GRAPHICAL SPLITTINGS OF GENERALIZED BESTVINA-BRADY GROUPS

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Abstract. We study generalized Bestvina-Brady groups, i.e. kernels of discrete characters of right-angled Artin groups, and we show that they decompose as graphs of groups in a way that can be explicitly computed from the underlying graph. When the underlying graph is chordal we show that every such subgroup either surjects to an infinitely generated free group or is a generalized Baumslag-Solitar group of variable rank. In particular for block graphs (e.g. trees), we obtain an explicit rank formula, and discuss some features of the space of fibrations of the associated right-angled Artin group.

Contents

1. Introduction

Given a finite simplicial graph Γ the associated right-angled Artin group (RAAG) is the group A_Γ generated by the vertices of Γ, with two generators commuting if and only if the corresponding vertices are connected by an edge in Γ; in other words, this is the group with the presentation:

$$A_\Gamma = \langle x \in V(\Gamma) \mid [x, y] \in E(\Gamma) \rangle$$

As such, RAAGs interpolate between free groups and free abelian groups. Moreover they enjoy many good algebraic properties (e.g. they are torsion free, linear over the integers, residually finite, ...). Despite their very simple definition, RAAGs turn out to display a surprisingly rich geometric behavior and a very diverse collection of subgroups, which has made them one of the main characters in recent breakthroughs in low-dimensional geometry, as in the work of Haglund and Wise on

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special cube complexes (see [HW08]) and of Agol on the Virtual Haken Conjecture (see [Ago13]).

One of the most appealing aspects of the theory of RAAGs is that many of their properties can be detected and computed directly from the structure of the underlying graph. For instance a RAAG A_Γ splits as a free product precisely when Γ is disconnected, and Clay has shown in [Cla14] that all non-trivial splittings of A_Γ over $\mathbb{Z}$ correspond to cut vertices of Γ, and more generally Groves and Hull have shown in [GH17] that A_Γ splits over an abelian subgroup if and only Γ is disconnected, or complete, or contains a separating clique (i.e. a full complete subgraphs whose removal disconnects Γ).

This paper deals with fibrations of RAAGs, i.e. non-trivial homomorphisms

$$1 \to K \to A_\Gamma \to \mathbb{Z} \to 1$$

with finitely generated kernel. When Γ is a tree, A_Γ is known to be the fundamental group of a compact 3-manifold (see [Dro87a]), and by classical result of Stallings (see [Sta62]) fibrations of A_Γ correspond to fibrations of the manifold over the circle, in a way which is well understood in terms of the Thurston norm on homology (see [Thu86]). For general graphs these fibrations can be conveniently organized by the so-called Bieri-Neumann-Strebel invariant of A_Γ, a tool introduced in [BNS87] as a generalization of Thurston's picture for general finitely generated groups, and explicitly computed for RAAGs by Meier and VanWyk in [MV95].

In [BB97] Bestvina and Brady considered the natural fibration obtained by sending all the vertices of a connected graph Γ to $1 \in \mathbb{Z}$, and studied the corresponding kernel, today known as the Bestvina-Brady group associated to Γ. They showed that its finiteness properties are completely determined by the topology of Γ; for instance, the Bestvina-Brady group is finitely presented if and only if the flag complex generated by Γ is simply connected. They used this correspondence to show that certain Bestvina-Brady groups provide counterexamples either to the Eilenberg-Ganea Conjecture or to the Whitehead Conjecture.

In this paper we consider generalized Bestvina-Brady groups, i.e. kernels of general homomorphisms $f : A_\Gamma \to \mathbb{Z}$, and we are interested in exploring their finiteness behavior in terms of their splittings induced by decompositions of the underlying graph Γ. A splitting of Γ is a decomposition of Γ into two full subgraphs $\Gamma = \Gamma_1 \cup \Gamma_2$ meeting along a full subgraph $\Gamma_3 = \Gamma_1 \cap \Gamma_2$; we denote the splitting by $(\Gamma_1, \Gamma_2, \Gamma_3)$ and emphasize that none of these graphs has to be connected. Our results concern the ways in which such a splitting can induce a splitting of the generalized Bestvina-Brady groups associated to Γ, and are inspired by the work of Cashen and Levitt on free-by-cyclic groups (see [CL16]), in which analogous tame/wild dichotomies and rank formulas are obtained. Our main result is the following (see Theorem 3.2 below for more details).

Theorem 1.1. *Let $(\Gamma_1, \Gamma_2, \Gamma_3)$ be a splitting of a connected graph Γ, let $f : A_\Gamma \to \mathbb{Z}$ be non-trivial, and let f_k denote the restriction of f to A_{Γ_k} for $k = 1, 2, 3$. Then*

1. *(wild) if $f_3 = 0$, then $\ker(f)$ surjects onto $\mathbb{F}_\infty$;*
2. *(tame) if $f_3 \neq 0$, then $\ker(f)$ splits as a finite graph of groups $\mathcal{G}$, with $[f(A_\Gamma) : f(A_{\Gamma_k})]$ vertex groups isomorphic to $\ker(f_k)$ for $k = 1, 2$, and $[f(A_\Gamma) : f(A_{\Gamma_3})]$ edge groups isomorphic to $\ker(f_3)$.*

Notice that the two cases here are not disjoint from each other, i.e. the same character f can display tame behavior with respect to one splitting, and wild behavior with respect to another (see Remark 3.3). In order to obtain a complete dichotomy, we restrict to chordal graphs in §3.2, where we prove the following (see Theorem 3.9).

Theorem 1.2. *Let Γ be a connected chordal graph, and let $f : A_\Gamma \to \mathbb{Z}$ be non-trivial. Then exactly one of the following holds.*

1. *(wild)* $\ker(f)$ *surjects onto* $\mathbb{F}_\infty$*;*
2. *(tame)* $\ker(f)$ *is a vGBS group (generalized Baumslag-Solitar group of variable rank), i.e. splits as a finite graph of finitely generated free abelian groups.*

In the tame case the graph of groups is obtained by applying the previous Theorem to a sequence of splittings that completely decompose Γ into the collection of its maximal cliques. It is worth mentioning that the vGBS groups appearing in the tame case are groups for which the multiple conjugacy problem is well understood (see Beeker [Bee15]). While it is not known in general if this is the only way a generalized Bestvina-Brady group decomposes over abelian subgroups, Chang has recently shown in [Cha20] that the classical Bestvina-Brady group over a general graph Γ splits over an abelian subgroup if and only if Γ is disconnected, or complete, or has a separating clique (the same statement holds for the RAAG A_Γ itself by [GH17]).

The above decomposition of the kernel as a vGBS group allows for a very explicit description of all the fibrations of a RAAG A_Γ when the underlying graph Γ is a block graph; in this case edge groups are trivial, and we can obtain a decomposition of the generalized Bestvina-Brady group as a free product of free abelian groups, whose rank can explicitly be computed, leading to the following rank formula (see Theorem 3.17, and compare the rank formula in [CL16, Theorem 1.1,6.1]). Here $\mathcal{B}(\Gamma)$ is the set of blocks (i.e. biconnected components, or equivalently maximal cliques) of the block graph Γ, and $\mathrm{bldeg}_\Gamma(v)$ is the number of blocks containing a vertex v.

Theorem 1.3. *Let Γ be a connected block graph, and $f : A_\Gamma \to \mathbb{Z}$ be non-zero on cut vertices. Then*

$$\mathrm{rk}(\ker(f)) = 1 + \sum_{B \in \mathcal{B}(\Gamma)} [f(A_\Gamma) : f(A_B)](|V(B)| - 2) + \sum_{v \in V(\Gamma)} \frac{(\mathrm{bldeg}_\Gamma(v) - 1)\,|f(v)|}{[\mathbb{Z} : f(A_\Gamma)]}$$

As an application of the explicit control that this formula provides, we show that when Γ is a block graph, the RAAG A_Γ admits fibrations with kernels of arbitrarily large rank (see Corollary 3.20), and we characterize those with kernel of minimal rank (see Corollary 3.19). Moreover in Corollary 3.21 we show how to construct sequences of tame fibrations

$$1 \to K_n \to A_\Gamma \xrightarrow{f_n} \mathbb{Z} \to 1$$

whose kernels have constant finite rank, but such that the sequence f_n converges to a homomorphism $f_\infty : A_\Gamma \to \mathbb{Z}$ whose kernel is wild (i.e. surjects onto $\mathbb{F}_\infty$). Here the convergence is best understood to be up to rescaling by (positive) real numbers, operation that of course does not modify the kernel; in this sense this shows an explicit way in which the function that records the rank of the kernel of a fibration is not a proper function on the BNS invariant of A_Γ.

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2. Background on right-angled Artin groups

Let Γ be a graph (by which we mean a finite 1-dimensional simplicial complex). We denote by $V(\Gamma)$ the set of its vertices and by $E(\Gamma)$ the set of its edges. The right-angled Artin group (RAAG in the following) associated to Γ is the group A_Γ generated by the vertices of Γ and such that two generators commute if and only if they are joined by an edge, i.e. A_Γ has the following presentation

$$A_\Gamma = \langle x \in V(\Gamma) \mid [x, y] \in E(\Gamma)\rangle$$

The RAAG associated to the empty graph is defined to be the trivial group. The RAAG associated to a clique (i.e. a complete graph) on n vertices is the free abelian group $\mathbb{Z}^n$. The RAAG associated to a totally disconnected graph on n vertices is the free group $\mathbb{F}_n$.

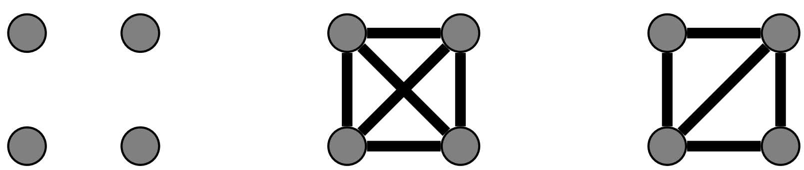

Figure 1. Graphs corresponding to $\mathbb{F}_4$, $\mathbb{Z}^4$, and an amalgamated product of two copies of $\mathbb{Z}^3$ over $\mathbb{Z}^2$.

More general RAAGs interpolate between these two examples, and can be quite rich despite their simple definition; for instance when Γ is a tree A_Γ is the fundamental group of a compact 3-manifold (with boundary) fibering over S^1 (see [Dro87a]).

It is well-known (see [Dro87b]) that the graph completely determines the group, in the sense that A_Γ is isomorphic to A_Λ if and only if the graphs Γ and Λ are isomorphic. Moreover given any full subgraph $\Lambda \subseteq \Gamma$ we have that the RAAG A_Λ embeds in a natural way as a subgroup of A_Γ.

Remark 2.1. Notice that any map of A_Γ into a group is completely determined by what it does on vertices. Moreover, since all the relations are just commutators, for any abelian group A any map $f_0 : V(\Gamma) \to A$ uniquely extends to a well-defined group homomorphism $f : A_\Gamma \to A$.

Bestvina and Brady in [BB97] considered the map $f : A_\Gamma \to \mathbb{Z}$ obtained by sending every vertex to $1 \in \mathbb{Z}$. The kernel of this map is known as the Bestvina-Brady subgroup of A_Γ. In this paper we consider the following generalization.

Definition 2.2. A generalized Bestvina-Brady group (or GBB group) is a normal subgroup of A_Γ with infinite cyclic quotient.

A practical way to construct such subgroups is of course to consider kernels of more general maps that send the vertices of Γ to more general collections of numbers (see Section 2.2 below). Notice that unlike the right-angled Artin group, Bestvina-Brady groups are not classified by their underlying graph, not even in the classical case: for instance the classical Bestvina-Brady group on any tree is a free group generated by the edges. Moreover while, as we will see, certain Bestvina-Brady groups are themselves RAAGs, this is not true for a general Bestvina-Brady group, as it can fail to be finitely presented (and for an example even in the finitely presented case see [PS07, Example 2.8]).

2.1. **Graphical splittings for RAAGs.** We will be interested in understanding the structure of generalized Bestvina-Brady groups induced by decompositions of the underlying graph Γ. We start by reviewing known results about splittings of RAAGs induced by decompositions of the underlying graph.

Recall that a group G splits over a subgroup $H \neq G$ if it can be realized as the free product of two subgroups $A, B \neq G$ amalgamated over H (i.e. $G \cong A *_H B$), or as an HNN extension of a subgroup $A \neq G$ over H (i.e. $G \cong A*_H$). For easy ways to obtain splittings of a RAAG one can notice the following: if the graph is disconnected then A_Γ splits as a free product of the RAAGs over the connected components, and if it is complete then A_Γ is free abelian, hence splits as an HNN extension over a codimension-1 subgroup. Less trivial splittings can be induced by decompositions of the graph, as follows.

Definition 2.3. For a subgraph $\Lambda \subseteq \Gamma$ we denote by $\Gamma \backslash \Lambda$ the full subgraph of Γ generated by $V(\Gamma)\backslash V(\Lambda)$, and call it the complement of Λ in Γ. We say Λ is separating when its complement is disconnected. A cut vertex is a separating vertex.

Definition 2.4. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be pairwise distinct non-empty full subgraphs of Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \Gamma_3$. We say that $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a splitting of Γ. If Γ_3 is connected, then we say $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a connected splitting of Γ.

For the sake of precision let us emphasize that here the condition $\Gamma = \Gamma_1 \cup \Gamma_2$ means that both $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2)$, and the condition $\Gamma_3 = \Gamma_1 \cap \Gamma_2$ means that both $V(\Gamma_3) = V(\Gamma_1) \cap V(\Gamma_2)$ and $E(\Gamma_3) = E(\Gamma_1) \cap E(\Gamma_2)$. In particular we have the following.

Lemma 2.5. *Let $(\Gamma_1, \Gamma_2, \Gamma_3)$ be a splitting of Γ. Then the following hold.*

1. *There is no edge between $V(\Gamma_1)\backslash V(\Gamma_3)$ and $V(\Gamma_2)\backslash V(\Gamma_3)$.*
2. *Γ_3 is separating.*
3. *If Γ and Γ_3 are connected, then Γ_1 and Γ_2 are connected too.*

Proof. Statement (1) follows from the fact that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \Gamma_3$, and statement (2) follows from (1). To prove (3), notice that since Γ is connected, we can join any vertex of Γ_1 to a vertex in Γ_3 by an edge-path; moreover by (1) this edge-path can be chosen to lie completely in Γ_1. Then connectedness of Γ_3 implies that any two vertices in Γ_1 can be joined by an edge-path entirely contained in Γ_1; the same applies to Γ_2. □

It is easy to check directly from the RAAG presentation that a splitting of Γ induces a splitting of A_Γ, in the following sense.

Lemma 2.6. *If $(\Gamma_1, \Gamma_2, \Gamma_3)$ split Γ, then $A_\Gamma = A_{\Gamma_1} \underset{A_{\Gamma_3}}{*} A_{\Gamma_2}$.*

For abelian splittings (i.e. splittings over abelian subgroups) actually more is true, as shown by Clay for splitting over $\mathbb{Z}$ (in [Cla14]), and Groves and Hull in the general case (see [GH17]). Notice that an abelian subgroup of a RAAG is necessarily torsion-free.

Proposition 2.7. *A_Γ splits over an abelian group if and only if one of the following occurs:*

1. *Γ is disconnected,*
2. *Γ is complete,*
3. *Γ has a separating clique.*

In particular one can see that when Γ is disconnected, A_Γ splits as a free product (indeed Lemma 2.6 holds even when $\Gamma_3 = \varnothing$), and the same holds for any generalized Bestvina-Brady group defined on the same graph. On the other hand when Γ is complete both A_Γ and all its generalized Bestvina-Brady subgroups are free abelian groups; as such they all split as HNN extensions over codimension-1 subgroups. Therefore in the rest of the paper we will be interested in the case in which Γ is connected and not complete, hence contains a separating subgraph.

2.2. Characters and BNS invariant for RAAGs.

Normal subgroups with infinite cyclic quotients can be conveniently defined as kernels of real characters, by embedding the quotient in $\mathbb{R}$. In this section we develop this point of view by reviewing a convenient space of real characters introduced by Bieri-Neumann-Strebel in [BNS87] for general finitely presented groups, and then studied by Meier and VanWyk in the specific case of RAAGs in [MV95]. We are going to use this to organize the collection of generalized Bestvina-Brady subgroups of a fixed RAAG, and to have a meaningful notion of convergence.

Let A_Γ be a RAAG defined by a connected graph Γ. The space of all real characters (i.e. group homomorphisms into $\mathbb{R}$) can be easily parametrized by recording the value of a character on the vertices of Γ, i.e.

$$\mathrm{Hom}(A_\Gamma, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}^{|V(\Gamma)|}, \quad f \mapsto (f(v_1), \dots, f(v_n))$$

where $V(\Gamma) = \{v_1, \dots, v_n\}$; notice this is as isomorphism by Remark 2.1.

We explicitly notice that if $f : A_\Gamma \to \mathbb{R}$ is a character whose image is a discrete non-trivial (equivalently, infinite cyclic) subgroup of $\mathbb{R}$, then $\ker(f)$ is a generalized generalized Bestvina-Brady group; conversely if $B \subset A_\Gamma$ is a generalized Bestvina-Brady group, then $A_\Gamma / B \cong \mathbb{Z}$, and any isomorphism provides a character of A_Γ. In other words, characters with discrete image correspond to generalized Bestvina-Brady groups. We find it useful to introduced the following terminology.

Definition 2.8. We say a character $f : A_\Gamma \to \mathbb{R}$ is respectively integral, rational or discrete if its image is an infinite cyclic subgroup of $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$.

Rescaling a character does not change its kernel, and a character with discrete image can always be rescaled to be integral, so that generalized Bestvina-Brady groups can actually be thought as primitive integral vectors in $\mathbb{R}^{|V(\Gamma)|}$ through the above isomorphism. Notice that the zero character is not discrete according to this definition. To avoid useless complications, unless otherwise specified, in the following we will implicitly assume that f is not the zero character.

As we are interested in characters only up to rescaling, we also find it convenient to introduce the character sphere

$$S(A_\Gamma) = (\mathrm{Hom}(A_\Gamma, \mathbb{R})\backslash\{0\})/\mathbb{R}^+$$

and to conflate a character and its class in $S(A_\Gamma)$, whenever no confusion arises. We will also extend the above terminology accordingly. We remark that here we are taking the quotient only with respect to positive scalars (as opposed to non-zero scalar) just for the sake of consistency with the literature.

Meier and VanWyk in [MV95] have obtained a simple condition for a discrete character of a RAAG to have a finitely generated kernel (notice they call rational what we call discrete here).

Definition 2.9. Given a character $f : A_\Gamma \to \mathbb{R}$ we define

- the living subgraph of f as the full subgraph $\mathcal{L}(f)$ of Γ generated by $\{v \in V(\Gamma) \mid f(v) \neq 0\}$;
- the dead subgraph of f as the full subgraph $\mathcal{D}(f)$ of Γ generated by $\{v \in V(\Gamma) \mid f(v) = 0\}$.

Notice these graphs depend only on the equivalence class of f. Moreover when Γ is connected we have that $\mathcal{L}(f)$ is connected if and only if $\mathcal{D}(f)$ is not separating.

Definition 2.10. A subgraph Λ of Γ is dominating if every vertex of Γ is adjacent to some vertex of Λ, i.e. Γ coincides with the star of Λ.

Notice that when Γ is connected we have that $\mathcal{L}(f)$ is connected and dominating if and only if $\mathcal{D}(f)$ does not contain a separating subgraph. As far as our discussion is concerned, the main result by Meier and VanWyk is the following.

Theorem 2.11 ([MV95, Theorem 6.1])**.** *Let $f : A_\Gamma \to \mathbb{R}$ be discrete. Then* $\ker(f)$ *is finitely generated if and only if $\mathcal{L}(f)$ is connected and dominating.*

The Bieri-Neumann-Strebel invariant (in the following BNS invariant) of a finitely generated group was introduced in [BNS87] as a device to capture information about finiteness properties of normal subgroups with abelian quotients, and therefore it is a convenient way to organize characters with finitely generated kernels. It is an open subset of the character sphere

$$\Sigma^1(A_\Gamma) \subseteq S(A_\Gamma)$$

which can be thought as a generalization of Thurston's polyhedron for 3-manifold groups (see [Thu86]). The original definition is not the most useful in our setting, and we prefer to use the following characterization (see [BNS87, §4] or [MV95, Corollary 1.2]) as a working definition of what it means for a discrete character to belong to the BNS invariant.

Theorem 2.12 (Bieri-Neumann-Strebel)**.** *Let $f : A_\Gamma \to \mathbb{R}$ be discrete. Then* $\ker(f)$ *is finitely generated if and only if both f and $-f \in \Sigma^1(A_\Gamma)$.*

To sum up we have the following characterization of discrete characters with finitely generated kernels, i.e. finitely generated generalized Bestvina-Brady subgroups of a fixed RAAG.

Corollary 2.13. *Let $f : A_\Gamma \to \mathbb{R}$ be discrete. Then the following are equivalent:*

(1) $\ker(f)$ *is finitely generated,*
(2) $\mathcal{L}(f)$ *is connected and dominating,*
(3) $f, -f \in \Sigma^1(A_\Gamma)$,
(4) $f \in \Sigma^1(A_\Gamma)$.

Proof. The equivalence of (1) and (2) is Theorem 2.11, and the equivalence of (1) and (3) is Theorem 2.12. The equivalence of (3) and (4) follows from the fact that $\Sigma^1(A_\Gamma)$ is invariant under automorphisms of A_Γ, and the fact that any RAAG has an involutive automorphism which sends every generator to its inverse; the action of such an automorphism on characters is of course $f \mapsto -f$. □

3. Splittings of generalized Bestvina-Brady groups

We now turn to the study of the finiteness properties of the generalized Bestvina-Brady subgroups of a fixed RAAGs in terms of the behavior of a defining character on separating subgraphs. Recall that in order to avoid trivial redundancies, by default we assume that $f : A_\Gamma \to \mathbb{R}$ is not the zero character, and that Γ is a finite connected non-complete graph, so that it admits non-trivial splittings.

After proving a general dichotomy relative to a single splitting in §3.1, in §3.2 we focus on graphs that can be iteratively decomposed into pieces which are easily understood, such as chordal graphs. In the last section §3.3 we obtain an explicit formula for the rank of the generalized Bestvina-Brady groups and discuss some features of the BNS invariant of A_Γ when Γ is a block graph.

3.1. Splitting dichotomy.

Here we prove a general splitting result along a separating subgraph, which will later be used as a key tool for more specific applications. More precisely we propose an extension to generalized Bestvina-Brady groups of the main Proposition in [Dro87a],which is about the classical Bestvina-Brady groups. This is inspired by an analogous dichotomy obtained by Cashen and Levitt in [CL16, Theorem 4.4] in the context of free-by-cyclic groups. It should be noted that classical Bestvina-Brady groups are free-by-cyclic precisely when the underlying graph is a tree. The main argument is based on the following orbit count lemma already used in [CL16], which we will apply to the action of generalized Bestvina-Brady groups on Bass-Serre trees associated to splittings of the underlying graph.

Lemma 3.1. *Let a group G act transitively on a set X; let $K \subseteq G$ be a normal subgroup and $f : G \to G/K$ be the quotient map. For $x \in X$ let G_x denote the stabilizer of x for the action $G \curvearrowright X$. Then the number of K-orbits in X is equal to the index $[G/K : f(G_x)]$.*

Here and in the following, for any subgraph Λ of Γ we let $\mathrm{I}_{\Gamma,\Lambda} = [f(A_\Gamma) : f(A_\Lambda)]$ be the index of $f(A_\Lambda)$ in $f(A_\Gamma)$. We begin with a statement that does not require the splitting to be connected. We also remark that by a finite graph of groups we just mean a graph of groups whose underlying graph is finite, but in which groups are allowed to be not finitely generated.

Theorem 3.2. *Let $(\Gamma_1, \Gamma_2, \Gamma_3)$ be a splitting of Γ, let $f : A_\Gamma \to \mathbb{R}$ be discrete, and let $f_k : A_{\Gamma_k} \to \mathbb{R}$ be its restriction for $k = 1, 2, 3$. Then $\ker(f)$ splits as a graph of groups $\mathcal{G}$ with*

- $\mathrm{I}_{\Gamma,\Gamma_k}$ *vertices with group isomorphic to* $\ker(f_k)$ *for $k = 1, 2$,*
- $\mathrm{I}_{\Gamma,\Gamma_3}$ *edges with group isomorphic to* $\ker(f_3)$.

In particular

(1) *if* $f_3 \neq 0$*, then* $\ker(f)$ *splits as a finite graph of groups* $\mathcal{G}$*,*
(2) *if* $f_3 = 0$*, then* $\ker(f)$ *surjects onto* $\mathbb{F}_\infty$*.*

Proof. By Lemma 2.6, we know that the splitting of Γ induces a splitting of A_Γ as

$$A_\Gamma = A_{\Gamma_1} \underset{A_{\Gamma_3}}{*} A_{\Gamma_2}$$

Let $\mathcal{T}$ be the Bass-Serre tree associated to this splitting. The group A_Γ acts on $\mathcal{T}$, and we restrict the action to the normal subgroup $\ker(f)$. Let $x \in \mathcal{T}$ be a vertex corresponding to a coset of A_{Γ_1}. We have that A_Γ acts transitively on the orbit $A_\Gamma.x$, with vertex stabilizers conjugate to A_{Γ_1}; by Lemma 3.1 this orbit splits in $\mathrm{I}_{\Gamma,\Gamma_1}$ orbits for the action of $\ker(f)$, and vertex stabilizers are conjugate to $\ker(f) \cap A_{\Gamma_1} = \ker(f_1)$. A similar argument applies to the other type of vertices, and to the edges. Then the quotient $\mathcal{G} = \mathcal{T}/\ker(f)$, labelled with the appropriate stabilizers, provides the desired graph of groups decomposition. Notice that this graph of groups is bipartite and has no loops. We are left to proving the dichotomy.

(1) If $f_3 \neq 0$, then of course also $f_k \neq 0$ for $k = 1, 2$, hence $\mathrm{I}_{\Gamma,\Gamma_k}$ is finite for $k = 1, 2, 3$, hence from the first part of the statement we get that $\mathcal{G}$ is finite.
(2) If $f_3 = 0$, then $\mathrm{I}_{\Gamma,\Gamma_3} = \infty$ the graph of groups has infinitely many edges. We distinguish two cases.
   (a) If $f_k \neq 0$ for $k = 1, 2$, then $\mathcal{G}$ has finitely many vertices, hence infinitely many bigons; in particular the underlying graph has fundamental group $\mathbb{F}_\infty$. Killing all the vertex groups provides a surjection to this infinitely generated group.
   (b) If $f_1 = 0$, then necessarily $f(A_{\Gamma_2}) = f(A_\Gamma) \neq 0$ because f is not the trivial character. In particular $\mathcal{G}$ has infinitely many vertices corresponding to Γ_1, and a single vertex corresponding to Γ_2, i.e. it is a star with infinitely many edges. Notice that since $\Gamma_3 \neq \Gamma_1$, each of the terminal vertex groups admits a non-trivial map to $\mathbb{Z}$ which kills (at least) the incoming edge group; pasting together these maps one obtains the desired surjection to an infinitely generated free group.

□

Remark 3.3. Notice that the separating graph does not have to be connected, and indeed in case (1) it is possible for either edge groups or vertex groups (or both) to be infinitely generated. Moreover it is possible for a fixed character f to behave as in case (1) with respect to one splitting and as in case (2) with respect to another one, i.e. $\ker(f)$ can both surject to $\mathbb{F}_\infty$ and split as a finite graph of groups (some vertex groups will then surject to $\mathbb{F}_\infty$). Concrete examples of these phenomena can be constructed on any cycle of at least four vertices, see Figure 2 for some concrete examples. We will see that this does not happen for chordal graphs in §3.2.

It is known by [MV95, Theorem 6.1] that if $f = 0$ on a separating subgraph then $\ker(f)$ is not finitely generated (compare Theorem 2.11). The above proof provides an independent proof of this fact, and actually shows more: as soon as $\ker(f)$ is not finitely generated it even surjects onto $\mathbb{F}_\infty$. The next statement is a sort of converse to case (2) of Theorem 3.2. Recall that the dead subgraph $\mathcal{D}(f)$ of f is defined to be the full subgraph generated by the vertices that f sends to 0.

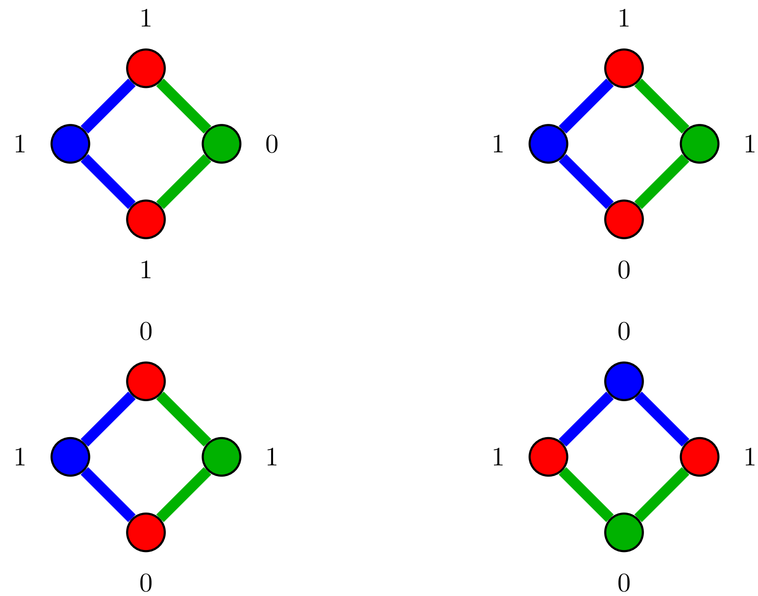

FIGURE 2. In the first three graphs the splitting subgraph is given by the union of the top and the bottom vertices, while in the last one it is given by the union of the left and the right vertices.

Corollary 3.4 (Wild GBB groups)**.** *Let $f : A_\Gamma \to \mathbb{R}$ be discrete. Then the following are equivalent:*

(1) $\exists \Lambda \subseteq \Gamma$ *separating subgraph such that* $\Lambda \subseteq \mathcal{D}(f)$ *(i.e.* $f = 0$ *on* Λ*).*
(2) $\ker(f)$ *surjects onto* $\mathbb{F}_\infty$*.*
(3) $\ker(f)$ *is not finitely generated.*

Proof. The fact that (1) implies (2) follows from Theorem 3.2, case (2); of course (2) implies (3). So we only need to show that (3) implies (1). Suppose $\ker(f)$ is not finitely generated. Then by Theorem 2.11 we know that $\mathcal{L}(f)$ is either not connected or not dominating. We distinguish two cases.

(1) If $\mathcal{L}(f)$ is not connected, then $\mathcal{D}(f)$ is separating.
(2) If $\mathcal{L}(f)$ is not dominating, then there is a vertex $v \in V(\Gamma)$ whose link entirely consists of vertices on which f is zero, i.e. $lk(v) \subseteq \mathcal{D}(f)$. By definition $lk(v)$ is separating.

□

In particular as soon as Γ is not complete, it has a separating subgraph, hence there is a generalized Bestvina-Brady subgroup $B \trianglelefteq A_\Gamma$ such that A_Γ / B is infinite cyclic and B surjects to $\mathbb{F}_\infty$.

So far the separating graphs have been allowed to be disconnected. On the other hand in order to obtain stronger finiteness properties for the tame case, we need to work with connected splittings. Recall that by Lemma 2.5 all the subgraphs in a connected splitting of a connected graph are connected. The next statement should be compared to the general treatment of BNS invariants for graphs of groups in

[CL16, §2]; in this regard, it should also be noted that non-abelian RAAGs are not slender because they contain $\mathbb{F}_2$ (recall that a group is said to be slender when every subgroup is finitely generated).

Lemma 3.5. *Let $(\Gamma_1, \Gamma_2, \Gamma_3)$ be a splitting of Γ, let $f : A_\Gamma \to \mathbb{R}$ be discrete, and let $f_k : A_{\Gamma_k} \to \mathbb{R}$ be its restriction for $k = 1, 2, 3$. Then the following hold.*

(1) *f_k is zero or discrete.*
(2) *If $\ker(f)$ is finitely generated, then f_3 is non-zero (and discrete).*
(3) *If $\ker(f)$ is finitely generated and $\ker(f_3)$ is finitely generated, then $\ker(f_k)$ is finitely generated for $k = 1, 2$.*

Proof. The restriction of a discrete character is clearly a discrete character as soon as it is not the zero character, so (1) is trivial. To prove (2), assume by contradiction that f_3 is identically 0 on Γ_3; since Γ_3 is separating, $\mathcal{L}(f)$ can not be connected, hence by Corollary 2.13 we get a contradiction.

Finally let us prove (3). Notice that the hypotheses imply that Γ and Γ_3 are connected. By Lemma 2.5 we get that Γ_1, Γ_2 are connected too. Moreover by Corollary 2.13 we get that $\mathcal{L}(f)$ (respectively $\mathcal{L}(f_3)$) is connected and dominating as a subgraph of Γ (respectively Γ_3). Notice that $\mathcal{L}(f_k) = \mathcal{L}(f) \cap \Gamma_k$. So, focusing on Γ_1 (the same proof works for Γ_2), we need to show that $\mathcal{L}(f_1)$ is connected and dominating as a subgraph of Γ_1.

Since $\mathcal{L}(f)$ is connected, any two vertices x, y in $\mathcal{L}(f_1)$ can be joined by an edge-path contained in $\mathcal{L}(f)$. We claim that such a path can be chosen to be also contained in Γ_1. By contradiction assume that any edge-path from x to y in $\mathcal{L}(f)$ makes an excursion into Γ_2; since Γ_3 is separating every such excursion starts and ends with a vertex on $\mathcal{L}(f_3)$. But $\mathcal{L}(f_3)$ is connected, so there is a shortcut entirely contained in $\mathcal{L}(f_3) \subseteq \mathcal{L}(f_1)$, leading to a contradiction.

To see that $\mathcal{L}(f_1)$ is dominating as a subgraph of Γ_1 we can argue as follows: if $x \in V(\Gamma_3)$ then we use that $\mathcal{L}(f_3)$ is dominating in Γ_3 to find a vertex in $\mathcal{L}(f_3)$ at distance 1 from x. If $x \in V(\Gamma_1) \backslash V(\Gamma_3)$, then since $\mathcal{L}(f)$ is dominating in Γ we find a vertex of $\mathcal{L}(f)$ at distance 1 from x; notice that by Lemma 2.5 this vertex can not live in $V(\Gamma_2) \backslash V(\Gamma_3)$. □

We then deduce the following statement. Notice its hypothesis are satisfied trivially when f does not vanish on any vertex, i.e. $\mathcal{D}(f) = \varnothing$.

Corollary 3.6 (Tame GBB groups)**.** *Let $(\Gamma_1, \Gamma_2, \Gamma_3)$ be a connected splitting of Γ, and let $f : A_\Gamma \to \mathbb{R}$ be a discrete character such that $\ker(f)$ and $\ker(f_3)$ are finitely generated. Then $\ker(f)$ splits as a finite graph of groups $\mathcal{G}$ as in Theorem 3.2 with finitely generated edge and vertex groups.*

Proof. From Theorem 3.2 we know that $\ker(f)$ splits as a finite graph of groups $\mathcal{G}$ in which vertex groups are isormorphic to $\ker(f_k)$ for $k = 1, 2$ and edge groups are isomorphic to $\ker(f_3)$. So the statement boils down to showing that $\ker(f_k)$ is finitely generated for $k = 1, 2, 3$, which follows from Lemma 3.5. □

3.2. **Chordal graphs.** In the previous section we have identified conditions under which a discrete character displays a wild behavior (i.e. surjects to $\mathbb{F}_\infty$) and conditions under which it enjoys a tame behavior (i.e. splits as a finite graph of groups with finitely generated edge and vertex groups, in a way which is controlled by the underlying graph, in the sense of Theorem 3.2).

In this section we show that when Γ is chordal every discrete character is subject to a complete tame-wild dichotomy; on the other hand when Γ is not chordal it is possible to construct characters whose kernel are not wild but at the same time are not tame in the above sense. We begin by recalling some definitions.

Definition 3.7. A graph Γ is chordal if every induced cycle of length at least four has chord.

These graphs are also known as triangulated graphs. By [Dro87a, Theorem 1] chordality is also equivalent to the fact that A_Γ is coherent (i.e. every finitely generated subgroup is finitely presented). Dirac characterized chordal graphs in terms of their splittings, in a way that we now review.

Definition 3.8. Let Γ be a graph and $a, b \in V(\Gamma)$. A subgraph $\Lambda \subset \Gamma$ is said to be an ab-separator if it is separating and a and b belong to different connected components of $\Gamma \backslash \Lambda$. We say Λ is a minimal ab-separator if it is minimal (with respect to inclusion) among ab-separators. We say that Λ is a minimal vertex separator if it is a minimal ab-separator for some $a, b \in V(\Gamma)$.

We explicitly notice that a minimal separating subgraph (i.e. a subgraph of Γ which is separating in the sense of Definition 2.3, and has no subgraph which separates Γ) is automatically a minimal vertex separator, but the converse is not true in general, i.e. a minimal vertex separator of Γ can contain a proper subgraph which separates Γ. According to a classical theorem by Dirac, a graph is chordal if and only if every minimal vertex separator is a clique (i.e. a complete graph). Notice that being chordal is an hereditary property, i.e. if Λ is a connected full subgraph of a chordal graph Γ then it is chordal too. In particular if Γ is chordal and $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a splitting, then each of the subgraphs appearing in it are chordal too. Using this characterization we obtain the following result.

Theorem 3.9. *Let Γ be a chordal graph and let $f : A_\Gamma \to \mathbb{R}$ be discrete. Then exactly one of the following holds.*

(1) *(wild)* $\ker(f)$ *surjects onto* $\mathbb{F}_\infty$*;*
(2) *(tame)* $\ker(f)$ *splits as a finite graph of groups* $\widehat{\mathcal{G}}$ *with finitely generated free abelian edge and vertex groups.*

Proof. Suppose $\ker(f)$ does not surject to $\mathbb{F}_\infty$; then by Corollary 3.4 we know that $\ker(f)$ is finitely generated, and that for any separating subgraph Λ we have that $\Lambda \nsubseteq \mathcal{D}(f)$. If Γ is complete, then the statement is trivial, because there is no separating subgraph and any subgroup of A_Γ is already a finitely generated free abelian group.

So assume Γ is not complete, in which case it admits a splitting $(\Gamma_1, \Gamma_2, \Gamma_3)$. As usual let us denote by f_k the restriction of f to Γ_k for $k = 1, 2, 3$. Up to replacing Γ_3 with a subgraph which still separates Γ, we can assume that Γ_3 is a minimal separator; since Γ is chordal, Γ_3 is a clique. As observed above, it is not possible that $f = 0$ on the entire Γ_3; in particular $\mathcal{L}(f_3)$ is connected and dominating, and therefore $\ker(f_3)$ is finitely generated. By Lemma 3.5 we get that $\ker(f_k)$ is finitely generated for $k = 1, 2$ as well. So the graph of groups $\mathcal{G}$ coming from Theorem 3.2 is finite and has finitely generated vertex and edge groups. Moreover Γ_1 and Γ_2 are still chordal graphs, so we can iterate the same argument on the vertex groups of $\mathcal{G}$. Notice that at each step the edge groups are finitely generated free abelian groups, because we use splittings along cliques; moreover the procedure halts when the pieces of the splitting are complete graphs, and the terminal graph of groups $\widehat{\mathcal{G}}$ satisfies the requirements. □

Remark 3.10. Groups that can be realized as the fundamental group of a finite graph of groups with finitely generated free abelian vertex groups have been studied by Beeker under the name of generalized Baumslag-Solitar groups of variable rank (vGBS groups). Beeker has studied JSJ-decompositions of such groups in [Bee13; Bee14], and the multiple conjugacy problem for them in [Bee15]. Notice that it follows from the above proof that the vertex groups of $\widehat{\mathcal{G}}$ correspond to maximal cliques of Γ, while edge groups correspond to the minimal vertex separators of Γ; in particular their rank can be easily computed. See Section 3.3 below for a detailed discussion of this point of view in the case of block graphs.

For completeness we show that when the graph is not chordal it is possible to construct characters that do not obey the tame-wild dichotomy.

Proposition 3.11. *Let Γ be connected but not chordal. Then there exists $f : A_\Gamma \to \mathbb{R}$ discrete such that $\ker(f)$ is neither wild nor tame (in the sense of Corollary 3.6).*

Proof. Since Γ is not chordal, there exists a splitting $(\Gamma_1, \Gamma_2, \Gamma_3)$ in which Γ_3 is a non-complete minimal separating subgraph. Since Γ_3 is minimal, no proper subgraph $\Lambda \subset \Gamma_3$ can separate Γ.

If Γ_3 is disconnected, then just set $f = 0$ on one component of Γ_3, and $f \neq 0$ on any other vertex of Γ. Then by construction $\mathcal{L}(f_3)$ is not dominating, hence $\ker(f_3)$ is not finitely generated and the graph of groups $\mathcal{G}$ from Theorem 3.2 does not have finitely generated edge groups so $\ker(f)$ is not tame. On the other hand $\mathcal{L}(f)$ is clearly connected by construction, and also dominating, because Γ_3 was chosen to be minimal, hence $\ker(f)$ is not wild.

If Γ_3 is connected but not complete, then it admits a separating subgraph Λ. Then set $f = 0$ on Λ and $f \neq 0$ on any other vertex of Γ. Since Λ does not separate Γ, $\mathcal{D}(f)$ does not contain any separating subgraph, hence $\ker(f)$ is finitely generated by Corollary 3.4, in particular $\ker(f)$ is not wild. However since Λ separates Γ_3, we have that $\mathcal{L}(f_3)$ is not dominating in Γ_3, hence $\ker(f_3)$ is not finitely generated, which implies that $\ker(f)$ is not tame, as above. □

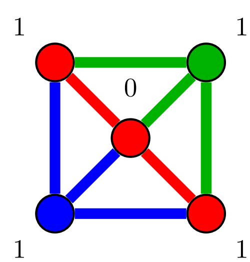

FIGURE 3. A non-chordal well-connected graph with a character as in Proposition 3.11

Notice that the statement of the above Proposition 3.11 is somewhat obvious for graphs admitting a non-connected splittings; for instance if Γ is a a cycle of at least four vertices, then the classical Bestvina-Brady group is finitely generated but not finitely presented, and the only graphical splitting available is such that the edge group is not finitely generated. On the other hand Proposition 3.11 also applies to graphs for which every separating subgraph is connected (but not necessarily complete), such as well-connected graphs (see [DLVM86]); these include for instance the cone or the suspension over any graph (see Figure 3 for an explicit example).

Remark 3.12. We conclude this section by explicitly observing that when we are in the tame case we obtain a decomposition of the group as a finite graph of groups in a way that can be computed from the underlying graph Γ, which is what we would call a graphical splitting. It is not clear whether this is the only way in which a generalized Bestvina-Brady group can split as a finite graph of groups with finitely generated vertex and edge groups. Obtaining such a result is outside the scope of the present work. It should be noted however that Chang has recently shown in [Cha20] that the classical Bestvina-Brady groups split over abelian subgroups if and only if Γ is disconnected, complete, or has a separating clique.

3.3. **Block graphs.** In this section we take a more computational point of view, and obtain explicit decompositions and rank formulas for tame generalized Bestvina-Brady groups over graphs with cut vertices, in the spirit of Remark 3.10. For simplicity, we will directly work with integral characters $f : A_\Gamma \to \mathbb{Z}$.

Definition 3.13. A graph Γ is biconnected if it is connected and it does not have any cut vertices. The blocks of Γ are its biconnected components (i.e. its maximal biconnected full subgraphs); we denote by $\mathcal{B}(\Gamma)$ the set of blocks of Γ.

In the following we write $\mathrm{I}_\Gamma = [\mathbb{Z} : f(A_\Gamma)]$; moreover for each block $B \in \mathcal{B}(\Gamma)$ we denote by f_B the restriction of f to the subgroup A_B, and we define $\mathrm{I}_{\Gamma,B} = [f(A_\Gamma) : f(A_B)]$. The block degree $\mathrm{bldeg}_\Gamma(v)$ of a vertex $v \in V(\Gamma)$ is defined to be the number of blocks of Γ meeting at v. Notice that if v is not a cut vertex, then $\mathrm{bldeg}_\Gamma(v) = 1$. We find it convenient to introduce the following number associated to Γ and to $f : V(\Gamma) \to \mathbb{Z}$:

$$m(\Gamma, f) = 1 - \sum_{B \in \mathcal{B}(\Gamma)} \mathrm{I}_{\Gamma,B} + \sum_{v \in V(\Gamma)} (\mathrm{bldeg}_\Gamma(v) - 1) \frac{|f(v)|}{\mathrm{I}_\Gamma}$$

Notice that it follows from the above general discussion that if f vanishes on a cut vertex, then we get a surjection to $\mathbb{F}_\infty$. When this is not the case, we can get the following result.

Proposition 3.14. *Let Γ be a connected graph, and $f : A_\Gamma \to \mathbb{Z}$ be non-zero on cut vertices. Then*

$$\ker(f) \cong \mathbb{F}_{m(\Gamma,f)} * \left(\mathop{\Huge *}_{B \in \mathcal{B}(\Gamma)} (\ker(f_B))^{* \mathrm{I}_{\Gamma,B}} \right)$$

Proof. We argue by induction on the number of blocks of Γ. The theorem is trivial in the case in which Γ is biconnected, as Γ consists of a single block and $m(\Gamma, f) = 0$. So let us assume there are at least two blocks (equivalently, at least a cut vertex).

Let us pick a block $B_0 \in \mathcal{B}(\Gamma)$ which contains only one of the cut vertices of Γ (this exists, because Γ is now assumed to be non-biconnected). Then let $\Lambda = \Gamma/B_0$ be the quotient graph, and let v_0 be the unique cut vertex of Γ separating B_0 from Λ. Notice (Λ, B_0, v_0) is a connected splitting of Γ Since $f(v_0) \neq 0$, all the indices $\mathrm{I}_{\Gamma,\Lambda}, \mathrm{I}_{\Gamma,v_0}$ and I_{Γ,B_0} are non-zero, hence by Theorem 3.2 $\ker(f)$ decomposes as a finite graph of groups $\mathcal{G}$; moreover $\mathcal{G}$ has $\mathrm{I}_{\Gamma,\Lambda}$ vertices with vertex groups isomorphic to $\ker(f_\Lambda)$, I_{Γ,B_0} vertices with vertex groups isomorphic to $\ker(f_{B_0})$, and $\mathrm{I}_{\Gamma,v_0} = \frac{|f(v_0)|}{\mathrm{I}_\Gamma}$ edges with trivial edge group. Since all edge groups are trivial, in particular we get

$$\ker(f) \cong \mathbb{F}_{b_1} * (\ker(f_\Lambda))^{*\,\mathrm{I}_{\Gamma,\Lambda}} * (\ker(f_{B_0}))^{*\,\mathrm{I}_{\Gamma,B_0}}$$

where $b_1 = 1 - (\mathrm{I}_{\Gamma,\Lambda} + \mathrm{I}_{\Gamma,B_0}) + \dfrac{|f(v_0)|}{\mathrm{I}_\Gamma}$ is the first Betti number of the underlying topological graph of $\mathcal{G}$. Now notice that $\mathcal{B}(\Gamma) = \{B_0\} \cup \mathcal{B}(\Lambda)$, and each cut vertex of Λ is also a cut vertex in Γ (even though v_0 might be a cut vertex in Λ or not). So by induction we get

$$\ker(f_\Lambda) \cong \mathbb{F}_{m(\Lambda,f)} * \left(\mathop{\scalebox{1.5}{$\ast$}}_{B \in \mathcal{B}(\Lambda)} (\ker(f_B))^{*\,\mathrm{I}_{\Lambda,B}} \right)$$

Therefore we obtain

$$\ker(f) \cong \mathbb{F}_{b_1} * \left(\mathbb{F}_{m(\Lambda,f)} * \left(\mathop{\scalebox{1.5}{$\ast$}}_{B \in \mathcal{B}(\Lambda)} (\ker(f_B))^{*\,\mathrm{I}_{\Lambda,B}} \right) \right)^{*\,\mathrm{I}_{\Gamma,\Lambda}} * (\ker(f_{B_0}))^{*\,\mathrm{I}_{\Gamma,B_0}} \cong$$

$$\cong \mathbb{F}_{b_1 + \mathrm{I}_{\Gamma,\Lambda}\, m(\Lambda,f)} * \left(\mathop{\scalebox{1.5}{$\ast$}}_{B \in \mathcal{B}(\Gamma)} (\ker(f_B))^{*\,\mathrm{I}_{\Gamma,B}} \right)$$

where we used that $\mathrm{I}_{\Gamma,\Lambda}\,\mathrm{I}_{\Lambda,B} = \mathrm{I}_{\Gamma,B}$ for all $B \in \mathcal{B}(\Gamma)$. We are therefore left with checking that the rank of the free group is the right one, i.e. with showing that $b_1 + \mathrm{I}_{\Gamma,\Lambda}\, m(\Lambda, f) = m(\Gamma, f)$. This can be checked by a direct computation, using the relation $\mathrm{I}_{\Gamma,\Lambda}\,\mathrm{I}_{\Lambda,B} = \mathrm{I}_{\Gamma,B}$ once again, and also that $\mathrm{I}_\Lambda = \mathrm{I}_\Gamma\,\mathrm{I}_{\Gamma,\Lambda}$, together with the fact that $\mathrm{bldeg}_\Gamma(v) = \mathrm{bldeg}_\Lambda(v)$ for all vertices $v \neq v_0$, while $\mathrm{bldeg}_\Lambda(v_0) = \mathrm{bldeg}_\Gamma(v_0) - 1$; the computation is as follows:

$$b_1 + \mathrm{I}_{\Gamma,\Lambda}\, m(\Lambda, f) =$$

$$= 1 - (\mathrm{I}_{\Gamma,\Lambda} + \mathrm{I}_{\Gamma,B_0}) + \frac{|f(v_0)|}{\mathrm{I}_\Gamma} + \mathrm{I}_{\Gamma,\Lambda} - \sum_{B \in \mathcal{B}(\Lambda)} \mathrm{I}_{\Gamma,\Lambda}\,\mathrm{I}_{\Lambda,B} + \sum_{v \in V(\Lambda)} (\mathrm{bldeg}_\Lambda(v) - 1)\,\frac{|f(v)|\,\mathrm{I}_{\Gamma,\Lambda}}{\mathrm{I}_\Lambda} =$$

$$= 1 - \sum_{B \in \mathcal{B}(\Gamma)} \mathrm{I}_{\Gamma,B} + \sum_{v \in V(\Gamma)} (\mathrm{bldeg}_\Gamma(v) - 1)\,\frac{|f(v)|}{\mathrm{I}_\Gamma} = m(\Gamma, f)$$

□

Remark 3.15. The fact that Γ can be decomposed into its blocks by cutting along its cut vertices, allows for a hierarchical decomposition of A_Γ. In the terminology of [CL16] this would be called a good $\mathbb{Z}$-hierarchy, with leaf groups corresponding to the blocks of Γ (i.e. the leaf groups are the RAAGs $A_B, B \in \mathcal{B}(\Gamma)$). Notice that a RAAG is slender if and only if its defining graph is complete: indeed it contains $\mathbb{F}_2$ otherwise. From this point of view, our Theorem 3.14 can be seen as a generalization of [CL16, Theorem 4.4 (2)] to this setting. In order for the aforementioned hierarchy to have slender leaf groups, one needs to work with graphs such that each block is a complete graph, which is the case on which we focus next.

Definition 3.16. A block graph is a graph whose blocks are complete subgraphs.

Notice that in this case blocks are automatically the maximal cliques of Γ. We obtain the following statement, involving an explicit rank formula, as a direct consequence of Theorem 3.14.

Theorem 3.17. *Let Γ be a connected block graph, and $f : A_\Gamma \to \mathbb{Z}$ be non-zero on cut vertices. Then*

$$\ker(f) \cong \mathbb{F}_{m(\Gamma,f)} * \left(\underset{B\in\mathcal{B}(\Gamma)}{\ast} \left(\mathbb{Z}^{|V(B)|-1}\right)^{* \mathrm{I}_{\Gamma,B}} \right)$$

In particular

$$\mathrm{rk}(\ker(f)) = 1 + \sum_{B\in\mathcal{B}(\Gamma)} \mathrm{I}_{\Gamma,B}(|V(B)|-2) + \sum_{v\in V(\Gamma)} (\mathrm{bldeg}_\Gamma(v)-1)\frac{|f(v)|}{\mathrm{I}_\Gamma}$$

Proof. It is enough to notice that when B is a clique with $|V(B)|$ vertices we have that $A_B \cong \mathbb{Z}^{|V(B)|}$ and therefore $\ker(f_B) \cong \mathbb{Z}^{|V(B)|-1}$. Computing the rank results in:

$$\mathrm{rk}(\ker(f)) = 1 - \sum_{B\in\mathcal{B}(\Gamma)} \mathrm{I}_{\Gamma,B} + \sum_{v\in V(\Gamma)} (\mathrm{bldeg}_\Gamma(v)-1)\frac{|f(v)|}{\mathrm{I}_\Gamma} + \sum_{B\in\mathcal{B}(\Gamma)} \mathrm{I}_{\Gamma,B}(|V(B)|-1)$$

□

Remark 3.18. When Γ is a tree, the group A_Γ is free-by-cyclic; indeed, as shown in [Dro87a], the classical Bestvina-Brady group associated to a tree is a free group generated by the edges of Γ. Therefore these RAAGs belong to the class of groups studied by Cashen and Levitt in [CL16]. When Γ is a more general block graph, the RAAG A_Γ still admits a natural good $\mathbb{Z}$-hierarchy with free abelian leaf groups coming from the blocks. The above statement can be seen as a generalization of [CL16, Theorem 6.1] to more general block graphs: the elements t_i which appear in the main statements of [CL16] can be identified to be the cut vertices of the underlying graph; their multiplicities correspond to their block degrees.

We will now discuss some applications of the above rank formula to the study of fibrations of a RAAG A_Γ associated to a block graph Γ, i.e. the collection of generalized Bestvina-Brady subgroups of A_Γ. In the spirit of Remark 3.18, the Corollaries 3.19, 3.20 and 3.21 should be compared to [CL16, Corollary 6.3, Corollary 6.4, Example 5.14].

Corollary 3.19 (Minimal rank)**.** *If Γ is a connected block graph, then*

1. *the minimal rank of the kernel of a homomorphism $f : A_\Gamma \to \mathbb{Z}$ is*

$$\mu_\Gamma = 1 + \sum_{B\in\mathcal{B}(\Gamma)} (|V(B)|-2) + \sum_{v\in V(\Gamma)} (\mathrm{bldeg}_\Gamma(v)-1)$$

2. $\mathrm{rk}(\ker(f)) = \mu_\Gamma$ *if and only if* $f(v) = 1$ *for each cut vertex.*

Proof. Recall that a necessary condition for a character to have a finitely generated kernel is that it does not send any cut vertex to 0. Hence $\mathrm{I}_{\Gamma,B}, |f(v)| \geqslant 1$ for each block and cut vertex. Moreover up to rescaling f we can always assume that $\mathrm{I}_\Gamma = 1$. The rank formula implies that μ_Γ is then a lower bound for the rank of any kernel. This lower bound is realized by characters that send all cut vertices to 1. □

Corollary 3.20 (Unbounded rank)**.** *If* Γ *is a connected block graph, then* A_Γ *admits integral characters with kernel of arbitrarily large rank.*

Proof. For each $n \in \mathbb{N}$ let f_n be the character that sends all cut vertices to n and the other vertices to 1. Then we have $\mathrm{I}_{\Gamma,B} = \mathrm{I}_\Gamma = 1$, hence

$$\mathrm{rk}(\ker(f_n)) = 1 + \sum_{B \in \mathcal{B}(\Gamma)} (|V(B)| - 2) + n \sum_{v \in V(\Gamma)} (\mathrm{bldeg}_\Gamma(v) - 1) \to \infty$$

□

For the next statement recall from §2.2 that $S(A_\Gamma)$ denotes the character sphere of A_Γ. This result can be interpreted as an example of exotic divergence in the BNS invariant of A_Γ.

Corollary 3.21 (Bounded divergence)**.** *Let* Γ *be a connected block graph, and let* f_n, f_∞ *be integral characters such that* $[f_n] \to [f_\infty]$ *in* $S(A_\Gamma)$*; then the following are equivalent*

(1) $\exists C > 0$ *such that* $\forall n$ *we have* $\mathrm{rk}(\ker(f_n)) \leqslant C$*, but* $\ker(f_\infty)$ *is not finitely generated;*
(2) $\exists C' > 0$ *such that for all* n *and all cut vertices* $v \in V(\Gamma)$ *we have* $|f_n(v)| \leqslant C'$*, but* $\exists w \in V(\Gamma)$ *such that* $|f_n(w)| \to \infty$*.*

Moreover in this situation $f_\infty(v) = 0$ *for all cut vertices* $v \in V(\Gamma)$*.*

Proof. Suppose (2) holds, and consider the character $g_n = \dfrac{f_n}{|f_n(w)|}$. Then $\ker(f_n) = \ker(g_n)$ and $[g_n] = [f_n] \to [f_\infty]$. Since we have $\mathrm{I}_{\Gamma,B} \leqslant f_n(v) \leqslant C'$ for each vertex $v \in V(B)$, the rank formula shows that there exists some $C > 0$ such that $\mathrm{rk}(\ker(f_n)) \leqslant C$ uniformly on n. However for each cut vertex v we have that $g_n(v) \to 0$, so that $f_\infty(v) = 0$ and $\ker(f_\infty)$ can not be finitely generated.

Suppose (1) holds. Then by the rank formula $f_n(v) \leqslant C$ for all cut vertices. If f_n was uniformly bounded on the entire $V(\Gamma)$, then f_n would converge to a limit character with finitely generated kernel; hence necessarily f_n must diverge on at least one vertex, which can not be a cut vertex. As a result also in this case $f_n(v) \to 0$ for all cut vertices.

□

For a concrete example which displays this behavior see the following.

Example 3.22. Pick a number $q \in \mathbb{Z}$ and an infinite sequence of numbers $p_n \in \mathbb{Z}$ diverging to infinity and each coprime with q. Consider the character f_n defined by sending all cut vertices of Γ to q and all other vertices to p_n. We have $\mathrm{I}_\Gamma = 1$ and $\mathrm{I}_{\Gamma,B} = 1$ for each block. As a result

$$\mathrm{rk}(\ker(f_n)) = 1 + \sum_{B \in \mathcal{B}(\Gamma)} (|V(B)| - 2) + q \sum_{v \in V(\Gamma)} (bledg_\Gamma(v) - 1)$$

does not depend on n (indeed by Proposition 3.14 all these groups are actually isomorphic). Also consider the character $g_n = \dfrac{f_n}{p_n}$, i.e. the one obtained by sending all cut vertices to $\dfrac{q}{p_n}$ and all other vertices to 1. We have $[f_n] = [g_n] \in S(A_\Gamma)$. However $g_n \to g_\infty$, where g_∞ is a character defined by sending all cut vertices to 0 and all other vertices to 1; by Theorem 2.11 $\ker(g_\infty)$ is not finitely generated.

Remark 3.23. It is natural to ask for a deeper understanding of this phenomenon, i.e. what makes these subgroups, which are in some explicit sense uniformly tame, abruptly converge to a wild subgroup. In terms of the graphical splittings studied in this paper it can be noticed that these subgroups are all isomorphic as vGBS groups, i.e. they can be realized as the fundamental group of a fixed graph of free abelian groups $\widehat{\mathcal{G}}$, determined by the structure of Γ itself. In other words no intrinsic algebraic invariant of these subgroups seem to be useful in understanding this phenomenon. From a more geometric point of view, one could try to understand the extrinsic properties of these as subgroups of A_Γ, e.g. if these tame subgroups are more and more distorted in A_Γ as they approach the wild limit. In this direction it should be noticed that the work of Tran shows that any finitely generated generalized Bestvina-Brady group has at worst quadratic distortion (see [Tra16, Corollary 1.5]), and at worst quadratic relative divergence (see [Tra17, Proposition 4.3]).

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