) is transverse to \( L \), so is \( \mathcal{W}^{cs}_{\epsilon} \) (for \( \epsilon \) small enough). Thus, \( \mathcal{W}^{cu}_{\epsilon} \) induces a 1-dimensional (non branching) foliation \( \mathcal{F}_{\epsilon} \) on \( L \), and hence its leaf space \( \mathcal{L}^c_{\epsilon} \) is a 1-dimensional, not necessarily Hausdorff, simply connected manifold.

The behavior described in Remark 10.12 above leads to the following issue: the unique center leaf \( c_1 = c_2 \) is approximated by two distinct leaves of \( \mathcal{F}_{\epsilon} \). Thus, the leaf space, \( \mathcal{L}^c_{\epsilon} \), of the center foliation on \( L \) is not in bijection with \( \mathcal{L}^c_{\epsilon} \). However, we still have a surjective, but not necessarily injective, projection \( \mathcal{A}_{\epsilon} : \mathcal{L}^c_{\epsilon} \rightarrow \mathcal{L}^c_{\epsilon} \) as in the previous subsection. Let \( \mathcal{A}_{\epsilon} \) be the quotient topology from the map \( \mathcal{A}_{\epsilon} \).

Just as in Proposition 10.11 one can prove the following:

Lemma 10.14. The set of center leaves in \( L \) through a point \( x \) is a closed interval. Let \( c_0 \) be a center leaf in \( L \). Let \( I = \mathcal{pr}^{-1}(c_0) \subset \mathcal{L}^c_{\epsilon} \). The set \( I \) is a closed interval. If \( \epsilon < \epsilon_0 \), then the topologies \( \mathcal{A} \) and \( \mathcal{A}_{\epsilon} \) are the same.

10.1.3. Leaf space of the center foliation in \( \tilde{M} \). Finally, we have to put a topology on the leaf space \( \mathcal{L}^c_{0} \) of the center foliation in \( \tilde{M} \).

Pick an \( \epsilon > 0 \), \( \epsilon < \epsilon_0 \) so that \( \mathcal{W}^{cs}_{\epsilon} \) and \( \mathcal{W}^{cu}_{\epsilon} \) are transverse to each other. Call \( \mathcal{F}_{\epsilon} \) the 1-dimensional foliation obtained as the intersection of \( \mathcal{W}^{cs}_{\epsilon} \) and \( \mathcal{W}^{cu}_{\epsilon} \). The leaf space \( \mathcal{L}^c_{\epsilon} \) of \( \mathcal{F}_{\epsilon} \) is now a simply connected, possibly non Hausdorff, 2-dimensional manifold. But as before, there is only a surjective, and not injective, projection \( g_{\epsilon} : \mathcal{L}^c_{\epsilon} \rightarrow \mathcal{L}^c_{0} \).

The map \( g_{\epsilon} \) is defined in the following way: If \( \tilde{c} \) is a leaf of \( \mathcal{F}_{\epsilon} \), then it is the intersection of a leaf \( \tilde{U} \) of \( \mathcal{W}^{cs}_{\epsilon} \) and a leaf \( \tilde{S} \) of \( \mathcal{W}^{cu}_{\epsilon} \). Then, there exists a unique connected component of \( \mathcal{L}^c_{\epsilon} \) that is at distance less than \( 2\epsilon \) from \( \tilde{c} \). We define \( g_{\epsilon}(\tilde{c}) = \tilde{c} \).

Once again, the topology \( \mathcal{B}_{\epsilon} \) put on \( \mathcal{L}^c_{0} \) is obtained by identifying elements of \( \mathcal{L}^c_{\epsilon} \) that project to the same element of \( \mathcal{L}^c_{0} \) and taking the quotient topology.

As done is the previous two subsections 10.1.1 and 10.1.2, in order to prove that the topology that we put on \( \mathcal{L}^c_{0} \) makes it a simply connected (not necessarily Hausdorff) 2-manifold, it is enough to show that the preimages of points by \( g_{\epsilon} \) are closed, simply connected sets contained in a local chart of \( \mathcal{L}^c_{\epsilon} \). In order to do that, first notice that \( \mathcal{L}^c_{\epsilon} \) is locally homeomorphic to \( \mathcal{L}^{cs}_{\epsilon} \times \mathcal{L}^{cu}_{\epsilon} \). Indeed, any \( \tilde{c}_0 \in \mathcal{L}^c_{\epsilon} \) is a connected component of \( \tilde{U}_0 \cap \tilde{S}_0 \), with \( \tilde{U}_0 \in \mathcal{L}^{cu}_{\epsilon} \) and \( \tilde{S}_0 \in \mathcal{L}^{cs}_{\epsilon} \). Now, if \( V_{\epsilon} \) is a small enough open interval in \( \mathcal{L}^{cu}_{\epsilon} \) and \( V_{\epsilon} \) is a small enough open interval in \( \mathcal{L}^{cs}_{\epsilon} \), then for any \( \tilde{U} \in V_{\epsilon} \) and \( \tilde{S} \in V_{\epsilon} \), the intersection \( \tilde{U} \cap \tilde{S} \) contains a unique connected component close to \( c_0 \). Using this local homeomorphism, the following lemma will imply that the topology \( \mathcal{L}^c_{0} \) is as we claimed.

Lemma 10.15. Let \( c_0 \) be in \( \mathcal{L}^c_{\epsilon} \). The set \( R = g_{\epsilon}^{-1}(c_0) \) is homeomorphic to a closed rectangle in \( \mathcal{L}^{cs}_{\epsilon} \times \mathcal{L}^{cu}_{\epsilon} \).

Proof. Let \( \tilde{c}_1, \tilde{c}_2 \in R \). Let \( \tilde{U}_1 \) be the leaf in \( \mathcal{L}^{cu}_{\epsilon} \) containing \( \tilde{c}_1 \) and let \( \tilde{S}_2 \) be the the leaf in \( \mathcal{L}^{cs}_{\epsilon} \) containing \( \tilde{c}_2 \). Let \( U_1 = g_{\epsilon,u}(\tilde{U}_1) \) and \( S_2 = g_{\epsilon,s}(\tilde{S}_2) \). Since \( \tilde{c}_1, \tilde{c}_2 \in R \), the center leaf \( c_0 \) is a connected component of \( U_1 \cap S_2 \). Thus \( U_1 \) and \( S_2 \) must intersect and the intersection contains a unique connected component \( \tilde{c}_3 \) at distance at most \( 2\epsilon \) from \( c_0 \).
Now, the proof of Lemma 10.14 shows that $\bar{c}_1$ and $\bar{c}_3$ are two ends of an interval in the leaf space of $\mathcal{F}$, restricted to $\tilde{U}_1$ that is entirely contained in $R$. Similarly, for $\bar{c}_2$ and $\bar{c}_3$ considered as elements of the leaf space of $\mathcal{F}_\epsilon$ restricted to $\tilde{S}_2$. In turns, the arguments of the proof of Lemma 10.14 imply that the set $R$ projects to a closed interval in both $\mathcal{L}_c^{cs}$ and $\mathcal{L}_c^{cu}$, i.e., it is a closed rectangle in $\mathcal{L}_c^{cs} \times \mathcal{L}_c^{cu}$. □

Just as in the previous two sections we can also put a topology $\mathcal{B}$ on $\mathcal{L}_c^b$ directly as follows:

**Definition 10.16.** (topology $\mathcal{B}$ on $\mathcal{L}_c^b$) In $M$ pick a collection of very small open rectangles $R_i$ which are almost perpendicular to the center bundle, and with boundary two arcs in a leaves of $\mathcal{W}^{cs}_{bran}$ and two arcs in leaves of $\mathcal{W}^{cu}_{bran}$. Consider all lifts $\tilde{R}$ of these to $\tilde{M}$. The set of center leaves intersecting $\tilde{R}$ is naturally bijective to an open rectangle and put the topology making this a local homeomorphism. The topology $\mathcal{B}$ is generated by these rectangles.

First we justify why the set of center leaves through $\tilde{R}$ is naturally an open rectangle. Let $L_1, L_2$ be the center stable leaves containing the two arcs in the boundary of $R$, and $U_1, U_2$ be the corresponding center unstable leaves. The set of center stable leaves between $L_1, L_2$ (not including $L_1, L_2$) is naturally ordered isomorphic to an open interval. This was proved in subsection 10.1.1. The same for the center unstable foliation. The product is an open rectangle. The set of center leaves intersecting $\tilde{R}$ is a quotient of this. The sets which are quotiented to a point are compact subrectangles. The proof is the same as the previous lemma. Hence the quotient is naturally a rectangle. In addition if a collection of center leaves intersects two such rectangles $R, R'$, then the identifications in $R$ also produce the same identifications in $R'$ and the order of the center stable and center unstable foliations in the subsets are the same whether in $R$ or $R'$. Hence in the identification, the topologies agree.

Just as in the previous sections one can prove:

**Lemma 10.17.** For $\epsilon < \epsilon_0$, the topologies $\mathcal{B}$ and $\mathcal{B}_\epsilon$ are the same.

The main property is to prove is exactly that of Lemma 10.15. The rest follows just as in the previous subsections.

11. General aspects without assuming dynamical coherence

In this section, $M$ is a closed 3-manifold, with non virtually solvable fundamental group, $f : M \to M$ is a partially hyperbolic diffeomorphism homotopic to the identity, and $\tilde{f}$ is a good lift of $f$ (Definition 2.3). We do not assume that $f$ is dynamically coherent.

We will assume throughout that the stable, center, and unstable bundles are oriented, and that $f$ preserves their orientations. This can be achieved by taking an iterate of $f$ and lifting to a finite cover of $M$. We will deal with the effects of replacing $f$ and $M$ in §12.

With this assumption, Burago-Ivanov’s Theorem 10.4 applies. We denote by $\mathcal{W}^{cs}_{bran}$ and $\mathcal{W}^{cu}_{bran}$ their center stable and center unstable branching foliations, and by $\tilde{\mathcal{W}}^{cs}_{bran}$ and by $\tilde{\mathcal{W}}^{cu}_{bran}$ the corresponding lifts to $\tilde{M}$.

11.1. First arguments. In this section, we will see that many of the results about the foliations from the dynamically coherent case work for branching foliations. From now on, we always assume that the branching foliations $\mathcal{F}_{bran}$ we consider are well-approximated by taut foliations $\mathcal{F}_\epsilon$. 
One of the first things to be careful with is the definition of \( f \)-minimality for a branching foliation. We first define the notion of saturation.

**Definition 11.1.** Let \( \mathcal{F}_{\text{bran}} \) be a branching foliation. A set \( C \subset M \) is \( \mathcal{F}_{\text{bran}} \)-saturated if, for every \( x \in C \), there is a leaf of \( \mathcal{F}_{\text{bran}} \) that contains \( x \) and is contained in \( C \).

Note that this is much weaker than asking that every leaf intersecting \( C \) is contained in \( C \). In particular, our notion of saturation has the peculiar property that the complement of a \( \mathcal{W}_{\text{cs}}^{\text{bran}} \)-saturated set need not be \( \mathcal{W}_{\text{cs}}^{\text{bran}} \)-saturated (see Figure 15). With this in mind, we make the following definition.

**Definition 11.2.** Let \( \mathcal{F}_{\text{bran}} \) be an \( f \)-invariant branching foliation. Then \( \mathcal{F}_{\text{bran}} \) is called \( f \)-minimal if the only \( \mathcal{F}_{\text{bran}} \)-saturated and \( f \)-invariant sets in \( M \) that are closed are the empty set or the whole manifold.

We emphasize here that closed in the above definition is meant as a set in \( M \), not as a set of leaves.

![Figure 15](image)

**Figure 15.** \( L_1 \) and \( L_2 \) are two leaves in \( C \), but the region \( R \) is not in \( C \). Then, in parts of \( R \), all the center stable leaves intersect the branch locus between \( L_1 \) and \( L_2 \), so have parts in \( C \) and parts not in \( C \) (and therefore \( M \setminus C \) is not saturated by center stable leaves).

**Remark 11.3.** Let \( C \) be an \( \mathcal{F}_{\text{bran}} \)-saturated set in \( M \) and \( \tilde{C} = \pi^{-1}(C) \). There are several, in general distinct, sets of leaves in \( \mathcal{L}_{\text{bran}} \), the leaf space of \( \tilde{\mathcal{F}}_{\text{bran}} \), that one can build from \( \tilde{C} \). This stems from the fact that there can be different ways of saturating a given set by leaves of \( \tilde{\mathcal{F}}_{\text{bran}} \).

More precisely, a saturation of \( \tilde{C} \) is a set \( \text{Sat}(\tilde{C}) \subset \mathcal{L}_{\text{bran}} \) such that, for all \( x \in \tilde{C} \), there exists \( L \in \text{Sat}(\tilde{C}) \) such that \( x \in L \) and \( L \subset C \). Such a set is not uniquely defined. However, there is a biggest such set: The full saturation of \( \tilde{C} \) is the set \( \text{FullSat}(\tilde{C}) \subset \mathcal{L}_{\text{bran}} \) defined by, if \( L \in \mathcal{L}_{\text{bran}} \) is such that \( L \subset C \), then \( L \in \text{FullSat}(\tilde{C}) \). Note that the image of both \( \text{Sat}(\tilde{C}) \) and \( \text{FullSat}(\tilde{C}) \) in \( \tilde{M} \) are just \( \tilde{C} \), since \( C \) is \( \mathcal{F}_{\text{bran}} \)-saturated.

Now, it could happen that a set \( C \) is closed in \( M \), but a saturation \( \text{Sat}(\tilde{C}) \) would fail to be closed in \( \mathcal{L}_{\text{bran}} \) (recall that the topology on \( \mathcal{L}_{\text{bran}} \) is defined in section 10.1.1). However, one can easily see that the following is true: The set \( C \) is a closed subset of \( M \) if and only if \( \text{FullSat}(\tilde{C}) \) is a closed subset of the leaf space \( \mathcal{L}_{\text{bran}} \).

A natural but less immediate result (see Lemma F.5) shows that if a saturation \( \text{Sat}(\tilde{C}) \) is closed in \( \mathcal{L}_{\text{bran}} \) and \( C = M \), then \( \text{Sat}(\tilde{C}) = \mathcal{L}_{\text{bran}} \) (so in particular, there is only one closed saturation in that case).
11.1.1. Complementary regions. Let $\mathcal{F}_{\text{bran}}$ be a branching foliation (assumed to be well-approximated by taut foliations) on a manifold $M$ that is not finitely covered by $S^2 \times S^1$. Then $\tilde{M} \simeq \mathbb{R}^3$, and each leaf of $\tilde{\mathcal{F}}_{\text{bran}}$ is a properly embedded plane that separates $\tilde{M}$ into two open balls.

As in §3.1.1, the complementary regions of a leaf $L \in \tilde{\mathcal{F}}_{\text{bran}}$ are the two connected components of $\tilde{M} \setminus L$. For each complementary region $U$ of a leaf $L$, the closure $\overline{U} = U \cup L$ is called a side of $L$.

As in §3.1.1, a coorientation of $\tilde{\mathcal{F}}_{\text{bran}}$ (defined as an orientation of the leaf space of $\tilde{\mathcal{F}}_{\text{bran}}$) determines, for each leaf $L \in \tilde{\mathcal{F}}_{\text{bran}}$, a positive and a negative complementary region which we denote by $L^+ = L^\oplus \cup L$ and $L^- = L^\ominus \cup L$.

To define the region between two leaves, it is best to work in the leaf space $\mathcal{L}_{\text{bran}}$, with the topology defined in §10.1.1. Let $K, L \in \tilde{\mathcal{F}}_{\text{bran}}$ be distinct leaves. Thinking of these as points in the leaf space, $\mathcal{L}_{\text{bran}} \setminus \{K, L\}$ consists of three open connected components. Only one of these components accumulates on both $K$ and $L$ — we call this the open $\mathcal{L}_{\text{bran}}$-region between $K$ and $L$. Its closure in $\mathcal{L}_{\text{bran}}$, which is obtained by adjoining $K$ and $L$, is called the closed $\mathcal{L}_{\text{bran}}$-region between $K$ and $L$.

Note that the subset of $\tilde{M}$ that corresponds to the open $\mathcal{L}_{\text{bran}}$-region between two leaves may not be open. However, the subset of $\tilde{M}$ that corresponds to the closed $\mathcal{L}_{\text{bran}}$-region between two leaves is closed. It is also connected, but its interior may not be. See Figure 16.

11.1.2. Translation-like behavior. Recall that $\mathcal{F}_{\text{bran}}$ is assumed to be well-approximated by taut foliations. Using this, we immediately obtain the Big Half-Space Lemma (Lemma 3.3).

**Lemma 11.4.** Let $L$ be a leaf of $\mathcal{F}_{\text{bran}}$. For any $R > 0$, there exists a ball of radius $R$ contained in each complementary region of $L$.

**Proof.** It suffices to apply Lemma 3.3 to a leaf corresponding to $L$ in the approximating foliation $\mathcal{F}_\epsilon$, and deduce that each complementary region of $L$ contains a ball of radius $R - \epsilon$ for any $R$. \(\square\)

The following is the equivalent of Proposition 3.5. The same proof applies if one considers complementary regions and regions between leaves as subsets of $\tilde{M}$ and $\mathcal{L}_{\text{bran}}$ as appropriate.

**Proposition 11.5.** Let $\mathcal{F}_{\text{bran}}$ be a branching foliation, $f : M \to M$ a diffeomorphism homotopic to the identity and preserving $\mathcal{F}_{\text{bran}}$, and $\tilde{f}$ be a good lift. If $L \in \tilde{\mathcal{F}}_{\text{bran}}$ is not fixed by $\tilde{f}$, then

(1) the closed $\mathcal{L}_{\text{bran}}$-region between $L$ and $\tilde{f}(L)$ is an interval,
(2) $\tilde{f}$ takes each coorientation at $L$ to the corresponding coorientation at $\tilde{f}(L)$, and

(3) the subset of $M$ corresponding to the closed $L_{\text{bran}}$-region between $L$ and $\tilde{f}(L)$ is contained in the closed $2R$-neighborhood of $L$, where $R = \max_{y \in \tilde{M}} d(y, \tilde{f}(y))$.

11.1.3. Uniform and $\mathbb{R}$-covered branching foliations. A branching foliation is once again called $\mathbb{R}$-covered if its leaf space $L_{\text{bran}}$ (see section 10.1.1) is homeomorphic to $\mathbb{R}$. Since the topology on $L_{\text{bran}}$ can be defined as a quotient of the leaf space of the approximating foliations $\tilde{F}_\epsilon$, the branching foliation is $\mathbb{R}$-covered if and only if the approximating one is, for $\epsilon$ small enough.

The definition of a uniform foliation (see Definition D.1) applies without any change to branching foliations. It is immediate to notice that a branching foliation is uniform if and only if the approximating foliations (see Definition 10.2) are uniform.

11.2. The dichotomy. Since Proposition 3.5 apply in the branching foliation context (and so does Lemma 3.6), we therefore also obtain the equivalent of Proposition 3.7.

Proposition 11.6. Let $M$ be a closed 3-manifold that is not finitely covered by $S^2 \times S^1$, $f: M \to M$ a homeomorphism homotopic to the identity that preserves a branching foliation $\tilde{F}_{\text{bran}}$, and $\tilde{f}$ a good lift.

Then the set $\Lambda \subset L_{\text{bran}}$ of leaves that are fixed by $\tilde{f}$ is closed and $\pi_1(M)$-invariant. Moreover, each connected component $I$ of $L_{\tilde{f}} \setminus \Lambda$ is an open interval that $\tilde{f}$ preserves and acts on as a translation, and every pair of leaves in $I$ are a finite Hausdorff distance apart.

In the above proposition, one has to be mindful again that “open” and “closed” refer to the topology on the leaf space $L_{\text{bran}}$ and not the topology on $\tilde{M}$.

From Proposition 11.6, we deduce as in §3.1.4 that, if the foliation is $f$-minimal, we get a dichotomy (Corollary 3.10):

Corollary 11.7. Let $M$ be a closed 3-manifold that is not finitely covered by $S^2 \times S^1$, $f: M \to M$ a homeomorphism homotopic to the identity that preserves a branching foliation $\tilde{F}_{\text{bran}}$, and $\tilde{f}$ a good lift.

If $\tilde{F}_{\text{bran}}$ is $f$-minimal, then either

1. $\tilde{f}$ fixes every leaf of $\tilde{F}_{\text{bran}}$, or

2. $\tilde{F}_{\text{bran}}$ is $\mathbb{R}$-covered and uniform, and $\tilde{f}$ acts as a translation on the leaf space of $\tilde{F}_{\text{bran}}$.

Proof. The proof is the same as that of Corollary 3.10. However, since the distinctions between the topology in the leaf space and that of corresponding sets in $\tilde{M}$ becomes essential, we redo the proof.

Let $\Lambda$ be the set of leaves that are fixed by $\tilde{f}$. Since $\tilde{f}$ commutes with deck transformation, each deck transformation preserves $\Lambda$. In particular, if $I$ is a component of $L \setminus \Lambda$ and $g \in \pi_1(M)$ then one has either $g(I) = I$ or $g(I) \cap I = \emptyset$.

So $\Lambda$ is invariant under $\tilde{f}$ and deck transformations, saturated by $\tilde{F}_{\text{bran}}$ and closed for the topology of $L_{\text{bran}}$ by Proposition 11.6.

Let $\tilde{B}$ be the set of points in $\tilde{M}$ contained in a leaf of $\Lambda$ and let $B = \pi(\tilde{B})$. Since $\Lambda$ is closed in $L_{\text{bran}}$, then $\tilde{B}$ is closed in $\tilde{M}$ and so is $B$ in $M$. In addition $B$ is $f$-invariant. Since $\tilde{F}_{\text{bran}}$ is $f$-minimal, $B$ is either empty or the whole of $M$.

If $B$ is empty, then $\Lambda$ is also empty, so Proposition 11.6 implies that we are in case (2).
Suppose instead that $B = M$, so $\tilde{B} = \tilde{M}$. Then we have to prove that $\Lambda = L_{\text{bran}}$. This follows from the more general Lemma F.5, but the proof in this case is easy so we give it:

Suppose $\Lambda \neq L_{\text{bran}}$. Let $I$ be a connected component of $L_{\text{bran}} \setminus \Lambda$. Let $J$ be the set of points of $\tilde{M}$ contained in a leaf in $I$. The set $I$ is open (in $L_{\text{bran}}$) and $\tilde{f}$ translates leaves in $I$. It follows that the interior in $\tilde{M}$ of $J$ is non-empty. These points in the interior of $J$ are not contained in $\tilde{B}$. This contradicts $\tilde{B} = \tilde{M}$. So $\Lambda = L_{\text{bran}}$ and we are in case (1).

From now on, we stop considering general well-approximated branching foliations and general branching foliations-preserving diffeomorphisms. Instead, we specialize to considering partially hyperbolic diffeomorphisms $f : M \to M$, homotopic to the identity, on a 3-manifold with non virtually solvable fundamental group and that admits a pair of center stable and center unstable branching foliations, $W^{cs}_{\text{bran}}$ and $W^{cu}_{\text{bran}}$.

11.2.1. Fixed points and fixed leaves. The non-existence of fixed points, given in Lemma 3.13, applies almost as stated, but one needs to have a stronger assumption.

**Lemma 11.8.** Let $L$ be a leaf of $W^{cs}_{\text{bran}}$ that is fixed by $\tilde{W}^{cs}_{\text{bran}}$. If, for any $y \in L$ there exists a leaf $L'$ of $W^{cs}_{\text{bran}}$ fixed by $\tilde{f}$ and intersecting the unstable leaf of $y$ in a point different from $y$, then there are no points in $L$ fixed by any non-trivial power of $\tilde{f}$.

**Proof.** The proof is the same as Lemma 3.13: Suppose $x$ was a fixed point of $\tilde{f}^n$, with $n > 0$, on $L$. Then, the unstable leaf through $x$ would intersect some other fixed stable leaf in a point distinct from $x$, and hence contain another fixed point of $\tilde{f}^n$, which is impossible. □

Note that we cannot just use the same condition as in Lemma 3.13, i.e., that $L$ is accumulated by a sequence of leaves $L_n$ fixed by $\tilde{f}$, because the $L_n$ could be a sequence of distinct leaves but that all share a part of $L$. Then, we could not exclude the existence of fixed points in the set $L \cap (\bigcap_n L_n)$ with that proof.

11.3. Good lifts and fixed points. We just showed that a good lift $\tilde{f}$ cannot have fixed (or periodic) points under the assumption that all leaves of $W^{cs}_{\text{bran}}$ are fixed. We will now exclude the existence of fixed or periodic points under a different assumption, namely $f$-minimality.

**Theorem 11.9.** Let $f$ be a partially hyperbolic diffeomorphism homotopic to the identity, and $\tilde{f}$ a good lift. If $W^{cs}_{\text{bran}}$ or $W^{cu}_{\text{bran}}$ is $f$-minimal, then $\tilde{f}$ does not have any periodic point.

**Proof.** We do the proof assuming $W^{cs}_{\text{bran}}$ is the $f$-minimal foliation. Note first that it is enough to show that $\tilde{f}$ has no fixed points. Indeed, for any fixed $n$, $W^{cs}_{\text{bran}}$ is also $f^n$-minimal and $f^n$ is a good lift of $f^n$.

By Corollary 11.7, either $\tilde{f}$ fixes every leaf of $W^{cs}_{\text{bran}}$ or it acts as a translation on $L^c_{\tilde{b}}$. If $\tilde{f}$ acts as a translation on $L^c_{\tilde{b}}$, then it cannot fix any point of $\tilde{M}$. This is because for any leaf $L$ of $W^{cs}_{\text{bran}}$, and $|\tilde{r}|$ big enough $\tilde{f}(L) \cap L = \emptyset$.

On the other hand, if $\tilde{f}$ fixes every leaves of $W^{cs}_{\text{bran}}$, then Lemma 11.8 implies that $\tilde{f}$ does not admit fixed points either. □
A noteworthy corollary of the above result is that a partially hyperbolic diffeomorphism homotopic to the identity that admits a \( f \)-minimal branching foliation cannot have so-called contractible periodic points.

**Definition 11.10.** Let \( g \) be a homeomorphism of a manifold homotopic to the identity. A point \( p \) is a contractible periodic point of \( g \) of period \( n \) if \( g^n(p) = p \) and there exists \( H : M \times [0, 1] \) a homotopy from the identity to \( g \), such that the closed path obtained by concatenation of the paths \( H(p, \cdot), H(g(p, \cdot)), \ldots, H(g^{n-1}(p), \cdot) \) is homotopically trivial.

Notice that if \( p \) is a contractible periodic point of \( g \) of period \( n \) then there exists a good lift \( \tilde{g} \) of \( g \) and a lift \( \tilde{p} \) of \( p \) such that \( \tilde{g}^n(\tilde{p}) = \tilde{p} \). Thus, Theorem 11.9 immediately yields:

**Corollary 11.11.** Let \( f \) be a partially hyperbolic diffeomorphism on a 3-manifold that is homotopic to the identity. Suppose that \( f \) admits a \( f \)-minimal branching center stable or center unstable foliation. Then \( f \) does not admit any contractible periodic points.

Notice that this completes the proof of Theorem 1.6 in the \( f \)-minimal case. For the hyperbolic and Seifert case, the proof is the same once the proof of Proposition 11.14 below is completed.

### 11.4. Fundamental group of leaves of \( \mathcal{W}_{\text{bran}}^{cs}, \mathcal{W}_{\text{bran}}^{cu} \)

The leaves of the branching foliations \( \mathcal{W}_{\text{bran}}^{cs} \) and \( \mathcal{W}_{\text{bran}}^{cu} \) given in Theorem 10.4 are only immersed manifolds. In particular, they may not be injectively immersed. However, in the universal cover, any leaf of \( \mathcal{W}_{\text{bran}}^{cs} \) or \( \mathcal{W}_{\text{bran}}^{cu} \) is a properly embedded plane (cf. section 10.1).

Thus, there might exists some closed loops in a leaf \( C \) of, say, \( \mathcal{W}_{\text{bran}}^{cs} \) such that no lift \( L \) of \( C \) is fixed by the element of the fundamental group of \( M \) that represents the loop. This type of elements of the fundamental group of \( C \) seen as a set of \( M \) are not useful for our purpose. So, we will remove them by convention:

**Convention.** Fix a lift \( L \) of a leaf \( C \) of \( \mathcal{W}_{\text{bran}}^{cs} \) (or \( \mathcal{W}_{\text{bran}}^{cu} \)). An element \( \gamma \in \pi_1(M) \) is said to be in the fundamental group of \( C \) if it is in the stabilizer of \( L \).

Notice that the fundamental group is only defined up to conjugation, hence the reason to fix a lift \( L \) of \( C \).

This convention seems to eliminate more than just the closed loops coming from self-intersections, as any potential closed loops that would be homotopically trivial in \( M \) but not in \( C \), would not be considered.

However, there is another way of seeing our notion of fundamental group arise: Recall (Theorem 10.4) that the branching foliations are approximated by true foliations \( \mathcal{W}^{cs}_\epsilon \) and \( \mathcal{W}^{cu}_\epsilon \) and that there exists maps, \( h^{cs}_\epsilon \) and \( h^{cu}_\epsilon \) mapping leaves of \( \mathcal{W}^{cs}_\epsilon \) (or \( \mathcal{W}^{cu}_\epsilon \)) to those of \( \mathcal{W}_{\text{bran}}^{cs} \) (or \( \mathcal{W}_{\text{bran}}^{cu} \)). Then, a loop is in the fundamental group of a leaf \( C \) of \( \mathcal{W}_{\text{bran}}^{cs} \) if and only if it is freely homotopic to a loop in a corresponding leaf \( C_\epsilon \) of \( \mathcal{W}^{cs}_\epsilon \), for every \( \epsilon \) small enough. Notice that if there are several leaves that project to \( C \), in the universal cover, take a lift \( L \) and it follows from Proposition 10.11 that the set of leaves that projects to \( L \) is an interval in the leaf space of \( \mathcal{W}^{cs}_\epsilon \). It follows that \( h^{cs}_\epsilon \) lifts to an equivariant (with respect to the defined fundamental group of \( C \)) diffeomorphism from the boundary leaves of the closed interval to \( L \). We call such a leaf \( L_\epsilon \) and denote \( C_\epsilon = \pi(L_\epsilon) \).

In other words, for us, the fundamental group of \( C \) based at \( y \) will be exactly \( (h^{cs}_\epsilon)_* (\pi_1(C_\epsilon, y_0)) \) where \( h^{cs}_\epsilon(y_0) = y \).
In particular, since $W_{cs}^\epsilon$ and $W_{cu}^\epsilon$ are taut foliations without Reeb components, each leaf is $\pi_1$-injective in $M$. Thus, this second interpretation helps explain our convention: the closed loops in a leaf of $W_{\text{bran}}^{cs}$ are either in the fundamental group as we defined it, or they are due to a self-intersection. In that case, they are not an essential feature of the leaf, as they stopped being closed when pulled-back to the approximating leaf.

Following our convention, we will then say that a leaf $C$ of the branching foliation is a plane, a cylinder, or a M"obius band if its corresponding approximated leaf $C_\epsilon$ is, respectively, a plane, a cylinder, or a M"obius band, for any small enough $\epsilon$.

Using these conventions, Proposition 3.14 holds for the leaves of the branching foliations whenever $\tilde{f}$ has no fixed points in the leaf (cf. Lemma 11.8). For ease of reference, we restate it here.

**Proposition 11.12.** Assume that $\tilde{f}$ fixes a leaf $L$ of $W_{\text{bran}}^{cs}$ then, $C = \pi(L)$ has cyclic fundamental group (thus it is either a plane, an annulus or a M"obius band), or $L$ has a point fixed by $\tilde{f}$.

**Remark 11.13.** Similarly, because of possible self-intersections, we need to be careful on how to define the path-metric on a leaf of $W_{\text{bran}}^{cs}$ or $W_{\text{bran}}^{cu}$.

If $C$ is a leaf of, say, $W_{\text{bran}}^{cs}$, we define a path on $C$ as a continuous curve $\eta$ that is the projection of a continuous curve $\tilde{\eta}$ in a lift $L$ of $C$ to $\tilde{M}$. We then define the path-metric on $C$ as usual, but considering only the paths as defined before.

Notice that not every continuous curve $\eta$ on $C$ is a path in the above sense, as there might not exists any lift of $\eta$ that stays on only one lift of $C$.

### 11.5. Minimality for Seifert and hyperbolic manifolds.

The goal of this subsection is to show an analogue of Proposition 3.15 in the context of a non necessarily dynamically coherent diffeomorphism.

**Proposition 11.14.** Suppose that $M$ is hyperbolic or Seifert. Suppose that $\tilde{f}$ fixes one leaf of $W_{\text{bran}}^{cs}$. Then $W_{\text{bran}}^{cs}$ is $f$-minimal (and therefore every leaf of $W_{\text{bran}}^{cs}$ is fixed by $\tilde{f}$). The same statement holds for $W_{\text{bran}}^{cu}$. In addition, every leaf of $W_{\text{bran}}^{cs}$, $W_{\text{bran}}^{cu}$, $W_{\text{bran}}^{cs}$ and $W_{\text{bran}}^{cu}$ is either a plane or an annulus.

We will need for this result to add some arguments to the proof of Proposition 3.15. Notice however that the proof of Proposition F.4 holds without change in the non dynamically coherent setting, thus if $f$ is transitive or volume preserving, then the branching foliations are $f$-minimal.

The main issue to extend the proof of Proposition 3.15 to the non dynamically coherent context is with the use of Lemma 3.13. We saw in section 11.2.1 that a weaker version of Lemma 3.13 (namely, Lemma 11.8) holds in this context. Unfortunately, the proof of Proposition 3.15 makes use of the strong version of the lemma, so Lemma 11.8 does not help us. Instead, we will replace that argument by the next lemma, whose proof will span the following six pages.

We first need a definition. So far, we only defined $f$-minimality for the whole foliations, but we can extend naturally the definition to a foliated subset: We say that a subset $\Lambda$ of $M$, saturated by $W_{\text{bran}}^{cs}$ (or $W_{\text{bran}}^{cu}$) is $f$-minimal if it is closed, non-empty, and invariant by $f$, and such that no proper saturated subset of $\Lambda$ verifies all these conditions.

We can now prove:
Lemma 11.15. Let \( \tilde{f} \) be a good lift of \( f \) to \( \tilde{M} \). Suppose that \( \Lambda \) is a non-empty \( f \)-minimal set of \( \mathcal{W}^c_{\text{bran}} \), such that every leaf \( L \) of \( \Lambda = \pi^{-1}(\Lambda) \) is fixed by \( \tilde{f} \). Then there are no fixed points of \( \tilde{f} \) in a leaf of \( \Lambda \).

Proof. During the proof of this lemma, we will use the expansion of stable length by \( \tilde{f}^{-1} \) a lot. To lighten the notation, we set \( g := \tilde{f}^{-1} \).

Suppose for a contradiction that there is a fixed point \( x_0 \) of \( \tilde{f} \) in a leaf \( L_0 \) of \( \Lambda \). This projects to a fixed point \( y = \pi(x_0) \) in \( M \). Notice that if a leaf \( L \) of \( \Lambda \) intersects \( u(x_0) \) then, since both are \( \tilde{f} \)-invariant, it follows that the intersection of \( L \) and \( u(x_0) \) has to be \( x_0 \).

We start with the following

Claim 11.16. There exists \( b > 0 \) such that any point in a leaf of \( \Lambda \) is at distance at most \( b \) (for the path metric on the leaf) from a fixed point of \( \tilde{f} \).

Proof. Indeed, suppose this was not the case. Then, for any \( b > 0 \), there exists a disk of radius \( b \) in a leaf of \( \Lambda \) that does not contain any fixed point of \( \tilde{f} \). Taking \( b \to +\infty \), up to deck transformations and considering a subsequence, these disks converge to a full leaf \( L_1 \) of \( \mathcal{W}^c_{\text{bran}} \) in \( \Lambda \). Here the convergence is with respect to the topology of the center stable leaf space, which also implies convergence as a set of \( M \). The leaf \( L_1 \) does not contain any fixed point of \( \tilde{f} \), because otherwise, since all leaves of \( \Lambda \) are fixed by \( \tilde{f} \), one would have some fixed points in the disks accumulating onto \( L_1 \).

Now consider \( \Lambda' \), the closure in \( M \) of the leaf \( A = \pi(L_1) \). Since \( \Lambda \) is closed, the set \( \Lambda' \) must be a (closed) subset of \( \Lambda \), foliated by \( \mathcal{W}^c_{\text{bran}} \). Moreover, by the previous remark, neither the leaf \( L_1 \) nor its translates by deck transformations can intersect \( u(x_0) \) as they do not have fixed points. It follows that \( \pi(x_0) \notin \Lambda' \) contradicting \( f \)-minimality of \( \Lambda \).

According to Lemma 3.11, together with Remark 3.12, there is a constant \( K_0 > 0 \) such that, for any \( z \in L_0 \), we have

\[
d_{L_0}(z, \tilde{f}(z)) \leq K_0,
\]

where \( d_L \) denotes the path-metric on \( L_0 \).

The rest of the proof will consist in proving that the fact that \( \tilde{f} \) moves points a bounded distance in \( L_0 \) contradicts the exponential contraction of length along the stable leaf \( s(x_0) \) of the fixed point \( x_0 \) of \( \tilde{f} \) in \( L_0 \). We will do that by building large metric balls with no fixed points of \( \tilde{f} \), in contradiction with Claim 11.16.

In order to obtain these fixed-point free sets, we will use compact simply connected domains such that their boundary is the union of a segment along the stable leaf \( s(x_0) \) and a geodesic segment in \( L_0 \). We will start by proving three claims about these domains. For that purpose, we introduce the following notations: given any \( y_1, y_2 \in s(x_0) \), we write

- \([y_1, y_2]^s\) is the closed segment along the stable leaf \( s(x_0) \) between \( y_1 \) and \( y_2 \),
- \([y_1, y_2]_{L_0}\) is the geodesic segment between them (for the path metric on \( L_0 \)).

Before moving on to the claims, notice also that, since the stable foliation is a true foliation, there exists \( \delta, \eta > 0 \) such that points in a same stable leaf that are at distance less than \( \delta \) in the path-metric of \( L_0 \), must be at distance less than \( \eta \) along the stable arc. Two consequences of this fact that will be used repeatedly are:
• points that are far enough away along \(s(x_0)\) must be at distance greater than \(\delta\) in \(L_0\), and
• the volume of a \(\delta/2\)-tubular neighborhood of a stable segment \([y_1, y_2]^s\) must go to infinity with the length of \([y_1, y_2]^s\).

Thus there exists domains bounded by stable segments \([y_1, y_2]^s\) and geodesics \([y_1, y_2]_{L_0}\) with arbitrarily large diameter. These domains with large diameters are the subject of the next three claims.

For \(y_1, y_2 \in s(x_0)\) we denote by \(D_{y_1,y_2}\) any of the closed topological disks bounded by arcs in \([y_1, y_2]^s\) and \([y_1, y_2]_{L_0}\). As mentioned before, there are disks \(D_{y_1,y_2}\) of arbitrarily large diameter if \(y_1\) is far from \(y_2\) in \(s(x_0)\). Given \(C > 0\), we let \(V_C\) be the open tubular neighborhood of \([y_1, y_2]_{L_0}\).

**Claim 11.17.** Let \(D' = D_{y_1,y_2}\) for \(y_1, y_2 \in s(x_0)\). Suppose that the length of \([y_1, y_2]_{L_0}\) is bounded above by \(d\). Then there exists a positive integer \(i\), with \(i \leq d/\delta\), such that either:

(i) \(D' \subset g^i(D')\), or,

(ii) \(g^i(D' \setminus V_C) \cap (D' \setminus V_C) = \emptyset\),

where \(C = K_0 d/\delta\) and \(g = \tilde{f}^{-1}\).

**Proof.** We assume first that the statement is not vacuously true, i.e., that \(D' \setminus V_C\) is not empty.

For simplicity, we will only consider positive \(i\). For any such \(i\), let \(C_i := iK_0\).

Assume that there is \(i\) such that \(g^i(D' \setminus V_{C_i}) \cap (D' \setminus V_{C_i}) \neq \emptyset\).

Then, in particular, \(g^i(D')\) and \(D'\) intersect. Hence, either \(g^i(D')\), or \(g^{-i}(D')\), is contained in \(D'\), or the boundaries must intersect.

First, notice that \(g^i(D')\) cannot be entirely contained in \(D'\). If that was the case, then, for all \(n > 0\), we would have \(g^{ni}(D') \subset D'\). But, as powers of \(g^i\) increase the length of the stable segment \([y_1, y_2]^s\), and these images would have to stay in the compact \(D'\), we would get an accumulation point for \(s(x_0)\) which is impossible.

Thus, either \(D' \subset g^i(D')\), or the boundaries of \(g^i(D')\) and \(D'\) must intersect.

Suppose for the moment that the boundaries intersect. Since \(g^i(D' \setminus V_{C_i}) \cap (D' \setminus V_{C_i}) \neq \emptyset\), it implies that there exists \(x_1^i \in g^i(\partial D') \cap (D' \setminus V_{C_i})\). See Figure 17. Moreover, \(g^i([y_1, y_2]_{L_0})\) is in the tubular neighborhood of \([y_1, y_2]_{L_0}\) of radius at most \(C_i = iK_0\). So \(x_1^i \in g^i([y_1, y_2]^s) \subset s(x_0)\).

Since no ray of \(s(x_0)\) can stay in \(D'\) nor can self-intersect, there exists two points \(z_1^i, z_2^i \in s(x_0) \cap [y_1, y_2]_{L_0}\) that we can choose in such a way that \(y_2 \leq z_1^i < x_1^i < z_2^i\) (for the order on \(s(x_0)\) given by an orientation). Since \(dL_0(x_1^i, z_2^i) \geq C_i = iK_0\), the distance between \(z_2^i\) and both \(y_1\) and \(y_2\) must be greater than \(\delta\) (if necessary, we take \(K_0\) bigger so that \(K_0 > \eta\), then the stable length between \(z_2^i\) and \(y_2\) is greater than \(\eta\), and thus their distance in \(L_0\) is greater than \(\delta\)).

So suppose that there exists \(n\) such that, \(D' \not\subset g^i(D')\) for all \(1 \leq i \leq n\), and all the sets \(g^i(D' \setminus V_{C_n})\) intersects \(D' \setminus V_{C_n}\), then we obtain \(n\) points \(z_2^1, z_2^2, \ldots, z_2^n\) on \([y_1, y_2]_{L_0}\), so that \(z_2^1, \ldots, z_2^n, y_1, y_2\) are pairwise at least \(\delta\) apart from each other. But the diameter of \([y_1, y_2]^s\) is at most \(d\), so there is a maximum of \(d/\delta - 1\) such points. Hence \(n \leq d/\delta - 1\), which proves the claim. □

Our next goal is going to be to eliminate possibility (i) in Claim 11.17, at least for the topological disks with large diameters.
Claim 11.18. Let \( D' = D_{y_1,y_2} \) for \( y_1, y_2 \in s(x_0) \). Suppose that there exists a positive integer \( i \) such that \( D' \subset g^i(D') \). If there exists \( u \in [y_1, y_2]^n \) such that
\[
d(u, [y_1, y_2]_{L_0}) \geq 10b + 3iK_0 \left( \frac{10b}{\delta} + 1 \right),
\]
then there exists a ball of radius \( 2b \) that does not contain any fixed point of \( \bar{f} \).

Proof. Since \( D' \subset g^i(D') \), where \( g = \bar{f}^{-1} \), the set \( S = \cup_{n \in \mathbb{N}} g^m(D') \setminus D' \) does not contain any fixed points. We will prove that \( S \) contains a ball of radius \( 2b \).

Let \( n \) be an integer such that \( 10b/\delta \leq n \leq 10b/\delta + 1 \). Consider the subset \( S_0 \) of \( S \) defined by
\[
S_0 = \bigcup_{k=1}^{2n} g^{ik}(D') \setminus D'.
\]
Let \( c \) be a path starting at \( g^{ni}(u) \). In order for \( c \) to escape \( S_0 \), either \( c \) must intersect \( g^{ki}([y_1, y_2]_{L_0}) \) for some \( 0 \leq k \leq 2n \), or \( c \) must intersects \( g^{ki}([y_1, y_2]^n) \) for all \( 0 \leq k \leq n-1 \) or all \( n+1 \leq k \leq 2n \).

If \( c \) intersects \( g^{ki}([y_1, y_2]_{L_0}) \), then its length is bounded below by
\[
d_{L_0} \left( g^{ni}(u), g^{ki}([y_1, y_2]_{L_0}) \right) \geq d_{L_0} \left( u, [y_1, y_2]_{L_0} \right) - (n+k)iK_0
\geq d_{L_0} \left( u, [y_1, y_2]_{L_0} \right) - 3iK_0 \left( \frac{10b}{\delta} + 1 \right) \geq 10b.
\]

On the other hand, since the stable segments \( g^{ki}([y_1, y_2]^n) \), \( 0 \leq k \leq n \) must be at least \( \delta \) apart, if \( c \) intersects \( g^{ki}([y_1, y_2]^n) \) for all \( 0 \leq k \leq n-1 \) or all \( n+1 \leq k \leq 2n \), then the length of \( c \) is bounded below by \( n\delta \geq 10b \).

So in either case, the length of \( c \) is greater than 10b. Thus the ball of radius \( 2b \) centered at \( g^{ni}(u) \) is contained in \( S_0 \), which does not contain any fixed points of \( \bar{f} \). \qed

As a consequence, we obtain

Claim 11.19. Let \( D' = D_{y_1,y_2} \) with \( y_1, y_2 \in s(x_0) \). Let \( d \) be the length of \([y_1, y_2]_{L_0}\). Suppose that there exists \( u \in [y_1, y_2]^n \cap \partial D' \) such that
\[
d(u, [y_1, y_2]_{L_0}) \geq 10b + 3K_0 \frac{d}{\delta} \left( \frac{10b}{\delta} + 1 \right).
\]
Then there exists $i$, with $i \leq d/\delta$, such that $g^i(D' \setminus V_C) \cap (D' \setminus V_C) = \emptyset$, where $V_C$ is the tubular neighborhood of the geodesic segment $[y_1, y_2]_{L_0}$ of radius $C = K_0d/\delta$ and $g = \tilde{f}^{-1}$.

In particular, $D' \setminus V_C$ contains no fixed points of $\tilde{f}$.

Proof. Since the conclusion of Claim 11.18 is in contradiction with Claim 11.16, it implies that only possibility (ii) in Claim 11.17 can arise for disks that have a large enough diameter. Our claim is just a reformulation of this. □

Now that we proved Claim 11.19, we can finish our proof of Lemma 11.15.

Since $g$ expands exponentially the stable lengths, we can pick $z \in s(x_0)$ such that the length of $[z, g(z)]^s$ is arbitrarily large as needed. In particular the set $L_0 \setminus ([z, g(z)]^s \cup [z, g(z)]_{L_0})$ contains at least one bounded connected component of arbitrarily large diameter. This is because the geodesic segment $[z, g(z)]_{L_0}$ has length bounded by $K_0$, whereas the length of $[z, g(z)]^s$, and therefore the volume of its $\delta/2$-tubular neighborhood, are arbitrarily large.

Hence, picking $z$ far enough in $s(x_0)$, we can assure that there exists $y_1, y_2 \in s(x_0)$ such that $[y_1, y_2]^s \subset [z, g(z)]^s$, $[y_1, y_2]_{L_0} \subset [z, g(z)]_{L_0}$, and such that there is a topological disk $D = D_{y_1, y_2}$ bounded by $[y_1, y_2]^s$ and $[y_1, y_2]_{L_0}$ that satisfies to the assumptions of Claim 11.19. We fix such a $z \in s(x_0)$ and a corresponding $D$.

Let $i_0$ be the positive integer given by Claim 11.19 applied to $D$. Notice that the length of $[y_1, y_2]_{L_0}$ is less than $K_0$, so $i_0 \leq K_0/\delta$.

Let $w$ be a point in $[y_1, y_2]^s$ that is farthest from $z$. Consider the closed domain $R$ bounded by the geodesics $[w, g^{i_0}(w)]_{L_0}$ and $[y_2, g^{i_0}(y_1)]_{L_0}$, and the stable segments $[w, y_2]^s$ and $[g^{i_0}(y_1), g^{i_0}(w)]^s$ (see Figure 18). To be precise, $R$ is obtained as the closure of the union of all the bounded connected components of $L_0$ minus the four curves.

![Figure 18. A depiction of case (ii) in Claim 11.17.](image)

Notice that the distance between $[w, g^{i_0}(w)]_{L_0}$ and $[y_2, g^{i_0}(y_1)]_{L_0}$ is as large as we want, because $g^{i_0}$ moves points a uniform bounded distance away (at most $i_0K_0$, so at most $K_0^2/\delta$), whereas the distance between $w$ and $[z, g(z)]_{L_0}$ is as large as we want.
Now, if necessary, we modify our choice of the original $z \in s(x_0)$ so that the diameter of $D$ is even larger in order to have a point $x \in R$ such that
\[
\min \left\{ d\left(x, [w, g^{i_0}(w)]_{L_0}\right), d\left(x, [y_1, g^{i_0}(y_2)]_{L_0}\right) \right\} \geq 10b + C + \left(1 + \frac{4b}{\delta}\right) \frac{K^2}{\delta}.
\]

Let $R_C := R \setminus V_C$, where $V_C$ is the union of the tubular neighborhoods of $[w, g^{i_0}(w)]_{L_0}$ and $[y_1, g^{i_0}(y_2)]_{L_0}$ of radius $C = 10b + 3 \frac{K^2}{\delta} \left(\frac{10b}{\delta} + 1\right)$.

By construction, $R$ can be covered by topological disks that are bounded by parts of the stable leaf $s(x_0)$ and parts of either $[w, g^{i_0}(w)]_{L_0}$ or $[y_1, g^{i_0}(y_2)]_{L_0}$. Moreover, the distance between $[w, g^{i_0}(w)]_{L_0}$ and $[y_1, g^{i_0}(y_2)]_{L_0}$ can be made arbitrarily large by choosing $z$ further in $s(x_0)$ if necessary. Hence, $R_C$ is not empty and, since $C$ is chosen big enough, any such topological disk that intersect $R_C$ will automatically satisfy the hypothesis of Claim 11.19.

Hence, $\tilde{f}$ admits no fixed points in $R_C$. Similarly, writing $D_C$ for the disk $D$ minus the $C$-tubular neighborhood of $[y_1, y_2]_{L_0}$, we know that $\tilde{f}$ admits no fixed points in $D_C$.

Now we consider $W_C$ to be the union $R_C \cup D_C$ minus the $C$-tubular neighborhood of $[w, g^{i_0}(w)]_{L_0}$. The set $W_C$ does not contain any fixed points of $\tilde{f}$ either. Hence, the set $S = \cup_{n \in \mathbb{Z}} g^{ni_0}(W_C)$ is also fixed-point free.

Moreover, the boundary of the set $D_C \cap W_C$ contains two disjoint sides made of subsegments of the stable segment $[y_1, y_2]^n$ (see Figure 18), and the distance between these two sides must be greater than $\delta$ (because the two sides are far enough apart in the stable leaf $s(x_0)$). Furthermore, since $g$ increases the stable length, for any $n \geq 0$, the distance in $L_0$ between the two stable sides of $g^{ni_0}(D_C \cap W_C)$ must also be greater than $\delta$ (having two distinct and far enough apart stable side is the reason we introduced $W_C$ instead of just considering $R_C \cup D_C$).

The proof of Lemma 11.15 then follows from the next claim, which directly contradicts Claim 11.16.

**Claim 11.20.** There is a ball of radius $2b$ in the set $S = \cup_{n \in \mathbb{Z}} g^{ni_0}(W_C)$.

**Proof.** Let $n_0$ be such that $2b/\delta - 1 < n_0 \leq 2b/\delta$. We will build a ball of radius $2b$ inside the subset $S_0$ of $S$ defined by
\[
S_0 = \cup_{k=0}^{2n_0+1} g^{ki_0}(W_C).
\]

Let $x$ be a point in $R$ such that
\[
\min \left\{ d\left(x, [w, g^{i_0}(w)]_{L_0}\right), d\left(x, [y_1, g^{i_0}(y_2)]_{L_0}\right) \right\} \geq 10b + C + \left(1 + \frac{4b}{\delta}\right) \frac{K^2}{\delta}.
\]

Then $x \in R_C$, so $g^{ni_0}(x) \in S_0$. We will show that the ball of radius $2b$ around $g^{ni_0}(x)$ is in $S_0$.

Let $c$ be a geodesic ray starting at $g^{ni_0}(x)$. In order for $c$ to exit $S_0$, it needs to intersect a boundary component of $S_0$. Now, by construction, the boundary of $S_0$ is composed of a stable segment $I_1^c$ in $\partial D_C$, a stable segment $I_2^c$ in $\partial g^{(2n_0+1)i_0}(R_C)$ (in fact $I_2^c = g^{(2n_0+2)i_0}(I_1^c)$ but we do not need that), and the images by powers of $g^{i_0}$ of two curves $\gamma_1$ and $\gamma_2$, which are curves at distance $C$ from, respectively, $[y_1, y_2]_{L_0} \cup [y_2, g^{i_0}(y_1)]_{L_0}$ and $[w, g^{i_0}(w)]_{L_0}$.

In the rest of the argument, the difference between $\gamma_1$ and $\gamma_2$ is irrelevant, so we will just write $\gamma$ to refer to either of them.

Thus, for $c$ to exit $S$, it needs to either intersect $I_1^c$, $I_2^c$ or $g^{ni_0}(\gamma)$ for some $0 \leq n \leq 2n_0 + 1$.

Suppose first that $c$ exits through $I_1^c$. Then it needs to have crossed the domains $W_C \cap D_C, g^{ni_0}(W_C \cap D_C), \ldots, g^{ni_0}(W_C \cap D_C)$. Here by cross we mean intersecting
the two stable sides. Now, as we noticed earlier the distance between the two stable sides of $g^{ki_0}(W_C \cap D_C)$ is greater than $\delta$ for any $k \geq 0$. Thus, if $c$ exits through $I^s_1$, its length needs to be at least $(n_0 + 1)\delta$, which is strictly greater than $2b$ by our choice of $n_0$.

Similarly, if $c$ exits through $I^s_2$. Then it needs to have crossed the domains $g^{(n_0+1)i_0}(W_C \cap D_C), \ldots, g^{(2n_0+1)i_0}(W_C \cap D_C)$, in which case, again, the length of $c$ is greater than $(n_0 + 1)\delta > 2b$.

Finally, suppose that $c$ exits through a $g^{ki_0}(\gamma)$ for some $0 \leq k \leq 2n_0 + 1$. Then, in order to prove our claim, all we have to do is to show that the distance between $g^{ni_0}(x)$ and $g^{ki_0}(\gamma)$ is larger than $2b$ for all $0 \leq k \leq 1 + 4b/\delta$.

Our condition on $x$ implies that

$$d(x, \gamma) \geq 10b + C + \left(1 + \frac{4b}{\delta}\right)K_0^2 - C = 10b + \left(1 + \frac{4b}{\delta}\right)K_0^2.$$

Hence, if $0 \leq k \leq 1 + 4b/\delta$, then we have

$$d(x, g^{ki_0}(\gamma)) \geq d(x, \gamma) - ki_0K_0 \geq d(x, \gamma) - \left(1 + \frac{4b}{\delta}\right)\frac{K_0^2}{\delta} \geq 10b.$$

Therefore, the ball of radius $2b$ centered at $g^{ni_0}(x)$ is entirely in $S$, proving Claim 11.20.

This ends the proof of Lemma 11.15.

An important consequence of Lemma 11.15 is that we can obtain an analog of Proposition 3.14:

**Corollary 11.21.** Suppose that $f$ is a partially hyperbolic diffeomorphism in $M$ that is homotopic to the identity. Let $\tilde{f}$ be a good lift of $f$ to $\tilde{M}$. Suppose that $\Lambda$ is a non empty (saturated) $f$-minimal subset of $W^{cs}_{\text{bran}}$ such that every leaf of the lift $\Lambda$ to $\tilde{M}$ is fixed by $\tilde{f}$. Then every leaf in the $f$-minimal set $\Lambda$ of $W^{cs}_{\text{bran}}$, is either a plane or an annulus.

**Proof.** Let $A$ be a leaf of $\Lambda$ and $L$ a lift in $\tilde{M}$. By Lemma 11.15, $L$ does not admit any fixed points of $\tilde{f}$. Hence, $\tilde{f}$ acts freely on the space of stable leaves in $L$.

Now, recall that $\pi_1(A)$ can be defined as the elements $\gamma \in \pi_1(M)$ that fix $L$ (see section 11.4). So if $\gamma \in \pi_1(A)$, it must also act freely on the space of stable leaves in $L$. As $\tilde{f}$ commutes with every deck transformation, Corollary E.4 (which still applies in our context, see Appendix E) implies that $\pi(A)$ is abelian, i.e., $A$ is either a plane or an annulus (again with the understanding that $A$ might actually only be an immersion of one of these manifolds in $M$ and recalling that all bundles were assumed to be orientable in this section, so in particular the leaves cannot be Möbius bands).

We are now ready to prove Proposition 11.14.

**Proof of Proposition 11.14.** This proof follows the same structure as the one of Proposition 3.15, and we will continuously refer to it. Recall the standing assumption that all bundles are orientable and their orientation is preserved by $f$.

Consider $\Lambda$ an $f$-minimal non empty subset. We need to show that $\Lambda = M$. We assume by contradiction that $\Lambda \neq M$. 

As in the proof of Proposition 3.15, we argue that since $\mathcal{W}^{cs}_{bran}$ has no closed leaves and $\Lambda$ is $f$-minimal, there cannot be any isolated leaves in $\Lambda$ (for the topology of the stable leaf space).

Now, Lemma 11.15 (instead of Lemma 3.13) allows us to assert that $\tilde{f}$ has no fixed points in leaves of $\Lambda$. Then, Corollary 11.21 (instead of Proposition 3.14) implies that each leaf of $\Lambda$ is either a plane or an annulus.

We fix an $\epsilon$ small enough and let $\Lambda'$ be the pull back of $\Lambda$ to the approximating foliation $\mathcal{W}^{cs}_{c}$. That is, $\Lambda' = (h^{cs}_{c})^{-1}(\Lambda)$. Let $V$ be a connected component of $\tilde{M} \setminus \Lambda'$.

Claim 3.16 of Proposition 3.15 applies to $V$, since it is just a general fact about codimension one foliations. So the projection $\pi(V)$ of $V$ to $\tilde{M}$ has only finitely many boundary leaves.

Now, we need to prove Claim 3.18 of Proposition 3.15, i.e.:

**Claim 11.22.** Let $L \in \partial V$. Then $\pi(L)$ is an annulus.

The proof of that claim is slightly different from the dynamically coherent case, as we now need to use both the foliation $\mathcal{W}^{cs}_{c}$ and the branching foliation $\mathcal{W}^{cs}_{bran}$.

**Proof.** Suppose that $\pi(L)$ is a plane. Recall (see [CC00]) that $\pi(V)$ has an octopus decomposition and a compact core. So for any $\delta > 0$, the subset of points in $\pi(L)$ that are at distance greater then $\delta$ from another boundary component of $\pi(V)$ is precompact. Since $\pi(L)$ is supposed to be a plane, that subset must be contained in a closed disk $D$. Then $\pi(L) \setminus D$ is an annulus that is $\delta$-close to another boundary component, $\pi(L')$ of $\pi(V)$. Moreover, the subset of $\pi(L')$ that is $\delta$-close to $\pi(L) \setminus D$ then also has to be an annulus. If $\pi_{1}(L')$ were not a plane it would be an annulus and its non-trivial curve corresponds to a curve homotopic to the boundary of the closed disk $D$ which is homotopically trivial in $M$. Since the leaves of $\mathcal{W}^{cs}_{c}$ are $\pi_{1}$-injective, this implies that $\pi(L')$ is also a plane.

Since $M$ is irreducible this implies that $\pi(V)$ is homeomorphic to an open disk times an interval. So $\pi(V)$ has only two boundary components, both of which are planes. In particular, the isotropy group of $V$ is trivial and $\pi(V)$ is homeomorphic to $V$.

We will now switch to the branching foliation to finish the proof. Let $A = h^{cs}_{c}(\pi(L))$ and $B = h^{cs}_{c}(\pi(L'))$. Since we chose $\epsilon$ small enough, up to taking $\delta$ small enough also, the unstable segments through $A \setminus h^{cs}_{c}(D)$ intersect $B$, and their length is uniformly bounded. Moreover, no unstable ray of $A$ can stay in $h^{cs}_{c}(\pi(V))$. This is because $\pi(V)$ is homeomorphic to an open disk times an interval. So, since $D$ is compact, the length of every unstable segment between $A$ and $B$ is bounded by a uniform constant. Notice that, since $\mathcal{W}^{cs}_{bran}$ is a branching foliation, we may have $A \cap B \neq \emptyset$, i.e., some of these unstable segments may be points.

Since $L$ and $L'$ are in $\partial V$, which is a connected component of $\tilde{M} \setminus \tilde{\Lambda}$, we have that $A, B \in \partial (\tilde{M} \setminus \Lambda)$. So in particular, $A$ and $B$ are fixed by $f$. Hence, the set of unstable segments between $A$ and $B$ is also invariant by $f$. Since the length of unstable segments between $A$ and $B$ are bounded above and $f$ expands the unstable length, all the unstable segments must have zero length, i.e., $A = B$. Which implies that $V$ is empty, which contradicts the assumption that $\Lambda \neq \tilde{M}$.

Thus we showed that every component of $\pi(\partial V)$ is an annulus. We can then apply without change the (topological) arguments of the proof of Proposition
3.15 to obtain a torus $T$, composed of annuli along leaves of $\mathcal{W}^{cs}_f$, together with annuli transverse to $\mathcal{W}^{cu}_f$, that bounds a solid torus $U'$ in $\pi(V)$.

Now consider $U = h^{cs}_f(U')$. Because of the collapsing of leaves, $U$ may not be a solid torus. If $U$ is empty for any any such component $U'$, this would directly contradict the assumption $\Lambda \neq M$. So for some such complementary component $U'$, the set $U$ is not empty and it is contained in a solid torus (the $\epsilon$-tubular neighborhood of $U'$ in $M$). We can then use the same “volume vs. length” argument on $U$, exactly as in the end of the proof of Proposition 3.15, to get a final contradiction. This ends the proof of Proposition 11.14. □

As a consequence, we get the following result that completes the proof of Theorem 1.6 as announced.

**Corollary 11.23.** Suppose that $f$ is a partially hyperbolic diffeomorphism homotopic to the identity. Suppose that $f$ is either volume preserving or transitive, or that $M$ is either hyperbolic or Seifert. Let $\tilde{f}$ be a good lift of $f$. Then $\tilde{f}$ has no periodic points. In particular, $f$ has no contractible periodic points.

**Proof.** Up to finite covers and iterates, we may assume that $f$ preserves the branching foliations $\mathcal{W}^{cs}_{\text{bran}}, \mathcal{W}^{cu}_{\text{bran}}$.

If $\tilde{f}$ acts as a translation on either $\mathcal{W}^{cs}_{\text{bran}}$ or $\mathcal{W}^{cu}_{\text{bran}}$, then it does not have periodic points.

Otherwise, depending on which assumption is verified, Proposition F.4 or Proposition 11.14, asserts that the branching foliations are $f$-minimal. The result then follows from Theorem 11.9. □

### 11.6. Gromov hyperbolicity of leaves

We now prove a version of Lemma 11.20 in the non dynamically coherent setting.

**Lemma 11.24.** Suppose that $f$ is a partially hyperbolic diffeomorphism in $M$ that is homotopic to the identity. Let $\tilde{f}$ be a good lift of $f$ to $\tilde{M}$. Suppose that $\tilde{f}$ fixes every leaf of $\mathcal{W}^{cs}_{\text{bran}}$, and that $\mathcal{W}^{cs}_{\text{bran}}$ is $f$-minimal. Then all the leaves of $\mathcal{W}^{cs}_{\text{bran}}$ are Gromov hyperbolic.

**Proof.** The foliation $\mathcal{W}^{cs}_\epsilon$ is taut. Thus, Candel’s theorem (Theorem C.1) asserts that either all the leaves of $\mathcal{W}^{cs}_\epsilon$ are Gromov hyperbolic or there is a holonomy invariant transverse measure (of zero Euler characteristic).

Assume for a contradiction that $\mu$ is a holonomy invariant transverse measure. Since $\mathcal{W}^{cs}_\epsilon$ is not $f$-invariant, we have to adjust the proof of Lemma 11.20.

The transverse measure $\mu$ lifts to a measure $\tilde{\mu}$ transverse to $\mathcal{W}^{cs}_\epsilon$. Thus, $\tilde{\mu}$ defines a measure on $L^{cs}_\epsilon$, the leaf space of $\mathcal{W}^{cs}_\epsilon$.

Let $g_{\epsilon,s}: L^{cs}_\epsilon \to L^{cs}_b$ be the canonical projection between the leaf spaces of $\mathcal{W}^{cs}_\epsilon$ and $\mathcal{W}^{cs}_{\text{bran}}$ (see section 10.1.1). Let $\tilde{\nu} := (g_{\epsilon,s})_* \tilde{\mu}$ be the corresponding measure on $L^{cs}_b$. Now $\tilde{\nu}$ is $\tilde{f}$-invariant since $\tilde{f}$ is the identity on $L^{cs}_b$, and it is also $\pi_1(M)$-invariant as $\tilde{\mu}$ is. The support of $\tilde{\nu}$ in $L^{cs}_b$ is a closed set $Z$ in $L^{cs}_b$ that is $\tilde{f}$-invariant and $\pi_1(M)$-invariant.

The measure $\tilde{\nu}$ on $L^{cs}_b$ can also be considered as a measure on the set of transversals to $\mathcal{W}^{cs}_{\text{bran}}$ in $\tilde{M}$: For any transversal $\tau$ to $\mathcal{W}^{cs}_{\text{bran}}$ in $\tilde{M}$, we define $\tilde{\nu}(\tau)$ as the $\tilde{\nu}$-measure of the set of leaves in $L^{cs}_b$ that intersects $\tau$. Notice that the measure of a point in $\tilde{M}$ (which can be thought of as a degenerate transversal) can be positive if the image of that point in $L^{cs}_b$ is an interval.

Note also that we refrained from calling $\tilde{\nu}$ a transverse measure to $\mathcal{W}^{cs}_{\text{bran}}$ because it is by no means holonomy invariant. In fact holonomy itself is not well
defined for a branching foliation. Still \( \tilde{\nu} \) satisfies the property that if \( \tau_1, \tau_2 \) are transversals and every leaf intersecting \( \tau_1 \), also intersects \( \tau_2 \), then \( \tilde{\nu}(\tau_1) \leq \tilde{\nu}(\tau_2) \).

Projecting down to \( M \), the measure \( \tilde{\nu} \) induces a measure \( \nu \) on the set of transversals to \( \mathcal{W}_{\text{bran}}^{cs} \) on \( M \).

Let \( \tau \) be any unstable segment in \( M \). Since \( \tilde{f} \) fixes every leaf of \( \mathcal{W}_{\text{bran}}^{cs} \), the measure of \( f^i(\tau) (= \nu(f^i(\tau))) \) is equal to \( \nu(\tau) \) for any integer \( i \). We can choose \( i \) very big and negative so that the length of \( f^i(\tau) \) is extremely small. Therefore it is contained in a small foliated box of \( \mathcal{W}_{\text{bran}}^{cs} \), which is the projection of a compact foliated box of \( \mathcal{W}^{cs} \). It follows that \( \nu(\tau) \) is uniformly bounded. In particular this implies that the \( \nu \)-measure of any unstable leaf in \( M \) is bounded above. In turns, it implies that for any \( j > 0 \) (assumed big enough), there is an unstable segment \( u_j \) of length \( > j \) which has \( \nu(u_j) \) measure \( < 1/j \). Taking the midpoint of these segments and a converging subsequence, we obtain a full unstable leaf, call it \( \zeta \), so that \( \zeta \) has \( \nu(\zeta) = 0 \) (since \( \nu(\zeta) < 1/j \) for all big enough \( j \)).

Let \( Y \) be the union of the leaves of \( \mathcal{W}_{\text{bran}}^{cs} \) that do not intersect \( \zeta \) or any of its iterates by \( f \). Then \( Y \) is a closed subset of \( M \) and clearly \( f \)-invariant. Let \( L \) be a leaf in \( \mathcal{W}_{\text{bran}}^{cs} \) which is in \( Z \), the support of \( \tilde{\nu} \). Then by definition of support of \( \tilde{\nu} \), it follows that \( \pi(L) \) cannot intersect \( \zeta \) or any of its iterates by \( f \). Hence \( \pi(L) \) is in \( Y \). In particular \( Y \) is not empty. This contradicts the fact that \( \mathcal{W}_{\text{bran}}^{cs} \) is \( f \)-minimal, and hence cannot happen.

This finishes the proof of the lemma. \( \square \)

11.6.1. **Perfect fits in branching foliations.** An essential tool for us has been the use of perfect fits between center leaves and stable (or unstable) leaves inside a center stable (resp. center unstable) leaf. Despite having branching foliations, the definitions of a \( \mathcal{CS} \)-perfect fits, \( \mathcal{SC} \)-perfect fits and perfect fits (see Definition 4.1) remains literally the same. However it is useful to add one precision on how to define what it means to be “on one side of \( c' \)” when \( c \) is a center leaf that may have branching loci for the definition of \( \mathcal{CS} \)-perfect fit. The definition of \( \mathcal{SC} \)-perfect fit does not even need this (because the stable foliation is a true foliation, not a branching one).

**Definition 11.25.** Let \( c \) be a center and \( s \) a stable leaf in a center stable leaf \( L \).

We denote by \( C^s \) the connected component of \( L \setminus c \) that contains \( s \).

The leaves \( c \) and \( s \) makes a \( \mathcal{CS} \)-**perfect fit** if there exists \( \tau \) an open transversal to the center foliation in \( L \) that intersects \( c \) and such that, for any center leaf \( c' \), if \( c' \) intersects \( \tau \) and \( c' \) intersects \( C^s \), then \( c' \) intersects \( s \).

Notice that the condition in the definition needs to apply to any \( c' \) that intersects the transversal \( \tau \). In particular, it needs to apply to any \( c' \) such that \( c' \cap \tau = c \cap \tau \), i.e., any center leaf that branches away from \( c \) after its intersection with the transversal \( \tau \).

One can also see the definition of a perfect fit at the leaf space level: Let \( s \) be a stable leaf in \( L \). The leaf \( s \) determines a set \( I_s \) in \( L_s^c \), the leaf space of the center branching foliation on \( L \) (see section 10.1.2), by considering all the center leaves that intersect \( s \). That is, \( c' \in I_s \) if and only if \( c' \cap s \neq \emptyset \). Then \( c \) and \( s \) makes a \( \mathcal{CS} \)-perfect fit if and only if \( c \in \partial I_s \).

Lemma 4.2 and its proof stays valid as written because the stable foliation is a true foliation. One can also show that if \( s \) and \( c \) make a \( \mathcal{SC} \)-perfect fit, then there exists \( c_0 \) that makes a perfect fit with \( s \) but one needs to modify the proof by going to the leaf space level.
11.7. Fixed center or coarse contraction. Proposition 4.4 gave a condition for the existence of center leaves that are fixed by a good lift \( \tilde{f} \). But the proof of Proposition 4.4 does not apply in the non dynamically coherent setting (see Remark 4.8). The next proposition will instead give a consequence to the non-existence of central leaves fixed by \( \tilde{f} \). First, we need a definition.

**Definition 11.26.** A fixed center leaf \( c \) of a partially hyperbolic diffeomorphism \( f : M \to M \) is called coarsely contracting if \( c \) is homeomorphic to the line, and it contains an non-empty maximal compact interval \( I \) such that:

1. \( I \) contains every fixed point of the restriction of \( f \) to \( c \);
2. For any compact interval \( J \) of \( c \) such that \( I \subseteq J \), we have \( f(J) \subseteq J \).

A fixed center leaf \( c \) of \( f \) is called coarsely expanding if \( c \) is coarsely contracting for \( f^{-1} \).

We also naturally extend the definition of coarsely expanding to leaves that are just periodic under \( f \).

**Proposition 11.27.** Let \( f : M \to M \) be a partially hyperbolic diffeomorphism. Let \( \tilde{f} : \tilde{M} \to \tilde{M} \) be a good lift of \( f \). Suppose that \( W^c_{\text{bran}} \) is \( f \)-minimal, that all the leaves of \( W^c_{\text{bran}} \) are fixed by \( \tilde{f} \), and that \( \tilde{f} \) does not fix any center leaf in \( \tilde{M} \).

If \( c \) is a periodic center leaf of \( f \) in \( M \), then \( c \) is coarsely contracting. In particular, \( c \) contains a periodic point of \( f \).

**Remark 11.28.** If \( \tilde{f} \) as above fixes every leaf of \( \tilde{W}^c_{\text{bran}} \) instead of \( W^c_{\text{bran}} \), the conclusion of the proposition gives a periodic center leaf that is coarsely expanding instead.

We start with a preliminary result.

**Lemma 11.29.** Assume that every leaf of \( W^c_{\text{bran}} \) is fixed by \( \tilde{f} \) and that \( \tilde{f} \) does not fix any center leaf. Then the same holds for \( f^n \), for every \( n \neq 0 \).

**Proof.** Suppose that there is \( n > 0 \) and \( c_0 \) a center leaf in a center stable leaf \( L \) such that \( f^n(c_0) = c_0 \).

The standing assumption in section 11 is that all bundles are oriented and that \( f \) preserves their orientations, in particular, \( \tilde{f} \) preserves the transverse orientation to the center and stable foliations on \( L \).

Let \( A^c \) be the axis of the action of \( \tilde{f} \) on the center leaf space in \( L \).

Since \( \tilde{f}^n(c_0) = c_0 \), the leaf \( c_0 \) is not in the axis \( A^c \). Thus, either \( c_0 \in \partial A^c \), or there exists a unique center leaf \( c_1 \in \partial A^c \) that separates \( c_0 \) from \( A^c \), in which case we must have \( \tilde{f}^n(c_1) = c_1 \).

Hence, up to renaming \( c_0 \), we assume that \( c_0 \in \partial A^c \).

Now, according to [Bar98, Proposition 2.15], the boundary \( \partial A^c \) splits into three disjoint sets: the center leaves \( c \) such that \( c \) and \( \tilde{f}(c) \) are non separated positively, the leaves \( c \) such that \( c \) and \( \tilde{f}(c) \) are non separated negatively, and the leaves that are non separated with a leaf in \( A^c \). Since \( c_0 \) is fixed by \( \tilde{f}^n \), it cannot be a leaf of the third type. Thus, \( c_0 \) and \( \tilde{f}(c_0) \) are non separated.

Hence, there exists a unique stable leaf \( s_0 \) that makes a perfect fit with \( c_0 \) and separates \( c_0 \) from \( \tilde{f}(c_0) \) (see section 11.6.1). This stable leaf is then fixed by \( \tilde{f}^n \), and thus admits a fixed point \( x \) of \( \tilde{f}^n \). Therefore, there exists a center leaf \( c_1 \) through \( x \) that is fixed by \( \tilde{f}^n \) (thanks to Lemma 10.14), and, in case there are several such leaves, we may chose the one that is in \( \partial A^c \).
Again using the description of $\partial A^c$, the leaf $c_1$ is non separated from $\tilde{f}(c_1)$. Then again, there exists a unique stable leaf $s_1$ making a perfect fit with $c_1$ and that separates $c_1$ and $\tilde{f}(c_1)$. Therefore, $\tilde{f}^n(s_1) = s_1$ and there exists a unique fixed point $y \in s_1$ of $\tilde{f}^n$.

But, any center leaf $c$ close enough to $c_1$ (and on the correct side of $c_1$) will intersect both $s_0$ and $s_1$, separate $x$ from $y$ and be attracted to both $x$ and $y$ under $\tilde{f}^n$, which is impossible.

Therefore $\tilde{f}^n$ also acts freely on the center leaf space for all $n > 0$. □

In order to obtain coarsely contracting center leaves we will use the following tool.

**Proposition 11.30.** Let $f : M \to M$ be a partially hyperbolic diffeomorphism homotopic to the identity. Let $\tilde{f}$ be a good lift of $f$ to $\tilde{M}$. Suppose that $\tilde{f}$ fixes each leaf of the branching foliation $\mathcal{W}_{\text{bran}}^{cs}$. Let $L$ be a center stable leaf fixed by $\gamma \in \pi_1(M) \setminus \{\text{Id}\}$.

Assume that there exists a properly embedded $C^1$-curve, $\hat{\eta}$, in $L$ that is transverse to the stable foliation and fixed by both $\gamma$ and $\tilde{f}$.

Then,

- If $\tilde{f}$ does not act freely on the center leaf space of $L$, then there is a center leaf in $L$ fixed by both $\tilde{f}$ and $\gamma$.
- If $\tilde{f}$ acts freely on the center leaf space of $L$, then every $f$ periodic center leaf in $\pi(L)$ is coarsely contracting.

Notice that in the first case, the center leaf projects to an $f$-invariant closed center leaf.

Remark also that the hypothesis of Proposition 11.30 are implied by the conclusion of the Graph Transform Lemma H.1.

**Proof.** Since $\tilde{f}$ fixes every leaf of $\mathcal{W}_{\text{bran}}^{cs}$, Lemma 11.8 implies that $\tilde{f}$ has no fixed points in $\tilde{M}$. Therefore, $\tilde{f}$ acts freely on the stable leaf space (recall that the stable foliation is a true, non branching foliation, so its leaf space is defined as usual with the quotient topology).

Let $S$ be the stable saturation of the curve $\hat{\eta}$. Let $\alpha = \pi(\hat{\eta})$. The curve $\alpha$ is closed, $f$-invariant, and tangent to the center bundle.

**Case 1** - We start by assuming that $\tilde{f}$ fixes a center leaf $c$ in $L$.

Suppose that $c$ and $\hat{\eta}$ do not intersect a common stable leaf. Then $c$ does not intersect the set $S$ and there is a unique stable leaf $s$ contained in the boundary of $S$ such that $s$ separates $S$ from $c$. Since both $S$ and $c$ are $\tilde{f}$-invariant, so is $s$. But then $\tilde{f}$ must admit a fixed point in $s$, contradiction$^9$.

Therefore there is a stable leaf $s$ intersecting in $y$ and $\hat{\eta}$ in $x$. Iterating forward by $\tilde{f}$, we deduce that $d(\tilde{f}^n(y), \tilde{f}^n(x))$ converges to zero as $y$ and $x$ are in the same stable leaf. Since both $c$ and $\hat{\eta}$ are $\tilde{f}$-invariant, it implies that $\pi(c)$ and $\alpha = \pi(\hat{\eta})$ are asymptotic. As $\alpha$ is closed and $\pi(c)$ is a center leaf, we deduce that $\alpha$ is also a center leaf. Hence $\hat{\eta}$ is the required center leaf of the first option of the proposition.

**Case 2** - Assume now that $\tilde{f}$ acts freely on the center leaf space of $L$.

According to Lemma 11.29, $\tilde{f}^n$ also acts freely on the center leaf space of $L$ for any $n \neq 0$.

$^9$Note the distinction of $c$ being fixed by $\tilde{f}$ as opposed to $\pi(c)$ periodic under $f$. It is the first property which creates a fixed point of $\tilde{f}$ and a contradiction.
We need to prove now that every center leaf in $\pi(L)$ that is periodic must be coarsely contracting.

Let then $c$ be a center leaf in $L$ such that $\pi(c) = e$ is periodic under $f$, say of period $m$. Then, for some $\gamma_1 \in \pi_1(M) \setminus \{\text{Id}\}$, we have $c = \gamma_1 \tilde{f}^m(c)$. (Note that one can show under our current assumptions that $\pi(L)$ projects to an annulus, so $\gamma$ and $\gamma_1$ are both powers of a particular deck transformation, but we do not need that fact for the proof.) Let

$$h := \gamma_1 \circ \tilde{f}^m.$$  

We now want to show that either $c$ intersect $\hat{\eta}$, or there exists another center leaf, also fixed by $h$, that does.

Notice that, if $c$ and $\hat{\eta}$ intersect a common stable leaf, then $c$ must intersect $\hat{\eta}$. Indeed, both $c$ and $\hat{\eta}$ are invariant by $h$, which contracts the stable length.

Suppose for an instant that $c$ does not intersect $\hat{\eta}$, and thus does not intersect $S$. Then, there exists a unique stable leaf $s$ in $\partial S$ that separates $\hat{\eta}$ from $c$. That leaf $s$ must then be invariant by $h$, so admits a fixed point for $h$. Then at least one center leaf, say $c_1$, through that fixed point must be fixed by $h$. Since $c_1$ intersects $S$ and is invariant by $h$, it must intersect $\hat{\eta}$.

Thus in any case, we have a center leaf $c_1$ that intersects $\hat{\eta}$, is invariant by $h$, and, by the above argument has both ends that escapes compacts sets of $L$.

Let $I$ be the projection of $c_1$ onto $\hat{\eta}$ along stable leaves.

Suppose first that $I$ is unbounded. Then, considering iterates by $f^m$, we deduce that $\pi(c_1)$ must be asymptotic to $\pi(\hat{\eta})$, so $\hat{\eta}$ must be a center leaf, which is not allowed, since $\tilde{f}$ is assumed to act freely on center leaves.

So $I$ is bounded in $\hat{\eta}$. Let $s_1$ and $s_2$ be the stable leaves through the two endpoints of the interval $I$. Since $I$ is fixed by $h$, so are $s_1$ and $s_2$. Moreover, the center leaf $c_1$, as well as $c$ if it is different from $c_1$, is in between $s_1$ and $s_2$.

Now, $\tilde{f}$ acts as a translation on $\hat{\eta}$, so there exists $k \in \mathbb{Z}$ such that $s_2$ separates $s_1$ from $\tilde{f}^k(s_1)$. By Lemma 3.11, $s_1$ and $\tilde{f}^k(s_1)$ are a bounded Hausdorff distance apart. Thus $s_1$ and $s_2$ are a bounded Hausdorff distance apart. So $c$ satisfies all the conditions for Lemma 4.15 to hold, thus it is coarsely contracting.

This finishes the proof of Proposition 11.30.

Now we are ready to prove the main result of this section:

Proof of Proposition 11.27. Let $e$ be a center leaf periodic under $f$ of period $m > 0$. Let $c$ be a lift of $e$ to $\tilde{M}$. Call $L$ a leaf of $W_{\text{bran}}^{cs}$ that contains $c$. Then $\tilde{f}^m(c)$ projects to the same center leaf in $M$ as $c$ does, so there exists $\gamma' \in \pi_1(M)$ with $\gamma'(\tilde{f}^m(c)) = c$. Clearly $\gamma'$ is in the stabilizer of $L$, because $\tilde{f}$ leaves invariant every leaf of $W_{\text{bran}}^{cs}$. Moreover, as $\tilde{f}^m$ also acts freely on the center leaf space (cf. Lemma 11.29), $\gamma'$ is not the identity.

Since $\tilde{f}$ does not have any fixed points, Proposition 11.12 implies that the stabilizer of $L$ in $\tilde{M}$ is infinite cyclic. Thus, there exists $\gamma \in \pi_1(M) \setminus \{\text{Id}\}$ such that $\gamma^n \circ \tilde{f}^m(c) = c$ for some $n \in \mathbb{Z}$, $n \neq 0$, and such that $\gamma$ generates the stabilizer of $L$. We call

$$h := \gamma^n \circ \tilde{f}^m.$$  

Notice that $h$ is still a partially hyperbolic diffeomorphism and has bounded derivatives.

Since $\tilde{f}$ acts freely on $\mathcal{L}_L^c$, the center leaf space in $L$, then it also acts freely on $\mathcal{L}_L^s$, the leaf space of the stable foliation on $L$. 

\[ \]
Let $A^s$ be the axis for the action of $\tilde{f}$ on the stable leaf space $L^s_\tilde{f}$. No stable leaf in $M$ can be closed, so $\gamma$ acts freely on $L^s_\tilde{f}$. Moreover, as $\gamma$ and $\tilde{f}$ commute, $A^s$ is also the axis for the action of $\gamma$ on $L^s_L$, the stable leaf space of $L$ (cf. Remark E.3). As always $A^s$ can be a line or a countable union of intervals.

Suppose first that $A^s$ is a line. Let $s$ be a stable leaf in $A^s$ and $p$ in $s$. Then $p$ and $\gamma p$ can be connected by a transversal to the stable foliation, chosen so that the projection to $\pi(L)$ is a smooth simple closed curve. Let $\eta$ be the union of the $\gamma$ iterates of this segment. Then $\eta$ satisfies the properties in the hypothesis of Proposition 11.30, which implies the result we sought.

So from now on we assume that the axis is a countable union of intervals, and we write

$$A^s = \bigcup_{i \in \mathbb{Z}} [s_i^-, s_i^+] = \bigcup_{i \in \mathbb{Z}} T_i.$$ 

Our first claim is that there exists $s \in A^s$, fixed by $h$, such that the center leaf $c$ is between $\gamma^{-1}s$ and $\gamma s$.

Suppose that $c$ intersects some stable leaf $s'$ in $A^s$, then $s'$ is in a unique $T_i$ for some $i$ (the center leaf $c$ cannot intersect two different intervals otherwise $c$ would intersect two non-separated leaves, which is impossible). Then, since $h$ fixes $c$, it also fixes the axis $A^s$ and preserves the transverse orientation. It follows that $h(T_j) = T_j$ for all $j$. In this case we set $s = s_i^\pm$. The leaf $s$ is fixed by $h$ and there exists $k \neq 0$ such that $\gamma \pm T_i = T_{i \pm k}$. Thus $T_i$ is in between $\gamma^{-1}s$ and $\gamma s$ and hence, so is $c$. Recall here that $h$ preserves orientation.

Now, suppose instead that $c$ does not intersect $A^s$. Hence, there is a unique $i$ such that $s_{i-1}^+ \cup s_i^-$ separates $c$ from all other stable leaves in $A^s$. We again set $s := s_i^\pm$. As before, since $h$ fixes both $c$ and $A^s$, and preserves the transverse orientation, it must fix $s$ also. The same argument as above also shows that $c$ is between $\gamma^{-1}s$ and $\gamma s$.

So in any case, we obtained a stable leaf $s$ (chosen as a positive endpoint of one of the closed intervals $T_i$), fixed by $h$, and such that $c$ is between $\gamma^{-1}s$ and $\gamma s$. Notice that both $\gamma s$ and $\gamma^{-1}s$ are also fixed by $h$.

The leaf $\gamma^{-1}s$ is between $\gamma s$ and $\tilde{f}^{2m}(\gamma s) = \gamma^{-2n+1}s$ (assuming $n \geq 1$, otherwise between $\gamma s$ and $f^{-2m}(\gamma s)$). Hence the Hausdorff distance between $\gamma^{-1}s$ and $\gamma s$ is bounded above by a uniform constant $C > 0$, depending only on $f$ and $m$.

Thus we obtained that the fixed center leaf $c$, fixed by $h$, is in between two stable leaves, $\gamma s$ and $\gamma^{-1}s$, also fixed by $h$ and a bounded Hausdorff distance apart. Moreover, the leaves of $\mathcal{W}^{cs}_{\text{bran}}$ are Gromov-hyperbolic by Lemma 11.24. These are all the conditions needed to apply Lemma 4.15, which states that $c$ is coarsely contracting for $h$. □

Remark 11.31. Notice that neither Proposition 11.27 nor 11.30 proves that there is a periodic center leaf. We prove this in the next result. While it is very easy to produce periodic center leaves in the dynamically coherent situation, in the next result we consider the non dynamically coherent situation, and also we produce a periodic center leaf in the projection $\pi(L)$ of the center stable leaf $L$ in question. This is much stronger than obtaining a generic periodic center leaf, which a priori could be in any center stable leaf.

**Proposition 11.32.** Let $f : M \to M$ be a partially hyperbolic diffeomorphism homotopic to the identity and let $\tilde{f}$ be a good lift to $\tilde{M}$. Suppose that $\tilde{f}$ fixes
every leaf of the branching foliation $\widetilde{W}_{\text{bran}}^{cs}$. Let $L$ be a center stable leaf fixed by $\gamma \in \pi_1(M) \setminus \{\text{Id}\}$. Then there is an $f$-periodic center leaf in $\pi(L)$.

**Proof.** First notice that if one can prove the above result for a finite cover of $M$ and a finite power of $f$, then the same result directly follows for the original map and manifold. Thus, we may assume that $M$ is orientable, $f$ is orientation-preserving, and the branching foliations are both transversely orientable.

Given these assumptions, $L$ projects to an annulus in $M$. Let $\gamma$ be a generator of the stabilizer of $L$. If $\tilde{f}$ fixes a center leaf in $L$, then it would project to a center leaf fixed by $f$, proving the claim. So we assume that $\tilde{f}$ acts freely on the center leaf space in $L$. This implies that $\tilde{f}$ also acts freely on the stable foliation in $L$, and we can thus consider the stable axis of $\tilde{f}$.

Suppose first that the stable axis of $\tilde{f}$ is a countable union of intervals $\bigcup_{i \in \mathbb{Z}} I_i$. Since $\gamma$ also acts freely on the stable leaves, and commutes with $\tilde{f}$, they have the same axis (see Remark E.3). Since the axis is a countable collection of intervals, there must exists a pair of integers $n, m$ such that $h := \gamma^n \tilde{f}^m$ fixes one of the intervals, and hence, a stable leaf. If $m = 0$, then $\gamma^n$ has a fixed stable leaf, which is impossible. So $m \neq 0$, and the stable leaf projects to a periodic stable leaf in $M$. This periodic stable leaf thus contain a periodic point, and at least one center leaf through that point is then periodic. So the proposition is proved in that case.

Suppose now that the stable axis (of $\gamma$ or $\tilde{f}$) is a line. Then the assumptions of the Graph Transform Lemma H.1 are verified. So there exists a properly embedded curve $\hat{\eta}$ in $L$ which is invariant under $\tilde{f}$ and $\gamma$. Then Lemma H.3 applies and give a periodic center leaf, as claimed.

\[\square\]

### 11.8. Regulating pseudo-Anosov flows and translations.

We now want to extend the results from sections 8 and 9. That is, we want to understand the dynamics of a homeomorphism acting by translation on a branching foliation.

In order to be able to do that, we first need to build a regulating pseudo-Anosov flow transverse to the branching foliation.

The existence of such a flow is a relatively immediate consequence of the construction of the regulating flow and the fact that the branching foliation is well-approximated by foliations.

**Proposition 11.33.** Let $M$ be a hyperbolic 3-manifold and $\mathcal{F}$ a branching foliation well-approximated by foliations $\mathcal{F}_\epsilon$ such that $\mathcal{F}$ (and thus also $\mathcal{F}_\epsilon$ for small $\epsilon$) are $\mathbb{R}$-covered and uniform. Then, there exists a transverse and regulating pseudo-Anosov flow $\Phi$ for $\mathcal{F}$.

**Proof.** By Theorem D.3, for any $\epsilon$, there exists a pseudo-Anosov flow $\Phi_\epsilon$ transverse to and regulating for $\mathcal{F}_\epsilon$.

Now, as $\epsilon$ get small, the angle between leaves of $\mathcal{F}_\epsilon$ and leaves of $\mathcal{F}$ becomes arbitrarily small.

Then, since both $\mathcal{F}$ and $\mathcal{F}_\epsilon$ are $\mathbb{R}$-covered and uniform, for any leaf $L \in \mathcal{F}$, there exists two leaves $L_1$ and $L_2$ such that $L$ is in between $L_1$ and $L_2$. As $\Phi_\epsilon$ is regulating for $\mathcal{F}_\epsilon$, every orbit intersects both $L_1$ and $L_2$, thus $L$. So every orbit of $\Phi$, intersect every leaf of $\mathcal{F}$, that is, $\Phi_\epsilon$ is regulating for $\mathcal{F}$.

The fact that the flow $\Phi_\epsilon$ can be chosen transverse to $\mathcal{F}$ follows from the construction of $\Phi_\epsilon$ (see [Thu, Cal00, Fen02]). The flow $\Phi_\epsilon$ is build by blowing down certain laminations transverse to $\mathcal{F}_\epsilon$. Moreover these laminations are transverse.
to any foliation that are close enough to $F$ for a uniform angle. Since the angle between $F$ and $F_\epsilon$ gets arbitrarily small, $\Phi_\epsilon$ will also be transverse. For a continuous family of $\mathbb{R}$-covered foliations, this property is Corollary 5.3.22 of [Cal00].

Using the regulating pseudo-Anosov flow given by Proposition 11.33, all of section 8 works for a branching foliation without change. Thus we obtain Proposition 11.34.

**Proposition 11.34.** Let $M$ be a hyperbolic 3-manifold. Let $f: M \to M$ be a homeomorphism homotopic to the identity that preserves a (branching) foliation $F$. Suppose that $F$ is uniform and $\mathbb{R}$-covered, and that a good lift $\tilde{f}$ of $f$ acts as a translation on the leaf space of $F$. Let $\Phi$ be a transverse regulating pseudo-Anosov flow to $F$.

Then, for every $\gamma \in \pi_1(M)$ associated with a periodic orbit of $\Phi$, there is a compact $\tilde{f}_\gamma$-invariant set $T_\gamma$ in $M_\gamma$ which intersects every leaf of $\tilde{F}_\gamma$, where $M_\gamma = \tilde{M}/\langle \gamma \rangle$ and $\tilde{f}_\gamma: M_\gamma \to M_\gamma$ is the corresponding lift of $f$.

Moreover, if an iterate $\tilde{f}_\gamma^k$ of $\tilde{f}_\gamma$ fixes a leaf $L$ of $\tilde{F}_\gamma$, and $\gamma$ fixes all the prongs of this orbit, then the fixed set of $\tilde{f}_\gamma^k$ in $L$ is contained in $T_\gamma \cap L$ and has negative Lefschetz index.

Almost without any change, we also obtain the corresponding version of Proposition 9.1.

**Proposition 11.35.** Let $f$ be partially hyperbolic diffeomorphism in a hyperbolic 3-manifold which preserves a branching foliation $W^{cs}_{bran}$ tangent to $E^{cs}$. Assume that a good lift $\tilde{f}$ of $f$ acts as a translation on the foliation $W^{cs}_{bran}$ and let $\Phi$ be a transverse regulating pseudo-Anosov flow. Then, for every $\gamma \in \pi_1(M)$ associated to the inverse periodic orbit of $\Phi$ there are $n > 0, m > 0$ such that $h = \gamma^n \circ \tilde{f}^m$ fixes a leaf $L$ of $\tilde{F}_\gamma$.

**Proof.** The only difference with Proposition 9.1 is that we cannot say that the action of $h$ in the leaf space is expanding since collapsing of leaves may change the behavior. However, the same proof gives the existence of an interval in the leaf space which is mapped inside itself by $h^{-1}$ giving a fixed leaf as desired. □

**Remark 11.36.** Note that in the non dynamically coherent situation, the proof of Theorem B (done in section 9) does not give a contradiction: it could happen (and indeed happens in a situation with similar properties, see e.g., [BGHP17]) that having a fixed point in a leaf of the foliation, does not force the dynamics on the leaf space to be repelling around the leaf in terms of the action on the leaf space. This issue has previously appeared in this section, in particular in Lemma 11.15.

Notice that if one assumes the existence of a periodic center leaf, then we can easily prove a version of Theorem B in the non dynamically coherent setting.

**Proposition 11.37.** Let $f: M \to M$ be a partially hyperbolic diffeomorphism on a hyperbolic 3-manifold. Suppose that there exists a closed center leaf $c$ that is periodic under $f$. Then $f$ is a discretized Anosov flow.

**Proof.** We start by replacing $f$ by a power, so that $f$ becomes homotopic to the identity.

Let $\tilde{f}$ be a good lift of $f$. We will show that $\tilde{f}$ fixes every leaf of $\tilde{W}^{cs}_{bran}$ and $\tilde{W}^{cu}_{bran}$. Then, section 12 below will imply that the original $f$ (before taking a power) is dynamically coherent, hence the result will follow from Theorem B.
Suppose that \( \tilde{f} \) does not fix every leaf of, say, \( \tilde{W}_{cs}^{\text{bran}} \). Then Corollary 11.7 implies that the leaf space of \( \tilde{W}_{cs}^{\text{bran}} \) is \( \mathbb{R} \) and that \( \tilde{f} \) acts as a translation on it.

Let \( \tilde{c} \) be a lift of the periodic closed center leaf \( c \). Since \( c \) is periodic and \( \tilde{f} \) acts as a translation, there exists \( \gamma \in \pi_1(M) \), non-trivial such that \( \gamma(\tilde{c}) = \tilde{f}^k(\tilde{c}) \) for some \( k \). Now \( c \) is also closed, so there exists \( g \) (distinct from any power of \( \gamma \), since they do not act in the same way on the leaf space of \( \tilde{W}_{cs}^{\text{bran}} \)) such that \( g(\tilde{c}) = \tilde{c} \). Thus \( g \) and \( \gamma \) produce a \( \mathbb{Z}^2 \) subgroup in \( \pi_1(M) \), which is impossible since \( M \) is hyperbolic.

\[ \square \]

12. Double invariance implies dynamical coherence

In this section we show that if the center-stable and center-unstable branching foliations are minimal and leafwise fixed by a good lift \( \tilde{f} : \tilde{M} \to \tilde{M} \), then, \( f \) has to be dynamically coherent (i.e., the branching foliations do not branch). Therefore, we will be able to apply the results from the dynamically coherent setting.

Recall that the universal cover \( \tilde{M} \) of \( M \) is homeomorphic to \( \mathbb{R}^3 \) (since it admits a partially hyperbolic diffeomorphism, see Appendix F). We do not assume anything further on \( M \) in this section.

Recall also that a center leaf is a connected component of the intersection of a leaf of \( \tilde{W}_{cs}^{\text{bran}} \) and one of \( \tilde{W}_{cu}^{\text{bran}} \) (cf. Definition 10.6).

This section (and the proof of dynamical coherence) is split in three parts. First, in subsection 12.1, we show that, for an appropriate lift of \( M \) and power of \( f \), double invariance of the foliations implies that the center leaves are fixed. The lift and power we need to consider here is in order to have everything orientable and coorientable. Then, in section 12.2, we show that if a good lift fixes every center leaf, then it must be dynamically coherent. Finally, in section 12.3, we show that if a lift and power of a partially hyperbolic diffeomorphism is dynamically coherent and fixes the center leaves, then the original diffeomorphism is itself dynamically coherent (and a good lift of a power of it will fix every center leaf).

12.1. Center leaves are all fixed. In this section we recover the results of section 6.1 in the context of branching foliations. This will be the key to obtaining dynamical coherence (in section 12.2).

Proposition 12.1. Let \( f : M \to M \) be a partially hyperbolic diffeomorphism homotopic to the identity and admitting branching foliations \( W_{cs}^{\text{bran}}, W_{cu}^{\text{bran}} \) that are \( f \)-minimal. Suppose that a good lift \( \tilde{f} \) of \( f \) to \( \tilde{M} \) fixes every leaf of \( \tilde{W}_{cs}^{\text{bran}}, \tilde{W}_{cu}^{\text{bran}} \). Then, every center leaf is fixed by \( \tilde{f} \).

We stress again that the assumption of \( f \)-minimality is automatic when \( f \) is transitive or when \( M \) is hyperbolic or Seifert, see section 11.5)

To prove Proposition 12.1, as in the dynamically coherent setting, we need the following result.

Lemma 12.2. Suppose that the hypothesis of Proposition 12.1 are satisfied. Then either every center leaf is fixed by \( \tilde{f} \) or no center leaf is fixed by \( \tilde{f} \).

Assuming this lemma, it is easy to prove Proposition 12.1:

Proof of Proposition 12.1. Suppose that \( \tilde{f} \) fixes no center leaf. By Proposition 11.32 (together with Proposition B.2) there are periodic center leaves in \( M \). Then we can apply Proposition 11.27 first to \( \tilde{W}_{cs}^{\text{bran}} \) and then \( \tilde{W}_{cu}^{\text{bran}} \). The conclusion is that for every \( f \) periodic center leaf \( M \), the center leaf must be first coarsely
contracting by $f$ and then coarsely expanding by $f$. This is a contradiction. Hence $\widetilde{f}$ fixes a center leaf and Lemma 12.2 implies the proposition.

To prove the lemma we will explain the modifications one has to make in the proof of Lemma 6.4 to adapt it to the non dynamically coherent setting.

Proof of Lemma 12.2. Let

$$\text{Fix}_{\widetilde{f}} := \{ c : \widetilde{f}(c) = c \}.$$  

The first difference from the dynamically coherent setting is that we will not directly regard this set as a subset of $\widetilde{M}$ (because center leaves may merge).

However, it is not hard to see that the argument of Lemma 6.3 holds: If $c$ is a fixed center leaf in a center stable leaf $L$ in $\widetilde{M}$, then for any center leaf $c'$ in $L$ close enough to $c$ (for the topology of the center leaf space in $L$), there exists a strong stable leaf that intersect $c$, $c'$ and $\widetilde{f}(c')$. Now, since $\widetilde{f}$ fixes the center unstable leaves, $c'$ and $\widetilde{f}(c')$ are on the same center unstable leaf. Since no transversal can intersect the same leaf twice, it implies that $c' = \widetilde{f}(c')$.

Thus, we obtained that if $c$ is a fixed center leaf in a center stable leaf $L$ in $\widetilde{M}$, center leaves near $c$ in $L$ are also fixed. This is in the center leaf space of $L$, which is a 1-dimensional manifold.

The same argument evidently applies for center leaves near $c$ in its center unstable leaf.

Note that since a good lift $\widetilde{f}$ fixes every leaf of $\widetilde{W}_{\text{cs}}^{\text{bran}}$, then $f$ fixes every leaf of $W_{\text{cs}}^{\text{bran}}$. In particular $f$-minimality of $W_{\text{cs}}^{\text{bran}}$ is equivalent to minimality of $W_{\text{cs}}^{\text{bran}}$. Hence $W_{\text{cs}}^{\text{bran}}$ is minimal. Similarly for $W_{\text{cu}}^{\text{bran}}$.

We now assume that the set of fixed center leaves is non-empty and we want to show that all the center leaves are fixed.

To do this, we proceed as in Lemma 6.4: We show first that every center leaf in a center stable leaf (resp. center unstable leaf) which projects to an annulus has to be fixed (due to our orientability assumptions, leaves cannot project to a Möbius band). Then the same argument as in Lemma 6.4 applies to show that every center leaf has to be fixed.

Let $L$ be any center stable leaf that projects to an annulus. Let $\gamma$ be a generator of the isotropy group of $L$.

Since the set of fixed center leaves is open in the center leaf spaces of any center unstable leaf, minimality of $W_{\text{cs}}^{\text{bran}}$ implies that $L$ must have some fixed center leaves.

We will first prove that, if $f$ does not fix all center leaves in $L$, then some center leaves in $\pi(L)$ are periodic under $f$. Then we will show, as in Proposition 11.30, that any periodic leaf in $\pi(L)$ must be coarsely contracting. The same argument applied to the center-unstable leaves yields that periodic center leaves must also be coarsely expanding, a contradiction.

Since $\widetilde{f}$ cannot have fixed points (as $\widetilde{f}$ fixes all the leaves of $\widetilde{W}_{\text{cs}}^{\text{bran}}$ and $\widetilde{W}_{\text{cu}}^{\text{bran}}$), then $\widetilde{f}$ acts freely on the space of stable leaves in $L$.

We assume, for a contradiction, that not all center leaves in $L$ are fixed. Let $\text{Fix}_L$ be the set (in, $L^*_\gamma$, the center leaf space on $L$) of center leaves fixed by $\gamma$.

The set $\text{Fix}_L$ is open, and assumed not to be the whole of $L$. So let $c_1$ be any leaf in $\partial \text{Fix}_L$.

\footnote{Note that $f$-minimality and minimality are in fact always equivalent as long as the branching foliation does not have compact leaf and without assumptions on $f$, see Lemma F.6.}
The leaf \( c_1 \) is not fixed by \( \tilde{f} \), so \( \tilde{f}(c_1) \) is non-separated from \( c_1 \). Hence, there exists a (unique) stable leaf \( s_1 \), which separates \( \tilde{f}(c_1) \) from \( c_1 \) and makes a perfect fit with \( c_1 \) (see section 11.6.1 for the definition of perfect fits in the non dynamically coherent setting). Then \( \tilde{f}(s_1) \) makes a perfect fit with \( \tilde{f}(c_1) \). Because \( c_1 \) and \( \tilde{f}(c_1) \) are non separated from each other, \( s_1 \) and \( \tilde{f}(s_1) \) intersect a common transversal to the stable foliation. It follows that the stable axis of \( \tilde{f} \) acting on \( L \) is a line. Thus, since \( \gamma \) commutes with \( \tilde{f} \), the stable axis of \( \gamma \) is that same line (see Remark E.3). Moreover, both the stable leaves \( s_1 \) and \( \tilde{f}(s_1) \) are in the axis of \( \tilde{f} \).

Since the stable axis of \( \tilde{f} \) acting on \( L \) is a line, the Graph Transform argument (Lemma H.1) applies and we obtain a curve \( \hat{\eta} \), tangent to the center direction, that is fixed by both \( \gamma \) and \( \tilde{f} \).

As \( s_1 \) makes a perfect fit with \( c_1 \) and \( s_1 \) intersects \( \hat{\eta} \), we deduce that there exists a stable leaf \( s \) that intersects both \( c_1 \) and \( \hat{\eta} \). Let \( x = s \cap \hat{\eta} \) and \( y = s \cap c_1 \). We denote by \( J \) the segment of \( s \) between \( x \) and \( y \).

Since \( \hat{\eta} \) projects down to a closed curve \( \pi(\hat{\eta}) \), and \( \tilde{f} \) decreases stable lengths, there exist \( n_1, n_2 \in \mathbb{Z} \) and \( m_1, m_2 \in \mathbb{N} \) as large as we want such that the four points \( \gamma^{n_1} \tilde{f}^{m_1}(x), \gamma^{n_1} \tilde{f}^{m_1}(y), \gamma^{n_2} \tilde{f}^{m_2}(x) \) and \( \gamma^{n_2} \tilde{f}^{m_2}(y) \) are all in a disk of radius as small as we want.

Suppose now that \( \gamma^{n_1} \tilde{f}^{m_1}(c_1) \neq \gamma^{n_2} \tilde{f}^{m_2}(c_1) \). Then, up to switching \( n_1, m_1 \) and \( n_2, m_2 \), we obtain that \( \gamma^{n_2} \tilde{f}^{m_2}(c_1) \) intersects \( \gamma^{n_1} \tilde{f}^{m_1}(J) \). This is in contradiction with the fact that \( c_1 \) is in \( \partial \text{Fix}_L \) which is invariant by both \( \tilde{f} \) and \( \gamma \).

Thus \( \gamma^{n_1} \tilde{f}^{m_1}(c_1) = \gamma^{n_2} \tilde{f}^{m_2}(c_1) \). In other words, \( c_1 \) is fixed by the map \( h = \gamma^n \tilde{f}^m \) for some \( n, m \) integers, \( m > 0 \). (Although not useful for the rest of the proof, one can further notice that \( \hat{\eta} \) and \( c_1 \) intersect, as \( h \) decreases the length of \( J \) by forward iterations and both \( c_1 \) and \( \hat{\eta} \) are fixed by \( h \).)

Now recall that we built above a stable leaf \( s_1 \) making a perfect fit with \( c_1 \). And, by our choice of \( s_1 \), the center leaf \( c_1 \) is in between \( s_1 \) and \( s_2 := \tilde{f}^{-1}(s_1) \).

The leaves \( s_1 \) and \( s_2 \) are both fixed by \( h \) (since \( c_1 \) is), and a bounded distance apart, so Lemma 4.15 holds and we deduce that \( c_1 \), as well as any other center leaf \( c \) that is in between \( s_1 \) and \( s_2 \) must be coarsely contracting.

Note now that any center leaf \( c \) in \( L \) that is fixed by some \( h' = \gamma^n \tilde{f}^m \) is separated from \( \text{Fix}_L \) by a center leaf \( c_1 \subset \partial \text{Fix}_L \) as above. Hence, we proved that every non-fixed periodic leaf in \( L \) is coarsely contracting.

Therefore, the same argument applied to the center unstable leaf containing \( c_1 \) shows that \( c_1 \) must also be coarsely expanding, a contradiction.

So we obtained that every center stable or center unstable leaf \( L \) which is fixed by some non trivial element of \( \pi_1(M) \) has all of its center leaves fixed by \( \tilde{f} \). Since \( \text{Fix}_f \) is open (in the center leaf space), minimality of the foliations implies that it contains every center leaf, as in the end of the proof of Lemma 6.4.

\[ \Box \]

12.2. Dynamical coherence. We now want to prove dynamical coherence provided that a good lift fixes every center leaf. We start with the following:

**Lemma 12.3.** Suppose that \( \tilde{f} \) fixes every leaf of the center foliation in \( \tilde{M} \). Then there is a global bound on the length from \( x \) to \( \tilde{f}(x) \) in any center leaf containing \( x \).

In the dynamically coherent case this was very easy as the center curves form an actual foliation and there is a local product picture near any compact segment. We have to be more careful in the non dynamically coherent setting.
Proof. We assume the conclusion of the lemma fails. Then there exists a sequence \( x_i \) of points in \( \widetilde{M} \) contained in center leaves \( c_i \) such that the length in \( c_i \) from \( x_i \) to \( \tilde{f}(x_i) \) diverges to infinity. Notice that this length depends not only on \( x_i \) but also on \( c_i \) since there may be many center leaves through \( x_i \). We denote by \( e_i \) the segment in \( c_i \) from \( x_i \) to \( \tilde{f}(x_i) \).

Up to acting by covering translations we can assume that the \( x_i \) converge to a point \( x \in \widetilde{M} \). Let \( L_i \) and \( U_i \) be respectively a center stable and center unstable leaves containing \( c_i \). Up to considering a subsequence, we may assume that \( L_i \) converges to a center stable leaf \( L \) containing \( x \) (see condition (iv) of Definition 10.2). Similarly, we can further assume that \( U_i \) converges to some center unstable leaf \( U \), with \( x \in U \).

For \( i \) large enough, all the leaves \( L_i \) intersect a small unstable segment in \( u(x) \). The set of center stable leaves intersecting this segment is a also a segment (even though many different leaves may intersect a given point in \( u(x) \)). Hence we may assume that \( L_i \) is weakly monotone, and so is \( U_i \). Let \( c \) be the center leaf through \( x \) contained in \( L \cap U \). Then \( \tilde{f}(x) \in c \), and we call \( e \) the segment in \( c \) from \( x \) to \( \tilde{f}(x) \).

Suppose first that \( L_i = L \) for all big \( i \). So we may assume \( L_i = L \) for all \( i \). Then the center leaves \( c_i \) are all in \( L \) and, for \( i \) big enough, intersect \( s(x) \). Hence the leaves \( c_i \) are, for \( i \) big enough, contained in an interval of the center leaf space in \( L \). In addition they are converging to \( c \) which is a center leaf through \( x \) and \( \tilde{f}(x) \). This implies that the length of \( e_i \) is converging to the length of \( e \) and hence the length of \( e_i \) is bounded in \( i \). Contradiction.

Suppose now that the \( L_i \) are all distinct from \( L \). Notice that the points \( x_i \), and \( \tilde{f}(x_i) \) are all in a compact region of \( \widetilde{M} \). Since \( L_i \) converges to \( L \), we have that \( u(x_i) \) intersects \( L \) for big enough \( i \). We call this nearby intersection \( y_i \). Likewise \( u(\tilde{f}(x_i)) \) intersects \( L \) in \( \tilde{f}(y_i) \). We want to push the center segments \( c_i \) contained in \( U_i \cap L_i \) along unstable segments to center segments in \( U_i \cap L \).

For \( i \) big enough, both \( x_i \) and \( \tilde{f}(x_i) \) are very near \( L \). Thus, their unstable leaves \( u(x_i) \) and \( u(\tilde{f}(x_i)) \) both intersect \( L \). Let \( y_i \) be the intersection of \( u(x_i) \) with \( L \) (recall that this intersection is unique as the center stable branching foliation is approximated by a taut foliation). Then \( \tilde{f}(y_i) \) is the intersection of \( u(\tilde{f}(x_i)) \) with \( L \) (since \( L \) is fixed by \( \tilde{f} \)). Then the intersection of the unstable saturation of \( e_i \) with \( L \) is a compact segment inside a center leaf between \( y_i \) and \( \tilde{f}(y_i) \) (since \( \tilde{f} \) fixes every center leaf). Let \( b_i \) be this segment between \( y_i \) and \( \tilde{f}(y_i) \). The segments \( b_i \) also converge to \( e \), so the previous paragraph shows that the lengths of the \( b_i \) are bounded. Since the distance between \( x_i \) and \( y_i \) converges to zero, this in turn implies that the lengths of the segments \( e_i \) are themselves bounded. Which contradicts our assumption and finishes the proof.

\[ \Box \]

**Lemma 12.4.** Suppose \( \tilde{f} \) fixes every leaf of the center foliation in \( \widetilde{M} \). Assume \( c_1, c_2 \) are different center leaves in the same leaf \( L \) of \( \widetilde{W}_{\mathrm{bran}}^{cs} \). Then, \( c_1 \cap c_2 = \emptyset \).

**Proof.** Suppose not, there is \( x \in c_1 \cap c_2 \) but \( c_1 \neq c_2 \). Then \( \tilde{f}(x) \) is also in \( c_1 \cap c_2 \). If \( c_1 \) coincides with \( c_2 \) in their respective segments from \( x \) to \( \tilde{f}(x) \), then applying iterates of \( \tilde{f} \) implies that \( c_1 = c_2 \), contrary to assumption.

So we may assume that \( x \) is a boundary point of an open interval \( I \) in, say, \( c_1 \) which is disjoint from \( c_2 \), but such that both endpoints are in \( c_2 \). Then \( c_1 \cup c_2 \) bounds a bigon \( B \) with endpoints \( x, y \) and a “side” in \( I \). All center segments in \( B \) pass through \( x \) and \( y \) and they have bounded length (by Lemma 12.3). Each
stable segment intersecting $I$ also intersects the other “boundary” component of $B$. See figure 19.

Figure 19. Two centers that merge. The bound on the distance between $x$ and $\tilde{f}(x)$ forces a behavior like the figure.

The stable lengths grow without bound under negative iterates of $\tilde{f}$. Hence, since a stable segment can intersect a local foliated disk of the stable foliation in $L$ only in a bounded length, it follows that the diameter in $\tilde{f}^n(L)$ of $\tilde{f}^n(B)$ grows without bound as $n$ goes to $-\infty$. But the length of the center segments in $\tilde{f}^n(B)$ are all bounded according to Lemma 12.3. Moreover, between any two points in $\tilde{f}^n(B)$ there exists a path along (at most) two center leaves (one just follows the center leaf to one of the endpoint and then switch to the appropriate other center leaf). Thus the diameter is bounded, which is a contradiction. □

Thus we deduce what we wanted to obtain in this section.

**Corollary 12.5.** If a good lift $\tilde{f}$ fixes every center leaf, then, $f$ is dynamically coherent.

**Proof.** By Proposition F.7 it is enough to show that the leaves of the branching foliations do not merge.

Assume that two center unstable leaves $U_1$ and $U_2$ merge. Let $L$ be a center stable leaf intersecting $U_1$ and $U_2$ at the merging, i.e., $L$ is a leaf through a point $x$ such that the unstable leaf through $x$ is a boundary component of $U_1 \cap U_2$. Then, connected components of $U_1 \cap L$ and $U_2 \cap L$ gives two center leaves that intersect but do not coincide. This contradicts Lemma 12.4. A symmetric argument gives that two center stable leaf cannot merge either, proving dynamical coherence of $f$. □

12.3. Dynamical coherence without taking lifts and iterates. We now want to prove that, if a finite lift and finite power of a partially hyperbolic diffeomorphism is dynamically coherent, then the original diffeomorphism is itself dynamically coherent. Although we do not know how to prove it in this generality, we show it when a good lift of the dynamically coherent lift fixes every center leaf, which is enough for our purposes.

We start by showing a uniqueness result for the pairs of the center stable and center unstable foliations under some conditions.
Lemma 12.6. Let $g : M \rightarrow M$ be a dynamically coherent partially hyperbolic diffeomorphism homotopic to the identity. Let $W^{cs}$ and $W^{cu}$ be $g$-invariant foliations tangent to $E^{cs}$ and $E^{cu}$ respectively. Let $W^c$ be the center foliation associated with $W^{cs}$ and $W^{cu}$ (defined as in Definition 10.6), and assume that there exists a good lift $\tilde{g}$ which fixes all the leaves of $W^c$.

Suppose that $W^{cs}_1$ and $W^{cu}_1$ are two $g$-invariant foliations tangent respectively to $E^{cs}$ and $E^{cu}$. Suppose that $\tilde{g}$ also fixes all the leaves of the center foliation $\tilde{W}^c_1$ associated with $W^{cs}_1$ and $W^{cu}_1$.

Then $W^{cs} = W^{cs}_1$ and $W^{cu} = W^{cu}_1$.

Note that if the foliations $W^{cs}$, $W^{cu}$, $W^{cs}_1$ and $W^{cu}_1$ are assumed to be $g$-minimal, then Proposition 6.2 imply that the hypothesis of the lemma are satisfied.

Proof. The argument is similar to the one made in Lemma 12.4.

Let $\tilde{W}^{cs}_1, \tilde{W}^{cu}_1$ be two $g$-equivariant foliations as in the lemma. Recall that the center foliation $\tilde{W}^c_1$ is defined by taking the connected components of intersections of leaves of $\tilde{W}^{cs}_1$ and $\tilde{W}^{cu}_1$.

Since every leaf of both $\tilde{W}^c$ and $\tilde{W}^c_1$ are fixed by $\tilde{g}$, Lemma 12.3 implies that $\tilde{g}$ moves points a uniformly bounded amount in both center foliations.

Consider, for a contradiction, a point $x \in \tilde{M}$ such that $\tilde{W}^c(x) \neq \tilde{W}^c_1(x)$ (note that we are dealing here with actual foliations, not branching ones, so this notation make sense). Without loss of generality, we can choose $x$ so that the leaves $L := \tilde{W}^{cs}(x)$ and $L_1 := \tilde{W}^{cs}_1(x)$ do not coincide in any neighborhood of $x$.

Let $c$ and $c_1$ be the center leaves obtained as the connected components of $L \cap F$ and $L_1 \cap F$ containing $x$.

By assumption, both $c$ and $c_1$ are fixed by $\tilde{g}$, so we are in the exact same set up as in the proof of Lemma 12.4. Thus we deduce that $c = c_1$, a contradiction. $\square$

We can now state and prove the aim of this section.

Proposition 12.7. Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism such that $f^k$ is homotopic to the identity for some $k > 0$. Let $\tilde{M}$ be a finite cover of $M$. Let $\tilde{g}$ be a lift to $\tilde{M}$ of a homotopy of $f^k$ to the identity.

Suppose that $\tilde{g}$ is dynamically coherent and that there exists a good lift $\tilde{g}$ of $g$ that fixes all the center leaves. Then, $f$ is dynamically coherent.

Proof. First we notice that the assumptions of the proposition will be verified for any further finite cover $\tilde{M}$ of $M$ (because one can take a further lift $\tilde{g}$ of $g$ to $\tilde{M}$, it is dynamically coherent and $\tilde{g}$ is a good lift of $\tilde{g}$ too). Hence, without loss of generality, we may and do assume that $\tilde{M}$ is a normal cover of $M$.

Let $\tilde{W}^{cs}$ and $\tilde{W}^{cu}$ be the lifts to $\tilde{M}$ of the center stable and center unstable foliations of $g$. Our goal is to show that these foliations are $\pi_1(M)$-invariant, thus decending to foliations in $M$, and that these projected foliations are $f$-invariant.

Notice that $\tilde{g}$ fixes each leaf of $\tilde{W}^{cs}$ and $\tilde{W}^{cu}$.

The map $g$ is obtained from a lift of a homotopy of $f^k$ to the identity. Lifting that homotopy further to $\tilde{M}$, we get a good lift $\tilde{f}^k$ of $f^k$ that is also a lift (and hence a good lift) of $g$ to $\tilde{M}$. As both $\tilde{g}$ and $\tilde{f}^k$ are good lifts of $g$, there exists $\beta \in \pi_1(\tilde{M}) \subset \pi_1(M)$ such that $\tilde{g} = \beta \tilde{f}^k$. (Note however that $\tilde{g}$ is not necessarily a good lift of $f^k$ as $\tilde{g}$ only commutes with elements of $\pi_1(\tilde{M})$ and not $\pi_1(M)$.)
Moreover, both $\tilde{g}$ and $f^k$ move points a bounded distance in $\tilde{M}$, hence so does $\beta = \tilde{g}(f^k)^{-1}$. Lemma A.3 then implies that either $\beta$ is the identity or $M$ is Seifert (and $\beta$ is either the identity or a power of a regular fiber).

We split the rest of the proof in these two cases.

**Case 1** – Suppose that $M$ is not a Seifert fibered space.

Then $\beta$ is the identity, which means that $\tilde{\gamma} = \tilde{f}^k$. Let $\gamma$ be a deck transformation in $\pi_1(M)$. Define the foliations $F^{cs}_\gamma := \gamma \tilde{W}^{cs}$, $F^{cu}_\gamma := \gamma \tilde{W}^{cu}$, and $F^c := \gamma \tilde{W}^c$. The leaves of these foliations are all fixed by $\tilde{g}$ because $\gamma$ commutes with $\tilde{f}^k = \tilde{g}$. In particular, Lemma 12.6 then implies that $\gamma \tilde{W}^{cs} = \tilde{W}^{cs}$ and $\gamma \tilde{W}^{cu} = \tilde{W}^{cu}$. Since this is true for any element of $\pi_1(M)$, these foliations descend to foliations $W^{cs}_M, W^{cu}_M$ in $M$.

Now we need to show that $W^{cs}_M, W^{cu}_M$ are also $f$-invariant. Equivalently, we need to show that $\tilde{W}^{cu}$ and $\tilde{W}^{cs}$ are invariant by any lift $f_1$ of $f$ to $\tilde{M}$.

Let $f_1$ be a lift of $f$ to $\tilde{M}$. Notice that $f$ may not be homotopic to the identity, so $f_1$ is not assumed to be a good lift. Let $F^{cs}_1 := f_1(\tilde{W}^{cs})$ and $F^{cu}_1 := f_1(\tilde{W}^{cu})$.

We will first show that $f_1$ and $\tilde{g}$ commute. Both $f_1 \tilde{g}$ and $\tilde{g} f_1$ are lifts of the map $f^{k+1}$ to $\tilde{M}$. So $(\tilde{g})^{-1}(f_1)^{-1}\tilde{g} f_1$ is a deck transformation $\gamma \in \pi_1(M)$. As $\tilde{g}$ moves points a bounded distance, we have that $d(f_1(y), \tilde{g} f_1(y))$ is bounded in $\tilde{M}$. In addition, $f_1$ has bounded derivatives so $d(y, (f_1)^{-1}\tilde{g} f_1(y))$ is also bounded in $\tilde{M}$. So using again that $\tilde{g}$ is a good lift, we deduce that $d(y, (\tilde{g})^{-1}(f_1)^{-1}\tilde{g} f_1(y))$ is bounded in $\tilde{M}$.

Hence $\gamma$ is a deck transformation that moves points a bounded distance. Applying Lemma A.3 again gives that $\beta$ is the identity (since $M$ is not Seifert). Hence $f_1$ and $\tilde{g}$ commute.

Since $\tilde{g}$ fixes every leaf of $\tilde{W}^c$ (the center foliation in $\tilde{M}$) and commutes with $f_1$, we deduce that $\tilde{g}$ fixes every leaf of $f_1(\tilde{W}^c)$. We can again apply Lemma 12.6 to get that $f_1(\tilde{W}^{cs}) = \tilde{W}^{cs}$ and $f_1(\tilde{W}^{cu}) = \tilde{W}^{cu}$. That is, the foliations $\tilde{W}^{cs}$ and $\tilde{W}^{cu}$ are $f_1$-invariant. Since this holds for any lift of $f$, it implies that $W^{cs}_M$ and $W^{cu}_M$ are $f$-invariant. Hence $f$ is dynamically coherent with foliations $W^{cs}_M, W^{cu}_M$. This completes the proof when $M$ is not Seifert fibered.

**Case 2** – Assume that $M$ is Seifert fibered.

In this case, Lemma A.3 implies that $\beta = \tilde{g}(f^k)^{-1}$ is either the identity or represent a power of a regular fiber of the Seifert fibration. In any case, $\beta$ is in a normal subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z}$. Moreover, as proved earlier, $\beta \in \pi_1(M)$.

Let $\gamma \in \pi_1(M)$ be any deck transformation. As before, consider the foliations $F^{cs}_\gamma := \gamma \tilde{W}^{cs}$ and $F^{cu}_\gamma := \gamma \tilde{W}^{cu}$.

We first claim that these foliations are $\tilde{g}$-invariant. We show this for $F^{cs}_\gamma$ the other being analogous. Let $L \in \tilde{W}^{cs}$. We have

$$\tilde{g}(\gamma L) = \beta f^k(\gamma L) = \beta \gamma f^k(L) = \gamma \beta^{\pm 1} f^k(L).$$

Notice that both $f^k$ (because it is a lift of $g$) and $\beta$ (because it belongs to $\pi_1(M)$ and the foliation $W^{cs}$ is defined in $\tilde{M}$) preserve the foliation $\tilde{W}^{cs}$. It follows that $\beta^{\pm 1} f^k(L) \in \tilde{W}^{cs}$, so

$$\tilde{g}(\gamma L) = \gamma \beta^{\pm 1} f^k(L) \in F^{cs}_\gamma.$$
Thus $F^c_γ$ is $\tilde{g}$-invariant.

We now want to show that the foliations $\mathcal{F}^{cs}_γ$, $\mathcal{F}^{cu}_γ$ and $\mathcal{F}^c_γ := γ\mathcal{W}^c$ are all leafwise fixed by $\tilde{g}$.

Since $\tilde{M}$ was chosen to be a normal cover of $M$, any element $γ ∈ π_1(M)$ can be thought of as a diffeomorphism of $\tilde{M}$. Hence we can consider the foliation $\tilde{F}^{cs}_γ := γ\mathcal{W}^{cs}$ in $\tilde{M}$. Note that $\tilde{F}^{cs}_γ$ is tangent to the center stable distribution $E^{cs} ⊂ T\tilde{M}$, since $γ$ preserves the tangent bundle decomposition, as it is defined by $f$ in $M$. The argument above shows that $\tilde{F}^{cs}_γ$ is $g$-invariant.

Thus, we can consider $g$ to be a dynamically coherent diffeomorphism for the pair of transverse foliations $\tilde{F}^{cs}_γ$ and $\mathcal{W}^{cu}$. Moreover, $g$ is homotopic to the identity and the good lift $\tilde{g}$ fixes every leaf of $\tilde{\mathcal{W}}^{cu}$. Since $\tilde{M}$ is Seifert, Theorem 5.1 implies that $\tilde{g}$ must also fix every leaf of $\tilde{F}^{cs}_γ$.

The symmetric argument show that $\tilde{F}^{cu}_γ$ is also fixed by $\tilde{g}$. So we can apply Proposition 3.15 to both $\tilde{F}^{cs}_γ$ and $\tilde{F}^{cu}_γ$, implying that they are $g$-minimal. Hence, the center foliation $\mathcal{F}^c_γ$ is fixed by $\tilde{g}$, thanks to Proposition 6.2.

Since all the leaves of $\mathcal{F}^c_γ$ are fixed by $\tilde{g}$, we can finally apply Lemma 12.6 to deduce that $\tilde{F}^{cs}_γ = \mathcal{W}^{cs}$ and $\tilde{F}^{cu}_γ = \mathcal{W}^{cu}$. As this is true for any $γ$, the foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ descend to foliations $\mathcal{W}^{cs}_M$ and $\mathcal{W}^{cu}_M$ on $M$ in this case too.

We now again have to show that $\mathcal{W}^{cs}_M$ and $\mathcal{W}^{cu}_M$ are $f$-invariant. The argument is the same for both foliations, so we only deal with $\mathcal{W}^{cs}_M$.

We start with a preliminary step. Let $f_∗$ be the automorphism of $π_1(M)$ induced by $f$. Let

$$A := π_1(\tilde{M}) \cap f_∗(π_1(\tilde{M})) \cap \cdots \cap (f_*)^{k-1}(π_1(\tilde{M})).$$

The set $A$ is a finite index, normal subgroup of $π_1(M)$. Moreover, as $f^k$ is homotopic to the identity, $f_*(A) = A$.

As we remarked at the beginning of the proof, we can without loss of generality prove the result for any further finite cover of $\tilde{M}$. Thus we choose if necessary a further cover so that $π_1(\tilde{M}) = A$. Since $f_*(A) = A$, the map $f$ lifts to a homeomorphism $\hat{f}$ of $\tilde{M}$.

As in the first case, we let $f_1$ be an arbitrary lift of $\hat{f}$ to $\tilde{M}$ and we define $\mathcal{F}^{cs}_1 := f_1(\mathcal{W}^{cs})$ and $\mathcal{F}^{cu}_1 := f_1(\mathcal{W}^{cu})$. (Note that $f_1$ is in particular also a lift of $f$.)

Note as before that both $\tilde{g}f_1$ and $f_1\tilde{g}$ are lifts of $f^{k+1}$, and $\tilde{g}f_1(\tilde{g})^{-1}(f_1)^{-1}$ is a bounded distance from the identity (because $\tilde{g}$ is and $f_1$ has bounded derivatives). So $δ := \tilde{g}f_1(\tilde{g})^{-1}(f_1)^{-1}$ is an element of $π_1(M)$ a bounded distance from identity.

By Lemma A.3, $δ$ represents a power of a regular fiber of the Seifert fibration, so is in the normal $\mathbb{Z}$ subgroup of $π_1(M)$ (note that since $π_1(M)$ is not virtually nilpotent, there exists a unique Seifert fibration on $M$, see Appendix A).

In addition $\tilde{g}f_1$ and $f_1\tilde{g}$ are also lifts of the homeomorphisms $g\hat{f}$ and $\hat{f}g$ in $\tilde{M}$ to $\tilde{M}$. Hence $δ$ is in $π_1(M)$.

Using once more the arguments above, we get that $(f_1)^{-1}δf_1(δ)^{-1}$ is a bounded distance from the identity, and projects to the identity in $M$ (and in $\tilde{M}$), hence it is a deck transformation $η$ also contained in the $\mathbb{Z}$ normal subgroup of $π_1(M)$. Thus $δ$ and $η$ commute. Moreover, $η$ is also in $π_1(M)$.

Now we can show that $\tilde{g}$ preserves $\mathcal{F}^{cs}_1$: Let $L$ in $\mathcal{W}^{cs}$. Then

$$\tilde{g}(f_1(L)) = δf_1(\tilde{g}(L)) = δf_1(L) = f_1(ηδ(L)).$$
Here $\eta \delta(L)$ is in $\tilde{W}^{cs}$, because $L$ is in $\tilde{W}^{cs}$ and $\eta \delta$ is in $\pi_1(M)$. Hence $\tilde{f}_1(\eta \delta L)$ is in $f_1(\tilde{W}^{cs})$ so $\tilde{g}$ preserves $F_1^{cs}$.

What we proved implies that $g$ preserves $\tilde{f}(W^{cs})$ in $\hat{M}$. Now consider the pair of foliations $\tilde{f}(W^{cs})$ and $W^{cu}$. They are both invariant by $g$, so $g$ is dynamically coherent for this particular pair of foliations, and $\tilde{g}$ fixes the leaves of $W^{cu}$. So once again, as $\hat{M}$ is Seifert, Theorem 5.1 implies that $\tilde{g}$ must also fix every leaf of $f_1(\tilde{W}^{cs})$.

The symmetric argument implies that $\tilde{g}$ fixes every leaf of $f_1(\tilde{W}^{cu})$. Once again, $\hat{M}$ being Seifert implies that all the foliations are $g$-minimal (Proposition 3.15). Hence $\tilde{g}$ also fixes the center foliation $f_1(\tilde{W}^{c})$ (Proposition 6.2). So Lemma 12.6 applies and we deduce that $f_1(\tilde{W}^{cs}) = W^{cs}$ and $f_1(\tilde{W}^{cu}) = W^{cu}$.

In particular, $f$ preserves the foliations $W^{cs}_M$ and $W^{cu}_M$ as wanted. So $f$ is dynamically coherent. □

13. Proof of Theorem A

In this section, we want to finish the proof of Theorem A. That is, $f : M \rightarrow M$ is assumed to be a partially hyperbolic diffeomorphism homotopic to identity in a Seifert manifold, and we need to show that a power of $f$ is a discretized Anosov flow.

We first fix a finite cover $\hat{M}$ of $M$ so that $\hat{M}$ is orientable, and so are all the bundles. Then, up to a finite power, a lift $g$ will preserve the orientations of the bundles. More precisely, there exists some integer $k > 0$ such that the lift $g$ obtained by lifting a homotopy of $f^k$ to the identity preserves the orientations.

Thanks to Theorem 10.4, there are branching foliations $W^{cs}_{bran}$ and $W^{cu}_{bran}$ in $\hat{M}$ that are preserved by $g$.

In order to finish the proof of Theorem A, we just need one more lemma.

Lemma 13.1. There exists a lift $\tilde{g}$ of an iterate of $g$ that fixes every leaf of $\tilde{W}^{cs}_{bran}$ and also fixes every leaf of $\tilde{W}^{cu}_{bran}$.

Postponing the proof of the lemma, we can finish the proof.

Proof of Theorem A. According to Lemma 13.1, there exists a lift $\tilde{g}$ of a power of $g^t$ of $g$ that fixes the leaves of both $\tilde{W}^{cs}_{bran}$ and $\tilde{W}^{cu}_{bran}$. Then Proposition 12.1 implies that $\tilde{g}$ fixes every center leaf. Thus Corollary 12.5 gives that $g^t$ is dynamically coherent. Then Proposition 12.7 tells us that $f$ is also dynamically coherent. So Theorem 7.3 applies to $f$ and gives that a finite power of $f$ is a discretized Anosov flow.

So all we have left to do is prove Lemma 13.1, which we now do.

13.1. Proof of Lemma 13.1. First, recall that in Section 7, we showed that it was always possible in a Seifert manifold to choose a convenient good lift. More precisely, we can restate Proposition 7.1 in our current setting and obtain

Proposition 13.2. There exists a good lift of an iterate of $g$ which fixes a leaf (and therefore every leaf) of $\tilde{W}^{cs}_{bran}$.

Proof. As stated in Remark 7.2, the proof of Proposition 7.1 works in the non dynamically coherent case. The only change needed is to replace the words foliations by branching foliations. Note also that Proposition 7.1 requires the Seifert fibration to be orientable. This is implied by our assumptions: Indeed,
\( \hat{M} \) is orientable, all the bundles are orientable and \( \mathcal{W}^{cs}_{\text{bran}} \) is a horizontal foliation (see Theorem F.3). Thus the Seifert fibration is orientable. □

Using Proposition 13.2, the lemma follows readily.

**Proof of Lemma 13.1.** First, using Proposition 13.2 we consider a good lift \( \tilde{g} \) of an iterate \( g^i \) that fixes every leaf of \( \mathcal{W}^{cs}_{\text{bran}} \). Suppose this lift fixes one center unstable leaf. Then Proposition 11.14 gives that \( \mathcal{W}^{cu}_{\text{bran}} \) is \( g^i \)-minimal. So Corollary 11.7 implies that \( \tilde{g}^i \) also fixes every leaf of \( \mathcal{W}^{cu}_{\text{bran}} \).

Thus we can suppose for a contradiction that \( \tilde{g}^i \) fixes no center unstable leaf. Therefore no center leaf can be fixed by \( \tilde{g} \). Applying Proposition 11.27 we deduce that every periodic center leaf of \( g \) has to be coarsely contracting.

Exchanging roles, and applying Proposition 13.2 to the center unstable branching foliation we deduce that every periodic center leaf for \( g \) must be coarsely expanding. Notice that, although the lifts may be different, the coarsely expanding and coarsely contracting behavior is for periodic center leaves of the original map \( g \) for both \( \mathcal{W}^{cs}_{\text{bran}} \) and \( \mathcal{W}^{cu}_{\text{bran}} \).

As there must be at least one such periodic center leaf (cf. Proposition 11.32) this gives a contradiction. So there exists a good lift of an iterate of \( g \) that fixes leaves of both \( \mathcal{W}^{cs}_{\text{bran}} \) and \( \mathcal{W}^{cu}_{\text{bran}} \). □

14. Translations in hyperbolic 3-manifolds

In this section, we will further the study started in Section 8 (and its generalization to the branching foliation case done in subsection 11.8) of a homeomorphism acting on a branching foliation by translation (when lifted to \( \hat{M} \)). This will allow us to prove that mixed behavior is impossible even for non dynamically coherent partially hyperbolic diffeomorphisms on a hyperbolic 3-manifold.

We start by recalling the setting. Let \( f : M \to M \) be a (not necessarily dynamically coherent) partially hyperbolic diffeomorphism on a hyperbolic 3-manifold. Up to replacing \( f \) by a power, we assume that it is homotopic to the identity. Up to taking a further iterate of \( f \) and a lift to a finite cover of \( M \), we can assume that \( f \) admits branching foliations, and that the good lift \( \tilde{f} \) acts as a translation on the leaf space of \( \mathcal{W}^{cs}_{\text{bran}} \).

Let \( \Phi^{cs} \) be a transverse regulating pseudo-Anosov flow to \( \mathcal{W}^{cs}_{\text{bran}} \) given by Proposition 11.33. This flow is fixed throughout the discussion.

Then Proposition 11.35 shows that, for any periodic orbit of \( \Phi^{cs} \), there exists a center stable leaf periodic by \( f \).

### 14.1. Periodic center rays

We will now produce rays in periodic center leaves which are expanding. A ray in \( L \) is a proper embedding of \([0, \infty)\) into \( L \). We say that a ray is a center ray if it is contained in a center leaf. So a center ray \( c_x \) is the closure in \( L \) of a connected component of \( c \setminus \{x\} \) where \( c \) is a center curve and \( x \in c \).

Let \( \gamma \) in \( \pi_1(M) \) be associated with a periodic orbit \( \delta_0 \) of the pseudo-Anosov flow \( \Phi^{cs} \). Let \( L \) be a leaf (given by Proposition 11.35) of \( \mathcal{W}^{cs}_{\text{bran}} \) fixed by \( h := \gamma^n \circ \tilde{f}^m \), with \( m > 0 \).

A center ray \( c_x \) is expanding if \( h(c_x) = c_x \) and \( x \) is the unique fixed point of \( h \) in \( c_x \) and every \( y \in c_x \setminus \{x\} \) verifies that \( h^{-n}(y) \to x \) as \( n \to +\infty \).

**Proposition 14.1.** Assume that a good lift \( \tilde{f} \) of \( f \) acts as a translation on the (branching) foliation \( \mathcal{W}^{cs}_{\text{bran}} \). Let \( \Phi^{cs} \) be a regulating transverse pseudo-Anosov flow. Let \( \gamma \) in \( \pi_1(M) \) associated with a periodic orbit \( \delta_0 \) of \( \Phi^{cs} \). Let \( L \) be a leaf
of \( \tilde{W}^{\text{cs}}_{\text{bran}} \) fixed by \( h = \gamma^n \circ \tilde{f}^m \), where \( m > 0 \). Assume that \( \gamma \) fixes all prongs of a lift of \( \delta_0 \) to \( \tilde{M} \). Then there are at least two center rays in \( L \), fixed by \( h \), which are expanding.

**Remark 14.2.** We should stress that we cannot guarantee to get a single center leaf with both rays expanding. For example, it is very easy to construct an example such that \( h \) has Lefschetz index \(-1\) in \( L \), it has exactly 3 fixed center leaves in \( L \), and only two fixed expanding rays, which are contained in distinct center leaves (see Figure 21). This situation occurs in the examples constructed in [BGHP17] in the unit tangent bundle of a surface.

We will use Proposition 14.1 and its proof to eliminate the mixed behavior in hyperbolic 3-manifolds. It should be noted that this proposition also gives some relevant information about the structure of the enigmatic double translations examples which are not ruled out by our study.

The key point is to understand how each fixed center leaf contributes to the total Lefschetz index of the map in a center-stable leaf which we can control. Since the dynamics preserves foliations and one of them has a well understood dynamical behavior (i.e., in the center stable foliation, the stable foliation is contracting) we can compute the index just by looking at the dynamics in the center foliation (see Figure 20).

As remarked above, one do have to be careful when computing the index as cancellations might happen with branching foliation (see Figure 21).

\[
\begin{array}{ccc}
\text{Index 1} & \text{Index 0} & \text{Index } -1 \\
\end{array}
\]

**Figure 20.** Contribution of index of a center arc depending on the center dynamics

We are now ready to give a proof of Proposition 14.1.

**Proof of Proposition 14.1.** By Proposition 11.34, we know that the fixed point set of \( h \) in \( L \) is contained in \( T_\gamma \) and has Lefschetz index \( 1 - p \) where \( p \) is the number of stable prongs at the fixed point (see Remark 8.2). In particular \( h \) has some fixed points in \( L \).

Let \( L_2 = \tilde{f}^m(L) \). We denote by \( \tau_{12} : L \to L_2 \) the flow along \( \tilde{\Phi}^{cs} \) map, as in section 8.

Let \( g := \gamma^n \circ \tau_{12} : L \to L \). The map \( g \) is a bounded distance away from \( h \) thanks to Lemma 8.8.
Claim 14.3. Let $c_1, c_2$ be two distinct center leaves in $L$ that have a non-trivial intersection. Suppose that both $c_1, c_2$ are fixed by $h$, and there exist two distinct points $z, y \in c_1 \cap c_2$ which are fixed by $h$. Then the center leaves $c_1$ and $c_2$ coincide on the segment between $z$ and $y$.

Proof of Claim 14.3. Let $[y, z]_{c_1}$ and $[y, z]_{c_2}$ be the center segments between $y$ and $z$ in $c_1$ and $c_2$ respectively.

Assume for a contradiction that $[y, z]_{c_1}$ and $[y, z]_{c_2}$ are distinct. Then, up to changing $y$ and $z$, we can assume that the intersection between the open intervals $(y, z)_{c_1}$ and $(y, z)_{c_2}$ is empty.

Thus, by construction, $[y, z]_{c_1}$ and $[y, z]_{c_2}$ intersect only at $z$ and $y$. We let $B$ be the bigon in $L$ bounded by $[y, z]_{c_1}$ and $[y, z]_{c_2}$.

Note that any stable leaf that enters the bigon $B$ must exit it (otherwise it would limit in a stable leaf entirely contained in $B$, which is impossible). Hence, $B$ is “product foliated” by stable leaves. Since $B$ is compact the length of the stable segments contained in $B$ is bounded.

Since $z, y$ are fixed by $h$ it follows that $B$ is also fixed by $h$. Let $s$ be one such stable segment connecting $(z, y)_{c_1}$ to $(z, y)_{c_2}$. Then, the images of $s$ under powers of $h^{-1}$ stay in $B$ but must also have unbounded length, contradiction. 

Let $x$ be a fixed point of $h$. Recall from Lemma 10.14 that the set of center leaves through $x$ in $L$ is a closed interval. In particular $h$ fixes the endpoints of this interval. Hence, $x$ is contained in a center leaf $c$ such that $h(c) = c$.

Claim 14.4. All the fixed points of $h$ in $L$ are contained in the union of finitely many compact segments of center leaves in $L$.

Proof of Claim 14.4. Let $c$ be a center leaf fixed by $h$. Since the fixed points are contained in a compact set $C$ (see Lemma 8.12), there is a minimal compact interval $J$ in $c$ which contains all the fixed points of $h$ in $c$.

Suppose that there exists infinitely many distinct such minimal intervals $J_i$ in center leaves $c_i$. Since the fixed points of $h$ in $L$ are in a compact set, we can choose $i, j$ large enough, so that $J_i$ is very close in the Hausdorff distance of $L$ to $J_j$. Let $z$ be an endpoint of $J_i$. Then the stable leaf $s(z)$ through $z$ intersects...
the center leaf $c_j$. As $z$ is fixed by $h$ and so is $c_j$, contraction of the stable length implies that $z \in c_j$, thus $z \in J_j$.

Hence, both endpoints of $J_i$ are on $J_j$. By Claim 14.3, it implies that $J_i \subset J_j$, and minimality of the interval $J_j$ implies $J_j = J_i$ which is a contradiction. □

Let $\{J_i, 1 \leq i \leq i_0\}$ be a finite family of compact intervals containing all the fixed point of $h$, as given by Claim 14.4. Note that we do not necessarily take the minimal intervals as constructed in the proof of Claim 14.4, as we want the following properties for that family.

**Claim 14.5.** We can choose the collection of intervals $\{J_i, 1 \leq i \leq i_0\}$, each in a center leaf fixed by $h$, satisfying the following properties:

1. The union $\bigcup_{1 \leq i \leq i_0} J_i$ contains all the fixed points of $h$.
2. The endpoints of each interval $J_i$ are fixed by $h$.
3. The intervals are pairwise disjoint.

**Proof of Claim 14.5.** Let $c_1, \ldots, c_n$ be a minimal collection of center leaves that contains all fixed points of $h$ in $L$, as given by Claim 14.4. Let $J_i$ be the minimal compact interval containing all fixed points of $h$ in $c_i$.

The family $J_i$ then satisfies conditions (1) and (2). So we only have to show that one can split the intervals $J_i$ further so that conditions (3) is also satisfied (while still satisfying the first two conditions).

Notice that $c_i, c_j$ intersect if and only if $J_i, J_j$ intersect. Thus, we can restrict our attention to each connected component of the union of the $c_i$’s separately.

Up to renaming, assume that $\bigcup_{1 \leq i \leq k} c_i$ is a connected component of $\bigcup_{1 \leq i \leq n} c_i$.

Now we can consider the union of the $J_1, \ldots, J_k$ as a graph, where the vertices are the endpoints of the segments $J_i$ together with the points where two segments merge, and the edge are the subsegments joining the vertices. With this convention, the union of the $J_1, \ldots, J_k$ is then a tree. Otherwise there would be a bigon in $L$ enclosed by the union, which is ruled out by Claim 14.3.

Let $\mathcal{B}$ be this tree. Our goal is to remove enough open segments from the $J_i$’s so that no vertex of this associated tree has degree 3 or more. Consider a vertex $p$ in $\mathcal{B}$ with degree 3 or more. Then there are two edges $e_1$ and $e_2$ abutting at $p$ on the same side of $p$. We claim that $e_1$ cannot have points fixed by $h$ arbitrarily close to $p$ (except for $p$ itself). Otherwise one would have a fixed point $y \in e_1$ such that $s(y)$ intersects $e_2$. Since $e_2$ is contained in a fixed leaf, $e_2 \cap s(y)$ is fixed by $h$. This implies (since $h$ decreases stable length) that $y$ is in $e_2$. Thus, by Claim 14.3, the intersection of $e_1$ and $e_2$ would contain the segment $[y, p]$, contradicting the fact that they are distinct edges.

Thus, we can remove an open interval $(p, z)$ from, say, $e_1$, where $z$ is fixed by $h$ but $(p, z)$ has no fixed points. In the new tree, $p$ has index one less than before and $z$ has index one.

Doing this recursively on each vertex of index strictly greater than 2, we will obtain, as sought, a disjoint collection of intervals that also satisfy conditions (1) and (2).

□

Now we will look at the index of $h$ on the fixed intervals $J_i$, $1 \leq i \leq i_0$ produced by Claim 14.5. Note that for each such interval $J_i$ there are no other fixed points of $h$ nearby in $L$. Let $c$ be a leaf fixed by $h$ containing $J_i$.

If $h$ is contracting on $c$ near both endpoints of $J_i$ on the outside then the index of $J_i$ is +1. This is because the stable foliation is contracting under $h = \gamma^n \circ \tilde{f}^m$ (since $m > 0$). Hence $h$ is contracting near $J_i$. If $h$ is expanding on both sides,
the index is \(-1\). If one side is contracting and the other is expanding then the index is zero.

The global index for \(h\) can then be computed by adding the indexes of \(h\) on each of the intervals \(J_i\), taking care of cancellations.

Let \(c_k, 1 \leq k \leq k_0\), be finitely many center leaves, fixed by \(h\) and containing all the \(J_i\). We choose this collection to have the minimum possible number of leaves.

Each leaf \(c_k\) contains finitely many segments \(J_i\), so there are exactly two infinite rays that do not contain any \(J_i\). The contribution of \(c_k\) to the global index of \(h\) (before possible cancellations) will then be \(-1\) if both rays are expanding, 0 if one is expanding while the other contracts and 1 if both are contracting.

Suppose for a contradiction, that there is at most one expanding ray in \(L\). So each \(c_k\), considered separately, has index either 0 or 1.

If there is an expanding ray, let \(c_k\) be a leaf with an expanding ray. Otherwise let \(c_k\) be any leaf. Now we need to consider how the other leaves and the possible cancellations impact the global index of \(h\). Let \(c_l\) be a leaf that intersect \(c_k\). If \(c_l\) shares an expanding ray with \(c_k\), then the other ray of \(c_l\) is contracting, and eventually disjoint from the corresponding ray of \(c_k\). The fixed set (if any) of this ray in \(c_l\) has index zero. If \(c_l\) does not share an expanding ray with \(c_k\), then both rays of \(c_l\) are contracting. The ray that is added to the same end as the expanding ray of \(c_k\) contributes index 1. The other ray contributes index 0. In any case the index, starting at 0 or 1, does not decrease.

Now, if \(c_m\) is another leaf that is disjoint from the set above, then both rays are contracting and it contributes an index 1. So again the index does not decrease.

Thus, if there is at most one expanding ray, then the index of \(h\) is at least 0. This contradicts the fact that the index of \(h\) is \(1 - p\) where \(p \geq 2\), and thus finishes the proof of Proposition 14.1. \(\square\)

14.2. Periodic rays and boundary dynamics. Proposition 14.1 gave the existence of periodic rays that are coarsely expanding. Here we will show that such a ray has a well-defined ideal point on the circle at infinity of the leaf, and that it corresponds to the endpoint of a prong of the transverse regulating pseudo-Anosov flow, \(\Phi^{cs}\).

As previously, we assume that we have a center stable leaf \(L \in \tilde{W}^{cs}_{\text{bran}}\) such that there is a deck transformation \(\gamma\) for which \(\gamma \circ \tilde{f}^m(L) = L\) for some \(m > 0\). We let \(L_2 = \tilde{f}^m(L)\) and define \(\tau_{12} : L \rightarrow L_2\) the flow along \(\Phi^{cs}\) map. We also take as before

\[ h := \gamma \circ \tilde{f}^m \quad \text{and} \quad g := \gamma \circ \tau_{12}. \]

Recall that \(h\) and \(g\) are maps of \(L\) that are a bounded distance from each other thanks to Lemma 8.8. Also \(g\) preserves the (singular) foliations \(G^s\) and \(G^u\). We again assume that if \(g\) has a fixed point \(x_0\) in \(L\) then \(\gamma\) is such that \(g\) preserves each of the prongs of \(G^s(x_0)\) (resp. \(G^u(x_0)\)).

The action of \(g\) on the circle at infinity \(S^1(L_1)\) has an even number of fixed points, which are alternately contracting and repelling. We denote by \(P\) the set of contracting fixed points and by \(N\) the set of repelling ones. With these notations, we get the following.

**Proposition 14.6.** Let \(\eta : [0, \infty) \rightarrow L\) be a contracting fixed ray for \(h\). Then \(\lim_{t \rightarrow \infty} \eta(t)\) exists in \(S^1(L)\) and it is a (unique) point in \(N\). (Symmetrically, if \(\eta\) is an expanding fixed ray, its limit point belongs to \(P\).)
Proof. Let $y$ in $P$ and $U$ a small neighborhood of $y$ in $L \cup S^1(L)$ as given by Lemma 8.12. If $\eta$ has a point $q$ in $U \cap L$, then $h^n(q)$ converges to $y$ as $n \to +\infty$, so $\eta$ could not be a contracting ray, a contradiction. So $\eta$ cannot limit on any point in $P$. If $z$ is in $S^1(L) \setminus \{N \cup P\}$, then $h^n(z)$ converges to a point in $P$ under forward iteration. Hence again a small neighborhood $Z$ of $z$ in $L \cup S^1(L)$ is sent under some iterate inside a neighborhood $U$ as in the first part of the proof. So any point in $Z \cap L$ converges to a point in $P$ under forward iteration. Hence $\eta$ cannot limit to a point in $S^1(L) \setminus \{N \cup P\}$ either. So $\eta$ can only limit on points in $N$. Since $\eta$ is properly embedded in $L$, the set of accumulations points of $\eta$ is connected, so it has to be a single point. 

15. Mixed case in hyperbolic manifolds

In this section we show that even in the non-dynamically coherent case, the mixed behavior is impossible for hyperbolic 3-manifolds. This will be done by using the study of translations in hyperbolic 3-manifolds developed in sections 11.8 and 14 to provide more information on the dynamics of general partially hyperbolic diffeomorphisms.

The main result of this section is the following.

**Theorem 15.1.** Let $f : M \to M$ be a partially hyperbolic diffeomorphism homotopic to the identity on a hyperbolic 3-manifold $M$. Suppose that $f$ preserves branching foliations $\mathcal{W}_{\text{bran}}^{cs}, \mathcal{W}_{\text{bran}}^{cu}$ and is such that a good lift $\tilde{f}$ fixes a leaf of $\mathcal{W}_{\text{bran}}^{cu}$. Then, $f$ is a discretized Anosov flow.

15.1. **The set up.** Consider a partially hyperbolic diffeomorphism $f$ as in Theorem 15.1.

Our goal is to show that the good lift $\tilde{f}$ of $f$ fixes every leaf of $\mathcal{W}_{\text{bran}}^{cs}, \mathcal{W}_{\text{bran}}^{cu}$. Indeed, this is enough to prove the theorem by applying Proposition 12.1 (and Corollary 12.5) which implies that $f$ is dynamically coherent. Once dynamical coherence is established, Theorem 6.1 completes the proof.

Thanks to Proposition 12.7, it is enough to prove dynamical coherence up to lifts and powers. Thus we assume that $\mathcal{W}_{\text{bran}}^{cs}$ and $\mathcal{W}_{\text{bran}}^{cu}$ are orientable and transversely orientable and that $f$ preserves their orientations.

Since $\tilde{f}$ is assumed to fix one leaf of $\mathcal{W}_{\text{bran}}^{cu}$, Proposition 11.14 implies that every leaf of $\mathcal{W}_{\text{bran}}^{cu}$ is fixed. We will prove that every leaf of $\mathcal{W}_{\text{bran}}^{cs}$ is fixed by $\tilde{f}$ by contradiction. So, by Proposition 11.14, we can assume that $\mathcal{W}_{\text{bran}}^{cs}$ is $\mathbb{R}$-covered and uniform and that $\tilde{f}$ acts as a translation on the leaf space of $\mathcal{W}_{\text{bran}}^{cs}$.

In particular, there are no center curves fixed by $\tilde{f}$.

Then, we can apply Proposition 11.27 to $\mathcal{W}_{\text{bran}}^{cu}$ to deduce that every periodic center leaf is coarsely expanding.

On the other hand, since $\tilde{f}$ acts as a translation on $\mathcal{W}_{\text{bran}}^{cs}$, we can use the results from sections 11.8 and 14. Let $\Phi_{cs}$ be a regulating pseudo-Anosov flow transverse to $\mathcal{W}_{\text{bran}}^{cs}$ given by Proposition 11.33.

The flow $\Phi_{cs}$ is a genuine pseudo-Anosov, that is it admits at least one periodic orbit which is a $p$-prong with $p \geq 3$ (see Proposition D.4).

Now, we choose $\gamma$ in $\pi_1(M)$, associated to this prong, and apply Proposition 11.35: Up to taking powers, we can assume that $h := \gamma \circ \tilde{f}^k$ for some $k > 0$ fixes a leaf $L$ of $\mathcal{W}_{\text{bran}}^{cs}$. Moreover, the dynamics in $L$ resembles that of the dynamics of a $p$-prong, and in particular fixes every prong.
Notice that Proposition 14.1 also provides some center rays which are expanding in \( L \) for \( h \). We will need to use some of the ideas involved in the proof of that proposition (even though the statement itself will not be used).

We summarize the discussion above in the following proposition.

**Proposition 15.2.** Let \( f: M \to M \) be a partially hyperbolic diffeomorphism homotopic to the identity of a hyperbolic 3-manifold \( M \) preserving branching foliations \( W_{\text{bran}}^{cs}, W_{\text{bran}}^{cu} \). Suppose that a good lift \( \tilde{f} \) fixes a leaf of \( W_{\text{bran}}^{cs} \) and acts as a translation on \( W_{\text{bran}}^{cs} \). Then, up to taking finite iterates and covers, there exists \( \gamma \in \pi_1(M) \) and \( k > 0 \) such that a center stable leaf \( L \in W_{\text{bran}}^{cs} \) is fixed by \( h := \gamma \circ \tilde{f}^k \) and its Lefschetz index is \( I_{\text{Fix}(\tilde{f})}(h) = 1 - p \) with \( p \geq 3 \). Moreover, every center curve fixed by \( h \) in \( L \) is coarsely expanding.

Let \( \gamma \) be as in the proposition. Let \( L \) be a center stable leaf fixed by \( h = \gamma \circ \tilde{f}^k \) and \( L_2 = \tilde{f}^k(L) \). As previously, we write \( \tau_{12}: L \to L_2 \) for the map obtained by flowing from \( L \) to \( L_2 \) along \( \tilde{f}^k \). We set \( g := \gamma \circ \tau_{12} \).

The map \( g \) acts on the compactification of \( L \) with its ideal circle \( L \cup S^1(L) \) the same way as \( h \) does (see sections 8, 11.8 and 14).

Let \( \delta \) be the unique orbit of \( \tilde{f}^k \) fixed by \( \gamma \) and let \( x \) be the (unique) intersection of \( \delta \) with \( L \). Note that \( x \) is the unique fixed point of \( g \). Since we assume that \( \gamma \) fixes the prongs of \( \delta \), then \( h \) has exactly \( 2p \) fixed points in \( S^1(L) \). These fixed points are contracting if they correspond to an ideal point of \( G^s(x) \) and expanding if they are ideal points of \( G^u(x) \).

**15.2. Proof of Theorem 15.1.** To prove Theorem 15.1 we will first show some properties. Recall from Proposition 14.6 that every proper ray in \( L \in W_{\text{bran}}^{cs} \), fixed by \( h \) has a unique limit point in \( S^1(L) \) (notice that by Lemma 8.12 the ray must be either expanding or contracting). We will show that the fixed rays associated to the center and stable (branching) foliations have different limit points at infinity.

**Lemma 15.3.** Let \( s \) be a stable leaf in \( L \) which is fixed by \( h \). Then the two rays of \( s \) limit to distinct ideal points of \( L \). The same holds if \( c \) is a center leaf in \( L \) fixed by \( h \).

**Proof.** We do the proof for the center leaf \( c \), the one for stable leaves is analogous, and a little bit easier (since there is no branching).

By hypothesis, \( c \) is fixed by \( h \), hence it is coarsely expanding under \( h \). It follows that there are fixed points of \( h \) in \( c \). By Proposition 14.6 each ray of \( c \) can only limit in a point in \( P \subset S^1(L) \), where, as previously, \( P \) is the set of attracting fixed points of \( h \) in \( S^1(L) \). Let \( q_1, q_2 \) be the ideal points of the rays. What we have to prove is that \( q_1 \) and \( q_2 \) are distinct.

Suppose that \( q_1 = q_2 \). Then \( c \) bounds a unique region \( S \) in \( L \) which limits only in \( q_1 \in S^1(L) \). The other complementary region of \( c \) in \( L \) limits to every point in \( S^1(L) \). Let \( z \) be a fixed point of \( h \) in \( c \). Then the stable leaf \( s(z) \) of \( z \) has a ray \( s_1 \) entering \( S \). It cannot intersect \( c \) again, and it is properly embedded in \( L \). Hence it has to limit in \( q_1 \) as well. See Figure 22.

But now this ray is contracting for \( h \). This contradicts Proposition 14.6 because this ray should limit in a point of \( N \). 

**Remark 15.4.** The proof used strongly that periodic center leaves are coarsely expanding, in order to induce a behavior at infinity. In the examples of [BGHP17] it does happen that different stable curves land in the same ideal point at infinity in their center stable leaf.
Now we show a sort of dynamical coherence for fixed center rays.

**Lemma 15.5.** Suppose that $c_1, c_2$ are distinct center leaves in $L$ which are fixed by $h$. Then $c_1, c_2$ cannot intersect.

Notice that since $f$ is not necessarily dynamically coherent, the distinct center leaves $c_1, c_2$ can a priori intersect each other. The proof will depend very strongly on the fact that center rays fixed by $h$ are coarsely expanding.

**Proof.** Suppose that $c_1, c_2$ intersect. Since $c_1, c_2$ are both fixed by $h$, so is their intersection. Since $h$ is coarsely expanding in each, then $c_1, c_2$ share a fixed point of $h$. In the the proof of Claim 14.3, we showed that $c_1$ and $c_2$ cannot form a bigon $B$.

It follows that there is a point $x$, fixed by $h$, which is an endpoint of all intersections of $c_1$ and $c_2$: On one side $x$ bounds a ray $e_1$ of $c_1$ and a ray $e_2$ of $c_2$ such that $e_1$ and $e_2$ are disjoint. For a point $y$ in $e_1$ near enough to $x$, we have that $s(y)$ must intersects $e_2$. Since stable lengths are contracting under powers of $h$, it implies that $e_1$ is contracting towards $x$ near $x$ and similarly for $e_2$ (see figure 23). But $e_1$ is coarsely expanding. Hence there must exist fixed points of $h$ in $e_1$. Let $y \in e_1$ be the closest point to $x$ which is fixed by $h$. Similarly, let $z$ in $e_2$ closest to $x$ fixed by $h$.

The leaves $s(y), s(z)$ are not separated from each other in the stable leaf space in $L$.

Let now $c$ be a center leaf through $x$, which is between $c_1$ and $c_2$ and which is the first center leaf not intersecting $s(y)$. Then $h(c) = c$. In addition $c$ has a ray $e$ with endpoint $x$ and intersecting only stable leaves which intersect $c_1$ between $x$ and $y$. It follows that this ray is contracting under $h$, contradicting Proposition 15.2, because this is fixed by $h$.

Thus far, we showed that distinct center leaves in $L$, which are fixed by $h$ do not intersect. Then, the proof of Claim 14.4 also implies that fixed center leaves cannot accumulate (as accumulation would imply that some fixed leaves intersect).

Figure 22. Rays have to land in different points of $S^1(L)$.
Figure 23. Showing the existence of fixed points below $x$ in Lemma 15.5.

We conclude that there are finitely many center leaves in $L$ that are fixed under $h$. Each such center leaf is coarsely expanding. For each such center leaf $c$, we consider a small enough open topological disk containing all the fixed points of $h$ in $c$, and no other fixed point of $h$ in $L$. Then, on such disks, the Lefschetz index of $h$ is $-1$. Since the total Lefschetz number of $h$ in $L$ is $1 - p$ it follows that:

**Lemma 15.6.** There are exactly $p - 1$ center leaves which are fixed by $h$ in $L$.

This together with the following lemma will allow us to make a counting argument to reach a contradiction.

**Lemma 15.7.** Let $c_1$, $c_2$ be two distinct center leaves in $L$ fixed by $h$. Let $y_1 \in c_1$ and $y_2 \in c_2$ be fixed points of $h$. Then $s(y_1)$ and $s(y_2)$ do not have common ideal points.

**Proof.** Suppose, for a contradiction, that there are distinct fixed center leaves $c_1$, $c_2$ satisfying the following: There are points $y_1 \in c_1$ and $y_2 \in c_2$, fixed by $h$, such that $s_1 = s(y_1)$ and $s_2 = s(y_2)$ share an ideal point in $S^1(L)$.

Let $q$ be the common ideal point of the corresponding rays of $s_1$ and $s_2$. Let $e_j$ be the ray in $s_j$ with endpoint $y_j$ and ideal point $q$. Suppose first that no center leaf intersecting $e_1$ intersects $e_2$. Let $c_0$ be a center leaf intersecting $e_1$. Iterate $c_0$ by powers of $h^{-1}$. It pushes points in $s_1$ away from $y_1$. Since the leaves $h^{-i}(c_0)$ all intersect $s_1$ and none of them intersect $s_2$, the sequence $(h^{-i}(c_0))$ converges to a collection of center leaves as $i \to +\infty$. Then there is only one center leaf in this limit, call it $c$, which separates all of $h^{-i}(c_0)$ from $s_2$. This $c$ is invariant under $h$, but it has an ideal point in $q$. Now $q$ is a repelling fixed point, so $c$ must have an attracting ray, a contradiction.

It follows that some center leaf intersecting $e_1$ also intersects $e_2$. Let $c_0$ be one such center leaf. Now iterate by positive powers of $h$. Then $(h^i(c_0))$ converges to a fixed center leaf $v_1$ through $y_1$ and a fixed center leaf $v_2$ through $y_2$. But then $v_1$ and $c_1$ are both fixed by $h$ and both contain $y_1$. Lemma 15.5 implies that $c_1 = v_1$ and $c_2 = v_2$. In particular $v_1 \neq v_2$, and they are non separated from each other. In this case, consider $s$ the unique stable leaf defined as the first leaf
Figure 24. A depiction of the main objects in the proof of Lemma 15.7.

not intersecting $c_1$ that separates $s_1$ from $s_2$. Then, as above, $h$ fixes $s$ and has a fixed point $y$ in $s$. But a center leaf $c$ through $y$ fixed by $h$ has to intersect the interior of the ray $e_1$. This intersection point is the intersection of $c$ fixed by $h$, and $s_1$ fixed by $h$. So this intersection point is fixed by $h$. But this is a contradiction, because $y_1$ is the only fixed point of $h$ in $s_1$. So Lemma 15.7 is proven.

We now can complete the proof of Theorem 15.1.

Proof of Theorem 15.1. By Lemma 15.6, there are $p - 1$ center leaves fixed by $h$ in $L$. We denote them by $c_1, \ldots, c_{p-1}$.

Each center leaf has at least one fixed point. Let $y_i, 1 \leq i \leq p - 1$ be a fixed point in $c_i$. Then, for each $i$, Lemma 15.3 states that $s(y_i)$ has two distinct ideal points $z_1^i$ and $z_2^i$.

Moreover, for every $i \neq j$, the ideal points of the stable leaves are distinct by Lemma 15.7. It follows that there are at least $2p - 2$ distinct points in $S^1(L)$ which are repelling.

But we also know that there are exactly $p$ points in $S^1(L)$ that are repelling under $h$. It follows that $2p - 2 \leq p$, which implies $p = 2$. However, we had that $p \geq 3$, thus obtaining a contradiction.

This finishes the proof of Theorem 15.1.

16. Absolutely partially hyperbolic diffeomorphisms

In this section, we explain how one can remove the need for dynamical coherence in Theorem 1.1 if we use a stronger version of partial hyperbolicity instead.

Definition 16.1. A partially hyperbolic diffeomorphism $f: M \to M$ on a 3-manifold is called absolutely partially hyperbolic if there exists constants $\lambda_1 < 1 < \lambda_2$ such that for some $\ell > 0$ and every $x \in M$, we have
\[ \|Df|_{E^s(x)}\| < \lambda_1 < \|Df|_{E^u(x)}\| < \lambda_2 < \|Df|_{E^c(x)}\|. \]

Notice that, although subtle, the difference between being absolutely partially hyperbolic versus just partially hyperbolic is far from trivial. Here, we just show that with this stronger property one can significantly simplify the arguments. However, some previous results have shown significant differences between the two notions, specifically with regard to the integrability of the bundles (see [BBI09, RHRHU16, Pot15]).

We will show the following

**Theorem 16.2.** Let \( f : M \to M \) be an absolutely partially hyperbolic diffeomorphism on a 3-manifold. Suppose that \( f \) is homotopic to the identity and preserves two branching foliations \( \mathcal{W}_{\text{bran}}^{cs} \) and \( \mathcal{W}_{\text{bran}}^{cu} \) that are both \( f \)-minimal. Then either

(i) \( f \) is a discretized Anosov flow, or,

(ii) \( \mathcal{W}_{\text{bran}}^{cs} \) and \( \mathcal{W}_{\text{bran}}^{cu} \) are \( \mathbb{R} \)-covered and uniform and a good lift \( \tilde{f} \) of \( f \) act as a translation on their leaf spaces.

In order to prove this theorem, the main step will be to show that, using absolute partial hyperbolicity, we can get Proposition 4.4 even without dynamical coherence. Recall that, in general, when we do not assume dynamical coherence, we only get Proposition 11.27.

**Proposition 16.3.** Let \( f : M \to M \) be an absolutely partially hyperbolic diffeomorphism homotopic to the identity and \( \tilde{f} \) a good lift of \( f \) to \( \tilde{M} \). Assume that every leaf of \( \mathcal{W}_{\text{bran}}^{cs} \) is fixed by \( \tilde{f} \). Let \( L \) be a leaf whose stabilizer is generated by \( \gamma \in \pi_1(M) \setminus \{\text{id}\} \). Then, there is a center leaf in \( L \) fixed by \( \tilde{f} \).

The proof is essentially the same as the one in [HPS18, Section 5.4] but we repeat it since the contexts are different.

**Proof.** The proof is by contradiction. Assume that \( \tilde{f} \) does not fix any center leaf in \( L \).

Proposition 11.32 gives that there exists a center leaf periodic by \( f \). Now, using the proof of Proposition 11.27 on the lift \( c \) of such a periodic leaf, we can be more precise: Let \( h := \gamma^n \circ f^m \), with \( m > 0 \) and \( \gamma \in \pi_1(M) \), be the diffeomorphism fixing \( c \). There exists two stable leaves \( s_1 \) and \( s_2 \) in \( L \) fixed by \( h \), a bounded distance apart in \( L \) and such that \( c \) separates \( s_1 \) from \( s_2 \) in \( L \) (as in Figure 7). We denote by \( B \) the band bounded by \( s_1 \) and \( s_2 \).

Since \( \gamma \) is an isometry, the diffeomorphism \( h \) is absolutely partially hyperbolic, and we can (modulo taking iterates) assume that there are constants \( \lambda_1 < \lambda_2 \) such that

\[ \|Dh|_{E^s}\| < \lambda_1 < \|Dh|_{E^c}\|. \]

Moreover, there is a constant \( R > 1 \) such that \( \|Dh^{-1}\| \leq R \) in all of \( L \).

For simplicity, we will assume that the distance between \( s_1 \) and \( s_2 \) is smaller than \( 1/2 \) so that the band \( B \) is contained in the neighborhood \( \hat{B} = \bigcup_{x \in S_1} B_1(x) \) of radius 1 around \( s_1 \).

For every positive \( d \) there is a constant \( r(d) > 0 \) such that for any set of diameter less than \( d \), the length of a stable leaf contained in this set is at most \( r(d) \). This is because in a foliated box only one segment of a stable segment can intersect it. This implies that stable leaves (and center leaves as well) are quasi-isometrically embedded in their neighborhoods of a fixed diameter. So there is
$K > 0$ so that for any stable segment $J$ contained in $\hat{B}$ with endpoints $z$ and $w$ we have

$$\text{length}(J) \leq K d_{\hat{B}}(z, w).$$

Now, choose $n > 0$ such that $K^2 \frac{\lambda_1^n}{\lambda_2} \ll 2\;\frac{1}{2}$ and once $n$ is fixed, choose $D > 0$ so that $\frac{D}{2} \gg 2R^n + \frac{2K}{\lambda_2^n}$.

We now pick points $z, w \in s_1$ such that $d_{\hat{B}}(z, w) = D$ and take $J^s$ an arc of $s_1$ joining these points. From the choice of $K$ and $D$ we know that $\text{length}(J^s) \leq KD$. So, it follows that $\text{length}(h^n(J^s)) \leq KD\lambda_1^n$.

Choose a center curve $J^c$ joining $B_1(h^n(z))$ with $B_1(h^n(w))$ (this can be done because $c$ separates $s_1$ from $s_2$) and call $z_n$ and $w_n$ the endpoints in each ball. It follows that $\text{length}(J^c) \leq K^2 D\lambda_1^n + 2K$.

Since the distance between the endpoints of $J^c$ and $h^n(z)$, $h^n(w)$ is less than 1, by iterating backwards by $h^{-n}$ we get that $d(h^{-n}(z_n), z)$ and $d(h^{-n}(w_n), w)$ are less than $R^n$.

This implies that

$$D \leq d_{\hat{B}}(z, w) \leq K^2 \frac{\lambda_1^n}{\lambda_2} D + 2R^n + \frac{2K}{\lambda_2^n},$$

a contradiction with the choices of $n$ and $D$. This completes the proof of the proposition. \qed

Using this proposition, we can prove Theorem 16.2 in the same way as Theorem 5.1.

\textbf{Proof of Theorem 16.2.} Let $\tilde{f}$ be a good lift of $f$. Since $\mathcal{W}_{\text{bran}}^{\text{cs}}$ and $\mathcal{W}_{\text{bran}}^{\text{cu}}$ are $f$-minimal, by Corollary 11.7, $\tilde{f}$ either fixes each leaf of $\mathcal{W}_{\text{bran}}^{\text{cs}}$ and $\mathcal{W}_{\text{bran}}^{\text{cu}}$, or acts as a translation on both leaf space (in which case the foliations are $\mathbb{R}$-covered and uniform and we are in case (ii) of the theorem), or $\tilde{f}$ translates one and fixes the other.

If $\tilde{f}$ fixes the leaves of both $\mathcal{W}_{\text{bran}}^{\text{cs}}$ and $\mathcal{W}_{\text{bran}}^{\text{cu}}$ then Proposition 12.1 and Corollary 12.5 imply that we are in case (i) of the theorem.

So we have to show that we cannot be in the mixed case. Suppose that $\tilde{f}$ fixes every leaf of $\mathcal{W}_{\text{bran}}^{\text{cs}}$.

Since $M$ is not $T^3$, there are leaves of $\mathcal{W}_{\text{bran}}^{\text{cs}}$ with non-trivial fundamental group (see Proposition 3.2). Consider the lift $L$ in $\mathcal{W}_{\text{bran}}^{\text{cs}}$ of such a leaf, with $L$ invariant by $\gamma$ in $\pi_1(M) \setminus \{\text{Id}\}$. We can apply Proposition 16.3 to conclude that there is a center leaf $c$ in $L$ that is fixed by $\tilde{f}$. So, in particular, $\tilde{f}$ needs to fix a center unstable leaf containing $c$. Thus $\tilde{f}$ has to also fix every leaf of $\mathcal{W}_{\text{bran}}^{\text{cu}}$.

\textbf{Appendix A. Some 3-manifold topology}

We collect here some concepts from 3-manifold topology that were used in this article. We refer the reader to [Hem76, Hat, Jac80] for more background.

A 3-manifold (which we always mean to be a smooth manifold) is irreducible, if every smoothly embedded sphere bounds a ball. It is well known that closed 3-manifolds admitting taut foliations are irreducible (see [Ros68]).

An irreducible compact, 3-manifold $M$ is said to be homotopically atoroidal if every $\pi_1$-injective map of a torus in $M$ is homotopic to a map into the boundary of $M$. The manifold is geometrically atoroidal if every $\pi_1$-injective, embedded smooth torus is homotopic to the boundary of $M$. 


If a manifold with exponential growth of fundamental group is homotopically atoroidal, then by the Geometrization Theorem of Perelman [Per02, Per03b, Per03a] it is hyperbolic, i.e., the interior of $M$ admits a complete, Riemannian metric of constant negative curvature. Notice that when $M$ is homotopically atoroidal, $\pi_1(M)$ does not contain any subgroup isomorphic to $\mathbb{Z}^2$.

A 3-manifold is called a Seifert manifold if it admits a partition by distinct circles such that a tubular neighborhood of each fiber is homeomorphic by a fiber-preserving homeomorphism to either:

- A fibered solid torus of type $(p,q)$. This is a torus obtained from $\mathbb{D}^2 \times [0,1]$ by identifying $\mathbb{D}^2 \times \{0\}$ to $\mathbb{D}^2 \times \{1\}$ via the map $(z,0) \mapsto (z \exp(2\pi i p/q), 1)$. The fiber $\{0\} \times S^1$ is called regular if $p = 0$ and exceptional otherwise.
- Or,
- A fibered solid Klein bottle, obtained from $\mathbb{D}^2 \times [0,1]$ by identifying $\mathbb{D}^2 \times \{0\}$ to $\mathbb{D}^2 \times \{1\}$ via the map $(z,0) \mapsto (\bar{z},1)$. The fibers $\{z\} \times S^1$, $z \in \mathbb{R}$, are also called exceptional.

The quotient of a Seifert manifold by the Seifert fibration, called the base, $B$, has a structure of a 2-orbifold (without corner reflectors). The exceptional fibers separate into two sets: The axis of the fibered solid torus projected to isolated points in the interior of $B$ (called conical points), while the exceptional fibers coming from fibered solid Klein bottles projects to a closed 1-submanifold of the boundary of $B$ (and each connected component is called a reflector curve).

Putting together work of Epstein [Eps72] and Tollefson [Tol78], one can notice that a 3-manifold is Seifert if and only if it admits a foliation by circles.

**Remark A.1.** The definition above is not the one originally taken by Seifert. Indeed, the fibered solid Klein bottles neighborhood were not allowed in the original definition. However, it is now more common to use this definition (see, e.g., [Sco83]). In particular with this definition then all 3-manifolds foliated by circles are Seifert.

Note that both the original definition and the one chosen here agree when the manifold is assumed orientable.

If a Seifert manifold has fundamental group with exponential growth, then it is finitely covered by a circle bundle over a surface of genus $\geq 2$. In particular, thanks to the classification of Seifert manifolds (see [Sco83, Theorem 3.8]), the Seifert fibration is unique in this case.

If a 3-manifold $M$ is geometrically atoroidal but not homotopically atoroidal then the proof of the Seifert fibered conjecture [CJ94, Gab92] implies that $M$ is closed and Seifert. The base surface has to be a sphere with 3-singular fibers. Unless the difference between geometric and homotopic atoroidal is essential we only refer to it as atoroidal.

The JSJ decomposition theorem implies that compact, irreducible, and orientable 3-manifolds admit a decomposition into finitely many pieces, which are either geometrically atoroidal or Seifert [Hem76, Hat].

The following lemma was used when establishing minimality of foliations in Seifert and hyperbolic 3-manifolds:

**Lemma A.2.** If $T$ is an embedded torus inside an orientable closed hyperbolic 3-manifold $M$, then $T$ either bounds a solid torus or is contained in a 3-dimensional ball.

**Proof.** This is standard result in 3-manifold topology, so we only sketch the proof. Since $M$ is orientable and hyperbolic, $T$ is two sided, and not $\pi_1$-injective. Since
$T$ is furthermore embedded, Dehn’s lemma [Hem76] implies that there is a compressing disk $D$. That is, $D$ is embedded and $D \cap T = \partial D$. Cutting $T$ along $D$ and capping with two copies of $D$ produces a sphere. Since $M$ is hyperbolic, it is irreducible [Ros68], so it follows that the sphere bounds a ball. This implies that either $T$ bounds a solid torus or $T$ is contained in a 3-ball [Hem76, Jac80].

The following lemma is important to understand certain particular deck transformations when one lifts to finite covers.

**Lemma A.3.** Let $M$ be a closed, irreducible 3-manifold with fundamental group that is not virtually nilpotent. Suppose that $\beta$ is a non trivial deck transformation so that $d(x, \beta(x))$ is bounded above in $\tilde{M}$. Then $M$ is a Seifert fibered space and $\beta$ represents a power of a regular fiber.

**Proof.** First we assume that $M$ is orientable. Then, the JSJ decomposition states that $M$ has a canonical decomposition into Seifert fibered and geometrically atoroidal pieces. We lift this to a decomposition of $\tilde{M}$ and construct a tree $T$ in the following way: The vertices are the lifts of components of the torus decomposition of $M$, and we associate an edge if two components intersect along the lift of a torus. Such a lift of a torus is called a wall. There is a minimum separation distance between any two walls.

The deck transformation $\beta$ acts on this tree. Let $W$ be a wall. Suppose that $\beta(W)$ is distinct from $W$. But, as subsets of $\tilde{M}$, the walls $W, \beta(W)$ are a finite Hausdorff distance from each other. Then $\pi(W), \pi(\beta(W))$ are tori in $M$, and the region $V$ in $\tilde{M}$ between $W, \beta(W)$ projects to $\pi(V)$ which is $\mathbb{T}^2 \times [0, 1]$ in $M$. If this happens then $\tilde{M}$ is a torus bundle over a circle. In that case, use that $\pi_1(M)$ is not virtually nilpotent, so the monodromy of the fibration is an Anosov map of $\mathbb{T}^2$. But then no $\beta$ as above could satisfy the bounded distance property. It follows that $\beta(W) = W$ for any wall, and in particular $\beta(P) = P$ for any vertex of $T$.

Now consider a vertex $P$. Suppose first that $\pi(P)$ is homotopically atoroidal. By the Geometrization Theorem, $\pi(P)$ is hyperbolic. If $\beta$ restricted to $P$ were to satisfy the bounded distance property, then it would have to be the identity on $P$. Hence $\beta$ itself is the identity, contradiction.

Hence all the pieces of the torus decomposition of $M$ are homotopically toroidal. Suppose now that there is one such piece $\pi(P)$ that is geometrically atoroidal (but not homotopically atoroidal). The proof of the Seifert fibered conjecture ([CJ94, Gab92]) shows that $\pi(P)$ has no boundary and $\pi(P)$ is Seifert. In other words, $M = \pi(P)$ is Seifert. So we can assume that all the pieces of the torus decomposition are geometrically toroidal. Then they are all Seifert fibered. Thus $M$ is a graph manifold.

We will show that the torus decomposition of $M$ is in fact trivial, proving that $M$ is Seifert fibered. Suppose it is not true. Then the tree $T$ is infinite. Let $P_1, P_2, P_3$ be three consecutive vertices in $T$. Let $W_1$ be the wall between $P_1$ and $P_2$. Then $\beta(W_1)$ (as a set in $\tilde{M}$) is a bounded distance from $W_1$ and sends the Seifert fibration of $P$ in $W_1$ to lifts of Seifert fibers. It follows that $\beta = \delta_1^k \alpha_1$ where $\delta_1$ represents a regular fiber in $\pi(P_1)$, and $\alpha_1$ is a loop in $\pi(W_1)$. Similarly if $W_2$ is the wall between $P_2$ and $P_3$ then $\beta = \delta_3^3 \alpha_3$ where $\alpha_3$ is a loop in $\pi(W_3)$. Then $\alpha_1, \alpha_3$ are both in the boundary of $\pi(P_2)$. The loops representing $\delta_1^k \alpha_1$, $\delta_3^3 \alpha_3$ are both in the boundary of $\pi(P_2)$. They represent the same element of $\pi_1(M)$ only when $k = i = 0$ and $\alpha_1, \alpha_3$ are freely homotopic. That means that $P_2$ is a torus times an interval, which is impossible in the torus decomposition in our situation as explained above.
It follows now that the torus decomposition of $M$ is trivial, which implies that $M$ is Seifert fibered. Moreover, if the base is not hyperbolic, then $\pi_1(M)$ is virtually nilpotent ([Sco83, Theorem 5.3]). But this contradicts the hypothesis of the lemma.

It follows that the base is hyperbolic. Also $\beta$ induces a transformation in the universal cover of the base that is a bounded distance from the identity. This can only happen if this transformation is the identity. Therefore $\beta$ represents a power of a regular Seifert fiber in $M$ (notice that non-regular fibers induce a finite symmetry on the base, thus not the identity, and not a bounded distance from the identity).

So the Lemma is proven when $M$ is orientable. If $M$ is not orientable, then it has a double cover $M_2$ which is orientable. Now $\beta^2$ lifts to an element of $\pi_1(M_2)$ that satisfies the assumption of the lemma. So we can apply the result to $M_2$ and obtain that $M_2$ is Seifert. Thus $M$ is doubly covered by a Seifert space, which, by a result of Tollefson [Tol78], implies that $M$ itself is Seifert fibered. It follows that $\beta$ corresponds to a power of a regular fiber. This finishes the proof of the lemma.

We also use the following consequence of Mostow rigidity [Mos68]

**Proposition A.4.** If $M$ is a hyperbolic 3-manifold and $f: M \to M$ a homeomorphism, then it has an iterate which is homotopic to identity.

**Proof.** Mostow rigidity [Mos68] implies that every homeomorphism is homotopic to an isometry. Isometries in a compact manifold have iterates which are close to the identity, so homotopic to them.

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**Appendix B. Taut foliations in 3-manifolds**

All the foliations considered in this article are continuous foliations, with $C^1$ leaves, tangent to a continuous distribution of a 3-manifold (so they are foliations of regularity $C^{0,1+}$ in the terminology of [CC00]). In this appendix, all foliations are 2-dimensional.

A foliation on $M$ is called *taut* if it admits a closed transversal that intersects every leaf of $T$.

An important consequence of Novikov’s theorem [Nov65] is that if a 2-dimensional foliation of a 3-manifold does not have compact leaves then it is taut (see, e.g., [CC00, CC03, CLN85, Cal07]).

Let $\widetilde{T}$ denote the lift of the foliation $T$ to $\widetilde{M}$. The leaf space of $T$ is defined as the set $\mathcal{L}_{\widetilde{T}} := \widetilde{M} / \widetilde{T}$ equipped with the quotient topology.

The following theorem gathers some known properties of taut foliations (see, for instance, [Cal07, Chapter 4] for the proofs) and relies particularly on the celebrated theorems by Novikov [Nov65] and Palmeira [Pal78].

**Theorem B.1.** A foliation without compact leaves in a 3-manifold $M$ is taut. If $M$ is a 3-manifold that is not finitely covered by $S^2 \times S^1$ and admitting a taut foliation $T$ then $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$. Moreover, every leaf of $T$ lifts to a plane $L \in \widetilde{T}$ which is properly tamely embedded in $\widetilde{M}$ and separates $\widetilde{M}$ in two half spaces.

The leaf space $\mathcal{L}_{\widetilde{T}}$ is a one dimensional (non necessarily Hausdorff), simply connected (separable) manifold. Furthermore, every point in $\mathcal{L}_{\widetilde{T}}$ is contained in the interior of an interval in $\mathcal{L}_{\widetilde{T}}$.

---

Note that since $M$ is not finitely covered by $S^2 \times S^1$ no leaves of $T$ can be a sphere or a projective plane.
In particular, if $\beta$ is a transversal to $\tilde{T}$, then $\beta$ intersects a leaf of $\tilde{T}$ at most once.

When $\mathcal{L}_\pi$ is Hausdorff, then it is homeomorphic to the real numbers $\mathbb{R}$. In this case, the foliation $\mathcal{T}$ is called $\mathbb{R}$-covered.

Since $\tilde{M}$ is simply connected, $\tilde{T}$ is transversely orientable (but deck transformations of $\tilde{M}$ may flip this transverse orientation).

For reference, we cite the following result that we used in this article.

**Proposition B.2** (Rosenberg [Ros68]). Let $M$ be a closed 3-manifold which is not $T^3$, and let $\mathcal{T}$ be a foliation on $M$. Then some leaf of $\mathcal{T}$ is not a plane.

**Appendix C. Uniformization of leaves**

The following result is very helpful to understand the action of deck transformations inside leaves of the (branching) foliations of a partially hyperbolic diffeomorphism.

**Theorem C.1** (Candel [Can93]). Let $\mathcal{F}$ be a taut foliation of a 3-manifold $M$ and assume that it has no transverse invariant measure. Then, there is a metric in $M$ which restricts to a (2-dimensional) hyperbolic metric in each leaf of $\mathcal{F}$.

A transverse invariant measure is an assignment of a non-negative number to each arc transverse to $\mathcal{F}$, such that it satisfies the properties of measures under countable union and restriction. In addition the measure is unchanged if we homotope the transverse arcs keeping each point in its respective leaf. The statement of [Can93] gives further properties on the transverse invariant measure (also called holonomy invariant transverse measure), stating that it has zero Euler characteristic, but we will avoid defining this (see [Cal07] for a detailed treatment).

It is well known that every taut foliation in a hyperbolic 3-manifold (see e.g., [Cal07]) or the horizontal foliations in a Seifert manifold used in this article satisfy the conclusion of Theorem C.1. We remark that it is possible to show that any minimal foliation on a manifold with non-virtually solvable fundamental group satisfies the hypothesis of Theorem C.1 (see [FP18, Section 5.1]).

**Appendix D. Uniform foliations and transverse pseudo-Anosov flows**

Uniform foliations were introduced by Thurston [Thu], and have been intensively studied, particularly when $M$ is a hyperbolic 3-manifold. They are intimately related to the notion of slitherings (see [Thu] or [Cal07, Chapter 9]).

**Definition D.1.** An $\mathbb{R}$-covered foliation $\mathcal{T}$ is called *uniform* if the Hausdorff distance between any pair of leaves $L$, $L'$ of $\tilde{T}$ is finite. That is, there exists $K > 0$ (depending on $L$ and $L'$) such that $L \subset B_K(L')$ and $L' \subset B_K(L)$ where $B_K(X)$ denotes the set of points at distance less than $K$ from $X \subset \tilde{M}$.

Thurston build a special pseudo-Anosov flow associated with a $\mathbb{R}$-covered foliation in a hyperbolic manifold.

**Definition D.2.** Let $\mathcal{F}$ be a foliation of a 3-manifold. A flow $\Psi: M \to M$ is called *regulating* for $\mathcal{F}$ if every orbit of the lifted flow $\tilde{\Psi}$ intersects every leaf of the lifted foliation $\tilde{\mathcal{F}}$ in the universal cover $\tilde{M}$. 
Theorem D.3 (Thurston, Calegari, Fenley [Thu, Cal00, Fen02]). A transversely oriented, $\mathbb{R}$-covered, uniform foliation in a hyperbolic 3-manifold admits a regulating transverse pseudo-Anosov flow $\Phi$. Moreover, $\Phi$ can be chosen so that the singular foliations have $C^1$ leaves outside the prongs.

Recall that a pseudo-Anosov flow $\Phi$ is a flow generated by a vector field $X$ which preserves two singular foliations $\Lambda^s$ and $\Lambda^u$ and such that, outside a finite number of singular orbits, the flow is locally modeled on a (topological) Anosov flow (see Appendix G). The foliations glue along the singularities forming $p$-prongs (with $p \geq 3$). We refer the reader to [Cal07] for more details. We note also that every expansive flow is orbit equivalent to a pseudo-Anosov flow [Pat93, IM90].

Work of Barbot and the second author implies that Thurston’s regulating flow is genuinely pseudo-Anosov:

Proposition D.4. If $\Phi$ is a pseudo-Anosov flow regulating and transverse to a uniform foliation in a non virtually solvable 3-manifold, then, $\Phi$ is not a topological Anosov flow. In particular, there are singular periodic orbits which are $p$-prongs with $p \geq 3$.

Proof. This fact can be found in [Fen13], but we recall the argument: every element of the fundamental group that represents a periodic orbit of $\Phi$ acts as a translation on the leaf space of $\tilde{F}$. However, this is inconsistent with the fact that every Anosov flow on a 3-manifold, except for suspensions of Anosov diffeomorphisms (which do not exist on non virtually solvable manifolds), admits pairs of periodic orbit that are freely homotopic to the inverse of each other (this fact follows from work of Barbot and the second author, see [BBGR17, Theorem 2.15]).

Appendix E. Axes

Here, we recall some needed results from the theory of axes for free actions on one-dimensional, non-Hausdorff, simply connected, manifolds. These results extend similar results for trees. We refer the reader to [Fen03, Bar98] for a more detailed account. All of the results we state are true for homeomorphisms of one-dimensional, non-Hausdorff, simply connected, separable manifolds. However, to keep the terminology close to the core of this article, we phrase our results in the setting of homeomorphisms preserving a (branching) foliation.

Let $L$ be a complete plane. Let $C$ be a (branching) foliation such that its leaf space $L_C$ is a one dimensional (not necessarily Hausdorff) simply connected manifold. (If $C$ is branching, we use Definition 10.13 to put a topology on $L_C$.)

Definition E.1 (Axis of a foliation-preserving homeomorphism). Let $g : L \to L$ be a homeomorphism which preserves $C$. The axis of $g$ (or, if one needs to be more precise, the $C$-axis of $g$) is the set of leaves $c \in C$ such that $g(c)$ separates $c$ from $g^2(c)$.

For the statement below, we recall that a $\mathbb{Z}$-union of intervals means an ordered set consisting of countably many closed (possibly degenerate) intervals which are ordered according to $\mathbb{Z}$.

Proposition E.2. Let $g : L \to L$ be a homeomorphism that preserves $C$ without leaving any leaf of $C$ fixed. Then, the $C$-axis for the action of $g$ in $L_C$ is non empty. In addition, the axis is either a line or an ordered $\mathbb{Z}$-union of intervals. In the second case, the axis is $\bigcup I_i$, where $I_i = [x_i, y_i]$ is a closed interval and $y_i$ is not separated from $x_{i+1}$ in the leaf space of $C$. 
Moreover, suppose that \( g, h : L \to L \) are two \( C \)-preserving homeomorphisms that do not fix any \( C \)-leaves, and that share the same axis. If the group generated by \( g \) and \( h \) acts freely\(^{12} \) on this axis, then it is abelian.

**Proof.** This is proven in section 3 of [Fen03]. See in particular Lemma 3.5, Theorem 3.8 and Proposition 3.10 there. The last statement uses Hölder’s Theorem (see e.g., [Nav11, Section 2.2.4]) to deduce that the group generated by \( g \) and \( h \) must be abelian. \( \square \)

**Remark E.3.** Two commuting homeomorphisms that act freely have the same axis (see [Bar98, Section 2] or [Fen03, Section 3]). We remark a couple of subtle points:

1. The fact that \( f \) acts freely does not imply that any power of \( f \) acts freely, and in fact there are easy counterexamples.
2. Unlike in the case of trees, \( f \) acting freely does not necessarily imply that the axis is properly embedded in the leaf space. If the axis is a bi-infinite union of intervals, then it is properly embedded (see Lemma E.5 below). If the axis is the reals, it may fail to be properly embedded, even if all powers of \( f \) act freely.

Notice that as a consequence we obtain:

**Corollary E.4.** Let \( f, g, h : L \to L \) be three \( C \)-preserving homeomorphisms such that the action of the group is free in the \( C \)-leaf space. Assume moreover that \( f \) commutes with \( g \) and with \( h \). Then, the group generated by \( f, g, h \) is abelian.

**Proof.** Notice that as \( f \) commutes with \( g \) and \( h \) then all three homeomorphisms of the leaf spaces share the same axis ([Bar98, Section 2]). Now the result follows from the previous proposition. \( \square \)

Another useful fact about axes is the following:

**Lemma E.5.** If \( A \) is the axis of a \( C \)-preserving homeomorphism \( f \) and \( A \) is a \( \mathbb{Z} \)-union of intervals, then \( A \) is properly embedded in the leaf space of \( C \).

**Proof.** Let \( A = \bigcup_{i} I_{i} \), where \( I_{i} = [x_{i}, y_{i}] \), with \( y_{i} \) and \( x_{i+1} \) not separated in the leaf space of \( C \).

If \( A \) is not properly embedded, then there exists a leaf \( c \in C \) such that \( (x_{i}) \) and \( (y_{i}) \) converges to \( c \) as, say, \( n \) goes to \( +\infty \). Now, for any \( i \), the interval \( I_{i} \) separates \( I_{i-1} \) from \( I_{i+1} \). Thus, if \( \tau \) is a transversal to \( C \) through \( c \), then \( \tau \) intersects every \( I_{i} \), for \( i \) big enough. So in particular, for some \( i \) big enough, \( \tau \) intersects both \( y_{i} \) and \( x_{i+1} \). But this is impossible since these two leaves are not separated. \( \square \)

**Appendix F. On partial hyperbolicity**

Here we state some facts about partial hyperbolicity that are used in the article but are well-known to the experts.

Recall that a \( C^{1} \)-diffeomorphism \( f : M \to M \) is partially hyperbolic if there exists a \( Df \)-invariant splitting \( TM = E^{s} \oplus E^{c} \oplus E^{u} \) into 1-dimensional bundles and an \( n > 0 \) such that for every \( x \in M \) we have

\[
\|Df^{n}|_{E^{s}(x)}\| < \min\{1, \|Df^{n}|_{E^{c}(x)}\|\} \leq \max\{1, \|Df^{n}|_{E^{u}(x)}\|\} < \|Df^{n}|_{E^{s}(x)}\|.\]

\(^{12}\)Recall that we say that a group acts freely if no element different from the identity has a fixed point.
By changing the Riemannian metric, one can always assume that $n = 1$ (see [Gou07, CP]).

A partially hyperbolic diffeomorphism is called *dynamically coherent* if there exists $f$-invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$. Taking the intersection of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ gives a one-dimensional foliation $\mathcal{W}^c$ tangent to $E^c$ and $f$-invariant. Note that these foliations are not assumed to be unique in any sense (see [BW08] for a discussion).

Partially hyperbolic diffeomorphisms need not be dynamically coherent, but when they are, the standard notion of equivalence (which goes back to [HPS77]) is that of *leaf conjugacy*: Two dynamically coherent partially hyperbolic diffeomorphisms $f: M \to M$ and $g: N \to N$ are said to be leaf conjugate if there exists a homeomorphism $h: M \to N$ that maps the center foliation $\mathcal{W}_f^c$ of $f$ to the center foliation $\mathcal{W}_g^c$ of $g$. More precisely, $h$ is such that $h(\mathcal{W}_f^c(f(x))) = \mathcal{W}_g^c(g(h(x)))$ for all $x \in M$. We refer the reader to [Pot18] and references therein for more discussions.

We state the following result of Hertz, Hertz and Ures in a way that fits our particular needs.

**Theorem F.1** ([RHRHU11]). Let $f: M \to M$ be a partially hyperbolic diffeomorphism admitting a compact manifold\(^{13}\) tangent to $E^{cs}$ (or $E^{cu}$). Then, $M$ has solvable fundamental group (indeed, it is a torus bundle over the circle).

In particular, if the fundamental group of $M$ is not virtually solvable, and $f$ is dynamically coherent, then the center-stable and center-unstable foliations are taut. The same holds for the approximating foliations to the branching foliations (cf. Definition 10.2) in the non-dynamically coherent case, i.e., if $\pi_1(M)$ is not virtually solvable, then the approximating foliations are taut.

Hence, using the fundamental results of Burago and Ivanov [BI08] (see Theorem 10.4), one gets:

**Corollary F.2** ([BI08, Par10]). Let $M$ be a $3$-manifold with non-solvable fundamental group. Suppose that $M$ admits a partially hyperbolic diffeomorphism $f$ such that the bundles $E^s, E^c, E^u$ are orientable and $Df$ preserves these orientations. Then $M$ admits a taut foliation. In particular, $M$ is irreducible, and aspherical.

We recall that when $\pi_1(M)$ is (virtually) solvable a complete classification of partially hyperbolic diffeomorphisms is known [HP14, HP15, HP19].

In the setting of Seifert manifolds we used the following result from [HPS18] which is partially based on the study of horizontal and vertical laminations in Seifert manifolds [Bri93].

**Theorem F.3** ([HPS18]). Let $f: M \to M$ be a partially hyperbolic diffeomorphism homotopic to the identity on a Seifert manifold $M$ whose fundamental group is not (virtually) solvable. Then, $M$ is a finite cover of $T^1S$ where $S$ is a $2$-dimensional hyperbolic orbifold, and the center-stable and center-unstable (branching) foliations of $f$ are horizontal. That is, there exists a Seifert fibration $p: M \to \Sigma$ for which every leaf of the center-stable and center-unstable ($2$-dimensional) foliations is transverse to the ($1$-dimensional) fibers.

Recall that an invariant (branching) foliation is called $f$-minimal if the only non-empty, saturated, closed set invariant by $f$ is the whole manifold. The following result motivates asking for $f$-minimality of the foliations as a hypothesis as it covers the most important (from a dynamical standpoint) cases.

\(^{13}\)Notice that a compact manifold tangent to $E^{cs}$ is necessarily a torus, see e.g., [RHRHU11].
**Proposition F.4** (see Lemma 1.1 of [BW05]). Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism. If $f$ is either volume preserving or transitive,\(^{14}\) then the center-stable and center-unstable foliations are $f$-minimal.

**Proof.** Assume that there is a compact, non-empty $f$-invariant set $\Lambda$ saturated by center-stable leaves. If $\Lambda \neq M, \emptyset$ then it must be a repeller, so $f$ cannot be transitive nor volume preserving. \qed

The same result with the same proof applies to branching foliations in the non-dynamically coherent setting (see [HPS18, Proposition 5.1]). We remark that the property of $f$-minimality of $W_{cs}$ and $W_{cu}$ is a strictly weaker hypothesis than (chain-)transitivity (as seen, for instance, in the examples of [BG10]).

We prove that in certain situations minimality is equivalent to $f$-minimality. We need the following result which is of interest in itself.

**Lemma F.5.** Let $\mathcal{L}_{cs}^b$ be the leaf space of $\tilde{\mathcal{W}}_{cs}$ bran. Let $B \subset \mathcal{L}_{cs}^b$ be a closed set of leaves. Suppose that, for all $x \in M$, there exists a leaf $L \in B$ containing $x$. Then $B = \mathcal{L}_{cs}^b$.

**Proof.** The lemma is obvious when $\mathcal{W}_{cs}$ bran is a true foliation (and one does not need to require $B$ to be closed). However, when $\mathcal{W}_{cs}$ bran has some branching, one could possibly have a union of leaves that cover all of $\tilde{M}$ without using all the leaves of $\tilde{\mathcal{W}}_{cs}$ bran. For closed sets of leaves we show this is not possible.

Let $L$ be a leaf of $\tilde{\mathcal{W}}_{cs}$ bran, $x$ a point in $L$ and $\tau$ an open unstable segment through $x$. The set of leaves of $\tilde{\mathcal{W}}_{cs}$ bran intersecting $\tau$ is isomorphic to an open interval. Using the transversal orientation to $\tilde{\mathcal{W}}_{cs}$ bran, we can put an order on this interval.

By our assumption, every point in $\tau$ intersects a leaf in $B$. Let $L'$ be the supremum of leaves in $B$, intersecting $\tau$ and smaller than or equal to $L$. Since $B$ is closed, we have $L' \in B$. Notice that $x$ is in both $L$ and $L'$.

We claim that $L' = L$. If $L$ is not equal to $L'$ then they branch out. Let $y$ be a boundary point of $L \cap L'$. Let $z \in L'$, with $z \notin L$ be close enough to $y$ so that its unstable leaf $u(z)$ intersects $L$. Now take any point $w \in u(z)$ in between $z$ and $L \cap u(z)$. Any leaf $L_1 \in \tilde{\mathcal{W}}_{cs}$ bran that contains $w$ must contain $y$. Hence (because leaves do not cross), $L_1$ also contains $x$. By definition, it is above $L'$, thus $L_1$ is not in $B$. Since this is true for any leaf through $w$, it contradicts our assumption. \qed

**Lemma F.6.** When $\mathcal{W}_{cs}$ bran does not have compact leaves, then $f$-minimality of $\mathcal{W}_{cs}$ bran is equivalent to minimality of $\mathcal{W}_{cs}$ bran.

**Proof.** Note that minimality obviously implies $f$-minimality, so we only need to show the other implication.

Suppose that $\mathcal{W}_{cs}$ bran is not minimal and let $C$ be the union of a set of $\mathcal{W}_{cs}$ bran leaves which is closed and not $M$. Let $\mathcal{W}_{cs}$ be an approximating foliation, with approximating map $h_{cs}^{\epsilon}$ sending leaves of $\mathcal{W}_{cs}$ to those of $\mathcal{W}_{cs}$. Then $(h_{cs}^{\epsilon})^{-1}(C)$ is a set which is a union of $\mathcal{W}_{cs}$ leaves, which is closed and not $M$. In particular it contains an exceptional minimal set $D$. By [HH87, Theorem 4.1.3] the actual foliation $\mathcal{W}_{cs}$ has finitely many exceptional minimal sets $B_1, \ldots, B_k$. The union $B$ of these is not $M$ because $D \neq M$. The set of leaves in $B$ is a closed set of leaves denoted by $\mathcal{B}$. Then $A = h_{cs}^{\epsilon}(B)$ is a closed subset of $M$, and $A = h_{cs}^{\epsilon}(B)$ is

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\(^{14}\)It in fact suffices that $f$ be chain-recurrent, that is, if a non-empty open set $U$ verifies that $f(U) \subset U$ then $U = M$, see [CP] for equivalences.
a closed set of leaves, being the image by $h^{cs}_\epsilon$ of the leaves in $\mathcal{B}$. Let $\widetilde{A} = \pi^{-1}(A)$, we stress that this is on the leaf space level, not in terms of sets. This is a closed subset of $\mathcal{L}^{cs}_b$.

Let $A_t := h^{cs}_\epsilon(B_t)$. Every leaf of $\mathcal{W}^{cs}_{\text{bran}}$ which is the image of a leaf in $B_t$ is dense in $A_t$. Using this, it is easy to see that $f(A) = A$. By $f$-minimality it follows that $A = M$.

Since $A = M$ then $\widetilde{A}$ is a closed subset of $\mathcal{L}^{cs}_b$, whose union of points in all leaves of $\widetilde{A}$ is $M$ as $A = M$. Lemma F.5 implies that $\widetilde{A} = \mathcal{L}^{cs}_b$. Hence for each leaf $E$ of $\mathcal{W}^{cs}_{\text{bran}}$, it is the image of a leaf $F$ in some $B_t$. Conversely every leaf of $\mathcal{W}^{cs}_\epsilon$ maps by $h^{cs}_\epsilon$ to a leaf of $\mathcal{W}^{cs}_{\text{bran}}$.

For each leaf $E$ of $\mathcal{W}^{cs}_{\text{bran}}$, its preimage $(h^{cs}_\epsilon)^{-1}(E)$ is a closed interval of leaves of $\mathcal{W}^{cs}_\epsilon$. No leaf in the interior of the interval can be in a $B_t$ as it is a minimal set. It follows that the complementary regions of $B$ in $M$ are $I$-bundles. These can be collapsed to generate another foliation $\mathcal{C}$. Since the $B_t$ were minimal sets of $\mathcal{W}^{cs}_\epsilon$, then the collapsing of each of these is a minimal set of $\mathcal{C}$. Since the union is all of $M$, there can be only one such minimal set, so $\mathcal{W}^{cs}_\epsilon$ is minimal.

But this contradicts the fact that $D$ is an exceptional minimal set of $\mathcal{W}^{cs}_\epsilon$. □

We state the following criteria for dynamical coherence (which in this setting is quite obvious).

**Proposition F.7** (Proposition 1.6 and Remark 1.10 in [BW05]). Assume that $f$ is a partially hyperbolic diffeomorphism admitting branching foliations $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ and assume that

- no two different leaves of $\mathcal{W}^{cs}_{\text{bran}}$ intersect,
- no two different leaves of $\mathcal{W}^{cu}_{\text{bran}}$ intersect.

Then, $f$ is dynamically coherent.

Finally, lets recall the classification of partially hyperbolic diffeomorphisms in manifolds with virtually solvable fundamental group under the assumption that $f$ is homotopic to identity (see [HP14, HP15, HP19] for the general case):

**Theorem F.8.** Let $f : M \to M$ be a partially hyperbolic diffeomorphism homotopic to identity in a $3$-manifold with virtually solvable fundamental group. Then $M$ is not Seifert fibered and moreover if there are no tori tangent to either $E^{cs}$ or $E^{cu}$, then $f$ is dynamically coherent and an iterate of $f$ is a discretised Anosov flow.

**Appendix G. Discretized Anosov flows**

Let $\varphi_t : M \to M$ be a continuous flow generated by a continuous vector field $X = \frac{\partial}{\partial t} |_{t=0}$. It is called a topological Anosov flow if it preserves two topologically transverse codimension one continuous\footnote{We emphasize here that we do not require a priori the foliations to have $C^1$-leaves.} foliations $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$ (called weak stable and weak unstable) such that:

(i) For every pair of points $x, y \in \mathcal{F}^{ws}$ (resp. $x, y \in \mathcal{F}^{wu}$), there exists an increasing continuous reparametrization $h : \mathbb{R} \to \mathbb{R}$ so that $d(\varphi_t(x), \varphi_{h(t)}(y)) \to 0$ as $t \to +\infty$ (resp. as $t \to -\infty$);

(ii) There exists $\epsilon > 0$ such that for every $x, y \in \mathcal{F}^{ws}$ (resp. $x, y \in \mathcal{F}^{wu}$) not on the same orbit, there exists $t \leq 0$ (resp. $t \geq 0$) such that $d(\varphi_t(x), \varphi_t(y)) > \epsilon$. 

\footnote{We emphasize here that we do not require a priori the foliations to have $C^1$-leaves.}
As mentioned earlier in Appendix D, thanks to the work of Paternain [Pat93] and Inaba and Matsumoto [IM90], the definition of topological Anosov flow can be replaced by asking for the flow to be expansive and to preserve two (non-singular, i.e., without prongs) foliations. Note also that just condition (i) is not enough for a flow to be topological Anosov as condition (i) does not imply condition (ii).

Conditions (i) and (ii) allow one to obtain the same classical results as for Anosov flows (e.g., there are no closed $F^s$ or $F^u$ leaves; the foliations $F^{ws}$ and $F^{wu}$ are taut; the leaves are planes, annuli or Möbius bands — these last two possibilities arising only when the leaves contain a periodic orbit; periodic points are dense in the non-wandering set, etc., see [Bar05] and references therein).

We say that a diffeomorphism $f: M \to M$ is a discretized Anosov flow if there exists a topological Anosov flow $\varphi_t: M \to M$ and a continuous function $\tau: M \to \mathbb{R}_{>0}$ such that $f(x) = \varphi_{\tau(x)}(x)$.

**Remark G.1.** Note that one should not expect a discretized Anosov flow to be the time-1 map of a reparametrization of the Anosov flow. To see this, one can for instance restrict to a periodic orbit of the flow: Then the discretized Anosov flow $f$ restrict to a diffeomorphism of the circle. But diffeomorphisms of the circle do not all come from the time-1 map of a non-singular flow on the circle.

The following result relates the notion of discretized Anosov flows with the usual form of equivalence between partially hyperbolic systems.

**Proposition G.2.** Let $f: M \to M$ be a partially hyperbolic diffeomorphism. The following are equivalent:

1. $f$ is a discretized Anosov flow;
2. $f$ is dynamically coherent, the center leaves are fixed by $f$ and the center foliation is the flow line foliation of a topological Anosov flow.

**Proof.** The fact that the second condition implies the first follows from arguments in [BW05], as was done in section 6.2.

The newer result is the other implication, which we now prove. Let $\varphi_t: M \to M$ be a topological Anosov flow and $\tau: M \to \mathbb{R}_{>0}$ be the positive continuous function such that $f(x) = \varphi_{\tau(x)}(x)$. Let $F$ be the distribution generated by the vector field $X$ generating $\varphi_t$. First, we claim that $F = E^c$. To prove this we will first show that $F$ cannot be equal to $E^s$ or $E^u$ at any point and then deduce that $F$ has to be $E^c$.

Suppose that there is $x \in M$ such that $F(x) = E^s(x)$. Then, the invariance of the orbits of the flow $\varphi_t$ by $f$ together with the uniqueness of the stable manifold implies that there is an arc $I = \varphi_{[-\epsilon,\epsilon]}(x)$ of the orbit of $x$ by $\varphi_t$ which is tangent to $E^s$. That fact is proven in [CP], we give here a brief explanation and the precise references. Consider a small cone field around $E^s$. Let $\alpha$ be the orbit of $\varphi^I$ through $x$. Since $F(x) = E^s(x)$, it follows that, near $f^n(x)$, the curves $f^n(\alpha)$ are uniformly Lipschitz and their tangent are inside the cone field around $E^s$. Notice also that the family of curves $\{f^n(\alpha)\}$ is also invariant under $f$. The uniqueness of the stable manifold implies that Lipschitz curves that are invariant and inside the cone have to be the stable manifold near the point (this is done in [CP, Sections 4.2 and 4.3]). Hence $f^n(\alpha)$ must contain an open interval inside the stable manifold near $x$.

Iterating $I$ backwards, we get that the length of $f^{-n}(I)$ grows exponentially, contradicting the continuity of $\tau: M \to \mathbb{R}_{>0}$.

Thus $F$ is never tangent to $E^s$. The same argument shows that it is never tangent to $E^u$. 


Now, suppose that there is a point $y$ such that $F$ is not inside $E^{cs}$ at $y$. Then applying $DF^n$ to $F(y)$ will get $F(f^n(y))$ closer and closer to $E^n(f^n(y))$. Hence, for any point $z$ in the $\omega$-limit set of $y$, one has that $F(z) = E^u(z)$, contradicting the above. So $F$ is everywhere inside $E^{cs}$ and, by the same argument, also inside $E^{cu}$. Thus $F = E^c$ everywhere.

The last step is to show that $f$ is dynamically coherent, for this, we use the fact that the strong stable saturation of a center curve is tangent to $E^s \oplus E^c$ (see [BI08, Proposition 3.1]). We stress that a center curve here means a curve whose tangent everywhere is in the center bundle $E^c$. In particular this implies that the weak stable foliation $F^{ws}$ of $\varphi_t$ (which a priori could be only continuous) is $C^1$ and everywhere tangent to $E^s \oplus E^c$. This establishes dynamical coherence and completes the proof. □

We remark that a long-standing conjecture (see [BW05]) states that every topological Anosov flow is orbit equivalent to an Anosov flow. If this conjecture is true, then condition (2) above is equivalent to saying that $f$ is dynamically coherent and leaf conjugate to the time one map of an Anosov flow. We remark here that it has been recently announced that this conjecture is true in the setting of transitive topological Anosov flows [Sha].

**Appendix H. The graph transform argument**

We give here an application of the general graph transform technique to the particular case we needed it in.

We call center stable plane any embedded $C^1$-plane tangent to $E^s \oplus E^c$ in $\hat{M}$. Notice that by unique integrability of $E^s$ there is always a stable foliation inside a center stable plane.

**Lemma H.1** (Graph Transform Lemma). Let $f$ be a partially hyperbolic diffeomorphism in $M$. Suppose that $L \subset \hat{M}$ is a center stable plane which is fixed by a lift $\hat{f}$ of $f$ to $\hat{M}$, and by some $\gamma \in \pi_1(M) \setminus \{id\}$. Assume that there is a properly embedded $C^1$ curve $\eta$ transverse to the stable foliation in $L$ and such that

$$\gamma \eta = \eta \quad \text{and} \quad \hat{f}(\eta) \subset \bigcup_{z \in \eta} \hat{W}^s(z).$$

Then in $L$ there is a curve $\hat{\eta}$ which is fixed by both $\hat{f}$ and $\gamma$ and is everywhere tangent to $E^c$.

Notice the subtlety in the conclusion of this lemma: The curve $\hat{\eta}$ produced is tangent to the center direction, however, it may not be a center leaf (as not all curves tangent to the center direction are part of a center leaf, see Definition 10.6 and Remark 10.7).

**Remark H.2.** The second hypothesis of the lemma is equivalent to saying that the union $\bigcup_{z \in \eta} \hat{W}^s(z)$ is invariant by $\hat{f}$. In particular, all positive and negative images of $\eta$ by powers of $\hat{f}$ are contained in this union. To see this, notice that the second condition implies that $f_1(\alpha)$ is freely homotopic to $\alpha$ in $L/\langle \gamma \rangle$, because it is a cylinder, and $\alpha$ (or $f_1(\alpha)$) is not null homotopic in this cylinder. Therefore $\bigcup_{z \in \eta} \hat{W}^s(z) = \bigcup_{z \in f(\eta)} \hat{W}^s(z)$. Thus $\bigcup_{z \in \eta} \hat{W}^s(z)$ is $\hat{f}$ invariant. The converse is immediate.

**Proof.** We work in the quotient $L/\langle \gamma \rangle$ which is an annulus on which $\eta$ projects to a closed $C^1$-circle transverse to the stable foliation, denoted by $\alpha$. Let $\pi_\eta: L \to L/\langle \gamma \rangle$ the quotient map. Let $f_1$ be the induced diffeomorphism on $L/\langle \gamma \rangle$. 
Up to a small modification of \( \alpha \) if necessary, we can assume that \( \alpha \) is simple, that is, it goes around the cylinder \( L/<\gamma> \) once.

We parametrize \( \alpha \) in \( L/<\gamma> \) by arclength (for the leaf-wise path metric on \( L/<\gamma> \)). Then, we parametrize \( \bigcup_{z \in \alpha} W^s(z) \) as a cylinder \( S^1 \times \mathbb{R} \) contained in \( L/<\gamma> \), where \( \alpha \) is the zero section and the stable leaves are parametrized by arclength.

Since all \( \hat{f}^n(\eta) \) are in \( \bigcup_{z \in \eta} W^s(z) \), we can express \( \eta_n := \hat{f}^n(\eta) \) as graphs of \( C^1 \)-functions\(^{16} \) from \( S^1 \) to \( \mathbb{R} \).

We want to show that \( \hat{f} \) acts as a contraction on curves transverse to the stable foliation in (at least a compact part of) \( \bigcup_{z \in \alpha} W^s(z) \simeq S^1 \times \mathbb{R} \). First we show that all the \( \eta_n \) stay in a compact subset of \( \bigcup_{z \in \eta} W^s(z) \) (and thus also of \( L/<\gamma> \)).

Our assumptions imply that there exists some \( a_0 > 0 \), such that \( f_1(\alpha) \) is contained in an \( a_0 \) stable neighborhood of \( \eta \). That is, in the union of some segment of length \( 2a_0 \) centered at \( \eta \).

Let \( \lambda < 1 \) be the smallest contraction factor for \( f_1 \) along stable leaves. It follows that \( f_1^2(\eta) \) is contained in the stable neighborhood of size \( a_0 + \lambda a_0 \) around \( \alpha \), and so on. Thus, we immediately get that, for all \( n \), \( f_1^n(\alpha) \) is contained in a compact subset of the annulus.

Now that we know all the \( \eta_n \) curves are contained in a compact subset, we can use the fact that \( f_1 \) contracts stable leaves more than centers to prove the following:

There exists some constant \( a_1 \) such that \( f_1 \) globally preserves the space of uniformly bounded (for some appropriately large bound) Lipschitz functions from \( S^1 \) to \( \mathbb{R} \) with Lipschitz constant less than \( a_1 \). By standard computations, one can see that this acts as a contraction on this complete metric space (this is usually called the graph transform technique see e.g., \cite{HPS77} or \cite[Section 4.2]{CP} for a more detailed study of this technique and the reason for considering Lipschitz functions).

Therefore, one obtains that there is a unique fixed point of this action which corresponds to the graph of a Lipschitz function from \( S^1 \) to \( \mathbb{R} \) which is the unique invariant Lipschitz graph under \( f_1 \). It is also standard to show that the tangent cones at each point must actually be degenerate (see \cite[Section 4.2]{CP}), i.e., the invariant curve is \( C^1 \). Moreover, since \( E^s \) is the only invariant bundle transverse to \( E^s \), the curve must be everywhere tangent to \( E^s \). The lift of this curve to \( L \) is the curve we sought. \( \square \)

Under the assumptions of the Graph Transform Lemma (Lemma H.1), another thing we easily deduce is that there must be a periodic center leaf of \( f \) in the projection of the leaf \( L \):

**Lemma H.3.** Let \( f \) be a partially hyperbolic diffeomorphism in \( M \). Suppose that \( L \subset \tilde{M} \) is a center stable plane which is fixed by a lift \( \hat{f} \) of \( f \) to \( \tilde{M} \), and by some \( \gamma \in \pi_1(M) \smallsetminus \{ id \} \).

Assume that there exists a curve \( \tilde{c} \) that is fixed by both \( \hat{f} \) and \( \gamma \). Then there exists a center leaf \( c \) in \( L \) and two integers \( n, m \), with \( m \neq 0 \), such that \( c = \gamma^n \hat{f}^m c \).

To prove this lemma, we need to use the center leaf space on \( L \). When the foliations are branching, the center leaf space was defined in Section 10.1.2.

\(^{16}\)In this specific case with one dimensional center, one can assume that the stable foliation is \( C^1 \) inside center stable leaves so that this makes sense, see \cite[Section 4.7]{CP}. If the stable foliation is less regular then one can go through with the proof by taking a smooth approximating foliation instead, and the arguments would be essentially the same.
Proof. Let $\eta = \pi(\hat{\eta})$ be the projection of $\hat{\eta}$ to $M$. Since $\hat{\eta}$ is invariant by $\gamma$ and $\hat{f}$, the curve $\eta$ is a circle on which $f$ acts.

Suppose first that $f$ has a periodic point on $\eta$. Then, there exists a center leaf through that point that is periodic, as claimed.

Otherwise, there exists a point in $\eta$ that is inside its $\omega$-limit set for $f$.

Lifting back to the universal cover, this means that there exists $x \in \hat{\eta}$ such that there exists integers $m, n$, with $m$ arbitrarily large, such that $x$ and $\gamma^n f^m(x)$ can be made arbitrarily close.

Let $\tau$ be a compact segment of the stable leaf through $x$. Since $\hat{f}$ contracts the length of stable segments, we can choose $m \in \mathbb{N}$ large enough so that every center leaf through $\gamma^n f^m(\tau)$ intersects the interior of $\tau$. (This is possible as $\gamma^n f^m(\tau)$ can be chosen arbitrarily small and arbitrarily close to $x$, which is in the interior of $\tau$.)

Let $L_\tau^c$ be the leaf space of the center foliation in $L$ (see section 10.1.2 for the definition when the center foliation is branching). Let

$$
\tau_c = \{ c \in L_\tau^c \mid c \cap \tau \neq \emptyset \}.
$$

Notice that $\tau_c$ is a compact interval in the 1-manifold $L_\tau^c$.

Then consider the function $h: \tau_c \to L_\tau^c$ defined by $h(c) = \gamma^n f^m(c)$. The map $h$ is continuous, and, thanks to our choice of $m$, $h(\tau_c)$ is contained in the interior of $\tau_c$. Hence, there exists $c_0 \in \tau_c$ that is fixed by $h$, as claimed. \hfill \Box

**Appendix I. The Lefschetz index**

Here we define the Lefschetz index and give the main property that we used. We refer to the monograph by Franks [Fra82, Section 5] for details and other references.

For any space $X$ and subset $A \subset X$, we denote by $H_k(X, A)$ the $k$-th relative homology group with coefficients in $\mathbb{Z}$.

**Definition I.1.** Let $V \subset \mathbb{R}^k$ be an open set and $F: V \subset \mathbb{R}^k \to \mathbb{R}^k$ be a continuous map such that the set of fixed point of $F$ is $\Gamma \subset V$, a compact set. Then the **Lefschetz index** of $F$, denoted by $I_L(F)$, is an element in $\mathbb{Z} \cong H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\})$, defined as follows. It is the image of the map $(\text{id} - F)_*: H_k(V, V - \Gamma) \to H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\})$ of the class $w_F$, where $w_F$ itself is the image of the generator 1 under the composite $H_k(\mathbb{R}^k, \mathbb{R}^k - D) \to H_k(\mathbb{R}^k, \mathbb{R}^k - \Gamma) \cong H_k(V, V - \Gamma)$. Here $D$ is a ball containing $\Gamma$.

It is easy to see that if $\Gamma = \text{Fix}(F) = \Gamma_1 \cup \cdots \cup \Gamma_j$, where $\Gamma_i$ are compact and disjoint then $I_L(F) = \sum I_L(F)$. Here $I_L(F)$ is the index restricted to an open set $V_i$ of $V$ which does not intersect the other $\Gamma_m$, see [Fra82, Theorem 5.8 (b)].

This technical definition works well with the standard examples. For a single hyperbolic fixed point $q$, the index at $q$ is exactly $\text{sgn}(\det(\text{id} - D_q F))$ (see [Fra82, Proposition 5.7]), where $\det$ is the determinant, and $\text{sgn}$ is the sign of the determinant. Hence in dimension 2 the index of a hyperbolic fixed point when the orientation of the bundles is preserved is $-1$. This can be generalized to a $p$-prong hyperbolic fixed point to obtain that the index is $1 - p$. This is because the index is invariant by homotopic changes. A $p$-prong can be easily split into $p - 1$ distinct hyperbolic points which are differentiable. In addition for any fixed set which behaves locally as a hyperbolic fixed point, the index is the same as the hyperbolic fixed point.

The main property we use is the following.
Proposition I.2 (Theorem 5.8(c) of [Fra82]). Let $P$ be a topological plane equipped with a metric $d$. Let $g, h : P \to P$ be two homeomorphisms. Suppose that there exists $R > 0$ such that:

- For every $x \in P$, one has that $d(g(x), h(x)) < R$;
- There is a disk $D$ such that, for every $x \notin D$, one has that $d(x, g(x)) > 2R$.

Then, the total index $I_{\text{Fix}(g)}(g) = I_{\text{Fix}(h)}(h)$.

See also [KH95, Section 8.6] for an alternate presentation of the Lefschetz index.

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