PARTIALLY HYPERBOLIC DIFFEOMORPHISMS HOMOTOPIC TO THE IDENTITY IN DIMENSION 3
PART II: BRANCHING FOLIATIONS

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ABSTRACT. Using branching foliations, and extending tools from foliation theory to this more general context, we remove the assumption of dynamical coherence from the first part of our work, [BFFP20b]. In particular, in Seifert manifolds, we finish the classification of partially hyperbolic diffeomorphisms homotopic to the identity. In hyperbolic manifolds, we obtain that a partially hyperbolic diffeomorphism is either dynamically coherent and, up to a power, a discretized Anosov flow; or it is of a special type that we call double translation, for which we can describe its branching foliations and understand its coarse dynamics. More general, albeit less complete results, are also obtained on general 3-manifolds.

Keywords: Partial hyperbolicity, 3-manifold topology, foliations, classification.

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CONTENTS

1. Introduction

This article is the second part of our work on the classification of partially hyperbolic diffeomorphisms homotopic to the identity in 3-dimensional manifolds started in [BFFP20b].

The standing assumption in [BFFP20b] was that the partially hyperbolic diffeomorphisms were *dynamically coherent*, i.e., left invariant a pair of transverse foliations tangent to the center stable and center unstable directions. This assumption has some great benefits because it allows one to work with true foliations, as opposed to the branching foliations which will be the main object we work with in the present paper. True foliations forbid a number of troublesome behaviors that branching foliations may have, making many arguments simpler, and some substantial shortcuts possible. Sometimes, new strategies had to be developped to deal with situations that cannot arise in the dynamically coherent case.

While assuming dynamical coherence makes a lot of sense from a pedagogical point of view, it is however a rather unnatural assumption. First of all, it is an extremely hard assumption to try to verify directly. For example, in the present article, we obtain dynamical coherence in certain situations only after having completely understood the global dynamical behavior of the partially hyperbolic diffeomorphism. In addition, while dynamical coherence was historically thought to be generally expected in partially hyperbolic systems, many recent works (see, e.g., [RHRHU16, BGHP17, BFFP20a]) have shattered that belief. For instance, in the unit tangent bundle of a hyperbolic surface, we proved in [BFFP20a] that most partially hyperbolic diffeomorphisms are not dynamically coherent.

Thus, it seems that dynamical coherence is more likely to be obtained under certain circumstances as a corollary of, instead of an initial step towards, a classification. Nonetheless, the overall scheme, as well as many intermediate results, that we developed in [BFFP20b] will be adapted here to the general setting. If nothing else, this shows that assuming dynamical coherence can help develop efficient strategies for the study of partial hyperbolicity and justify our decision to split our work in two. Another advantage is that we thus help bring to light the differences between properties that stem from dynamical coherence as opposed to those coming only from partial hyperbolicity.

The two main consequences of this second part are the extension of [BFFP20b, Theorem A], and a description of the only two possible behaviors of partially hyperbolic diffeomorphisms in hyperbolic 3-manifolds:

Theorem A. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism on a closed Seifert fibered 3-manifold. If f is homotopic to the identity, then it is dynamically coherent, and some iterate of f is a discretized Anosov flow.*

Theorem B. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism on a closed hyperbolic 3-manifold. Then, either*

(i) *the diffeomorphism f is dynamically coherent and has an iterate which is a discretized Anosov flow, or*

(ii) *the diffeomorphism f is not dynamically coherent, and up to a finite cover[1] and finite iterate, M admits a pair of transversely orientable $\mathbb{R}$-covered, uniform, branching foliations. A a lift of f to the universal cover translates the lifts of these foliations.*

A discretized Anosov flow is a map of the form $\phi_{t(x)}(x)$ where ϕ is a topological Anosov flow, see next subsection.

The starting point in order to deal with the general case and obtain the theorems above is the foundational work of Burago and Ivanov [BI08] proving the existence of structures called *branching foliations* assuming the orientability of the bundles (see Section 3 for proper definitions). These conditions are always satisfied in finite covers, but the existence of branching foliations without taking finite lifts is still an open question. However, we prove that if a finite lift is a discretized Anosov flow then the original partially hyperbolic diffeomorphism is already a discretized Anosov flow (see section 7.3). Hence we are sometimes able to remove the finite cover hypothesis.

The existence of a partially hyperbolic diffeomorphism satisfying condition (ii) in Theorem B is still unknown and is probably the most pressing open question that follows from this work. Let us note that we did not find any reasons yet, neither in this work nor in current further research, to dismiss this type of example. Indeed, while their existence would be conceptually new, the behavior that we find such example would need to have with respect to their branching foliations is very similar to those of examples built in [BGHP17]. Therefore, the fact that they are necessarily non dynamically coherent does not make their existence less likely.

Both of the theorems above follow from more general, albeit less complete, results that we now present after recalling some necessary definitions.

1.1. **Results.** We always assume our manifolds to have non virtually solvable fundamental group.

Definition 1.1. A C^1-diffeomorphism $f\colon M \to M$ on a 3-manifold M is *partially hyperbolic* if there is a Df-invariant splitting of the tangent bundle TM into three 1-dimensional bundles

$$TM = E^s \oplus E^c \oplus E^u$$

such that for some $n > 0$, one has

$$\|Df^n|_{E^s(x)}\| < 1,$$
$$\|Df^n|_{E^u(x)}\| > 1, \text{ and}$$
$$\|Df^n|_{E^s(x)}\| < \|Df^n|_{E^c(x)}\| < \|Df^n|_{E^u(x)}\|,$$

for all $x \in M$.

[1]Taking a finite cover is only needed in order to get the existence of branching foliations preserved by f.

Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism on a closed 3-manifold M. When f is homotopic to the identity, we denote by $\widetilde{f}$ the specific lift to the universal cover $\widetilde{M}$ that is obtained by lifting such a homotopy.

We assume that f leaves invariant two *branching foliations*, $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$, tangent to the center stable (E^{cs}) and center unstable (E^{cu}) bundles respectively (see Definition 3.2). By [BI08], up to taking a finite cover of M that orients the three bundles and a power of f preserving these orientations, such branching foliations always exist (see Theorem 3.4). These behave like foliations, but different leaves are allowed to merge together, while not allowed to topologically cross (see section 3). Their lifts to $\widetilde{M}$ are denoted by $\widetilde{\mathcal{W}^{cs}_{\text{bran}}}$ and $\widetilde{\mathcal{W}^{cu}_{\text{bran}}}$, these are branching foliations by topological planes in $\widetilde{M}$.

An obvious but fundamental difference between true foliations and their branching counterparts, is that a point does not necessarily determine a unique leaf.

Recall [BFFP20b, Definition 2.2] the following definition:

Definition 1.2. A *discretized Anosov flow* is a partially hyperbolic diffeomorphism $g\colon M \to M$ on a 3-manifold M that is of the form $g(p) = \Phi_{t(p)}(p)$ for a topological Anosov flow Φ and a map $t\colon M \to (0, \infty)$.

The most general theorem we obtain in this work is the following.

Theorem 1.3. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism on a closed 3-manifold M that is homotopic to the identity. If f preserves two branching foliations $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ that are f-minimal, then either*

(i) *f is a discretized Anosov flow (and in particular dynamically coherent),*
(ii) *$\widetilde{f}$ fixes each of the leaves of one of the lifted branching foliations in $\widetilde{M}$, and the other branching foliation is $\mathbb{R}$-covered, uniform, and $\widetilde{f}$ acts as a translation on its leaf space in the universal cover, or*
(iii) *$\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ are $\mathbb{R}$-covered and uniform, and $\widetilde{f}$ acts as a translation on the leaf spaces of $\widetilde{\mathcal{W}^{cs}_{\text{bran}}}$ and $\widetilde{\mathcal{W}^{cu}_{\text{bran}}}$.*

In the previous result, we say that a branching foliation $\mathcal{W}^{cs}_{\text{bran}}$ (or $\mathcal{W}^{cu}_{\text{bran}}$) is f-minimal when a closed, non empty, f-invariant set which is a union of leaves of $\mathcal{W}^{cs}_{\text{bran}}$ must be M itself. We emphasize that in this definition, the sets considered are saturated by leaves of $\mathcal{W}^{cs}_{\text{bran}}$, but do not a priori contain all the leaves of $\mathcal{W}^{cs}_{\text{bran}}$ intersecting the given set. When f is either transitive or volume-preserving, and admits branching foliations, then they are f-minimal (see [BW05]).

We will show that case (ii) of Theorem 1.3 cannot occur when M is hyperbolic or Seifert fibered (in §12 and §8 respectively), in which case we can also eliminate the hypothesis of f-minimality, obtaining the following:

Theorem 1.4. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism on a closed hyperbolic or Seifert fibered 3-manifold that is homotopic to the identity. Then either*

(i) *f is a discretized Anosov flow, or*
(ii) *up to a finite iterate and a lift to a finite cover, f admits center stable and center unstable branching foliations, which are $\mathbb{R}$-covered and uniform, and $\widetilde{f}$ acts as a translation on their leaf spaces in $\widetilde{M}$.*

As was already pointed out in [BFFP20b, Remark 7.4], case (ii) of Theorem 1.4 can occur in Seifert manifolds, but a finite power of such diffeomorphisms is a discretized Anosov flow, leading to Theorem A. Moreover, since every diffeomorphism of a hyperbolic 3-manifold has an iterate homotopic to the identity one also deduces[2] Theorem B from it.

We believe that Theorem 1.4 may be proven, following the same strategy as here, under the more general assumptions of f-minimality together with the existence of an atoroidal piece in the JSJ decomposition of M.

Remark 1.5. Case (ii) of Theorem 1.3 may also be ruled out under the assumption of *absolute partial hyperbolicity* (cf. §9).

We end the introduction by stating a dynamical consequence of our results and analysis.

Theorem 1.6. *Let $f: M \to M$ be a partially hyperbolic diffeomorphism of a closed 3-manifold M homotopic to the identity and assume that one of the following conditions is verified:*

- *M is hyperbolic or Seifert fibered, or,*
- *the (branching) center stable foliation is f-minimal,*

then f has no contractible periodic points.

This result will be proven as Corollary 4.11.

1.2. **Remarks and references.** We refer to [CRRU15, HP18, Pot18] for surveys on the problem of classification of partially hyperbolic diffeomorphisms in dimension 3 and to the introduction of [BFFP20b] for a wider introduction to our results. Instead, we will emphasize here the new tools developed in the present article as well as put it in perspective with respect to previous work in the quest for the classification of partially hyperbolic diffeomorphisms in 3-manifolds.

One important feature of the present article is to not assume dynamical coherence. This has certainly been done before (see [HP18]), but previous works tended to have two simplifying characteristics: Their study took place on manifolds where taut foliations are well understood and amenable to classification, and on which known partially hyperbolic models where available to compare to. Typically, dynamical coherence was established under the assumption of non-existence of invariant tori by using the fact that coarse dynamics separates leaves of the branching foliations. However, for the manifolds considered in this article, neither of these features exists, and dynamical incoherence does appear in several different ways.

For instance, we obtain dynamical coherence in Section 7 when the lift of the partially hyperbolic diffeomorphism fixes each leaf of the lifted branching foliations. We also manage to obtain information on the structure of the branching foliation in the non dynamically coherent case, leading, in particular, to case (ii) of Theorem B. This structure also allows us to better understand the dynamical properties of the system, even when the manifold is not hyperbolic, as can be seen in Theorem 1.6.

More generally, the framework that we develop for the study of non dynamically coherent partially hyperbolic diffeomorphism should find its use outside of the homotopic to the identity case that is the focus of the analysis here.

The first tool that we need to develop in this article, is the basic theory of the topological study of branching foliations. This includes the definition of the leaf spaces and their behavior under diffeomorphisms which preserve them (see Section 3).

Among the other tools that we develop, we wish to emphasize:

[2]To be precise, one needs to apply Theorem 3.4.

(1) In §5 we introduce the notion of coarsely contracting and coarsely repelling periodic rays. This should be useful for the study of all, i.e., not necessarily homotopic to the identity, partially hyperbolic diffeomorphisms in 3-manifolds.
(2) In §6 we study the dynamics of certain lifts inside a fixed center stable leaf and show that, under some assumptions, it cannot admit fixed points. This involves understanding the behavior of strong stable manifolds of fixed points under iteration which may find applications in other contexts.
(3) In §7 we prove uniqueness of (branching) foliations under certain conditions. This is fundamental in order to prove results that do not require taking finite lifts and finite power. As such, it may also be relevant for the study of topological obstructions for partially hyperbolic diffeomorphisms since for instance, the topological obstructions for the existence of Anosov flows can depend on taking finite lifts (see, e.g., [Cal07]). Note that uniqueness of branching foliations was previously proven in other works, but always in a setting where there was an understood model partially hyperbolic diffeomorphism to compare with.
(4) Finally, in §11, 12 we develop some tools to analyze the geometric structure of (branching) foliations. This analysis combines tools from Lefschetz index theory, hyperbolic geometry, and the notion of coarsely expanding and contracting rays mentioned in item (1).

Note that the tools developed in (4) are used in a different setting in [BFFP20a] to prove that a large class of partially hyperbolic diffeomorphisms in Seifert manifolds are dynamically incoherent. Together with (1), it has also successfully been applied in [FP18] to obtain fine dynamical consequences of partial hyperbolicity in 3-manifolds.

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2. Outline and discussion

2.1. **Setup.** We will now set some basic definitions and outline our major arguments. Throughout this article, we assume a good familiarity with the firt part of our work, [BFFP20b].

As in [BFFP20b], our running hypothesis in this article is that M is a closed 3-manifold with non-solvable fundamental group[3], and $f\colon M \to M$ will be a partially hyperbolic diffeomorphism that is homotopic to the identity.

A lot of the analysis is done for general partially hyperbolic diffeomorhisms homotopic to the identity and preserving branching foliations. Then we specialize to obtain the specific results in Seifert manifolds and hyperbolic manifolds.

[3]The reason being that the case of manifolds with solvable fundamental group is well understood, as discussed in [BFFP20b].

Some results, particularly in the appendix, are for general partially hyperbolic diffeomorphisms, without assuming homotopic to the identity.

2.2. **Structure of the proof.** Many of the constructions and arguments in [BFFP20b] adapt directly to the non dynamically coherent case. On the other hand, some subresults are much more complex or need completely different proofs. There are three main places where the general non dynamically coherent situation diverges significantly from the dynamically coherent case:

- The first is that in general there may be annular center leaves which do not contained a closed center leaf inside (see condition ($\star\star$) in [BFFP20b, §2]). In the non-dynamically coherent case we prove a weaker statement that serves the same purpose in many situations.
- A second and more important difference is that we cannot deduce the impossibility of double translations from the general version of the existence of cores that "shadow" the periodic orbits of the transverse pseudo-Anosov flow (see condition ($\star\star\star$) in [BFFP20b, §2]). This means that here we need to analyze that case further to see what structure we can obtain from it.
- In hyperbolic and Seifert manifolds, to avoid the need to assume f-minimality of the foliations in the dynamically coherent case it was enough to remark that a minimal invariant set lifted to $\widetilde{M}$ could not have fixed points under a good lift of f. When there is branching this no longer holds, and we need here to make a very delicate analysis that takes place in §6.

One of the most important questions left open by our work is the following:

Question. *Does there exists a partially hyperbolic diffeomorphism of a hyperbolic 3-manifold which acts as a double translation?*

We do obtain (in §11) some strong dynamical properties that would have to be satisfied by such an example. This behavior is akin to what is seen in the examples of [BGHP17], but we refrain from giving a conjectural answer to our question.

Let us now take $f\colon M \to M$ to be a partially hyperbolic diffeomorphism, not necessarily dynamically coherent. In §3, we review Burago–Ivanov's [BI08] construction of branching center stable and center unstable foliations. We also show that these branching foliations have leaf spaces that behave like the leaf spaces of true foliations.

2.2.1. *Dichotomies for branching foliations.* In §4, we extend the results from [BFFP20b] and adapt them to the branching foliation case.

In particular, in §4.1–4.4 we show that the dichotomy obtained in [BFFP20b] (see condition ($\star$) there) holds without assuming dynamical coherence, so we can arrange our arguments around the same trichotomy: If $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$ are f-minimal, or M is hyperbolic or Seifert-fibered (this is handled in §6), then one of the following holds, where $\widetilde{f}$ is the lift of a homotopy from f to the identity:

1. **double invariance:** $\widetilde{f}$ fixes every leaf of both $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$;
2. **mixed behavior:** $\widetilde{f}$ fixes every leaf of either $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ or $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$, and acts as a translation on the leaf space of the other, which is $\mathbb{R}$-covered and uniform; or
3. **double translation:** $\widetilde{f}$ acts as a translation on both $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$, which are $\mathbb{R}$-covered and uniform.

This allows to prove Theorem 1.6 (see §4.3 and Corollary 4.11).

2.2.2. *Center dynamics in fixed leaves.* In §5, we work under the assumption that $\widetilde{f}$ fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$, and study the dynamics within each center stable leaf. Although the statements that are obtained in [BFFP20b] fail without dynamical coherence, we are able to obtain the following which is sometimes enough (Proposition 5.2):

($\star\star'$) Suppose that $\mathcal{W}^{cs}_{\text{bran}}$ is f-minimal, that all the leaves of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ are fixed by $\widetilde{f}$, and that $\widetilde{f}$ does not fix any center leaf in $\widetilde{M}$. If c is a periodic center leaf of f in M, then c is coarsely contracted by h. In particular, c contains a periodic point of f.

This fact, together with the fact that periodic center leaves exist on any leaf with non-trivial fundamental group (see Proposition 5.8) gives us the tool to continue the analysis of the above trichotomy.

At this point, the reader interested in *absolutely* partially hyperbolic diffeomorphisms can fast forward to §9 to see how one can recover the same statement of [BFFP20b] (where one gets a closed fixed center curve) under that stronger dynamical assumption (see Proposition 9.3).

2.2.3. *Double invariance implies dynamical coherence.* With ($\star\star'$) in hand, we show in §7 that the existence of a good lift $\widetilde{f}$ with doubly invariant behavior implies that f is dynamically coherent. By the work of [BFFP20b], we get that f is a discretized Anosov flow.

Let us stress here that in this case the dynamical coherence of f (a very strong property) is obtained at the very end of a long and complicated analysis.

There is one additional unsavory and very non trivial issue that we have to address in this section: The theorem of Burago–Ivanov gives the existence of branching foliations under some orientability conditions (see Theorem 3.4). These conditions can always be achieved by taking an appropriate lift of M and power of f. However, in order not to have these conditions appear in Theorem A or Theorem B, we need to show that if a lift and power of a partially hyperbolic diffeomorphism is dynamically coherent, then so is the original one. We do not know if this statement is true in general, but we prove it (in §7.3) when the lift is further assumed to be doubly invariant.

The work up to section 7 implies Theorem 1.3.

2.2.4. *General version of Theorem A.* In §8 we finish the proof of Theorem A by ruling out both mixed behavior and double translations when M is a Seifert-fibered manifold.

This uses a combination of the good lift trick, which allows to take one good lift that fixes one of the foliations, and Proposition 5.2. If a good lift (of a power) does not fix both branching foliations, then we obtain periodic center leaves that must be both coarsely expanding and contracting, a contradiction.

2.2.5. *No mixed behavior in hyperbolic manifolds.* Sections 11 and 12 deal with the last property we want to show in order to obtain Theorem 1.4 in the hyperbolic case. That is, we want to eliminate mixed behavior.

To reach this goal, we first get, in §11, a better understanding of homeomorphisms that act as a translation on a branching foliation. In §10 we prove that the dynamics of such a homeomorphism resembles the one of a regulating pseudo-Anosov flow transverse to the foliation. We push the understanding of that resemblance further and show (see Proposition 11.1) that, on periodic center stable leaves, at least some center rays that are fixed must be expanding, i.e., act in a similar way as the strong unstable foliation of the pseudo-Anosov regulating flow.

This property is then used in §12 to rule out mixed behavior, but it does not rule out double translations.

In the next section we develop the basic theory of branching foliations and their leaf spaces.

3. Branching foliations and leaf spaces

Many non dynamically coherent partially hyperbolic examples have been constructed in recent years, hence in general one cannot assume dynamical coherence when trying to classify these diffeomorphisms on a given manifold or within a homotopy class. The role of the foliations we used in [BFFP20b], will then be replaced by *branching foliations*, that were constructed by Burago and Ivanov [BI08] for general partially hyperbolic diffeomorphisms under some orientability conditions.

Remark 3.1. Notice that the term *branching* is sometimes used with a different meaning in the study of codimension one foliations (to describe non-separated leaves in the leaf space). Here, branching means that two leaves may merge (and this is irrespective of whether the leaf space in $\widetilde{M}$ is Hausdorff or not).

We start with a proper definition and refer the reader to [HP18] for a detailed explanation on this tool as well as contexts where they are used.

Definition 3.2. A *branching foliation* $\mathcal{F}_{\mathrm{bran}}$ of a 3-manifold M is a collection of C^1-immersed surfaces complete for the pull-back metric and satisfying:

(i) Every point $x \in M$ belongs to at least one surface (called *leaf*) of $\mathcal{F}_{\mathrm{bran}}$;
(ii) An immersed leaf of $\mathcal{F}_{\mathrm{bran}}$ does not topologically cross itself;
(iii) Different leaves of $\mathcal{F}_{\mathrm{bran}}$ do not topologically cross;
(iv) If L_n are leaves of $\mathcal{F}_{\mathrm{bran}}$ and $x_n \in L_n$ is a sequence that converges to x, then, up to taking a subsequence, L_n converges to a leaf L[4] of $\mathcal{F}_{\mathrm{bran}}$ with $x \in L$.

Moreover, we say that a branching foliation is *well-approximated by foliations* if there exists a family of foliations $\mathcal{F}_\epsilon$, with C^1 leaves, and a family of continuous maps $h_\epsilon \colon M \to M$, with $\epsilon > 0$, such that, for a fixed Riemannian metric, we have:

(v) The angle between a leaf of $\mathcal{F}_{\mathrm{bran}}$ and $\mathcal{F}_\epsilon$ is less than ϵ;
(vi) The map h_ϵ is at C^0-distance less than ϵ from the identity;
(vii) The map h_ϵ maps leaves of $\mathcal{F}_\epsilon$ to leaves of $\mathcal{F}_{\mathrm{bran}}$ by a local diffeomorphism (so in particular, the restriction of h_ϵ to any leaf is C^1);
(viii) For every leaf L of $\mathcal{F}_{\mathrm{bran}}$, there exists a leaf L_ϵ of $\mathcal{F}_\epsilon$ such that $h_\epsilon(L_\epsilon) = L$.

Notice that, as a branching foliation has C^1 leaves and that all possible intersections are not topological crossings, it makes sense to talk about the tangent distribution to a branching foliation.

Remark 3.3. When $\mathcal{F}_{\mathrm{bran}}$ is a branching foliation but not a true foliation, then the map h_ϵ is never a local diffeomorphism, even though it restricts to a local diffeomorphism on each leaf: There are open sets where leaves are collapsed

[4]Here convergence should be understood in the pointed compact-open topology, i.e., given a compact set K in L containing x, there is a sequence of compact subsets K_n of L_n containing x_n such that K_n converges to K in the Hausdorff topology and x_n convereges to x.

transversely by h_ϵ. In fact, even when restricted to a leaf, it may fail to be a global diffeomorphism as leaves of $\mathcal{F}_{\text{bran}}$ can self intersect, forming branching locus.

As is the case with foliations, there exists a small enough scale at which the branching foliation is "trivially product (branched) foliated". Let us be more precise: We fix a Riemannian metric. Then there exists $\epsilon_0 > 0$, such that any open set B of diameter less than ϵ_0 satisfies the following. The set B is contained in a smooth chart $\mathbb{D}^2 \times [0,1]$ such that the local leaves of $\mathcal{F}_{\text{bran}}$ through B intersects the chart in sets transverse to the $[0,1]$-fibration in $D^2 \times [0,1]$, each local leaf intersects every $[0,1]$-fiber and they are close to being horizontal. This fact readily follows from the fact that the branching foliation are tangent to a continuous distribution.

We call the scale $\epsilon_0 > 0$ above the *local product structure size*.

The foundational result of Burago and Ivanov states that, under some orientability conditions, a partially hyperbolic diffeomorphism always admits a pair of branching foliations tangent to the center stable and center unstable distributions. We naturally say that a branching foliation is f-invariant if the image of any leaf by f is again a leaf.

Theorem 3.4 (Burago-Ivanov [BI08])**.** *Let f be a partially hyperbolic diffeomorphism of a 3-manifold M. Suppose that the bundles E^s, E^u and E^c are orientable and that Df preserves these orientations.*

Then there exists two f-invariant branching foliations $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ tangent respectively to E^{cs} and E^{cu}. Moreover, these branching foliations are well-approximated by foliations $\mathcal{W}^{cs}_\epsilon$ and $\mathcal{W}^{cu}_\epsilon$, with associated maps denoted by h^{cs}_ϵ and h^{cu}_ϵ.

The collections of surfaces $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ are called the center stable and center unstable branching foliations.

There is one property that the center stable and center unstable branching foliations have which will be very useful to us: Since the stable bundle E^s is uniquely integrable, if a point p is in a center stable leaf L, then the entire stable leaf $s(p)$ through p is also contained in L. As a consequence intersections between distinct center stable leaves are saturated by stable leaves.

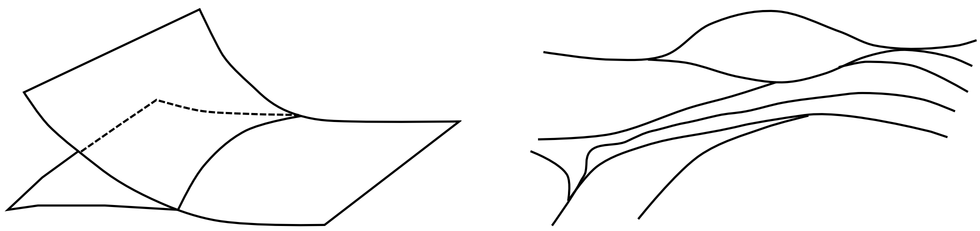
Remark 3.5. Since the manifolds we consider in this article are not virtually solvable, no leaf of the approximating foliation, is compact (cf. [RHRHU11]). Thus the approximating foliations $\mathcal{W}^{cs}_\epsilon$ and $\mathcal{W}^{cu}_\epsilon$ are always taut.

Using branching foliations, we can still define center leaves:

Definition 3.6. A *center leaf* c of a partially hyperbolic diffeomorphism is the projection to M of a connected component of the intersection between a leaf of the central stable branching foliation $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and a leaf of the central unstable branching foliation $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$ (lifts to $\widetilde{M}$).

Even though the collection of center leaves is not a foliation, we will also define a leaf space of center leaves in section 3.1.

Remark 3.7. Notice that a center leaf c is automatically tangent to the central direction E^c. However, complete curves that are tangent to the central direction may fail to be center leaves for our definition. Indeed, even when the diffeomorphism is dynamically coherent, the central direction may not be *uniquely integrable*, thus, some complete curves may be tangent to E^c, but are *not* the intersection of a central stable and central unstable (such an example is constructed in [RHRHU16]).



(A) Two center-stable leaves sharing a region

(B) Distinct center leaves inside a center-stable leaf

FIGURE 1. The branching of center and center-stable leaves.

3.1. **Leaf Spaces.** When $\mathcal{F}$ is a foliation, the *leaf space* of $\mathcal{F}$ is the collection of distinct leaves of the lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ to $\widetilde{M}$. Moreover, it comes naturally equipped with a quotient topology. Indeed, the leaf space of $\mathcal{F}$ can be defined as the set $\widetilde{M}$ quotiented by the relation "being on the same leaf of $\widetilde{\mathcal{F}}$".

When $\mathcal{F}$ is a *branching* foliation, we want to define the leaf space again as the collection of distinct leaves of the lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ to $\widetilde{M}$. However, this space does *not* necessarily come from a quotient. Indeed, some points $x \in \widetilde{M}$ may belong to more than one (in which case x belongs to uncountably many) distinct leaves, thus one cannot define a quotient projection from $\widetilde{M}$.

In the next three sections, we will explain how to put a topology on the leaf spaces of each of the branching foliations. More importantly, we show that these topologies make the leaf spaces of the branching foliations homeomorphic to those of the approximating foliations, for small enough ϵ.

3.1.1. *Leaf spaces of the center stable and center unstable foliations.* Recall that, by Theorem 3.4, the branching foliations $\mathcal{W}^{cs}_{\mathrm{bran}}$ and $\mathcal{W}^{cu}_{\mathrm{bran}}$ are well-approximated by foliations $\mathcal{W}^{cs}_{\epsilon}$ and $\mathcal{W}^{cu}_{\epsilon}$. Now property (viii) of Definition 3.2 implies that for ϵ sufficiently small (which is assumed from now on), there is a canonical surjection between the leaf spaces of $\widetilde{\mathcal{W}}^{cs}_{\epsilon}$ and $\widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$ and the leaf spaces of $\widetilde{\mathcal{W}}^{cu}_{\epsilon}$ and $\widetilde{\mathcal{W}}^{cu}_{\mathrm{bran}}$.

It is possible to modify the proof of [BI08, Theorem 7.2], where the foliations $\mathcal{W}^{cs}_{\epsilon}$ and the map h^{cs}_{ϵ} are constructed, so that the map between leaf spaces given by h^{cs}_{ϵ} is also injective. With this result on hand, we could define the topology on the leaf space of $\widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$ as the one making that map a homeomorphism. However, proving the injectivity would require to redo the whole proof of [BI08, Theorem 7.2]. So instead, we use a simpler fact which can be easily extracted from the proof of [BI08, Theorem 7.2]: The map h^{cs}_{ϵ} is "monotone" meaning that, in local charts, where there is a well defined linear order between leaves, this order is preserved by h^{cs}_{ϵ}.

Definition 3.8. We denote by:

- $\mathcal{L}^{cs}_b$ the leaf space of the center stable branching foliation $\widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$;
- $\mathcal{L}^{cu}_b$ the leaf space of the center unstable branching foliation $\widetilde{\mathcal{W}}^{cu}_{\mathrm{bran}}$;
- $\mathcal{L}^{cs}_{\epsilon}$ the leaf space of the approximating center stable foliation $\widetilde{\mathcal{W}}^{cs}_{\epsilon}$;
- $\mathcal{L}^{cu}_{\epsilon}$ the leaf space of the approximating center unstable foliation $\widetilde{\mathcal{W}}^{cu}_{\epsilon}$.

Furthermore, we denote the surjections between the leaf spaces of the branching foliations and the approximating foliations by

$$g_{\epsilon,s} \colon \mathcal{L}^{cs}_{\epsilon} \to \mathcal{L}^{cs}_b, \text{ and } g_{\epsilon,u} \colon \mathcal{L}^{cu}_{\epsilon} \to \mathcal{L}^{cu}_b.$$

Since $\mathcal{W}^{cs}_\epsilon$ is a true foliation, its leaf space $\mathcal{L}^{cs}_\epsilon$ has a natural topology making it a simply connected, but perhaps non Hausdorff, 1-manifold (cf. [BFFP20b, Appendix B].

Each leaf L of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ is a properly embedded plane in $\widetilde{M}$. Using this one defines as before L^+ to be the closure of the connected component of $\widetilde{M} \smallsetminus L$ on the "positive side of L", and similarly for L^-. To define positive side pick an orientation to the unstable bundle in $\widetilde{M}$.

Topology of $\mathcal{L}^{cs}_b$. The topology in $\mathcal{L}^{cs}_b$ is defined as follows: Consider a finite collection of transversals τ_i to $\mathcal{W}^{cs}_{\text{bran}}$ such that:

(i) Each transversal τ_i is open.
(ii) τ_i is perpendicular to E^{cs} everywhere.
(iii) Every leaf of $\mathcal{W}^{cs}_{\text{bran}}$ intersects at least one of the τ_i.

Let β be a lift to $\widetilde{M}$ of some τ_i. Consider the collection of leaves of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ intersecting β. Each such leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ is a properly embedded plane and intersects β only once.

Claim 3.9. *Let $x \in \beta$. Let I be the collection of leaves I intersecting x. Then I is a singleton or order isomorphic to a closed interval.*

Proof. Suppose that I is not a singleton. Then, given any leaves $L \neq E$ in I, either $L \subset E^+$ or $E \subset L^+$ and only one option occurs (this is thanks to property (iii) of Definition 3.2). We say $L > E$ in the first case and $L < E$ in the second case, which gives a total order on I. By property (iv), this order is complete. Moreover, there are no gaps in this order: Let $L \neq E$ two leaves in I such that $L < E$. We want to show that there exists a leaf $L' \in I$, with $L < L' < E$. Let y be a boundary point of the connected component of $L \cap E$ containing x. Then consider a neighborhood B of y of diameter smaller than ϵ_0, the local product structure size of the branching foliation $\mathcal{W}^{cs}_{\text{bran}}$. Since $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ has a trivially product foliated structure in B, every leaf that intersects $B \cap (L^+ \cap E^-)$ must intersect y, and since leaves of $\mathcal{W}^{cs}_{\text{bran}}$ do not cross, they must intersect x also. Thus there is $L' \in I$ such that $L < L' < E$.

So I is order isomorphic to a closed interval in $\mathbb{R}$. □

The claim implies that putting the order topology on the set of leaves of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ intersecting a lift β of τ_i makes it homeomorphic to an open interval in $\mathbb{R}$.

Notice the following: suppose that β_1, β_2 are lifts of τ_1, τ_2, and L, E are leaves of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ intersecting both β_1, β_2. Then the order induced by β_1 is the same as the order induced by β_2 (in the set of leaves intersecting both transversals). Hence the order topology is well defined when there are intersections.

Definition 3.10 (topology of $\mathcal{L}^{cs}_b$)**.** The topology $\mathcal{T}$ in $\mathcal{L}^{cs}_b$ is the one generated by the open intervals defined above. This topology makes $\mathcal{L}^{cs}_b$ a simply-connected 1-manifold.

Proposition 3.11. *For ϵ small enough (smaller than the local product sizes of $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$), the preimage of a point in $\mathcal{L}^{cs}_b$ (resp. $\mathcal{L}^{cu}_b$) by $g_{\epsilon,s}$ (resp. $g_{\epsilon,u}$) is a closed interval. Moreover, the space $\mathcal{L}^{cs}_\epsilon$ (resp. $\mathcal{L}^{cu}_\epsilon$) is homeomorphic to $\mathcal{L}^{cs}_b$ (resp. $\mathcal{L}^{cu}_b$). The maps $g_{\epsilon,s} \colon \mathcal{L}^{cs}_\epsilon \to \mathcal{L}^{cs}_b$ are continuous.*

Proof. We work with $\mathcal{L}^{cs}_b$ as the proof for $\mathcal{L}^{cu}_b$ is identical. The key property is to show that the preimage by $g_{\epsilon,s}$ of points are closed intervals in the leaf space $\mathcal{L}^{cs}_\epsilon$, the rest will follow rather easily.

We let $\mathcal{T}_\epsilon$ be the quotient topology induced by $g_{\epsilon,s}$ on $\mathcal{L}^{cs}_b$. Our goal is to show that $\mathcal{T}_\epsilon = \mathcal{T}$.

Let ϵ_0 be the local product sizes of $\mathcal{W}^{cs}_{\text{bran}}$. Let $\epsilon < \epsilon_0/2$.

It is in order to prove this proposition that we will use the remark made above that the map h^{cs}_ϵ is monotone[5].

Let I be the preimage of a leaf $L \in \mathcal{L}^{cs}_b$. Suppose that I contains two leaves $\hat{L}_1$ and $\hat{L}_2$, we want to show that every leaf in between $\hat{L}_1$ and $\hat{L}_2$ is mapped by $\widetilde{h}^{cs}_\epsilon$ to L. From property (vi) of Definition 3.2, we have that the Hausdorff distance between $\hat{L}_1$ and $\hat{L}_2$ is $< 2\epsilon$. Now, as 2ϵ is chosen smaller than the local product structure size ϵ_0, it follows that the region between the leaves $\hat{L}_1$ and $\hat{L}_2$ has leaf space which is a closed interval.

Because of the property of monotonicity of $\widetilde{h}^{cs}_\epsilon$ it follows that $g_{\epsilon,s}$ maps the region between $\hat{L}_1$ and $\hat{L}_2$ to L. This implies that the preimage of L is an interval. It remains to show that it is closed, but this is just a consequence of the continuity of $\widetilde{h}^{cs}_\epsilon$.

So the preimage of any point is a closed interval. We now proceed with proving the other needed properties.

Let J be an open interval J in $\mathcal{L}^{cs}_b$ for the topology $\mathcal{T}$. Up to taking J smaller, we can assume that J is the set of branching leaves that intersects a small open transversal β. We want to show that $g^{-1}_{\epsilon,s}(J)$ is open in $\mathcal{L}^{cs}_\epsilon$. Let $\hat{L}_1$ be a leaf in $g^{-1}_{\epsilon,s}(J)$. Then $\hat{L}_1$ intersects β (or a slightly bigger transversal), so all the leaves of $\widetilde{\mathcal{W}}^{cs}_\epsilon$ close enough to $\hat{L}_1 \cap \beta$ intersect β. Thus an open neighborhood of $\hat{L}_1$ is contained in $g^{-1}_{\epsilon,s}(J)$.

Hence the interval J is also open in the topology $\mathcal{T}_\epsilon$. It follows that $\mathcal{T} \subset \mathcal{T}_\epsilon$. In particular this shows that $g_{\epsilon,s}$ is continuous.

Now for the other inclusion. Suppose W is an open set in $\mathcal{T}_\epsilon$ and y is in W. Hence $(g_{\epsilon,s})^{-1}(W)$ is open and contains $(g_{\epsilon,s})^{-1}(y)$, which is an interval I with boundary leaves L, E. Since $(g_{\epsilon,s})^{-1}(W)$ is open, it contains and interval of leaves around, say, L. Consider the part of this interval made up of L and the side outside $(g_{\epsilon,s})^{-1}(y)$. This projects to an interval in $\mathcal{L}^{cs}_b$, which is not just y by definition of I. Hence W contains an open interval around y, and therefore W is open in $\mathcal{T}$. This shows that $\mathcal{T} = \mathcal{T}_\epsilon$.

We already proved that the preimage of a point in $\mathcal{L}^{cs}_b$ is a a closed interval in $\mathcal{L}^{cs}_\epsilon$. This implies that $\mathcal{L}^{cs}_b$, $\mathcal{L}^{cs}_\epsilon$ are homeomorphic. This is because the only collapsing from $\mathcal{L}^{cs}_\epsilon$ to $\mathcal{L}^{cs}_b$ is done along closed intervals I. If L, E are the endpoints of I, then there is no other leaf in the region between L and E besides those leaves that are in I.

This finishes the proof of the proposition. □

Notice that the leaf spaces $\mathcal{L}^{cs}_b, \mathcal{L}^{cs}_\epsilon$ are homeomorphic, however the natural map $g_{\epsilon,s} \colon \mathcal{L}^{cs}_\epsilon \to \mathcal{L}^{cs}_b$ is not necessarily a homeomorphism, as it may collapse points. In the sequel, we fix some ϵ small enough so that the previous proposition applies.

3.1.2. *Leaf spaces of the center foliation in a center stable or center unstable leaf.* We now define a topology on the leaf space of the center branching foliation, restricted to a particular center stable or center unstable leaf.

[5] Otherwise, the preimage could be disconnected. One can recover the rest of the statements, but that would need to construct new maps $\hat{g}_{\epsilon,s}$ by collapsing closed intervals in both spaces and see that these induce the same topology.

Remark 3.12. Recall from Definition 3.6 that a center leaf in $\widetilde{M}$ is defined as a connected component of the intersection between a leaf of $\widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$ and a leaf of $\widetilde{\mathcal{W}}^{cu}_{\mathrm{bran}}$. Now, the following situation may arise (see Figure 2): Two leaves U_1, U_2 of $\widetilde{\mathcal{W}}^{cu}_{\mathrm{bran}}$ and a leaf L of $\widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$ such that the triple intersection $U_1 \cap L \cap U_2$ contains a connected component of c_1 of $U_1 \cap L$ as well as a connected component c_2 of $U_2 \cap L$. That is, the center leaves c_1 and c_2 represents the same set in $\widetilde{M}$. In this case, we also consider c_1 and c_2 as the *same* leaf of the center foliation in L.

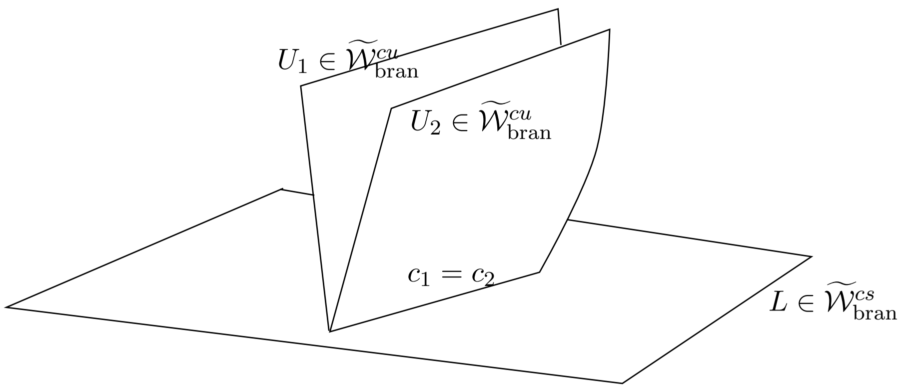


FIGURE 2. Different center unstable leaves may intersect a given center stable leaf in the same center leaf.

We will describe the topology of the center leaf space $\mathcal{L}^c_L$ on a given leaf L of $\widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$. The center leaf space $\mathcal{L}^u_U$ on a leaf U of $\widetilde{\mathcal{W}}^{cu}_{\mathrm{bran}}$ is defined in the same manner, so we do not explicit it.

Definition 3.13 (topology $\mathcal{A}$ in $\mathcal{L}^c_L$)**.** Consider a countable set of open transversals τ_i which are perpendicular to the center bundle in L, and so that the union intersects every center leaf in L. Put the order topology in the set I_i of center leaves intersecting τ_i. This induces the topology $\mathcal{A}$ in $\mathcal{L}^c_L$.

Let L be a fixed leaf of $\widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$. We again fix an $\epsilon > 0$ and consider the approximating foliation $\widetilde{\mathcal{W}}^{cu}_{\epsilon}$. Since $\widetilde{\mathcal{W}}^{cu}_{\mathrm{bran}}$ is transverse to L, so is $\widetilde{\mathcal{W}}^{cu}_{\epsilon}$ (for ϵ small enough). Thus, $\widetilde{\mathcal{W}}^{cu}_{\epsilon}$ induces a 1-dimensional (non branching) foliation $\mathcal{F}_\epsilon$ on L, and hence its leaf space $\mathcal{L}^c_\epsilon$ is a 1-dimensional, not necessarily Hausdorff, simply connected manifold.

The behavior described in Remark 3.12 above leads to the following issue: the unique center leaf $c_1 = c_2$ is approximated by two distinct leaves of $\mathcal{F}_\epsilon$. Thus, the leaf space, $\mathcal{L}^c_L$, of the center foliation on L is not in bijection with $\mathcal{L}^c_\epsilon$. However, we still have a surjective, but not necessarily injective, projection $\mathrm{pr}_\epsilon \colon \mathcal{L}^c_\epsilon \to \mathcal{L}^c_L$ as in the previous subsection. Let $\mathcal{A}_\epsilon$ be the quotient topology from the map pr_ϵ.

Just as in Proposition 3.11 one can prove the following:

Lemma 3.14. *The set of center leaves in L through a point x is a closed interval. Let c_0 be a center leaf in L. Let $I = \mathrm{pr}^{-1}(c_0) \subset \mathcal{L}^c_\epsilon$. The set I is a closed interval. If $\epsilon < \epsilon_0$, then the topologies $\mathcal{A}$ and $\mathcal{A}_\epsilon$ are the same.*

3.1.3. *Leaf space of the center foliation in $\widetilde{M}$.* Finally, we have to put a topology on the leaf space $\mathcal{L}^c_b$ of the center foliation in $\widetilde{M}$.

Pick an $0 < \epsilon < \epsilon_0$ so that $\widetilde{\mathcal{W}^{cs}_\epsilon}$ and $\widetilde{\mathcal{W}^{cu}_\epsilon}$ are transverse to each other. Call $\mathcal{F}_\epsilon$ the 1-dimensional foliation obtained as the intersection of $\widetilde{\mathcal{W}^{cs}_\epsilon}$ and $\widetilde{\mathcal{W}^{cu}_\epsilon}$. The leaf space $\mathcal{L}^c_\epsilon$ of $\mathcal{F}_\epsilon$ is now a simply connected, possibly non Hausdorff, 2-dimensional manifold. But as before, there is only a surjective, and not injective, projection $g_\epsilon \colon \mathcal{L}^c_\epsilon \to \mathcal{L}^c_b$.

The map g_ϵ is defined in the following way: If $\bar{c}$ is a leaf of $\mathcal{F}_\epsilon$, then it is the intersection of a leaf $\bar{U}$ of $\widetilde{\mathcal{W}^{cu}_\epsilon}$ and a leaf $\bar{S}$ of $\widetilde{\mathcal{W}^{cs}_\epsilon}$. Then, there exists a unique connected component c of $g_{\epsilon,u}(\bar{U}) \cap g_{\epsilon,s}(\bar{S})$ that is at distance less than 2ϵ from $\bar{c}$. We define $g_\epsilon(\bar{c}) = c$.

Once again, the topology $\mathcal{B}_\epsilon$ we put on $\mathcal{L}^c_b$ is obtained by identifying elements of $\mathcal{L}^c_\epsilon$ that project to the same element of $\mathcal{L}^c_b$ and taking the quotient topology.

As done is the previous two subsections 3.1.1 and 3.1.2, in order to prove that the topology that we put on $\mathcal{L}^c_b$ makes it a simply connected (not necessarily Hausdorff) 2-manifold, it is enough to show that the preimages of points by g_ϵ are closed, simply connected sets contained in a local chart of $\mathcal{L}^c_\epsilon$. In order to do that, first notice that $\mathcal{L}^c_\epsilon$ is locally homeomorphic to $\mathcal{L}^{cs}_\epsilon \times \mathcal{L}^{cu}_\epsilon$. Indeed, any $\bar{c}_0 \in \mathcal{L}^c_\epsilon$ is a connected component of $\bar{U}_0 \cap \bar{S}_0$, with $\bar{U}_0 \in \mathcal{L}^{cu}_\epsilon$ and $\bar{S}_0 \in \mathcal{L}^{cs}_\epsilon$. Now, if V_u is a small enough open interval in $\mathcal{L}^{cu}_\epsilon$ and V_s is a small enough open interval in $\mathcal{L}^{cs}_\epsilon$, then for any $\bar{U} \in V_u$ and $\bar{S} \in V_s$, the intersection $\bar{U} \cap \bar{S}$ contains a unique connected component close to c_0. Using this local homeomorphism, the following lemma will imply that the topology $\mathcal{L}^c_b$ is as we claimed.

Lemma 3.15. *Let c_0 be in $\mathcal{L}^c_b$. The set $R = g_\epsilon^{-1}(c_0)$ is homeomorphic to a closed rectangle in $\mathcal{L}^{cs}_\epsilon \times \mathcal{L}^{cu}_\epsilon$.*

Proof. Let $\bar{c}_1, \bar{c}_2 \in R$. Let $\bar{U}_1$ be the leaf in $\mathcal{L}^{cu}_\epsilon$ containing $\bar{c}_1$ and let $\bar{S}_2$ be the the leaf in $\mathcal{L}^{cs}_\epsilon$ containing $\bar{c}_2$. Let $U_1 = g_{\epsilon,u}(\bar{U}_1)$ and $S_2 = g_{\epsilon,s}(\bar{S}_2)$. Since $\bar{c}_1, \bar{c}_2 \in R$, the center leaf c_0 is a connected component of $U_1 \cap S_2$. Thus $\bar{U}_1$ and $\bar{S}_2$ must intersect and the intersection contains a unique connected component $\bar{c}_3$ at distance at most 2ϵ from c_0.

Now, the proof of Lemma 3.14 shows that $\bar{c}_1$ and $\bar{c}_3$ are two ends of an interval in the leaf space of $\mathcal{F}_\epsilon$ restricted to $\bar{U}_1$ that is entirely contained in R. Similarly, for $\bar{c}_2$ and $\bar{c}_3$ considered as elements of the leaf space of $\mathcal{F}_\epsilon$ restricted to $\bar{S}_2$. In turns, the arguments of the proof of Lemma 3.14 imply that the set R projects to a closed interval in both $\mathcal{L}^{cs}_\epsilon$ and $\mathcal{L}^{cu}_\epsilon$, i.e., it is a closed rectangle in $\mathcal{L}^{cs}_\epsilon \times \mathcal{L}^{cu}_\epsilon$. ☐

Just as in the previous two sections we can also put a topology $\mathcal{B}$ on $\mathcal{L}^c_b$ directly as follows:

Definition 3.16. (topology $\mathcal{B}$ on $\mathcal{L}^c_b$) In M pick a collection of very small open rectangles R_i which are almost perpendicular to the center bundle, and with boundary two arcs in a leaves of $\mathcal{W}^{cs}_{bran}$ and two arcs in leaves of $\mathcal{W}^{cu}_{bran}$. Consider all lifts R of these to $\widetilde{M}$. The set of center leaves intersecting R is naturally bijective to an open rectangle and put the topology making this a local homeomorphism. The topology $\mathcal{B}$ is generated by these rectangles.

First we justify why the set of center leaves through R is naturally an open rectangle. Let L_1, L_2 be the center stable leaves containing the two arcs in the boundary of R, and U_1, U_2 be the corresponding center unstable leaves. The set of center stable leaves between L_1, L_2 (not including L_1, L_2) is naturally ordered isomorphic to an open interval. This was proved in subsection 3.1.1. The same for the center unstable foliation. The product is an open rectangle. The set of center leaves intersecting R is a quotient of this. The sets which are quotiented to a point are compact subrectangles. The proof is the same as the previous lemma. Hence the quotient is naturally a rectangle. In addition if a collection of center leaves intersects two such rectangles R, R', then the identifications in R also produce the same identifications in R' and the order of the center stable and center unstable foliations in the subsets are the same whether in R or R'. Hence in the identification, the topologies agree.

Just as in the previous sections one can prove:

Lemma 3.17. *For $\epsilon < \epsilon_0$, the topologies $\mathcal{B}$ and $\mathcal{B}_\epsilon$ are the same.*

The main property is to prove is exactly that of Lemma 3.15. The rest follows just as in the previous subsections.

4. General aspects without assuming dynamical coherence

In this section, M is a closed 3-manifold, with non virtually solvable fundamental group, $f\colon M \to M$ is a partially hyperbolic diffeomorphism homotopic to the identity, and $\widetilde{f}$ is a lift of f obtained by lifting a homotopy from the identity to f. Such a lift of f is called a *good lift* of f. Notice that a good lift commutes with every deck transformation of $\widetilde{M}$. We do *not* assume that f is dynamically coherent.

We will assume throughout this section that the stable, center, and unstable bundles are oriented, and that f preserves their orientations. This can be achieved by taking an iterate of f and lifting to a finite cover of M. We will deal with the effects of replacing f and M in §7.

With this assumption, Burago-Ivanov's Theorem 3.4 applies. We denote by $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ their center stable and center unstable branching foliations, and by $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and by $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$ the corresponding lifts to $\widetilde{M}$.

4.1. First arguments.

In this section, we will see that many of the results about the foliations from the dynamically coherent case work for branching foliations. From now on, we always assume that the branching foliations $\mathcal{F}_{\text{bran}}$ we consider are well-approximated by taut foliations $\mathcal{F}_\epsilon$.

One of the first things to be careful with is the definition of f-minimality for a branching foliation. We first define the notion of saturation.

Definition 4.1. Let $\mathcal{F}_{\text{bran}}$ be a branching foliation. A set $C \subset M$ is $\mathcal{F}_{\text{bran}}$-*saturated* if, for every $x \in C$, there is a leaf of $\mathcal{F}_{\text{bran}}$ that contains x and is contained in C.

Note that this is much weaker than asking that *every* leaf intersecting C is contained in C. In particular, our notion of saturation has the peculiar property that the complement of a $\mathcal{W}^{cs}_{\text{bran}}$-saturated set need not be $\mathcal{W}^{cs}_{\text{bran}}$-saturated (see Figure 3). With this in mind, we make the following definition.

Definition 4.2. Let $\mathcal{F}_{\text{bran}}$ be an f-invariant branching foliation. Then $\mathcal{F}_{\text{bran}}$ is called *f-minimal* if the only $\mathcal{F}_{\text{bran}}$-saturated and f-invariant sets in M that are closed are the empty set or the whole manifold.

We emphasize here that closed in the above definition is meant as a set in M, not as a set of leaves.

Remark 4.3. Let C be an $\mathcal{F}_{\text{bran}}$-saturated set in M and $\widetilde{C} = \pi^{-1}(C)$. There are several, in general distinct, sets of leaves in $\mathcal{L}_{\text{bran}}$, the leaf space of $\widetilde{\mathcal{F}}_{\text{bran}}$, that

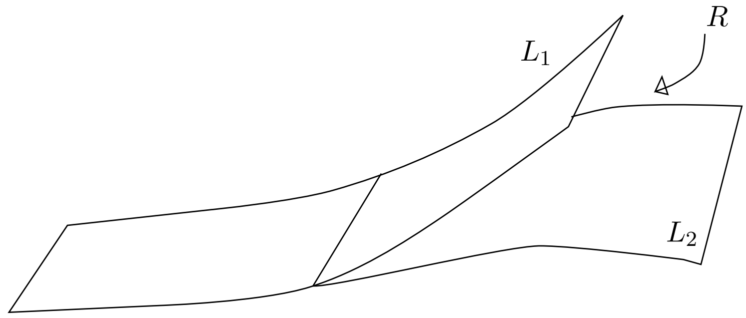


FIGURE 3. L_1 and L_2 are two leaves in C, but the region R is not in C. Then, in parts of R, all the center stable leaves intersect the branch locus between L_1 and L_2, so have parts in C and parts not in C (and therefore $M \setminus C$ is not saturated by center stable leaves).

one can build from $\widetilde{C}$. This stems from the fact that there can be different ways of saturating a given set by leaves of $\widetilde{\mathcal{F}}_{\mathrm{bran}}$.

More precisely, a saturation of $\widetilde{C}$ is a set $\mathrm{Sat}(\widetilde{C}) \subset \mathcal{L}_{\mathrm{bran}}$ such that, for all $x \in \widetilde{C}$, there exists $L \in \mathrm{Sat}(\widetilde{C})$ such that $x \in L$ and $L \subset C$. Such a set is not uniquely defined. However, there is a biggest such set: The *full saturation* of $\widetilde{C}$ is the set $\mathrm{FullSat}(\widetilde{C}) \subset \mathcal{L}_{\mathrm{bran}}$ defined by, if $L \in \mathcal{L}_{\mathrm{bran}}$ is such that $L \subset C$, then $L \in \mathrm{FullSat}(\widetilde{C})$. Note that the image of both $\mathrm{Sat}(\widetilde{C})$ and $\mathrm{FullSat}(\widetilde{C})$ in $\widetilde{M}$ are just $\widetilde{C}$, since C is $\mathcal{F}_{\mathrm{bran}}$-saturated.

Now, it could happen that a set C is closed in M, but a saturation $\mathrm{Sat}(C)$ would fail to be closed in $\mathcal{L}_{\mathrm{bran}}$ (recall that the topology on $\mathcal{L}_{\mathrm{bran}}$ is defined in section 3.1.1). However, one can easily see that the following is true: The set C is a closed subset of M if and only if $\mathrm{FullSat}(\widetilde{C})$ is a closed subset of the leaf space $\mathcal{L}_{\mathrm{bran}}$.

A natural but less immediate result (see Lemma B.1) shows that if a saturation $\mathrm{Sat}(\widetilde{C})$ is closed in $\mathcal{L}_{\mathrm{bran}}$ *and* $C = M$, then $\mathrm{Sat}(\widetilde{C}) = \mathcal{L}_{\mathrm{bran}}$ (so in particular, there is only one closed saturation in that case).

4.1.1. *Complementary regions.* Let $\mathcal{F}_{\mathrm{bran}}$ be a branching foliation (assumed to be well-approximated by taut foliations) on a manifold M that is not finitely covered by $S^2 \times S^1$. Then $\widetilde{M} \simeq \mathbb{R}^3$, and each leaf of $\widetilde{\mathcal{F}}_{\mathrm{bran}}$ is a properly embedded plane that separates $\widetilde{M}$ into two open balls.

The *complementary regions* of a leaf $L \in \widetilde{\mathcal{F}}_{\mathrm{bran}}$ are the two connected components of $\widetilde{M} \smallsetminus L$ (cf. [BFFP20b, §3.1.1]). For each complementary region U of a leaf L, the closure $\overline{U} = U \cup L$ is called a *side* of L.

A coorientation of $\widetilde{\mathcal{F}}_{\mathrm{bran}}$ (defined as an orientation of the leaf space of $\widetilde{\mathcal{F}}_{\mathrm{bran}}$) determines, for each leaf $L \in \widetilde{\mathcal{F}}_{\mathrm{bran}}$, a *positive* and a *negative* complementary region which we denote by $L^{\oplus}$ and $L^{\ominus}$. The corresponding sides are denoted by $L^+ = L^{\oplus} \cup L$ and $L^- = L^{\ominus} \cup L$.

To define the region between two leaves, it is best to work in the leaf space $\mathcal{L}_{\mathrm{bran}}$, with the topology defined in §3.1.1. Let $K, L \in \widetilde{\mathcal{F}}_{\mathrm{bran}}$ be distinct leaves. Thinking of these as points in the leaf space, $\mathcal{L}_{\mathrm{bran}} \smallsetminus \{K, L\}$ consists of three open connected components. Only one of these components accumulates on both K and L — we call this the *open* $\mathcal{L}_{\mathrm{bran}}$*-region between* K *and* L. Its closure in

$\mathcal{L}_{\mathrm{bran}}$, which is obtained by adjoining K and L, is called the *closed $\mathcal{L}_{\mathrm{bran}}$-region between K and L.*

Note that the subset of $\widetilde{M}$ that corresponds to the open $\mathcal{L}_{\mathrm{bran}}$-region between two leaves may not be open. However, the subset of $\widetilde{M}$ that corresponds to the closed $\mathcal{L}_{\mathrm{bran}}$-region between two leaves is closed. It is also connected, but its interior may not be. See Figure 4.

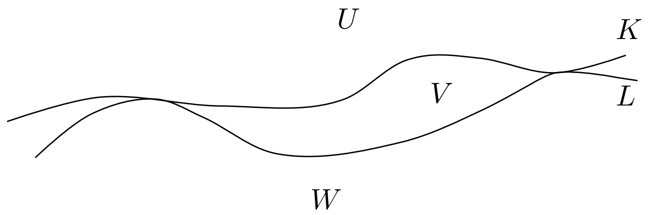


FIGURE 4. The interior of the closed region between leaves may not be connected.

4.1.2. *Translation-like behavior.* Recall that $\mathcal{F}_{\mathrm{bran}}$ is assumed to be well-approximated by taut foliations. Using this, we immediately obtain the Big Half-Space Lemma ([BFFP20b, Lemma 3.3]).

Lemma 4.4. *Let L be a leaf of $\mathcal{F}_{\mathrm{bran}}$. For any $R > 0$, there exists a ball of radius R contained in each complementary region of L.*

Proof. It suffices to apply [BFFP20b, Lemma 3.3] to a leaf corresponding to L in the approximating foliation $\mathcal{F}_\epsilon$, and deduce that each complementary region of L contains a ball of radius $R - \epsilon$ for any R. ☐

The following is the equivalent of [BFFP20b, Proposition 3.5]. The same proof applies if one considers complementary regions and regions between leaves as subsets of $\widetilde{M}$ and $\mathcal{L}_{\mathrm{bran}}$ as appropriate.

Proposition 4.5. *Let $\mathcal{F}_{\mathrm{bran}}$ be a branching foliation, $f\colon M \to M$ a diffeomorphism homotopic to the identity and preserving $\mathcal{F}_{\mathrm{bran}}$, and $\widetilde{f}$ be a good lift. If $L \in \widetilde{\mathcal{F}}_{\mathrm{bran}}$ is not fixed by $\widetilde{f}$, then*

(1) the closed $\mathcal{L}_{\mathrm{bran}}$-region between L and $\widetilde{f}(L)$ is an interval,
(2) $\widetilde{f}$ takes each coorientation at L to the corresponding coorientation at $\widetilde{f}(L)$, and
(3) the subset of M corresponding to the closed $\mathcal{L}_{\mathrm{bran}}$-region between L and $\widetilde{f}(L)$ is contained in the closed $2R$-neighborhood of L, where $R = \max_{y\in\widetilde{M}} d(y, \widetilde{f}(y))$.

4.1.3. *Uniform and $\mathbb{R}$-covered branching foliations.* A branching foliation is once again called $\mathbb{R}$-covered if its leaf space $\mathcal{L}_{\mathrm{bran}}$ (see section 3.1.1) is homeomorphic to $\mathbb{R}$. Since the topology on $\mathcal{L}_{\mathrm{bran}}$ can be defined as a quotient of the leaf space of the approximating foliations $\widetilde{\mathcal{F}}_\epsilon$, the branching foliation is $\mathbb{R}$-covered if and only if the approximating one is, for ϵ small enough.

The definition of a uniform foliation applies without any change to branching foliations. It is immediate to notice that a branching foliation is uniform if and only if the approximating foliations (see Definition 3.2) are uniform (pairs of leaves in the universal cover are bounded Hausdorff distance apart).

4.2. **The dichotomy.** Using Proposition 4.5 we therefore also obtain the equivalent of [BFFP20b, Proposition 3.7].

Proposition 4.6. *Let M be a closed 3-manifold that is not finitely covered by $S^2 \times S^1$, $f\colon M \to M$ a homeomorphism homotopic to the identity that preserves a branching foliation $\mathcal{F}_{\text{bran}}$, and $\widetilde{f}$ a good lift.*

Then the set $\Lambda \subset \mathcal{L}_{\text{bran}}$ of leaves that are fixed by $\widetilde{f}$ is closed and $\pi_1(M)$-invariant. Moreover, each connected component I of $\mathcal{L}_{\widetilde{\mathcal{T}}} \setminus \Lambda$ is an open interval that $\widetilde{f}$ preserves and acts on as a translation, and every pair of leaves in I are a finite Hausdorff distance apart.

In the above proposition, one has to be mindful again that "open" and "closed" refer to the topology on the leaf space $\mathcal{L}_{\text{bran}}$, and not the topology on $\widetilde{M}$.

From Proposition 4.6, we deduce as in [BFFP20b, §3] that, if the foliation is f-minimal, we get a dichotomy:

Corollary 4.7. *Let M be a closed 3-manifold that is not finitely covered by $S^2 \times S^1$, $f\colon M \to M$ a homeomorphism homotopic to the identity that preserves a branching foliation $\mathcal{F}_{\text{bran}}$, and $\widetilde{f}$ a good lift.*

If $\mathcal{F}_{\text{bran}}$ is f-minimal, then either

(1) $\widetilde{f}$ fixes every leaf of $\widetilde{\mathcal{F}}_{\text{bran}}$, or

(2) $\mathcal{F}_{\text{bran}}$ is $\mathbb{R}$-covered and uniform, and $\widetilde{f}$ acts as a translation on the leaf space of $\widetilde{\mathcal{F}}_{\text{bran}}$.

Proof. The proof is the same as that in [BFFP20b]. However, since the distinctions between the topology in the leaf space and that of corresponding sets in $\widetilde{M}$ becomes essential, we redo the proof.

Let Λ be the set of leaves that are fixed by $\widetilde{f}$. Since $\widetilde{f}$ commutes with deck transformation, each deck transformation preserves Λ. In particular, if I is a component of $\mathcal{L} \setminus \Lambda$ and $g \in \pi_1(M)$ then one has either $g(I) = I$ or $g(I) \cap I = \emptyset$.

So Λ is invariant under $\widetilde{f}$ and deck transformations, saturated by $\widetilde{\mathcal{F}}_{\text{bran}}$ and closed for the topology of $\mathcal{L}_{\text{bran}}$ by Proposition 4.6.

Let $\widetilde{B}$ be the set of points in $\widetilde{M}$ contained in a leaf of Λ and let $B = \pi(\widetilde{B})$. Since Λ is closed in $\mathcal{L}^{cs}_b$, then $\widetilde{B}$ is closed in $\widetilde{M}$ and so is B in M. In addition B is f-invariant. Since $\mathcal{F}_{\text{bran}}$ is f-minimal, B is either empty or the whole of M.

If B is empty, then Λ is also empty, so Proposition 4.6 implies that we are in case (2).

Suppose instead that $B = M$, so $\widetilde{B} = \widetilde{M}$. Then we have to prove that $\Lambda = \mathcal{L}_{\text{bran}}$. This follows from the more general Lemma B.1, but the proof in this case is easy so we give it:

Suppose $\Lambda \neq \mathcal{L}_{\text{bran}}$. Let I be a connected component of $\mathcal{L}_{\text{bran}} \smallsetminus \Lambda$. Let J be the set of points of $\widetilde{M}$ contained in a leaf in I. The set I is open (in $\mathcal{L}_{\text{bran}}$) and $\widetilde{f}$ translates leaves in I. It follows that the interior in $\widetilde{M}$ of J is non-empty. These points in the interior of J are not contained in $\widetilde{B}$. This contradicts $\widetilde{B} = \widetilde{M}$. So $\Lambda = \mathcal{L}_{\text{bran}}$ and we are in case (1). $\square$

From now on, we stop considering general well-approximated branching foliations and general branching foliations-preserving diffeomorphisms. Instead, we specialize to considering partially hyperbolic diffeomorphisms $f\colon M \to M$, homotopic to the identity, on a 3-manifold with non virtually solvable fundamental group and that admits a pair of center stable and center unstable branching foliations, $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$.

4.2.1. *Fixed points and fixed leaves.* The non-existence of fixed points, applies almost the same as in [BFFP20b] (see also [BW05]) but one needs to have a stronger assumption.

Lemma 4.8. *Let L be a leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ that is fixed by $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$. If, for any $y \in L$ there exists a leaf L' of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ fixed by $\widetilde{f}$ and intersecting the unstable leaf of y in a point different from y, then there are no points in L fixed by any non-trivial power of $\widetilde{f}$.*

Proof. Suppose x was a fixed point of $\widetilde{f}^n$, with $n > 0$, on L. Then, the unstable leaf through x would intersect some other fixed stable leaf in a point distinct from x, and hence contain another fixed point of $\widetilde{f}^n$, which is impossible. □

Here we see an essential difference with the dynamically coherent setting: If L is accumulated by a sequence of leaves L_n fixed by $\widetilde{f}$ these sequence may intersect L at a fixed point for all n. Then, we cannot not exclude the existence of fixed points in the set $L \cap (\bigcap_n L_n)$ with that proof. This will be the main endeavor in §6.

4.3. **Good lifts and fixed points.** We just showed that a good lift $\widetilde{f}$ cannot have fixed (or periodic) points under the assumption that all leaves of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ are fixed. We will now exclude the existence of fixed or periodic points under a different assumption, namely f-minimality.

Theorem 4.9. *Let f be a partially hyperbolic diffeomorphism homotopic to the identity, and $\widetilde{f}$ a good lift. If $\mathcal{W}^{cs}_{\text{bran}}$ or $\mathcal{W}^{cu}_{\text{bran}}$ is f-minimal, then $\widetilde{f}$ does not have any periodic point.*

Proof. We do the proof assuming $\mathcal{W}^{cs}_{\text{bran}}$ is the f-minimal foliation. Note first that it is enough to show that $\widetilde{f}$ has no fixed points. Indeed, for any fixed n, $\mathcal{W}^{cs}_{\text{bran}}$ is also f^n-minimal and $\widetilde{f}^n$ is a good lift of f^n.

By Corollary 4.7, either $\widetilde{f}$ fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ or it acts as a translation on $\mathcal{L}^{cs}_b$. If $\widetilde{f}$ acts as a translation on $\mathcal{L}^{cs}_b$, then it cannot fix any point of $\widetilde{M}$. This is because for any leaf L of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$, and $|i|$ big enough $\widetilde{f}^i(L) \cap L = \emptyset$.

On the other hand, if $\widetilde{f}$ fixes every leaves of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$, then Lemma 4.8 implies that $\widetilde{f}$ does not admit fixed points either. □

A noteworthy corollary of the above result is that a partially hyperbolic diffeomorphism homotopic to the identity that admits a f-minimal branching foliation cannot have so-called contractible periodic points.

Definition 4.10. Let g be a homeomorphism of a manifold homotopic to the identity. A point p is a *contractible periodic point* of g of period n if $g^n(p) = p$ and there exists $H \colon M \times [0,1]$ a homotopy from the identity to g, such that the closed path obtained by concatenation of the paths $H(p,\cdot), H(g(p),\cdot), \dots, H(g^{n-1}(p),\cdot)$ is homotopically trivial.

Notice that if p is a contractible periodic point of g of period n then there exists a good lift $\widetilde{g}$ of g and a lift $\widetilde{p}$ of p such that $\widetilde{g}^n(\widetilde{p}) = \widetilde{p}$. Thus, Theorem 4.9 immediately yields:

Corollary 4.11. *Let f be a partially hyperbolic diffeomorphism on a 3-manifold that is homotopic to the identity. Suppose that f admits a f-minimal branching center stable or center unstable foliation. Then f does not admit any contractible periodic points.*

Notice that this completes the proof of Theorem 1.6 in the f-minimal case. For the hyperbolic and Seifert case, the proof is the same once the proof of Proposition 6.1 below is completed.

4.4. **Fundamental group of leaves of $\mathcal{W}^{cs}_{\text{bran}}, \mathcal{W}^{cu}_{\text{bran}}$.** The leaves of the branching foliations $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ given in Theorem 3.4 are only immersed manifolds. In particular, they may not be injectively immersed. However, in the universal cover, any leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ or $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$ is a properly embedded plane (cf. section 3.1).

Thus, there might exists some closed loops in a leaf C of, say, $\mathcal{W}^{cs}_{\text{bran}}$ such that no lift L of C is fixed by the element of the fundamental group of M that represents the loop. This type of elements of the fundamental group of C seen as a set of M are not useful for our purpose. So, we will remove them by convention:

Convention. Fix a lift L of a leaf C of $\mathcal{W}^{cs}_{\text{bran}}$ (or $\mathcal{W}^{cu}_{\text{bran}}$). An element $\gamma \in \pi_1(M)$ is said to be in the fundamental group of C if it is in the stabilizer of L.

Notice that the fundamental group is only defined up to conjugation, hence the reason to fix a lift L of C.

This convention seems to eliminate more than just the closed loops coming from self-intersections, as any potential closed loops that would be homotopically trivial in M but not in C, would not be considered.

However, there is another way of seeing our notion of fundamental group arise: Recall (Theorem 3.4) that the branching foliations are approximated by true foliations $\mathcal{W}^{cu}_\epsilon$ and $\mathcal{W}^{cs}_\epsilon$ and that there exists maps, h^{cs}_ϵ and h^{cu}_ϵ mapping leaves of $\mathcal{W}^{cs}_\epsilon$ (or $\mathcal{W}^{cu}_\epsilon$) to those of $\mathcal{W}^{cs}_{\text{bran}}$ (or $\mathcal{W}^{cu}_{\text{bran}}$). Then, a loop is in the fundamental group of a leaf C of $\mathcal{W}^{cs}_{\text{bran}}$ if and only if it is freely homotopic to a loop in a corresponding leaf C_ϵ of $\mathcal{W}^{cs}_\epsilon$, for every ϵ small enough. Notice that if there are several leaves that project to C, in the universal cover, take a lift L and it follows from Proposition 3.11 that the set of leaves that projects to L is an interval in the leaf space of $\widetilde{\mathcal{W}}^{cs}_\epsilon$. It follows that h^{cs}_ϵ lifts to a equivariant (with respect to the defined fundamental group of C) diffeomorphism from the boundary leaves of the closed interval to L. We call such a leaf L_ϵ and denote $C_\epsilon = \pi(L_\epsilon)$.

In other words, for us, the fundamental group of C based at y will be exactly $(h^{cs}_\epsilon)_*(\pi_1(C_\epsilon, y_0))$ where $h^{cs}_\epsilon(y_0) = y$.

In particular, since $\mathcal{W}^{cs}_\epsilon$ and $\mathcal{W}^{cu}_\epsilon$ are taut foliations without Reeb components, each leaf is π_1-injective in M. Thus, this second interpretation helps explain our convention: the closed loops in a leaf of $\mathcal{W}^{cs}_{\text{bran}}$ are either in the fundamental group as we defined it, or they are due to a self-intersection. In that case, they are not an essential feature of the leaf, as they stopped being closed when pulled-back to the approximating leaf.

Following our convention, we will then say that a leaf C of the branching foliation is a plane, a cylinder, or a Mobiüs band if its corresponding approximated leaf C_ϵ is, respectively, a plane, a cylinder, or a Mobiüs band, for any small enough ϵ.

Using these conventions, [BFFP20b, Proposition 3.14] holds for the leaves of the branching foliations whenever $\widetilde{f}$ has no fixed points in the leaf (cf. Lemma 4.8). For ease of reference, we restate it here.

Proposition 4.12. *Assume that $\widetilde{f}$ fixes a leaf L of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ then, $C = \pi(L)$ has cyclic fundamental group (thus it is either a plane, an annulus or a Möbius band), or L has a point fixed by $\widetilde{f}$.*

Remark 4.13. Similarly, because of possible self-intersections, we need to be careful on how to define the path-metric on a leaf of $\mathcal{W}^{cs}_{\text{bran}}$ or $\mathcal{W}^{cu}_{\text{bran}}$.

If C is a leaf of, say, $\mathcal{W}^{cs}_{\text{bran}}$, we define a *path* on C as a continuous curve η that is the projection of a continuous curve $\widetilde{\eta}$ in a lift L of C to $\widetilde{M}$. We then define the path-metric on C as usual, but considering only the paths as defined before.

Notice that not every continuous curve η on C is a path in the above sense, as there might not exists any lift of η that stays on only one lift of C.

Still the analogous of [BFFP20b, Lemma 3.11] holds:

Lemma 4.14. *If $\widetilde{f}$ fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ (resp. $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$) then there is $K > 0$ such that for every $L \in \widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ (resp. $L \in \widetilde{\mathcal{W}}^{cu}_{\text{bran}}$) we have that $d_L(x, \widetilde{f}(x)) < K$.*

4.5. **Gromov hyperbolicity of leaves.** We now prove a version of [BFFP20b, Lemma 3.20] in the non dynamically coherent setting.

Lemma 4.15. *Suppose that f is a partially hyperbolic diffeomorphism in M that is homotopic to the identity. Let $\widetilde{f}$ be a good lift of f to $\widetilde{M}$. Suppose that $\widetilde{f}$ fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$, and that $\mathcal{W}^{cs}_{\text{bran}}$ is f-minimal.*

Then all the leaves of $\mathcal{W}^{cs}_{\text{bran}}$ are Gromov hyperbolic.

Proof. The foliation $\mathcal{W}^{cs}_{\epsilon}$ is taut. Thus, Candel's theorem [Can93] asserts that either all the leaves of $\mathcal{W}^{cs}_{\epsilon}$ are Gromov hyperbolic or there is a holonomy invariant transverse measure (of zero Euler characteristic).

Assume for a contradiction that μ is a holonomy invariant transverse measure. Since $\mathcal{W}^{cs}_{\epsilon}$ is not f-invariant, we have to adjust the proof given in [BFFP20b].

The transverse measure μ lifts to a measure $\widetilde{\mu}$ transverse to $\widetilde{\mathcal{W}}^{cs}_{\epsilon}$. Thus, $\widetilde{\mu}$ defines a measure on $\mathcal{L}^{cs}_{\epsilon}$, the leaf space of $\mathcal{W}^{cs}_{\epsilon}$.

Let $g_{\epsilon,s} \colon \mathcal{L}^{cs}_{\epsilon} \to \mathcal{L}^{cs}_{b}$ be the canonical projection between the leaf spaces of $\mathcal{W}^{cs}_{\epsilon}$ and $\mathcal{W}^{cs}_{\text{bran}}$ (see section 3.1.1). Let $\widetilde{\nu} := (g_{\epsilon,s})_* \widetilde{\mu}$ be the corresponding measure on $\mathcal{L}^{cs}_{b}$. Now $\widetilde{\nu}$ is $\widetilde{f}$-invariant since $\widetilde{f}$ is the identity on $\mathcal{L}^{cs}_{b}$, and it is also $\pi_1(M)$-invariant as $\widetilde{\mu}$ is. The support of $\widetilde{\nu}$ in $\mathcal{L}^{cs}_{b}$ is a closed set Z in $\mathcal{L}^{cs}_{b}$ that is $\widetilde{f}$-invariant and $\pi_1(M)$-invariant.

The measure $\widetilde{\nu}$ on $\mathcal{L}^{cs}_{b}$ can also be considered as a measure on the set of transversals to $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ in $\widetilde{M}$: For any transversal τ to $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ in $\widetilde{M}$, we define $\widetilde{\nu}(\tau)$ as the $\widetilde{\nu}$-measure of the set of leaves in $\mathcal{L}^{cs}_{b}$ that intersects τ. Notice that the measure of a point in $\widetilde{M}$ (which can be thought of as a degenerate transversal) can be positive if the image of that point in $\mathcal{L}^{cs}_{b}$ is an interval.

Note also that we refrained from calling $\widetilde{\nu}$ a transverse measure to $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ because it is by no means holonomy invariant. In fact holonomy itself is not well defined for a branching foliation. Still $\widetilde{\nu}$ satisfies the property that if τ_1, τ_2 are transversals and every leaf intersecting τ_1, also intersects τ_2, then $\widetilde{\nu}(\tau_1) \leq \widetilde{\nu}(\tau_2)$.

Projecting down to M,the measure $\widetilde{\nu}$ induces a measure ν on the set of transversals to $\mathcal{W}^{cs}_{\text{bran}}$ on M.

Let τ be any unstable segment in M. Since $\widetilde{f}$ fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$, the measure of $f^i(\tau)$ $(= \nu(f^i(\tau)))$ is equal to $\nu(\tau)$ for any integer i. We can choose i very big and negative so that the length of $f^i(\tau)$ is extremely small. Therefore it is contained in a small foliated box of $\mathcal{W}^{cs}_{\text{bran}}$, which is the projection of a compact foliated box of $\mathcal{W}^{cs}_{\epsilon}$. It follows that $\nu(\tau)$ is uniformly bounded. In particular this implies that the ν-measure of any unstable leaf in M is bounded above. In turns, it implies that for any $j > 0$ (assumed big enough), there is an unstable segment u_j of length $> j$ which has $\nu(u_j)$ measure $< 1/j$. Taking the midpoint of these segments and a converging subsequence, we obtain a full unstable leaf, call it ζ, so that ζ has $\nu(\zeta) = 0$ (since $\nu(\zeta) < 1/j$ for all big enough j).

Let Y be the union of the leaves of $\mathcal{W}^{cs}_{\text{bran}}$ that do not intersect ζ or any of its iterates by f. Then Y is a closed subset of M and clearly f-invariant. Let L be a leaf in $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ which is in Z, the support of $\widetilde{\nu}$. Then by definition of support of $\widetilde{\nu}$, it follows that $\pi(L)$ cannot intersect ζ or any of its iterates by f. Hence $\pi(L)$ is in Y. In particular Y is not empty. This contradicts the fact that $\mathcal{W}^{cs}_{\text{bran}}$ is f-minimal, and hence cannot happen.

This finishes the proof of the lemma. □

4.6. **Perfect fits in branching foliations.** An essential tool for us has been the use of perfect fits between center leaves and stable (or unstable) leaves inside a center stable (resp. center unstable) leaf. Despite having branching foliations, the definitions of a $\mathcal{CS}$-perfect fits, $\mathcal{SC}$-perfect fits and perfect fits (cf [BFFP20b, §4.1]) remains literally the same. However it is useful to add one precision on how to define what it means to be "on one side of c" when c is a center leaf that may have branching loci for the definition of $\mathcal{CS}$-perfect fit. The definition of $\mathcal{SC}$-perfect fit does not even need this (because the stable foliation is a true foliation, not a branching one).

Definition 4.16. Let c be a center and s a stable leaf in a center stable leaf L. We denote by C^s the connected component of $L \smallsetminus c$ that contains s.

The leaves c and s makes a $\mathcal{CS}$-*perfect fit* if there exists τ an open transversal to the center foliation in L that intersects c and such that, for any center leaf c', if c' intersects τ and c' intersects C^s, then c' intersects s.

Notice that the condition in the definition needs to apply to any c' that intersects the transversal τ. In particular, it needs to apply to any c' such that $c' \cap \tau = c \cap \tau$, i.e., any center leaf that branches away from c after its intersection with the transversal τ.

One can also see the definition of a perfect fit at the leaf space level: Let s be a stable leaf in L. The leaf s determines a set I_s in $\mathcal{L}^c_L$, the leaf space of the center branching foliation on L (see section 3.1.2), by considering all the center leaves that intersect s. That is, $c' \in I_s$ if and only if $c' \cap s \neq \emptyset$. Then c and s makes a $\mathcal{CS}$-perfect fit if and only if $c \in \partial I_s$.

[BFFP20b, Lemma 4.2] and its proof stays valid as written because the stable foliation is a true foliation. One can also show that if s and c make a $\mathcal{SC}$-perfect fit, then there exists c_0 that makes a perfect fit with s but one needs to modify the proof by going to the leaf space level.

5. Fixed center or coarse contraction

This section deals with one of the central difficulties in the non-dynamically coherent setting.

In [BFFP20b, Proposition 4.4] we gave a condition for the existence of center leaves that are fixed by a good lift $\widetilde{f}$. But the proof of that result does not apply in the non dynamically coherent setting (see [BFFP20b, Remark 4.8]).

The next proposition will instead give a consequence to the *non-existence* of central leaves fixed by $\widetilde{f}$. First, we need a definition.

Definition 5.1. A fixed center leaf c of a partially hyperbolic diffeomorphism $f\colon M \to M$ is called *coarsely contracting* if c is homeomorphic to the line, and it contains an non-empty maximal compact interval I such that:

(1) I contains every fixed point of the restriction of f to c;
(2) For any compact interval J of c such that $I \subset \mathring{J}$, we have $f(J) \subset \mathring{J}$.

A fixed center leaf c of f is called *coarsely expanding* if c is coarsely contracting for f^{-1}.

We also naturally extend the definition of coarsely expanding to leaves that are just periodic under f.

Proposition 5.2. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism. Let $\widetilde{f}\colon \widetilde{M} \to \widetilde{M}$ be a good lift of f. Suppose that $\mathcal{W}^{cs}_{\text{bran}}$ is f-minimal, that all the leaves of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ are fixed by $\widetilde{f}$, and that $\widetilde{f}$ does not fix any center leaf in $\widetilde{M}$.*

If c is a periodic center leaf of f in M, then c is coarsely contracting. In particular, c contains a periodic point of f.

Remark 5.3. If $\widetilde{f}$ as above fixes every leaf of $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$ instead of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$, the conclusion of the proposition gives a periodic center leaf that is coarsely *expanding* instead.

We start with a preliminary result.

Lemma 5.4. *Assume that every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ is fixed by $\widetilde{f}$ and that $\widetilde{f}$ does not fix any center leaf. Then the same holds for $\widetilde{f}^n$, for every $n \neq 0$.*

Proof. Suppose that there is $n > 0$ and c_0 a center leaf in a center stable leaf L such that $\widetilde{f}^n(c_0) = c_0$.

The standing assumption in section 4 is that all bundles are oriented and that f preserves their orientations, in particular, $\widetilde{f}$ preserves the transverse orientation to the center and stable foliations on L.

Let A^c be the axis of the action of $\widetilde{f}$ on the center leaf space in L.

Since $\widetilde{f}^n(c_0) = c_0$, the leaf c_0 is not in the axis A^c. Thus, either $c_0 \in \partial A^c$, or there exists a unique center leaf $c_1 \in \partial A^c$ that separates c_0 from A^c, in which case we must have $\widetilde{f}^n(c_1) = c_1$.

Hence, up to renaming c_0, we assume that $c_0 \in \partial A^c$.

Now, according to [Bar98, Proposition 2.15], the boundary ∂A^c splits into three disjoint sets: the center leaves c such that c and $\widetilde{f}(c)$ are non separated positively, the leaves c such that c and $\widetilde{f}(c)$ are non separated negatively, and the leaves that are non separated with a leaf in A^c. Since c_0 is fixed by $\widetilde{f}^n$, it cannot be a leaf of the third type. Thus, c_0 and $\widetilde{f}(c_0)$ are non separated.

Hence, there exists a unique stable leaf s_0 that makes a perfect fit with c_0 and separates c_0 from $\widetilde{f}(c_0)$ (see section 4.6). This stable leaf is then fixed by $\widetilde{f}^n$, and thus admits a fixed point x of $\widetilde{f}^n$. Therefore, there exists a center leaf c_1 through x that is fixed by $\widetilde{f}^n$ (thanks to Lemma 3.14), and, in case there are several such leaves, we may chose the one that is in ∂A^c.

Again using the description of ∂A^c, the leaf c_1 is non separated from $\widetilde{f}(c_1)$. Then again, there exists a unique stable leaf s_1 making a perfect fit with c_1 and that separates c_1 and $\widetilde{f}(c_1)$. Therefore, $\widetilde{f}^n(s_1) = s_1$ and there exists a unique fixed point $y \in s_1$ of $\widetilde{f}^n$.

But, any center leaf c close enough to c_1 (and on the correct side of c_1) will intersect both s_0 and s_1, separate x from y and be attracted to both x and y under $\widetilde{f}^n$, which is impossible.

Therefore $\widetilde{f}^n$ also acts freely on the center leaf space for all $n > 0$. ∎

In order to obtain coarsely contracting center leaves we will use the following tool.

Proposition 5.5. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism homotopic to the identity. Let $\widetilde{f}$ be a good lift of f to $\widetilde{M}$. Suppose that $\widetilde{f}$ fixes each leaf of the branching foliation $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$. Let L be a center stable leaf fixed by $\gamma \in \pi_1(M) \smallsetminus \{\mathrm{Id}\}$.*

Assume that there exists a properly embedded C^1-curve, $\hat{\eta}$, in L that is transverse to the stable foliation and fixed by both γ and $\widetilde{f}$.

Then,

- *If $\widetilde{f}$ does not act freely on the center leaf space of L, then there is a center leaf in L fixed by both $\widetilde{f}$ and γ.*
- *If $\widetilde{f}$ acts freely on the center leaf space of L, then every f periodic center leaf in $\pi(L)$ is coarsely contracting.*

Notice that in the first case, the center leaf projects to an f-invariant closed center leaf.

Remark also that the hypothesis of Proposition 5.5 are implied by the conclusion of the Graph Transform Lemma [BFFP20b, Appendix H].

We will need to apply the following result from [BFFP20b] whose proof works equally well in the non dynamically coherent case:

Lemma 5.6 (Lemma 4.15 in [BFFP20b])**.** *Let c be a center leaf in a center stable leaf $L \subset \widetilde{M}$. Suppose that L is Gromov-hyperbolic, and fixed by $\widetilde{f}$ and some nontrivial $\gamma \in \pi_1(M)$. Moreover, assume that there exist two stable leaves s_1, s_2 on L such that:*

(1) The center leaf c is in the region between s_1 and s_2;
(2) The leaves s_1 and s_2 are a bounded Hausdorff distance apart;
(3) The leaves c, s_1 and s_2 are all fixed by $h = \gamma^n \circ \widetilde{f}^m$, $m \neq 0$.

Then, there exists a compact segment $I \subset c$, such that h (if $m > 0$) or h^{-1} (if $m < 0$) acts as a contraction on $c \smallsetminus \mathring{I}$.

Proof of Proposition 5.5. Since $\widetilde{f}$ fixes every leaf of $\mathcal{W}^{cs}_{\text{bran}}$, Lemma 4.8 implies that $\widetilde{f}$ has no fixed points in $\widetilde{M}$. Therefore, $\widetilde{f}$ acts freely on the stable leaf space (recall that the stable foliation is a true, non branching foliation, so its leaf space is defined as usual with the quotient topology).

Let S be the stable saturation of the curve $\hat{\eta}$. Let $\alpha = \pi(\hat{\eta})$. The curve α is closed, f-invariant, and tangent to the center bundle.

Case 1 - We start by assuming that $\widetilde{f}$ fixes a center leaf c in L.

Suppose that c and $\hat{\eta}$ do not intersect a common stable leaf. Then c does not intersect the set S and there is a unique stable leaf s contained in the boundary of S such that s separates S from c. Since both S and c are $\widetilde{f}$-invariant, so is s. But then $\widetilde{f}$ must admit a fixed point in s, contradiction[6].

Therefore there is a stable leaf s intersecting c in y and $\hat{\eta}$ in x. Iterating forward by $\widetilde{f}$, we deduce that $d(\widetilde{f}^n(y), \widetilde{f}^n(x))$ converges to zero as y and x are in the same stable leaf. Since both c and $\hat{\eta}$ are $\widetilde{f}$-invariant, it implies that $\pi(c)$ and $\alpha = \pi(\hat{\eta})$ are asymptotic. As α is closed and $\pi(c)$ is a center leaf, we deduce that α is also a center leaf. Hence $\hat{\eta}$ is the required center leaf of the first option of the proposition.

Case 2 - Assume now that $\widetilde{f}$ acts freely on the center leaf space of L.

[6]Note the distinction of c being fixed by $\widetilde{f}$ as opposed to $\pi(c)$ periodic under f. It is the first property which creates a fixed point of $\widetilde{f}$ and a contradiction.

According to Lemma 5.4, $\widetilde{f}^n$ also acts freely on the center leaf space of L for any $n \neq 0$.

We need to prove now that every center leaf in $\pi(L)$ that is periodic must be coarsely contracting.

Let then c be a center leaf in L such that $\pi(c) = e$ is periodic under f, say of period m. Then, for some $\gamma_1 \in \pi_1(M) \smallsetminus \{\mathrm{Id}\}$, we have $c = \gamma_1 \widetilde{f}^m(c)$. (Note that one can show under our current assumptions that $\pi(L)$ projects to an annulus, so γ and γ_1 are both powers of a particular deck transformation, but we do not need that fact for the proof). Let

$$h := \gamma_1 \circ \widetilde{f}^m.$$

We now want to show that either c intersect $\hat{\eta}$, or there exists another center leaf, also fixed by h, that does.

Notice that, if c and $\hat{\eta}$ intersect a common stable leaf, then c must intersect $\hat{\eta}$. Indeed, both c and $\hat{\eta}$ are invariant by h, which contracts the stable length.

Suppose for an instant that c does not intersect $\hat{\eta}$, and thus does not intersect S. Then, there exists a unique stable leaf s in ∂S that separates $\hat{\eta}$ from c. That leaf s must then be invariant by h, so admits a fixed point for h. Then at least one center leaf, say c_1, through that fixed point must be fixed by h. Since c_1 intersects S and is invariant by h, it must intersect $\hat{\eta}$.

Thus in any case, we have a center leaf c_1 that intersects $\hat{\eta}$, is invariant by h, and, by the above argument has both ends that escapes compacts sets of L.

Let I be the projection of c_1 onto $\hat{\eta}$ along stable leaves.

Suppose first that I is unbounded. Then, considering iterates by f^m, we deduce that $\pi(c_1)$ must be asymptotic to $\pi(\hat{\eta})$, so $\hat{\eta}$ must be a center leaf, which is not allowed, since $\widetilde{f}$ is assumed to act freely on center leaves.

So I is bounded in $\hat{\eta}$. Let s_1 and s_2 be the stable leaves through the two endpoints of the interval I. Since I is fixed by h, so are s_1 and s_2. Moreover, the center leaf c_1, as well as c if it is different from c_1, is in between s_1 and s_2.

Now, $\widetilde{f}$ acts as a translation on $\hat{\eta}$, so there exists $k \in \mathbb{Z}$ such that s_2 separates s_1 from $\widetilde{f}^k(s_1)$. By Lemma 4.14, s_1 and $\widetilde{f}^k(s_1)$ are a bounded Hausdorff distance apart. Thus s_1 and s_2 are a bounded Hausdorff distance apart. So c satisfies all the conditions for Lemma 5.6 to hold, thus it is coarsely expanding.

This finishes the proof of Proposition 5.5. □

Now we are ready to prove the main result of this section:

Proof of Proposition 5.2. Let e be a center leaf periodic under f of period $m > 0$. Let c be a lift of e to $\widetilde{M}$. Call L a leaf of $\widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$ that contains c. Then $\widetilde{f}^m(c)$ projects to the same center leaf in M as c does, so there exists $\gamma' \in \pi_1(M)$ with $\gamma'(\widetilde{f}^m(c)) = c$. Clearly γ' is in the stabilizer of L, because $\widetilde{f}$ leaves invariant every leaf of $\widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$. Moreover, as $\widetilde{f}^m$ also acts freely on the center leaf space (cf. Lemma 5.4), γ' is not the identity.

Since $\widetilde{f}$ does not have any fixed points, Proposition 4.12 implies that the stabilizer of L in $\widetilde{M}$ is infinite cyclic. Thus, there exists $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ such that $\gamma^n \circ \widetilde{f}^m(c) = c$ for some $n \in \mathbb{Z}$, $n \neq 0$, and such that γ generates the stabilizer of L. We call

$$h := \gamma^n \circ \tilde{f}^m.$$

Notice that h is still a partially hyperbolic diffeomorphism and has bounded derivatives.

Since $\widetilde{f}$ acts freely on $\mathcal{L}^c_L$, the center leaf space in L, then it also acts freely on $\mathcal{L}^s_L$ the leaf space of the stable foliation on L.

Let A^s be the axis for the action of $\widetilde{f}$ on the stable leaf space $\mathcal{L}^s_L$. No stable leaf in M can be closed, so γ acts freely on $\mathcal{L}^s_L$. Moreover, as γ and $\widetilde{f}$ commute, A^s is also the axis for the action of γ on $\mathcal{L}^s_L$, the stable leaf space of L. As always A^s can be a line or a countable union of intervals.

Suppose first that A^s is a line. Let s be a stable leaf in A^s and p in s. Then p and γp can be connected by a transversal to the stable foliation, chosen so that the projection to $\pi(L)$ is a smooth simple closed curve. Let η be the union of the γ iterates of this segment. Then η satisfies the properties in the hypothesis of Proposition 5.5, which implies the result we sought.

So from now on we assume that the axis is a countable union of intervals, and we write

$$A^s = \bigcup_{i\in\mathbb{Z}} [s_i^-, s_i^+] = \bigcup_{i\in\mathbb{Z}} T_i.$$

Our first claim is that there exists $s \in A^s$, fixed by h, such that the center leaf c is between $\gamma^{-1}s$ and γs.

Suppose that c intersects some stable leaf s' in A^s, then s' is in a unique T_i for some i (the center leaf c cannot intersect two different intervals otherwise c would intersect two non-separated leaves, which is impossible). Then, since h fixes c, it also fixes the axis A^s and preserves the transverse orientation. It follows that $h(T_j) = T_j$ for all j. In this case we set $s = s_i^+$. The leaf s is fixed by h and there exists $k \neq 0$ such that $\gamma^{\pm 1} T_i = T_{i\pm k}$. Thus T_i is in between $\gamma^{-1}s$ and γs and hence, so is c. Recall here that h preserves orientation.

Now, suppose instead that c does not intersect A^s. Hence, there is a unique i such that $s_{i-1}^+ \cup s_i^-$ separates c from all other stable leaves in A^s. We again set $s := s_i^+$. As before, since h fixes both c and A^s, and preserves the transverse orientation, it must fix s also. The same argument as above also shows that c is between $\gamma^{-1}s$ and γs.

So in any case, we obtained a stable leaf s (chosen as a positive endpoint of one of the closed intervals T_i), fixed by h, and such that c is between $\gamma^{-1}s$ and γs. Notice that both γs and $\gamma^{-1}s$ are also fixed by h.

The leaf $\gamma^{-1}s$ is between γs and $\widetilde{f}^{2m}(\gamma s) = \gamma^{-2n+1}s$ (assuming $n \geq 1$, otherwise between γs and $f^{-2m}(\gamma s)$). Hence the Hausdorff distance between $\gamma^{-1}s$ and γs is bounded above by a uniform constant $C > 0$, depending only on f and m.

Thus we obtained that the fixed center leaf c, fixed by h, is in between two stable leaves, γs and $\gamma^{-1}s$, also fixed by h and a bounded Hausdorff distance apart. Moreover, the leaves of $\mathcal{W}^{cs}_{\text{bran}}$ are Gromov-hyperbolic by Lemma 4.15. These are all the conditions needed to apply Lemma 5.6, which states that c is coarsely contracting for h. $\square$

Remark 5.7. Notice that neither Proposition 5.2 nor 5.5 proves that there is a periodic center leaf. We prove this in the next result. While it is very easy to produce periodic center leaves in the dynamically coherent situation, in the next result we consider the non dynamically coherent situation, and also we produce a periodic center leaf in the projection $\pi(L)$ of the center stable leaf L in question. This is much stronger than obtaining a generic periodic center leaf, which a priori could be in any center stable leaf.

Proposition 5.8. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism homotopic to the identity and let $\widetilde{f}$ be a good lift to $\widetilde{M}$. Suppose that $\widetilde{f}$ fixes every leaf of the branching foliation $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$. Let L be a center stable leaf fixed by $\gamma \in \pi_1(M) \smallsetminus \{\text{Id}\}$. Then there is an f-periodic center leaf in $\pi(L)$.*

Proof. First notice that if one can prove the above result for a finite cover of M and a finite power of f, then the same result directly follows for the original map and manifold. Thus, we may assume that M is orientable, f is orientation-preserving, and the branching foliations are both transversely orientable.

Given these assumptions, L projects to an annulus in M. Let γ be a generator of the stabilizer of L.

If $\widetilde{f}$ fixes a center leaf in L, then it would project to a center leaf fixed by f, proving the claim. So we assume that $\widetilde{f}$ acts freely on the center leaf space in L. This implies that $\widetilde{f}$ also acts freely on the stable foliation in L, and we can thus consider the stable axis of $\widetilde{f}$.

Suppose first that the stable axis of $\widetilde{f}$ is a countable union of intervals $\bigcup_{i\in\mathbb{Z}} I_i$. Since γ also acts freely on the stable leaves, and commutes with $\widetilde{f}$, they have the same axis. Since the axis is a countable collection of intervals, there must exists a pair of integers n, m such that $h := \gamma^n \widetilde{f}^m$ fixes one of the intervals, and hence, a stable leaf. If $m = 0$, then γ^n has a fixed stable leaf, which is impossible. So $m \neq 0$, and the stable leaf projects to a periodic stable leaf in M. This periodic stable leaf thus contain a periodic point, and at least one center leaf through that point is then periodic. So the proposition is proved in that case.

Suppose now that the stable axis (of γ or $\widetilde{f}$) is a line. Then the assumptions of the Graph Transform Lemma [BFFP20b, Appendix H] are verified. So there exists a properly embedded curve $\hat{\eta}$ in L which is invariant under $\widetilde{f}$ and γ. Then [BFFP20b, Lemma H.3] applies and give a periodic center leaf, as claimed. □

6. Minimality for Seifert and hyperbolic manifolds

The goal of this section is to show that when M is hyperbolic or Seifert, then if there are fixed leaves of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ for $\widetilde{f}$, then every leaf is fixed. In the dynamically coherent case this was obtained in [BFFP20b, Proposition 3.15] but here we face substancial new difficulties. This will be a consequence of the following:

Proposition 6.1. *Suppose that M is hyperbolic or Seifert. Suppose that $\widetilde{f}$ fixes one leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$. Then $\mathcal{W}^{cs}_{\text{bran}}$ is f-minimal (and therefore every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ is fixed by $\widetilde{f}$). The same statement holds for $\mathcal{W}^{cu}_{\text{bran}}$. In addition, every leaf of $\mathcal{W}^{cs}_{\epsilon}$, $\mathcal{W}^{cu}_{\epsilon}$, $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ is either a plane or an annulus.*

The main issue to extend the proof of [BFFP20b] to the non dynamically coherent context is that here we cannot ensure the non-existence of fixed points of $\widetilde{f}$ since Lemma 4.8 does not exclude fixed points when the branching foliation is not f-minimal.

We first need a definition. So far, we only defined f-minimality for the whole foliations, but we can extend naturally the definition to a foliated subset: We say that a subset Λ of M, saturated by $\mathcal{W}^{cs}_{\text{bran}}$ (or $\mathcal{W}^{cu}_{\text{bran}}$) is *$f$-minimal* if it is closed, non-empty, and invariant by f, and such that no proper saturated subset of Λ verifies all these conditions.

Using this definition, we prove a lemma that holds without assuming M to be hyperbolic or Seifert.

Lemma 6.2. *Let $\widetilde{f}$ be a good lift of f to $\widetilde{M}$. Suppose that Λ is a non empty f-minimal set of $\mathcal{W}^{cs}_{\text{bran}}$, such that every leaf L of $\widetilde{\Lambda} = \pi^{-1}(\Lambda)$ is fixed by $\widetilde{f}$. Then there are no fixed points of $\widetilde{f}$ in a leaf of $\widetilde{\Lambda}$.*

Proof. During the proof of this lemma, we will use the expansion of stable length by $\widetilde{f}^{-1}$ a lot. To lighten the notation, we set $g := \widetilde{f}^{-1}$.

Suppose for a contradiction that there is a fixed point x_0 of $\widetilde{f}$ in a leaf L_0 of $\widetilde{\Lambda}$. This projects to a fixed point $y = \pi(x_0)$ in M. Notice that if a leaf L of $\widetilde{\Lambda}$ intersects $u(x_0)$ then, since both are $\widetilde{f}$-invariant, it follows that the intersection of L and $u(x_0)$ has to be x_0.

We start with the following

Claim 6.3. *There exists $b > 0$ such that any point in a leaf of $\widetilde{\Lambda}$ is at distance at most b (for the path metric on the leaf) from a fixed point of $\widetilde{f}$.*

Proof. Indeed, suppose this was not the case. Then, for any $b > 0$, there exists a disk of radius b in a leaf of $\widetilde{\Lambda}$ that does not contain any fixed point of $\widetilde{f}$. Taking $b \to +\infty$, up to deck transformations and considering a subsequence, these disks converge to a full leaf L_1 of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ in $\widetilde{\Lambda}$. Here the convergence is with respect to the topology of the center stable leaf space, which also implies convergence as a set of $\widetilde{M}$. The leaf L_1 does not contain any fixed point of $\widetilde{f}$, because otherwise, since all leaves of $\widetilde{\Lambda}$ are fixed by $\widetilde{f}$, one would have some fixed points in the disks accumulating onto L_1.

Now consider Λ', the closure in M of the leaf $A = \pi(L_1)$. Since Λ is closed, the set Λ' must be a (closed) subset of Λ, foliated by $\mathcal{W}^{cs}_{\text{bran}}$. Moreover, by the previous remark, neither the leaf L_1 nor its translates by deck transformations can intersect $u(x_0)$ as they do not have fixed points. It follows that $\pi(x_0) \notin \Lambda'$ contradicting f-minimality of Λ. $\square$

According[7] to Lemma 4.14 there is a constant $K_0 > 0$ such that, for any $z \in L_0$, we have

$$d_{L_0}(z, \widetilde{f}(z)) \leq K_0,$$

where d_L denotes the path-metric on L_0.

The rest of the proof will consist in proving that the fact that $\widetilde{f}$ moves points a bounded distance in L_0 contradicts the exponential contraction of length along the stable leaf $s(x_0)$ of the fixed point x_0 of $\widetilde{f}$ in L_0. We will do that by building large metric balls with no fixed points of $\widetilde{f}$, in contradiction with Claim 6.3.

In order to obtain these fixed-point free sets, we will use compact simply connected domains such that their boundary is the union of a segment along the stable leaf $s(x_0)$ and a geodesic segment in L_0. We will start by proving three claims about these domains. For that purpose, we introduce the following notations: given any $y_1, y_2 \in s(x_0)$, we write

- $[y_1, y_2]^s$ is the closed segment along the stable leaf $s(x_0)$ between y_1 and y_2,
- $[y_1, y_2]_{L_0}$ is the geodesic segment between them (for the path metric on L_0).

Before moving on to the claims, notice also that, since the stable foliation is a true foliation, there exists $\delta, \eta > 0$ such that points in a same stable leaf that are at distance less than δ in the path-metric of L_0, must be at distance less than η

[7]It is not hard to see that the proof applies to the fixed sublamination.

along the stable arc. Two consequences of this fact that will be used repeatedly are:

- points that are far enough away along $s(x_0)$ must be at distance greater than δ in L_0, and
- the volume of a $\delta/2$-tubular neighborhood of a stable segment $[y_1, y_2]^s$ must go to infinity with the length of $[y_1, y_2]^s$.

Thus there exists domains bounded by stable segments $[y_1, y_2]^s$ and geodesics $[y_1, y_2]_{L_0}$ with arbitrarily large diameter. These domains with large diameters are the subject of the next three claims.

For $y_1, y_2 \in s(x_0)$ we denote by D_{y_1,y_2} any of the closed topological disks bounded bounded by arcs in $[y_1, y_2]^s$ and $[y_1, y_2]_{L_0}$. As mentioned before, there are disks D_{y_1,y_2} of arbitrarily large diameter if y_1 is far from y_2 in $s(x_0)$. Given $C > 0$, we let V_C be the open tubular neighborhood of $[y_1, y_2]_{L_0}$.

Claim 6.4. *Let $D' = D_{y_1,y_2}$ for $y_1, y_2 \in s(x_0)$. Suppose that the length of $[y_1, y_2]_{L_0}$ is bounded above by d. Then there exists a positive integer i, with $i \leq d/\delta$, such that either:*

(i) $D' \subset g^i(D')$, *or,*

(ii) $g^i(D' \smallsetminus V_C) \cap (D' \smallsetminus V_C) = \emptyset$,

where $C = K_0 d/\delta$ and $g = \widetilde{f}^{-1}$.

Proof. We assume first that the statement is not vacuously true, i.e., that $D' \smallsetminus V_C$ is not empty.

For simplicity, we will only consider positive i. For any such i, let $C_i := iK_0$.

Assume that there is i such that $g^i(D' \smallsetminus V_{C_i}) \cap (D' \smallsetminus V_{C_i}) \neq \emptyset$.

Then, in particular, $g^i(D')$ and D' intersect. Hence, either $g^i(D')$, or $g^{-i}(D')$, is contained in D', or the boundaries must intersect.

First, notice that $g^i(D')$ cannot be entirely contained in D'. If that was the case, then, for all $n > 0$, we would have $g^{ni}(D') \subset D'$. But, as powers of g^i increase the length of the stable segment $[y_1, y_2]^s$, and these images would have to stay in the compact D', we would get an accumulation point for $s(x_0)$ which is impossible.

Thus, either $D' \subset g^i(D')$, or the boundaries of $g^i(D')$ and D' must intersect.

Suppose for the moment that the boundaries intersect. Since $g^i(D' \smallsetminus V_{C_i}) \cap (D' \smallsetminus V_{C_i}) \neq \emptyset$, it implies that there exists $x_1^i \in g^i(\partial D') \cap (D' \smallsetminus V_{C_i})$. See Figure 5. Moreover, $g^i([y_1, y_2]_{L_0})$ is in the tubular neighborhood of $[y_1, y_2]_{L_0}$ of radius at most $C_i = iK_0$. So $x_1^i \in g^i([y_1, y_2]^s) \subset s(x_0)$.

Since no ray of $s(x_0)$ can stay in D' nor can self-intersect, there exists two points $z_1^i, z_2^i \in s(x_0) \cap [y_1, y_2]_{L_0}$ that we can choose in such a way that $y_2 \leq z_1^i < x_1^i < z_2^i$ (for the order on $s(x_0)$ given by an orientation). Since $d_{L_0}(x_1^i, z_2^i) \geq C_i = iK_0$, the distance between z_2^i and both y_1 and y_2 must be greater than δ (if necessary, we take K_0 bigger so that $K_0 > \eta$, then the stable length between z_2^i and y_2 is greater than η, and thus their distance in L_0 is greater than δ).

So suppose that there exists n such that, $D' \not\subset g^i(D')$ for all $1 \leq i \leq n$, and all the sets $g^1(D' \smallsetminus V_{C_n}), \dots, g^n(D' \smallsetminus V_{C_n})$ intersects $D' \smallsetminus V_{C_n}$, then we obtain n points $z_2^1, \dots, z_2^n$ on $[y_1, y_2]_{L_0}$, so that $z_1^n, \dots, z_2^n, y_1, y_2$ are pairwise at least δ apart from each other. But the diameter of $[y_1, y_2]^s$ is at most d, so there is a maximum of $d/\delta - 1$ such points. Hence $n \leq d/\delta - 1$, which proves the claim. ☐

Our next goal is going to be to eliminate possibility (i) in Claim 6.4, at least for the topological disks with large diameters.

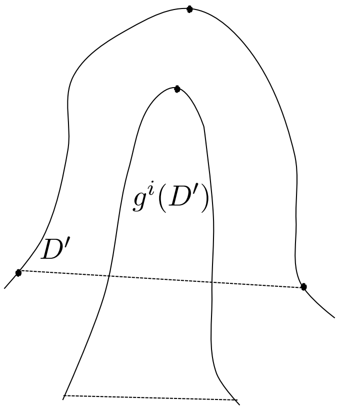


FIGURE 5. What happens when neither (i) nor (ii) is verified for a given i.

Claim 6.5. *Let $D' = D_{y_1,y_2}$ for $y_1, y_2 \in s(x_0)$. Suppose that there exists a positive integer i such that $D' \subset g^i(D')$. If there exists $u \in [y_1, y_2]^s$ such that*

$$d(u, [y_1, y_2]_{L_0}) \geq 10b + 3iK_0 \left(\frac{10b}{\delta} + 1 \right),$$

then there exists a ball of radius $2b$ that does not contain any fixed point of $\widetilde{f}$.

Proof. Since $D' \subset g^i(D')$, where $g = \widetilde{f}^{-1}$, the set $S = \cup_{n\in\mathbb{N}} g^{in}(D') \smallsetminus D'$ does not contain any fixed points. We will prove that S contains a ball of radius $2b$.

Let n be an integer such that $10b/\delta \leq n \leq 10b/\delta + 1$. Consider the subset S_0 of S defined by

$$S_0 = \bigcup_{k=1}^{2n} g^{ik}(D') \smallsetminus D'.$$

Let c be a path starting at $g^{ni}(u)$. In order for c to escape S_0, either c must intersect $g^{ki}([y_1, y_2]_{L_0})$ for some $0 \leq k \leq 2n$, or c must intersects $g^{ki}([y_1, y_2]^s)$ for all $0 \leq k \leq n-1$ or all $n+1 \leq k \leq 2n$.

If c intersects $g^{ki}([y_1, y_2]_{L_0})$, then its length is bounded below by

$$\begin{aligned} d_{L_0}\left(g^{ni}(u), g^{ki}([y_1, y_2]_{L_0})\right) &\geq d_{L_0}(u, [y_1, y_2]_{L_0}) - (n+k)iK_0 \\ &\geq d_{L_0}(u, [y_1, y_2]_{L_0}) - 3iK_0 \left(\frac{10b}{\delta} + 1 \right) \geq 10b. \end{aligned}$$

On the other hand, since the stable segments $g^{ki}([y_1, y_2]^s)$, $0 \leq k \leq n$ must be at least δ apart, if c intersects $g^{ki}([y_1, y_2]^s)$ for all $0 \leq k \leq n-1$ or all $n+1 \leq k \leq 2n$, then the length of c is bounded below by $n\delta \geq 10b$.

So in either case, the length of c is greater than $10b$. Thus the ball of radius $2b$ centered at $g^{ni}(u)$ is contained in S_0, which does not contain any fixed points of $\widetilde{f}$. □

As a consequence, we obtain

Claim 6.6. *Let $D' = D_{y_1,y_2}$ with $y_1, y_2 \in s(x_0)$. Let d be the length of $[y_1, y_2]_{L_0}$. Suppose that there exists $u \in [y_1, y_2]^s \cap \partial D'$ such that*

$$d(u, [y_1, y_2]_{L_0}) \geq 10b + 3K_0 \frac{d}{\delta} \left(\frac{10b}{\delta} + 1 \right).$$

Then there exists i, with $i \leq d/\delta$, such that $g^i(D' \smallsetminus V_C) \cap (D' \smallsetminus V_C) = \emptyset$, where V_C is the tubular neighborhood of the geodesic segment $[y_1, y_2]_{L_0}$ of radius $C = K_0 d/\delta$ and $g = \widetilde{f}^{-1}$.

In particular, $D' \smallsetminus V_C$ contains no fixed points of $\widetilde{f}$.

Proof. Since the conclusion of Claim 6.5 is in contradiction with Claim 6.3, it implies that only possibility (ii) in Claim 6.4 can arise for disks that have a large enough diameter. Our claim is just a reformulation of this. □

Now that we proved Claim 6.6, we can finish our proof of Lemma 6.2.

Since g expands exponentially the stable lengths, we can pick $z \in s(x_0)$ such that the length of $[z, g(z)]^s$ is arbitrarily large as needed. In particular the set $L_0 \smallsetminus ([z, g(z)]^s \cup [z, g(z)]_{L_0})$ contains at least one bounded connected component of arbitrarily large diameter. This is because the geodesic segment $[z, g(z)]_{L_0}$ has length bounded by K_0, whereas the length of $[z, g(z)]^s$, and therefore the volume of its $\delta/2$-tubular neighborhood, are arbitrarily large.

Hence, picking z far enough in $s(x_0)$, we can assure that there exists $y_1, y_2 \in s(x_0)$ such that $[y_1, y_2]^s \subset [z, g(z)]^s$, $[y_1, y_2]_{L_0} \subset [z, g(z)]_{L_0}$, and such that there is a topological disk $D = D_{y_1,y_2}$ bounded by $[y_1, y_2]^s$ and $[y_1, y_2]_{L_0}$ that satisfies to the assumptions of Claim 6.6. We fix such a $z \in s(x_0)$ and a corresponding D.

Let i_0 be the positive integer given by Claim 6.6 applied to D. Notice that the length of $[y_1, y_2]_{L_0}$ is less than K_0, so $i_0 \leq K_0/\delta$.

Let w be a point in $[y_1, y_2]^s$ that is farthest from z. Consider the closed domain R bounded by the geodesics $[w, g^{i_0}(w)]_{L_0}$ and $[y_2, g^{i_0}(y_1)]_{L_0}$, and the stable segments $[w, y_2]^s$ and $[g^{i_0}(y_1), g^{i_0}(w)]^s$ (see Figure 6). To be precise, R is obtained as the closure of the union of *all* the bounded connected components of L_0 minus the four curves.

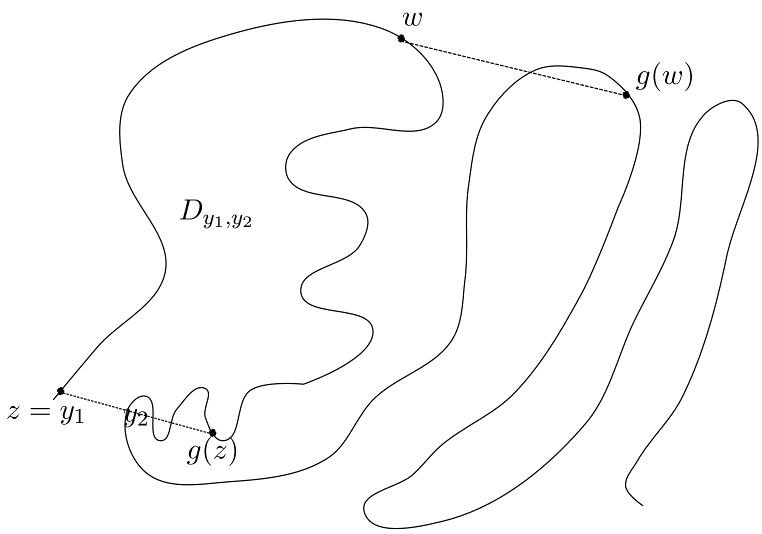


FIGURE 6. A depiction of case (ii) in Claim 6.4.

Notice that the distance between $[w, g^{i_0}(w)]_{L_0}$ and $[y_2, g^{i_0}(y_1)]_{L_0}$ is as large as we want, because g^{i_0} moves points a uniform bounded distance away (at most $i_0 K_0$, so at most K_0^2/δ), whereas the distance between w and $[z, g(z)]_{L_0}$ is as large as we want.

Now, if necessary, we modify our choice of the original $z \in s(x_0)$ so that the diameter of D is even larger in order to have a point $x \in R$ such that

$$\min\left\{d\left(x,[w,g^{i_0}(w)]_{L_0}\right), d\left(x,[y_1,g^{i_0}(y_2)]_{L_0}\right)\right\} \geq 10b + C + \left(1+\frac{4b}{\delta}\right)\frac{K_0^2}{\delta}.$$

Let $R_C := R \smallsetminus V_C$, where V_C is the union of the tubular neighborhoods of $[w,g^{i_0}(w)]_{L_0}$ and $[y_1,g^{i_0}(y_2)]_{L_0}$ of radius $C = 10b + 3\frac{K_0^2}{\delta}\left(\frac{10b}{\delta}+1\right)$.

By construction, R can be covered by topological disks that are bounded by parts of the stable leaf $s(x_0)$ and parts of either $[w,g^{i_0}(w)]_{L_0}$ or $[y_1,g^{i_0}(y_2)]_{L_0}$. Moreover, the distance between $[w,g^{i_0}(w)]_{L_0}$ and $[y_1,g^{i_0}(y_2)]_{L_0}$ can be made arbitrarily large by choosing z further in $s(x_0)$ if necessary. Hence, R_C is not empty and, since C is chosen big enough, any such topological disk that intersect R_C will automatically satisfy the hypothesis of Claim 6.6.

Hence, $\widetilde{f}$ admits no fixed points in R_C. Similarly, writing D_C for the disk D minus the C-tubular neighborhood of $[y_1,y_2]_{L_0}$, we know that $\widetilde{f}$ admits no fixed points in D_C.

Now we consider W_C to be the union $R_C \cup D_C$ minus the C-tubular neighborhood of $[w,g^{i_0}(w)]_{L_0}$. The set W_C does not contain any fixed points of $\widetilde{f}$ either. Hence, the set $S = \cup_{n\in\mathbb{Z}} g^{ni_0}(W_C)$ is also fixed-point free.

Moreover, the boundary of the set $D_C \cap W_C$ contains two disjoint sides made of subsegments of the stable segment $[y_1,y_2]^s$ (see Figure 6), and the distance between these two sides must be greater than δ (because the two sides are far enough apart in the stable leaf $s(x_0)$). Furthermore, since g increases the stable length, for any $n \geq 0$, the distance in L_0 between the two stable sides of $g^{ni_0}(D_C \cap W_c)$ must also be greater than δ (having two distinct and far enough apart stable side is the reason we introduced W_C instead of just considering $R_C \cup D_C$).

The proof of Lemma 6.2 then follows from the next claim, which directly contradicts Claim 6.3.

Claim 6.7. *There is a ball of radius $2b$ in the set $S = \cup_{n\in\mathbb{Z}} g^{ni_0}(W_C)$.*

Proof. Let n_0 be such that $2b/\delta - 1 < n_0 \leq 2b/\delta$. We will build a ball of radius $2b$ inside the subset S_0 of S defined by

$$S_0 = \cup_{k=0}^{2n_0+1} g^{ki_0}(W_C).$$

Let x be a point in R such that

$$\min\left\{d\left(x,[w,g^{i_0}(w)]_{L_0}\right), d\left(x,[y_1,g^{i_0}(y_2)]_{L_0}\right)\right\} \geq 10b + C + \left(1+\frac{4b}{\delta}\right)\frac{K_0^2}{\delta}.$$

Then $x \in R_C$, so $g^{n_0}(x) \in S_0$. We will show that the ball of radius $2b$ around $g^{n_0 i_0}(x)$ is in S_0.

Let c be a geodesic ray starting at $g^{n_0}(x)$. In order for c to exit S_0, it needs to intersect a boundary component of S_0. Now, by construction, the boundary of S_0 is composed of a stable segment I_1^s in ∂D_C, a stable segment I_2^s in $\partial g^{(2n_0+1)i_0}(R_C)$ (in fact $I_2^s = g^{(2n_0+2)i_0}(I_1^s)$ but we do not need that), and the images by powers of g^{i_0} of two curves γ_1 and γ_2, which are curves at distance C from, respectively, $[y_1,y_2]_{L_0} \cup [y_2,g^{i_0}(y_1)]_{L_0}$ and $[w,g^{i_0}(w)]_{L_0}$.

In the rest of the argument, the difference between γ_1 and γ_2 is irrelevant, so we will just write γ to refer to either of them.

Thus, for c to exit S, it needs to either intersect I_1^s, I_2^s or $g^{ni_0}(\gamma)$ for some $0 \leq n \leq 2n_0+1$.

Suppose first that c exits through I_1^s. Then it needs to have crossed the domains $W_C \cap D_C, g^{i_0}(W_C \cap D_C), \ldots, g^{n_0 i_0}(W_C \cap D_C)$. Here by cross we mean intersecting the two stable sides. Now, as we noticed earlier the distance between the two stable sides of $g^{ki_0}(W_C \cap D_C)$ is greater than δ for any $k \geq 0$. Thus, if c exits through I_1^s, its length needs to be at least $(n_0+1)\delta$, which is strictly greater than $2b$ by our choice of n_0.

Similarly, if c exits through I_2^s. Then it needs to have crossed the domains $g^{(n_0+1)i_0}(W_C \cap D_C), \dots, g^{(2n_0+1)i_0}(W_C \cap D_C)$, in which case, again, the length of c is greater than $(n_0+1)\delta > 2b$.

Finally, suppose that c exits through a $g^{ki_0}(\gamma)$ for some $0 \leq k \leq 2n_0+1$. Then, in order to prove our claim, all we have to do is to show that the distance between $g^{n_0 i_0}(x)$ and $g^{ki_0}(\gamma)$ is larger than $2b$ for all $0 \leq k \leq 1 + 4b/\delta$.

Our condition on x implies that

$$d(x,\gamma) \geq 10b + C + \left(1 + \frac{4b}{\delta}\right)\frac{K_0^2}{\delta} - C = 10b + \left(1 + \frac{4b}{\delta}\right)\frac{K_0^2}{\delta}.$$

Hence, if $0 \leq k \leq 1 + 4b/\delta$, then we have

$$\begin{aligned} d(x, g^{ki_0}(\gamma)) &\geq d(x,\gamma) - ki_0 K_0 \\ &\geq d(x,\gamma) - \left(1 + \frac{4b}{\delta}\right)\frac{K_0^2}{\delta} \\ &\geq 10b. \end{aligned}$$

Therefore, the ball of radius $2b$ centered at $g^{n_0 i_0}(x)$ is entirely in S, proving Claim 6.7. □

This ends the proof of Lemma 6.2. □

An important consequence of Lemma 6.2 is the following:

Corollary 6.8. *Suppose that f is a partially hyperbolic diffeomorphism in M that is homotopic to the identity. Let $\widetilde{f}$ be a good lift of f to $\widetilde{M}$. Suppose that Λ is a non empty (saturated) f-minimal subset of $\mathcal{W}^{cs}_{\text{bran}}$ such that every leaf of the lift $\widetilde{\Lambda}$ to $\widetilde{M}$ is fixed by $\widetilde{f}$. Then every leaf in the f-minimal set Λ of $\mathcal{W}^{cs}_{\text{bran}}$, is either a plane or an annulus.*

Proof. Let A be a leaf of Λ and L a lift in $\widetilde{M}$. By Lemma 6.2, L does not admit any fixed points of $\widetilde{f}$. Hence, $\widetilde{f}$ acts freely on the space of stable leaves in L.

Now, recall that $\pi_1(A)$ can be defined as the elements $\gamma \in \pi_1(M)$ that fix L (see section 4.4). So if $\gamma \in \pi_1(A)$, it must also act freely on the space of stable leaves in L. As $\widetilde{f}$ commutes with every deck transformation, Corollary E.4 of [BFFP20b] (which still applies in the context of branching foliation, as does all of [BFFP20b, Appendix E]) implies that $\pi(A)$ is abelian, i.e., A is either a plane or an annulus (again with the understanding that A might actually only be an immersion of one of these manifolds in M and recalling that all bundles were assumed to be orientable in this section, so in particular the leaves cannot be Möbius bands). □

We are now ready to prove Proposition 6.1.

Proof of Proposition 6.1. This proof follows the same structure as the one of [BFFP20b, Proposition 3.15] and we will continuously refer to it. Recall the standing assumption that all bundles are orientable and their orientation is preserved by f.

Consider Λ an f-minimal non empty subset. We need to show that $\Lambda = M$. We assume by contradiction that $\Lambda \neq M$.

Since $\mathcal{W}^{cs}_{\text{bran}}$ has no closed leaves and Λ is f-minimal, there cannot be any isolated leaves in Λ (for the topology of the stable leaf space).

Now, Lemma 6.2 allows us to assert that $\widetilde{f}$ has no fixed points in leaves of $\widetilde{\Lambda}$. Then, Corollary 6.8 implies that each leaf of Λ is either a plane or an annulus.

We fix an ϵ small enough and let Λ' be the pull back of Λ to the approximating foliation $\mathcal{W}^{cs}_{\epsilon}$. That is, $\Lambda' = (h^{cs}_{\epsilon})^{-1}(\Lambda)$. Let V be a connected component of $\widetilde{M} \setminus \widetilde{\Lambda}'$.

[BFFP20b, Claim 3.16] applies to V, since it is just a general fact about codimension one foliations. So the projection $\pi(V)$ of V to M has only finitely many boundary leaves.

Now, we need to prove the analogous to [BFFP20b, Claim 3.18]:

Claim 6.9. *Let $L \in \partial V$. Then $\pi(L)$ is an annulus.*

The proof of that claim is slightly different from the dynamically coherent case, as we now need to use both the foliation $\mathcal{W}^{cs}_{\epsilon}$ and the branching foliation $\mathcal{W}^{cs}_{\text{bran}}$.

Proof. Suppose that $\pi(L)$ is a plane. Recall (see [CC00]) that $\pi(V)$ has an octopus decomposition and a compact core. So for any $\delta > 0$, the subset of points in $\pi(L)$ that are at distance greater then δ from another boundary component of $\pi(V)$ is precompact. Since $\pi(L)$ is supposed to be a plane, that subset must be contained in a closed disk D. Then $\pi(L) \smallsetminus D$ is an annulus that is δ-close to another boundary component, $\pi(L')$ of $\pi(V)$. Moreover, the subset of $\pi(L')$ that is δ-close to $\pi(L) \smallsetminus D$ then also has to be an annulus. If $\pi_1(L')$ were not a plane it would be an annulus and its non-trivial curve corresponds to a curve homotopic to the boundary of the closed disk D which is homotopically trivial in M. Since the leaves of $\mathcal{W}^{cs}_{\epsilon}$ are π_1-injective, this implies that $\pi(L')$ is also a plane.

Since M is irreducible this implies that $\pi(V)$ is homeomorphic to an open disk times an interval. So $\pi(V)$ has only two boundary components, both of which are planes. In particular, the isotropy group of V is trivial and $\pi(V)$ is homeomorphic to V.

We will now switch to the branching foliation to finish the proof. Let $A = h^{cs}_{\epsilon}(\pi(L))$ and $B = h^{cs}_{\epsilon}(\pi(L'))$. Since we chose ϵ small enough, up to taking δ small enough also, the unstable segments through $A \smallsetminus h^{cs}_{\epsilon}(D)$ intersect B, and their length is uniformly bounded. Moreover, no unstable ray of A can stay in $h^{cs}_{\epsilon}(\pi(V))$. This is because $\pi(V)$ is homeomorphic to an open disk times an interval. So, since D is compact, the length of every unstable segment between A and B is bounded by a uniform constant. Notice that, since $\mathcal{W}^{cs}_{\text{bran}}$ is a branching foliation, we may have $A \cap B \neq \emptyset$, i.e., some of these unstable segments may be points.

Since L and L' are in ∂V, which is a connected component of $\widetilde{M} \smallsetminus \widetilde{\Lambda}'$, we have that $A, B \in \partial (M \smallsetminus \Lambda)$. So in particular, A and B are fixed by f. Hence, the set of unstable segments between A and B is also invariant by f. Since the length of unstable segments between A and B are bounded above and f expands the unstable length, all the unstable segments must have zero length. i.e., $A = B$. Which implies that V is empty, which contradicts the assumption that $\Lambda \neq M$. ☐

Thus we showed that every component of $\pi(\partial V)$ is an annulus. We can then apply without change the (topological) arguments of the proof of [BFFP20b, Proposition 3.15] to obtain a torus T, composed of annuli along leaves of $\mathcal{W}^{cs}_{\epsilon}$, together with annuli transverse to $\mathcal{W}^{cs}_{\epsilon}$, that bounds a solid torus U' in $\pi(V)$.

Now consider $U = h^{cs}_{\epsilon}(U')$. Because of the collapsing of leaves, U may not be a solid torus. If U is empty for any any such component U', this would directly contradict the assumption $\Lambda \neq M$. So for some such complementary component U', the set U is not empty and it is contained in a solid torus (the ϵ-tubular neighborhood of U' in M). We can then use the same "volume vs. length" argument on U, exactly as in the end of the proof of [BFFP20b, Proposition 3.15], to get a final contradiction. This ends the proof of Proposition 6.1. $\square$

As a consequence, we get the following result that completes the proof of Theorem 1.6 as announced.

Corollary 6.10. *Suppose that f is a partially hyperbolic diffeomorphism homotopic to the identity. Suppose that f is either volume preserving or transitive, or that M is either hyperbolic or Seifert. Let $\widetilde{f}$ be a good lift of f. Then $\widetilde{f}$ has no periodic points. In particular, f has no contractible periodic points.*

Proof. Up to finite covers and iterates, we may assume that f preserves the branching foliations $\mathcal{W}^{cs}_{\text{bran}}, \mathcal{W}^{cu}_{\text{bran}}$.

If $\widetilde{f}$ acts as a translation on either $\mathcal{W}^{cs}_{\text{bran}}$ or $\mathcal{W}^{cu}_{\text{bran}}$, then it does not have periodic points.

Otherwise, since we showed that under our assumptions the branching foliations are f-minimal. The result then follows from Theorem 4.9. $\square$

7. Double invariance implies dynamical coherence

In this section we show that if the center-stable and center-unstable branching foliations are minimal and leafwise fixed by a good lift $\widetilde{f} \colon \widetilde{M} \to \widetilde{M}$, then, f has to be dynamically coherent (i.e., the branching foliations do not branch). Therefore, we will be able to apply the results from the dynamically coherent setting.

The universal cover $\widetilde{M}$ of M is homeomorphic to $\mathbb{R}^3$ (since it admits a partially hyperbolic diffeomorphism, see Appendix B of [BFFP20b]). We do not assume anything further on M in this section.

Recall also that a center leaf is a connected component of the intersection of a leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and one of $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$ (cf. Definition 3.6).

This section (and the proof of dynamical coherence) is split in three parts. First, in subsection 7.1, we show that, for an appropriate lift of M and power of f, double invariance of the foliations implies that the center leaves are fixed. The lift and power we need to consider here is in order to have everything orientable and coorientable. Then, in section 7.2, we show that if a good lift fixes every center leaf, then it must be dynamically coherent. Finally, in section 7.3, we show that if a lift and power of a partially hyperbolic diffeomorphism is dynamically coherent and fixes the center leaves, then the original diffeomorphism is itself dynamically coherent (and a good lift of a power of it will fix every center leaf).

7.1. Center leaves are all fixed.

In this section we recover the results of [BFFP20b, §6] in the context of branching foliations. This will be the key to obtaining dynamical coherence (in section 7.2).

Proposition 7.1. *Let $f \colon M \to M$ be a partially hyperbolic diffeomorphism homotopic to the identity and admitting branching foliations $\mathcal{W}^{cs}_{\text{bran}}, \mathcal{W}^{cu}_{\text{bran}}$ that are f-minimal. Suppose that a good lift $\widetilde{f}$ of f to $\widetilde{M}$ fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}, \widetilde{\mathcal{W}}^{cu}_{\text{bran}}$. Then, every center leaf is fixed by $\widetilde{f}$.*

We stress again that the assumption of f-minimality is automatic when f is transitive or when M is hyperbolic or Seifert, see section 6)

To prove Proposition 7.1, as in the dynamically coherent setting, we need the following result.

Lemma 7.2. *Suppose that the hypothesis of Proposition 7.1 are satisfied. Then either every center leaf is fixed by $\widetilde{f}$ or no center leaf is fixed by $\widetilde{f}$.*

Assuming this lemma, it is easy to prove Proposition 7.1:

Proof of Proposition 7.1. Suppose that $\widetilde{f}$ fixes no center leaf. By Proposition 5.8 and the fact that there must be some non-planar leaf there are periodic center leaves in M. Then we can apply Proposition 5.2 first to $\widetilde{\mathcal{W}}^{cs}_{bran}$ and then $\widetilde{\mathcal{W}}^{cu}_{bran}$. The conclusion is that for every f periodic center leaf M, the center leaf must be first coarsely contracting by f and then coarsely expanding by f. This is a contradiction. Hence $\widetilde{f}$ fixes a center leaf and Lemma 7.2 implies the proposition. □

To prove the lemma we will explain the modifications one has to make in the proof of [BFFP20b, Lemma 6.4] to adapt it to the non dynamically coherent setting.

Proof of Lemma 7.2. Let

$$Fix^c_{\widetilde{f}} := \{c \ : \ \widetilde{f}(c) = c\}.$$

The first difference from the dynamically coherent setting is that we will not directly regard this set as a subset of $\widetilde{M}$ (because center leaves may merge).

However, it is not hard to see that the argument of Lemma 6.3 of [BFFP20b] holds: If c is a fixed center leaf in a center stable leaf L in $\widetilde{M}$, then for any center leaf c' in L close enough to c (for the topology of the center leaf space in L), there exists a strong stable leaf that intersect c, c' and $\widetilde{f}(c')$. Now, since $\widetilde{f}$ fixes the center unstable leaves, c' and $\widetilde{f}(c')$ are on the same center unstable leaf. Since no transversal can intersect the same leaf twice, it implies that $c' = \widetilde{f}(c')$.

Thus, we obtained that if c is a fixed center leaf in a center stable leaf L in $\widetilde{M}$, center leaves near c in L are also fixed. This is in the center leaf space of L, which is a 1-dimensional manifold.

The same argument evidently applies for center leaves near c in its center unstable leaf.

Note that since a good lift $\widetilde{f}$ fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$, then f fixes every leaf of $\mathcal{W}^{cs}_{\mathrm{bran}}$. In particular f-minimality of $\mathcal{W}^{cs}_{\mathrm{bran}}$ is equivalent to minimality of $\mathcal{W}^{cs}_{\mathrm{bran}}$. Hence $\mathcal{W}^{cs}_{\mathrm{bran}}$ is minimal. Similarly for $\mathcal{W}^{cu}_{\mathrm{bran}}$.[8]

We now assume that the set of fixed center leaves is non-empty and we want to show that all the center leaves are fixed.

To do this, we proceed as in [BFFP20b, Lemma 6.4]: We show first that every center leaf in a center stable leaf (resp. center unstable leaf) which projects to an annulus has to be fixed (due to our orientability assumptions, leaves cannot project to a Möbius band). Then the same argument as in [BFFP20b, Lemma 6.4] applies to show that every center leaf has to be fixed.

Let L be any center stable leaf that projects to an annulus. Let γ be a generator of the isotropy group of L.

[8]Note that f-minimality and minimality are in fact always equivalent as long as the branching foliation does not have compact leaf and without assumptions on f, see Lemma B.2.

Since the set of fixed center leaves is open in the center leaf spaces of any center unstable leaf, minimality of $\mathcal{W}^{cs}_{\mathrm{bran}}$ implies that L must have some fixed center leaves.

We will first prove that, if f does not fix all center leaves in L, then some center leaves in $\pi(L)$ are periodic under f. Then we will show, as in Proposition 5.5, that any periodic leaf in $\pi(L)$ must be coarsely contracting. The same argument applied to the center-unstable leaves yields that periodic center leaves must also be coarsely expanding, a contradiction.

Since $\widetilde{f}$ cannot have fixed points (as $\widetilde{f}$ fixes all the leaves of $\widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$ and $\widetilde{\mathcal{W}}^{cu}_{\mathrm{bran}}$), then $\widetilde{f}$ acts freely on the space of stable leaves in L.

We assume, for a contradiction, that not all center leaves in L are fixed. Let Fix_L be the set (in, $\mathcal{L}^c_L$, the center leaf space on L) of center leaves fixed by γ.

The set Fix_L is open, and assumed not to be the whole of L. So let c_1 be any leaf in $\partial \mathrm{Fix}_L$.

The leaf c_1 is not fixed by $\widetilde{f}$, so $\widetilde{f}(c_1)$ is non-separated from c_1. Hence, there exists a (unique) stable leaf s_1, which separates $\widetilde{f}(c_1)$ from c_1 and makes a perfect fit with c_1 (see section 4.6 for the definition of perfect fits in the non dynamically coherent setting). Then $\widetilde{f}(s_1)$ makes a perfect fit with $\widetilde{f}(c_1)$. Because c_1 and $\widetilde{f}(c_1)$ are non separated from each other, s_1 and $\widetilde{f}(s_1)$ intersect a common transversal to the stable foliation. It follows that the stable axis of $\widetilde{f}$ acting on L is a line. Thus, since γ commutes with $\widetilde{f}$, the stable axis of γ is that same line. Moreover, both the stable leaves s_1 and $\widetilde{f}(s_1)$ are in the axis of $\widetilde{f}$.

Since the stable axis of $\widetilde{f}$ acting on L is a line, the Graph Transform argument [BFFP20b, Appendix H] applies and we obtain a curve $\hat{\eta}$, tangent to the center direction, that is fixed by both γ and $\widetilde{f}$.

As s_1 makes a perfect fit with c_1 and s_1 intersects $\hat{\eta}$, we deduce that there exists a stable leaf s that intersects both c_1 and $\hat{\eta}$. Let $x = s \cap \hat{\eta}$ and $y = s \cap c_1$. We denote by J the segment of s between x and y.

Since $\hat{\eta}$ projects down to a closed curve $\pi(\hat{\eta})$, and $\widetilde{f}$ decreases stable lengths, there exist $n_1, n_2 \in \mathbb{Z}$ and $m_1, m_2 \in \mathbb{N}$ as large as we want such that the four points $\gamma^{n_1}\widetilde{f}^{m_1}(x)$, $\gamma^{n_1}\widetilde{f}^{m_1}(y)$, $\gamma^{n_2}\widetilde{f}^{m_2}(x)$ and $\gamma^{n_2}\widetilde{f}^{m_2}(y)$ are all in a disk of radius as small as we want.

Suppose now that $\gamma^{n_1}\widetilde{f}^{m_1}(c_1) \neq \gamma^{n_2}\widetilde{f}^{m_2}(c_1)$. Then, up to switching n_1, m_1 and n_2, m_2, we obtain that $\gamma^{n_2}\widetilde{f}^{m_2}(c_1)$ intersects $\gamma^{n_1}\widetilde{f}^{m_1}(J)$. This is in contradiction with the fact that c_1 is in $\partial \mathrm{Fix}_L$ which is invariant by both $\widetilde{f}$ and γ.

Thus $\gamma^{n_1}\widetilde{f}^{m_1}(c_1) = \gamma^{n_2}\widetilde{f}^{m_2}(c_1)$. In other words, c_1 is fixed by the map $h = \gamma^n \widetilde{f}^m$ for some n, m integers, $m > 0$. (Although not useful for the rest of the proof, one can further notice that $\hat{\eta}$ and c_1 intersect, as h decreases the length of J by forward iterations and both c_1 and $\hat{\eta}$ are fixed by h.)

Now recall that we built above a stable leaf s_1 making a perfect fit with c_1. And, by our choice of s_1, the center leaf c_1 is in between s_1 and $s_2 := \widetilde{f}^{-1}(s_1)$.

The leaves s_1 and s_2 are both fixed by h (since c_1 is), and a bounded distance apart, so Lemma 5.6 holds and we deduce that c_1, as well as any other center leaf c that is in between s_1 and s_2 must be coarsely contracting. Note now that any center leaf c in L that is fixed by some $h' = \gamma^{n'}\widetilde{f}^{m'}$ is separated from Fix_L by a center leaf $c_1' \subset \partial \mathrm{Fix}_L$ as above. Hence, we proved that every non-fixed periodic leaf in L is coarsely contracting.

Therefore, the same argument applied to the center *unstable* leaf containing c_1 shows that c_1 must also be coarsely expanding, a contradiction.

So we obtained that every center stable or center unstable leaf L which is fixed by some non trivial element of $\pi_1(M)$ has all of its center leaves fixed by $\widetilde{f}$. Since $\mathrm{Fix}^c_{\widetilde{f}}$ is open (in the center leaf space), minimality of the foliations implies that it contains every center leaf, as in the end of the proof of [BFFP20b, Lemma 6.4]. ☐

7.2. **Dynamical coherence.** We now want to prove dynamical coherence provided that a good lift fixes every center leaf. We start with the following:

Lemma 7.3. *Suppose that $\widetilde{f}$ fixes every leaf of the center foliation in $\widetilde{M}$. Then there is a global bound on the length from x to $\widetilde{f}(x)$ in any center leaf containing x.*

In the dynamically coherent case this was very easy as the center curves form an actual foliation and there is a local product picture near any compact segment. We have to be more careful in the non dynamically coherent setting.

Proof. We assume the conclusion of the lemma fails. Then there exists a sequence x_i of points in $\widetilde{M}$ contained in center leaves c_i such that the length in c_i from x_i to $\widetilde{f}(x_i)$ divverges to infinity. Notice that this length depends not only on x_i but also on c_i since there may be many center leaves through x_i. We denote by e_i the segment in c_i from x_i to $\widetilde{f}(x_i)$.

Up to acting by covering translations we can assume that the x_i converge to a point $x \in \widetilde{M}$. Let L_i and U_i be respectively a center stable and center unstable leaves containing c_i. Up to considering a subsequence, we may assume that L_i converges to a center stable leaf L containing x (see condition (iv) of Definition 3.2). Similarly, we can further assume that U_i converges to some center unstable leaf U, with $x \in U$.

For i large enough, all the leaves L_i intersect a small unstable segment in $u(x)$. The set of center stable leaves intersecting this segment is a also a segment (even though many different leaves may intersect a given point in $u(x)$). Hence we may assume that L_i is weakly monotone, and so is U_i. Let c be the center leaf through x contained in $L \cap U$. Then $\widetilde{f}(x) \in c$, and we call e the segment in c from x to $\widetilde{f}(x)$.

Suppose first that $L_i = L$ for all big i. So we may assume $L_i = L$ for all i. Then the center leaves c_i are all in L and, for i big enough, intersect $s(x)$. Hence the leaves c_i are, for i big enough, contained in an interval of the center leaf space in L. In addition they are converging to c which is a center leaf through x and $\widetilde{f}(x)$. This implies that the length of e_i is converging to the length of e and hence the length of e_i is bounded in i. Contradiction.

Suppose now that the L_i are all distinct from L. Notice that the points x_i, and $\widetilde{f}(x_i)$ are all in a compact region of $\widetilde{M}$. Since L_i converges to L, we have that $u(x_i)$ intersects L for big enough i. We call this nearby intersection y_i. Likewise $u(\widetilde{f}(x_i))$ intersects L in $\widetilde{f}(y_i)$. We want to push the center segments c_i contained in $U_i \cap L_i$ along unstable segments to center segments in $U_i \cap L$.

For i big enough, both x_i and $\widetilde{f}(x_i)$ are very near L. Thus, their unstable leaves $u(x_i)$ and $u(\widetilde{f}(x_i))$ both intersect L. Let y_i be the intersection of $u(x_i)$ with L (recall that this intersection is unique as the center stable branching foliation is approximated by a taut foliation). Then $\widetilde{f}(y_i)$ is the intersection of $u(\widetilde{f}(x_i))$ with L (since L is fixed by $\widetilde{f}$). Then the intersection of the unstable saturation of e_i with L is a compact segment inside a center leaf between y_i and $\widetilde{f}(y_i)$ (since $\widetilde{f}$ fixes every center leaf). Let b_i be this segment between y_i and $\widetilde{f}(y_i)$. The segments b_i also converge to e, so the previous paragraph shows that the lengths of the b_i are bounded. Since the distance between x_i and y_i converges to zero, this in turn implies that the lengths of the segments e_i are themselves bounded. Which contradicts our assumption and finishes the proof. $\square$

Lemma 7.4. *Suppose $\widetilde{f}$ fixes every leaf of the center foliation in $\widetilde{M}$. Assume c_1, c_2 are different center leaves in the same leaf L of $\widetilde{\mathcal{W}}^{cs}_{\rm bran}$. Then, $c_1 \cap c_2 = \emptyset$.*

Proof. Suppose not, there is $x \in c_1 \cap c_2$ but $c_1 \neq c_2$. Then $\widetilde{f}(x)$ is also in $c_1 \cap c_2$. If c_1 coincides with c_2 in their respective segments from x to $\widetilde{f}(x)$, then applying iterates of $\widetilde{f}$ implies that $c_1 = c_2$, contrary to assumption.

So we may assume that x is a boundary point of an open interval I in, say, c_1 which is disjoint from c_2, but such that both endpoints are in c_2. Then $c_1 \cup c_2$ bounds a bigon B with endpoints x, y and a "side" in I. All center segments in B pass through x and y and they have bounded length (by Lemma 7.3). Each stable segment intersecting I also intersects the other "boundary" component of B. See figure 7.

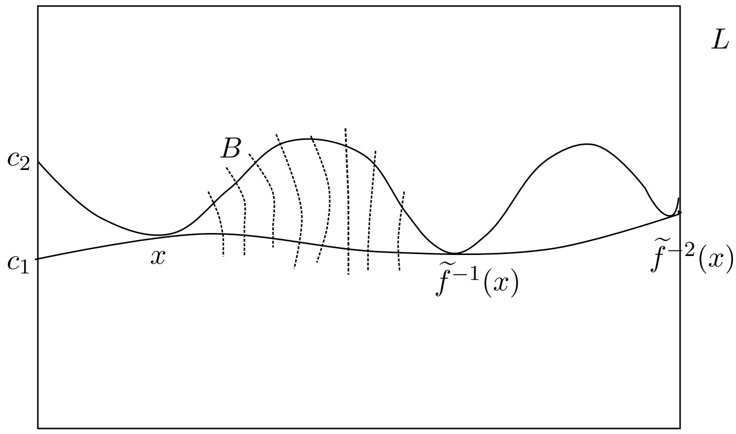


FIGURE 7. Two centers that merge. The bound on the distance between x and $\widetilde{f}(x)$ forces a behavior like the figure.

The stable lengths grow without bound under negative iterates of $\widetilde{f}$. Hence, since a stable segment can intersect a local foliated disk of the stable foliation in L only in a bounded length, it follows that the diameter in $\widetilde{f}^n(L)$ of $\widetilde{f}^n(B)$ grows without bound as n goes to $-\infty$. But the length of the center segments in $\widetilde{f}^n(B)$ are all bounded according to Lemma 7.3. Moreover, between any two points in $\widetilde{f}^n(B)$ there exists a path along (at most) two center leaves (one just follows the center leaf to one of the endpoint and then switch to the appropriate other center leaf). Thus the diameter is bounded, which is a contradiction. $\square$

Thus we deduce what we wanted to obtain in this section.

Corollary 7.5. *If a good lift $\tilde{f}$ fixes every center leaf, then, f is dynamically coherent.*

Proof. By Proposition B.3 it is enough to show that the leaves of the branching foliations do not merge.

Assume that two center unstable leaves U_1 and U_2 merge. Let L be a center stable leaf intersecting U_1 and U_2 at the merging, i.e., L is a leaf through a point x such that the unstable leaf through x is a boundary component of $U_1 \cap U_2$. Then, connected components of $U_1 \cap L$ and $U_2 \cap L$ gives two center leaves that intersect but do not coincide. This contradicts Lemma 7.4. A symmetric argument gives that two center stable leaf cannot merge either, proving dynamical coherence of f. □

7.3. **Dynamical coherence without taking lifts and iterates.** We now want to prove that, if a finite lift and finite power of a partially hyperbolic diffeomorphism is dynamically coherent, then the original diffeomorphism is itself dynamically coherent. Although we do not know how to prove it in this generality, we show it when a good lift of the dynamically coherent lift fixes every center leaf, which is enough for our purposes.

We start by showing a uniqueness result for the pairs of the center stable and center unstable foliations under some conditions.

Lemma 7.6. *Let $g\colon M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism homotopic to the identity. Let $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ be g-invariant foliations tangent to E^{cs} and E^{cu} respectively. Let $\mathcal{W}^c$ be the center foliation associated with $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ (defined as in Definition 3.6), and assume that there exists a good lift $\widetilde{g}$ which fixes all the leaves of $\widetilde{\mathcal{W}}^c$.*

Suppose that $\mathcal{W}^{cs}_1$ and $\mathcal{W}^{cu}_1$ are two g-invariant foliations tangent respectively to E^{cs} and E^{cu}. Suppose that $\widetilde{g}$ also fixes all the leaves of the center foliation $\widetilde{\mathcal{W}}^c_1$, associated with $\mathcal{W}^{cs}_1$ and $\mathcal{W}^{cu}_1$.

Then $\mathcal{W}^{cs} = \mathcal{W}^{cs}_1$ and $\mathcal{W}^{cu} = \mathcal{W}^{cu}_1$.

Note that if the foliations $\mathcal{W}^{cs}$, $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}_1$ and $\mathcal{W}^{cu}_1$ are assumed to be g-minimal, then Proposition 7.1 implies that the hypothesis of the lemma are satisfied.

Proof. The argument is similar to the one made in Lemma 7.4.

Let $\widetilde{\mathcal{W}}^{cs}_1, \widetilde{\mathcal{W}}^{cu}_1$ be two g-equivariant foliations as in the lemma. Recall that the center foliation $\widetilde{\mathcal{W}}^c_1$ is defined by taking the connected components of intersections of leaves of $\widetilde{\mathcal{W}}^{cs}_1$ and $\widetilde{\mathcal{W}}^{cu}_1$.

Since every leaf of both $\widetilde{\mathcal{W}}^c$ and $\widetilde{\mathcal{W}}^c_1$ are fixed by $\widetilde{g}$, Lemma 7.3 implies that $\widetilde{g}$ moves points a uniformly bounded amount in both center foliations.

Consider, for a contradiction, a point $x \in \widetilde{M}$ such that $\widetilde{\mathcal{W}}^c(x) \neq \widetilde{\mathcal{W}}^c_1(x)$ (note that we are dealing here with actual foliations, not branching ones, so this notation make sense). Without loss of generality, we can choose x so that the leaves $L := \widetilde{\mathcal{W}}^{cs}(x)$ and $L_1 := \widetilde{\mathcal{W}}^{cs}_1(x)$ do not coincide in any neighborhood of x.

Let c and c_1 be the center leaves obtained as the connected components of $L \cap F$ and $L_1 \cap F$ containing x.

By assumption, both c and c_1 are fixed by $\widetilde{g}$, so we are in the exact same set up as in the proof of Lemma 7.4. Thus we deduce that $c = c_1$, a contradiction. □

We can now state and prove the aim of this section.

Proposition 7.7. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism such that f^k is homotopic to the identity for some $k > 0$. Let $\hat{M}$ be a finite cover of M. Let g be a lift to $\hat{M}$ of a homotopy of f^k to the identity.*

Suppose that g is dynamically coherent and that there exists a good lift $\widetilde{g}$ of g that fixes all the center leaves. Then, f is dynamically coherent.

Proof. First we notice that the assumptions of the proposition will be verified for any further finite cover $\bar{M}$ of $\hat{M}$ (because one can take a further lift $\bar{g}$ of g to $\bar{M}$, it is dynamically coherent and $\widetilde{g}$ is a good lift of $\bar{g}$ too). Hence, without loss of generality, we may and do assume that $\hat{M}$ is a normal cover of M.

Let $\widetilde{\mathcal{W}^{cs}}$ and $\widetilde{\mathcal{W}^{cu}}$ be the lifts to $\widetilde{M}$ of the center stable and center unstable foliations of g. Our goal is to show that these foliations are $\pi_1(M)$-invariant, thus decending to foliations in M, and that these projected foliations are f-invariant.

Notice that $\widetilde{g}$ fixes each leaf of $\widetilde{\mathcal{W}^{cs}}$ and $\widetilde{\mathcal{W}^{cu}}$.

The map g is obtained from a lift of a homotopy of f^k to the identity. Lifting that homotopy further to $\widetilde{M}$, we get a good lift $\widetilde{f^k}$ of f^k that is also a lift (and hence a good lift) of g to $\widetilde{M}$. As both $\widetilde{g}$ and $\widetilde{f^k}$ are good lifts of g, there exists $\beta \in \pi_1(\hat{M}) \subset \pi_1(M)$ such that $\widetilde{g} = \beta \widetilde{f^k}$. (Note however that $\widetilde{g}$ is not necessarily a good lift of f^k as $\widetilde{g}$ only commutes with elements of $\pi_1(\hat{M})$ and not $\pi_1(M)$.)

Moreover, both $\widetilde{g}$ and $\widetilde{f^k}$ move points a bounded distance in $\widetilde{M}$, hence so does $\beta = \widetilde{g}(\widetilde{f^k})^{-1}$. Lemma A.1 then implies that either β is the identity or M is Seifert (and β is either the identity or a power of a regular fiber).

We split the rest of the proof in these two cases.

Case 1 – Suppose that M is *not* a Seifert fibered space.

Then β is the identity, which means that $\widetilde{g} = \widetilde{f^k}$.

Let γ be a deck transformation in $\pi_1(M)$. Define the foliations $\mathcal{F}^{cs}_\gamma := \gamma\widetilde{\mathcal{W}^{cs}}$, $\mathcal{F}^{cu}_\gamma := \gamma\widetilde{\mathcal{W}^{cu}}$, and $\mathcal{F}^{c}_\gamma := \gamma\widetilde{\mathcal{W}^{c}}$. The leaves of these foliations are all fixed by $\widetilde{g}$ because γ commutes with $\widetilde{f^k} = \widetilde{g}$. In particular, Lemma 7.6 then implies that $\gamma\widetilde{\mathcal{W}^{cs}} = \widetilde{\mathcal{W}^{cs}}$ and $\gamma\widetilde{\mathcal{W}^{cu}} = \widetilde{\mathcal{W}^{cu}}$. Since this is true for any element of $\pi_1(M)$, these foliations descend to foliations $\mathcal{W}^{cs}_M, \mathcal{W}^{cu}_M$ in M.

Now we need too show that $\mathcal{W}^{cs}_M, \mathcal{W}^{cu}_M$ are also f-invariant. Equivalently, we need to show that $\widetilde{\mathcal{W}^{cu}}$ and $\widetilde{\mathcal{W}^{cs}}$ are invariant by any lift f_1 of f to $\widetilde{M}$.

Let f_1 be a lift of f to $\widetilde{M}$. Notice that f may not be homotopic to the identity, so f_1 is not assumed to be a good lift. Let $\mathcal{F}^{cs}_1 := f_1(\widetilde{\mathcal{W}^{cs}})$ and $\mathcal{F}^{cu}_1 := f_1(\widetilde{\mathcal{W}^{cu}})$.

We will first show that f_1 and $\widetilde{g}$ commute. Both $f_1\widetilde{g}$ and $\widetilde{g}f_1$ are lifts of the map f^{k+1} to $\widetilde{M}$. So $(\widetilde{g})^{-1}(f_1)^{-1}\widetilde{g}f_1$ is a deck transformation $\gamma \in \pi_1(M)$. As $\widetilde{g}$ moves points a bounded distance, we have that $d(f_1(y), \widetilde{g}f_1(y))$ is bounded in $\widetilde{M}$. In addition, f_1 has bounded derivatives so $d(y, (f_1)^{-1}\widetilde{g}f_1(y))$ is also bounded in $\widetilde{M}$. So using again that $\widetilde{g}$ is a good lift, we deduce that $d(y, (\widetilde{g})^{-1}(f_1)^{-1}\widetilde{g}f_1(y))$ is bounded in $\widetilde{M}$.

Hence γ is a deck transformation that moves points a bounded distance. Applying Lemma A.1 again gives that β is the identity (since M is not Seifert). Hence f_1 and $\widetilde{g}$ commute.

Since $\widetilde{g}$ fixes every leaf of $\widetilde{\mathcal{W}^{c}}$ (the center foliation in $\widetilde{M}$) and commutes with f_1, we deduce that $\widetilde{g}$ fixes every leaf of $f_1(\widetilde{\mathcal{W}^{c}})$. We can again apply Lemma 7.6 to get that $f_1(\widetilde{\mathcal{W}^{cs}}) = \widetilde{\mathcal{W}^{cs}}$ and $f_1(\widetilde{\mathcal{W}^{cu}}) = \widetilde{\mathcal{W}^{cu}}$. That is, the foliations $\widetilde{\mathcal{W}^{cs}}$ and $\widetilde{\mathcal{W}^{cu}}$ are f_1-invariant. Since this holds for any lift of f, it implies that $\mathcal{W}^{cs}_M$ and $\mathcal{W}^{cu}_M$ are f-invariant. Hence f is dynamically coherent with foliations $\mathcal{W}^{cs}_M, \mathcal{W}^{cu}_M$. This completes the proof when M is not Seifert fibered.

Case 2 – Assume that M is Seifert fibered.

In this case, Lemma A.1 implies that $\beta = \widetilde{g}(\widetilde{f^k})^{-1}$ is either the identity or represent a power of a regular fiber of the Seifert fibration. In any case, β is in a normal subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z}$. Moreover, as proved earlier, $\beta \in \pi_1(\hat{M})$.

Let $\gamma \in \pi_1(M)$ be any deck transformation. As before, consider the foliations $\mathcal{F}^{cs}_\gamma := \gamma\widetilde{\mathcal{W}}^{cs}$ and $\mathcal{F}^{cu}_\gamma := \gamma\widetilde{\mathcal{W}}^{cu}$.

We first claim that these foliations are $\widetilde{g}$-invariant. We show this for $\mathcal{F}^{cs}_\gamma$ the other being analogous. Let $L \in \widetilde{\mathcal{W}}^{cs}$. We have

$$\widetilde{g}(\gamma L) = \beta\widetilde{f^k}(\gamma L) = \beta\gamma\widetilde{f^k}(L) = \gamma\beta^{\pm 1}\widetilde{f^k}(L).$$

Notice that both $\widetilde{f^k}$ (because it is a lift of g) and β (because it belongs to $\pi_1(\hat{M})$ and the foliation $\mathcal{W}^{cs}$ is defined in $\hat{M}$) preserve the foliation $\widetilde{\mathcal{W}}^{cs}$. It follows that $\beta^{\pm 1}\widetilde{f^k}(L) \in \widetilde{\mathcal{W}}^{cs}$, so

$$\widetilde{g}(\gamma L) = \gamma\beta^{\pm 1}\widetilde{f^k}(L) \in \mathcal{F}^{cs}_\gamma.$$

Thus $\mathcal{F}^{cs}_\gamma$ is $\widetilde{g}$-invariant.

We now want to show that the foliations $\mathcal{F}^{cs}_\gamma$, $\mathcal{F}^{cu}_\gamma$ and $\mathcal{F}^{c}_\gamma := \gamma\widetilde{\mathcal{W}}^{c}$ are all leafwise fixed by $\widetilde{g}$.

Since $\hat{M}$ was chosen to be a normal cover of M, any element $\gamma \in \pi_1(M)$ can be thought of as a difeomorphism of $\hat{M}$. Hence we can consider the foliation $\hat{\mathcal{F}}^{cs}_\gamma := \gamma\mathcal{W}^{cs}$ in $\hat{M}$. Note that $\hat{\mathcal{F}}^{cs}_\gamma$ is tangent to the center stable distribution $E^{cs} \subset T\hat{M}$, since γ preserves the tangent bundle decomposition, as it is defined by f in M. The argument above shows that $\hat{\mathcal{F}}^{cs}_\gamma$ is g-invariant.

Thus, we can consider g to be a dynamically coherent diffeomorphism for the pair of transverse foliations $\hat{\mathcal{F}}^{cs}_\gamma$ and $\mathcal{W}^{cu}$. Moreover, g is homotopic to the identity and the good lift $\widetilde{g}$ fixes every leaf of $\widetilde{\mathcal{W}}^{cu}$. Since $\hat{M}$ is Seifert, mixed behaviour is excluded (cf. [BFFP20b, Theorem 5.1]) and this implies that $\widetilde{g}$ must also fix every leaf of $\mathcal{F}^{cs}_\gamma$.

The symmetric argument show that $\mathcal{F}^{cu}_\gamma$ is also fixed by $\widetilde{g}$. So we can apply Proposition 6.1 to both $\hat{\mathcal{F}}^{cs}_\gamma$ and $\hat{\mathcal{F}}^{cu}_\gamma$, implying that they are g-minimal. Hence, the center foliation $\mathcal{F}^{c}_\gamma$ is fixed by $\widetilde{g}$, thanks to Proposition 7.1.

Since all the leaves of $\mathcal{F}^{c}_\gamma$ are fixed by $\widetilde{g}$, we can finally apply Lemma 7.6 to deduce that $\mathcal{F}^{cs}_\gamma = \widetilde{\mathcal{W}}^{cs}$ and $\mathcal{F}^{cu}_\gamma = \widetilde{\mathcal{W}}^{cu}$. As this is true for any γ, the foliations $\widetilde{\mathcal{W}}^{cs}$ and $\widetilde{\mathcal{W}}^{cu}$ descends to foliations $\mathcal{W}^{cs}_M$ and $\mathcal{W}^{cu}_M$ on M in this case too.

We now again have to show that $\mathcal{W}^{cs}_M$ and $\mathcal{W}^{cu}_M$ are f-invariant. The argument is the same for both foliations, so we only deal with $\mathcal{W}^{cs}_M$.

We start with a preliminary step. Let f_* be the automorphism of $\pi_1(M)$ induced by f. Let

$$A := \pi_1(\hat{M}) \cap f_*(\pi_1(\hat{M})) \cap \cdots \cap (f_*)^{k-1}(\pi_1(\hat{M})).$$

The set A is a finite index, normal subgroup of $\pi_1(M)$. Moreover, as f^k is homotopic to the identity, $f_*(A) = A$.

As we remarked at the beginning of the proof, we can without loss of generality prove the result for any further finite cover of $\hat{M}$. Thus we choose if necessary a further cover so that $\pi_1(\hat{M}) = A$. Since $f_*(A) = A$, the map f lifts to a homeomorphism $\hat{f}$ of $\hat{M}$.

As in the first case, we let f_1 be an arbitrary lift of $\hat{f}$ to $\widetilde{M}$ and we define $\mathcal{F}^{cs}_1 := f_1(\widetilde{\mathcal{W}}^{cs})$ and $\mathcal{F}^{cu}_1 := f_1(\widetilde{\mathcal{W}}^{cu})$. (Note that f_1 is in particular also a lift of f.)

Note as before that both $\widetilde{g}f_1$ and $f_1\widetilde{g}$ are lifts of f^{k+1}, and $\widetilde{g}f_1(\widetilde{g})^{-1}(f_1)^{-1}$ is a bounded distance from the identity (because $\widetilde{g}$ is and f_1 has bounded derivatives). So $\delta := \widetilde{g}f_1(\widetilde{g})^{-1}(f_1)^{-1}$ is an element of $\pi_1(M)$ a bounded distance from identity. By Lemma A.1, δ represents a power of a regular fiber of the Seifert fibration, so is in the normal $\mathbb{Z}$ subgroup of $\pi_1(M)$ (note that since $\pi_1(M)$ is not virtually nilpotent, there exists a unique Seifert fibration on M, see Appendix A).

In addition $\widetilde{g}f_1$ and $f_1\widetilde{g}$ are also lifts of the homeomorphisms $g\hat{f}$ and $\hat{f}g$ in $\hat{M}$ to $\widetilde{M}$. Hence δ is in $\pi_1(\hat{M})$.

Using once more the arguments above, we get that $(f_1)^{-1}\delta f_1(\delta)^{-1}$ is a bounded distance from the identity, and projects to the identity in M (and in $\hat{M}$), hence it is a deck transformation η also contained in the $\mathbb{Z}$ normal subgroup of $\pi_1(M)$. Thus δ and η commute. Moreover, η is also in $\pi_1(\hat{M})$.

Now we can show that $\widetilde{g}$ preserves $\mathcal{F}_1^{cs}$: Let L in $\widetilde{\mathcal{W}^{cs}}$. Then

$$\widetilde{g}(f_1(L)) = \delta f_1(\widetilde{g}(L)) = \delta f_1(L) = f_1(\eta\delta(L)).$$

Here $\eta\delta(L)$ is in $\widetilde{\mathcal{W}^{cs}}$, because L is in $\widetilde{\mathcal{W}^{cs}}$ and $\eta\delta$ is in $\pi_1(\hat{M})$. Hence $\widetilde{f_1}(\eta\delta L)$ is in $f_1(\widetilde{\mathcal{W}^{cs}})$ so $\widetilde{g}$ preserves $\mathcal{F}_1^{cs}$.

What we proved implies that g preserves $\hat{f}(\mathcal{W}^{cs})$ in $\hat{M}$. Now consider the pair of foliations $\hat{f}(\mathcal{W}^{cs})$ and $\mathcal{W}^{cu}$. They are both invariant by g, so g is dynamically coherent for this particular pair of foliations, and $\widetilde{g}$ fixes the leaves of $\widetilde{\mathcal{W}^{cu}}$. So once again, as $\hat{M}$ is Seifert, we get that $\widetilde{g}$ must also fix every leaf of $f_1(\widetilde{\mathcal{W}^{cs}})$ (cf. [BFFP20b, Theorem 5.1]).

The symmetric argument implies that $\widetilde{g}$ fixes every leaf of $f_1(\widetilde{\mathcal{W}^{cu}})$. Once again, $\hat{M}$ being Seifert implies that all the foliations are g-minimal (Proposition 6.1). Hence $\widetilde{g}$ also fixes the center foliation $f_1(\widetilde{\mathcal{W}^{c}})$ (Proposition 7.1). So Lemma 7.6 applies and we deduce that $f_1(\widetilde{\mathcal{W}^{cs}}) = \widetilde{\mathcal{W}^{cs}}$ and $f_1(\widetilde{\mathcal{W}^{cu}}) = \widetilde{\mathcal{W}^{cu}}$.

In particular, f preserves the foliations $\mathcal{W}^{cs}_M$ and $\mathcal{W}^{cu}_M$ as wanted. So f is dynamically coherent. ☐

8. Proof of Theorem A

In this section, we finish the proof of Theorem A. That is, $f\colon M \to M$ is assumed to be a partially hyperbolic diffeomorphism homotopic to identity in a Seifert manifold, and we need to show that a power of f is a discretized Anosov flow.

We first fix a finite cover $\hat{M}$ of M so that $\hat{M}$ is orientable, and so are all the bundles. Then, up to a finite power, a lift g will preserve the orientations of the bundles. More precisely, there exists some integer $k > 0$ such that the lift g obtained by lifting a homotopy of f^k to the identity preserves the orientations.

Thanks to Theorem 3.4, there are branching foliations $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ in $\hat{M}$ that are preserved by g.

In order to finish the proof of Theorem A, we just need one more lemma.

Lemma 8.1. *There exists a lift $\widetilde{g}$ of an iterate of g that fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and also fixes every leaf of $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$.*

Postponing the proof of the lemma, we can finish the proof.

Proof of Theorem A. According to Lemma 8.1, there exists a lift $\widetilde{g}$ of a power of g^i of g that fixes the leaves of both $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$. Then Proposition 7.1 implies that $\widetilde{g}$ fixes every center leaf. Thus Corollary 7.5 gives that g^i is dynamically coherent. Then Proposition 7.7 tells us that f is also dynamically coherent. So we can now use the results of [BFFP20b, §7] to conclude. $\square$

So all we have left to do is prove Lemma 8.1, which we now do.

8.1. **Proof of Lemma 8.1.** In [BFFP20b, §7] we showed that it was always possible in a Seifert manifold to choose a convenient good lift.

Proposition 8.2. *There exists a good lift of an iterate of g which fixes a leaf (and therefore every leaf) of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$.*

Proof. As stated in [BFFP20b, Remark 7.2], the proof of [BFFP20b, Proposition 7.1] works in the non-dynamically coherent case.

The only change needed is to replace the words foliations by branching foliations. Note also that [BFFP20b, Proposition 7.1] requires the Seifert fibration to be orientable. This is implied by our assumptions: Indeed, $\hat{M}$ is orientable, all the bundles are orientable and $\mathcal{W}^{cs}_{\text{bran}}$ is a horizontal foliation (see [BFFP20b, Theorem F.3]). Thus the Seifert fibration is orientable. $\square$

Using Proposition 8.2, the lemma follows readily.

Proof of Lemma 8.1. First, using Proposition 8.2 we consider a good lift $\widetilde{g^i}$ of an iterate g^i that fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$. Suppose this lift fixes one center unstabe leaf. Then Proposition 6.1 gives that $\mathcal{W}^{cu}_{\text{bran}}$ is g^i-minimal. So Corollary 4.7 implies that $\widetilde{g^i}$ also fixes every leaf of $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$.

Thus we can suppose for a contradiction that $\widetilde{g^i}$ fixes no center unstable leaf. Therefore no center leaf can be fixed by $\widetilde{g}$. Applying Proposition 5.2 we deduce that every periodic center leaf of g has to be coarsely contracting.

Exchanging roles, and applying Proposition 8.2 to the center unstable branching foliation we deduce that every periodic center leaf for g must be coarsely expanding. Notice that, althought the lifts may be different, the coarsely expanding and coarsely contracting behavior is for periodic center leaves of the original map g for both $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$.

As there must be at least one such periodic center leaf (cf. Proposition 5.8) this gives a contradiction. So there exists a good lift of an iterate of g that fixes leaves of both $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$. $\square$

9. Absolutely partially hyperbolic diffeomorphisms

In this section, we explain how one can improve Theorem 1.3 if one uses a strong version of partial hyperbolicity.

Definition 9.1. A partially hyperbolic diffeomorphism $f\colon M \to M$ on a 3-manifold is called *absolutely partially hyperbolic* if there exists constants $\lambda_1 < 1 < \lambda_2$ such that for some $\ell > 0$ and every $x \in M$, we have

$$\|Df^\ell|_{E^s(x)}\| < \lambda_1 < \|Df^\ell|_{E^c(x)}\| < \lambda_2 < \|Df^\ell|_{E^u(x)}\|.$$

Notice that, although subtle, the difference between being absolutely partially hyperbolic versus just partially hyperbolic is far from trivial. Here, we just show that with this stronger property one can significantly simplify the arguments. However, some previous results have shown significant differences between the two notions, specifically with regard to the integrability of the bundles (see [BBI09, RHRHU16, Pot15]).

We will show the following

Theorem 9.2. *Let $f\colon M \to M$ be an absolutely partially hyperbolic diffeomorphism on a 3-manifold. Suppose that f is homotopic to the identity and preserves two branching foliations $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ that are both f-minimal. Then either*

- *(i) f is a discretized Anosov flow, or,*
- *(ii) $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ are $\mathbb{R}$-covered and uniform and a good lift $\widetilde{f}$ of f act as a translation on their leaf spaces.*

In order to prove this theorem, the main step will be to show that, using absolute partial hyperbolicity, we have an improvement of Proposition 5.2.

Proposition 9.3. *Let $f\colon M \to M$ be an absolutely partially hyperbolic diffeomorphism homotopic to the identity and $\widetilde{f}$ a good lift of f to $\widetilde{M}$. Assume that every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ is fixed by $\widetilde{f}$. Let L be a leaf whose stabilizer is generated by $\gamma \in \pi_1(M) \setminus \{\text{id}\}$. Then, there is a center leaf in L fixed by $\widetilde{f}$.*

The proof is essentially the same as the one in [HPS18, Section 5.4] but we repeat it since the contexts are different.

Proof. The proof is by contradiction. Assume that $\widetilde{f}$ does not fix any center leaf in L.

Proposition 5.8 gives that there exists a center leaf periodic by f. Now, using the proof of Proposition 5.2 on the lift c of such a periodic leaf, we can be more precise: Let $h := \gamma^n \circ f^m$, with $m > 0$ and $\gamma \in \pi_1(M)$, be the diffeomorphism fixing c. There exists two stable leaves s_1 and s_2 in L fixed by h, a bounded distance apart in L and such that c separates s_1 from s_2 in L. We denote by B the band bounded by s_1 and s_2.

Since γ is an isometry, the diffeomorphism h is absolutely partially hyperbolic, and we can (modulo taking iterates) assume that there are constants $\lambda_1 < \lambda_2$ such that

$$\|Dh|_{E^s}\| < \lambda_1 < \lambda_2 < \|Dh|_{E^c}\|.$$

Moreover, there is a constant $R > 1$ such that $\|Dh^{-1}\| \leq R$ in all of L.

For simplicity, we will assume that the distance between s_1 and s_2 is smaller than $1/2$ so that the band B is contained in the neighborhood $\hat{B} = \bigcup_{x\in S_1} B_1(x)$ of radius 1 around s_1.

For every positive d there is a constant $r(d) > 0$ such that for any set of diameter less than d, the length of a stable leaf contained in this set is at most $r(d)$. This is because in a foliated box only one segment of a stable segment can intersect it. This implies that stable leaves (and center leaves as well) are quasi-isometrically embedded in their neighborhoods of a fixed diameter. So there is $K > 0$ so that for any stable segment J contained in $\hat{B}$ with endpoints z and w we have

$$\text{length}(J) \leq K d_{\hat{B}}(z, w).$$

Now, choose $n > 0$ such that $K^2 \frac{\lambda_1^n}{\lambda_2^n} \ll \frac{1}{2}$ and once n is fixed, choose $D > 0$ so that $\frac{D}{2} \gg 2R^n + \frac{2K}{\lambda_2^n}$.

We now pick points $z, w \in s_1$ such that $d_{\hat{B}}(z, w) = D$ and take J^s an arc of s_1 joining these points. From the choice of K and D we know that $\text{length}(J^s) \leq KD$. So, it follows that $\text{length}(h^n(J^s)) \leq KD\lambda_1^n$.

Choose a center curve J^c joining $B_1(h^n(z))$ with $B_1(h^n(w))$ (this can be done because c separates s_1 from s_2) and call z_n and w_n the endpoints in each ball. It follows that $\text{length}(J^c) \leq K^2 D\lambda_1^n + 2K$.

Since the distance between the endpoints of J^c and $h^n(z)$, $h^n(w)$ is less than 1, by iterating backwards by h^{-n} we get that $d(h^{-n}(z_n), z)$ and $d(h^{-n}(w_n), w)$ are less than R^n.

This implies that

$$D \leq d_{\hat{B}}(z, w) \leq K^2 \frac{\lambda_1^n}{\lambda_2^n} D + 2R^n + \frac{2K}{\lambda_2^n},$$

a contradiction with the choices of n and D. This completes the proof of the proposition. □

Using this proposition, we can prove Theorem 9.2 in the same way as [BFFP20b, Theorem 5.1]

Proof of Theorem 9.2. Let $\widetilde{f}$ be a good lift of f. Since $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ are f-minimal, by Corollary 4.7, $\widetilde{f}$ either fixes each leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$, or act as a translation on both leaf space (in which case the foliations are $\mathbb{R}$-covered and uniform and we are in case (ii) of the theorem), or $\widetilde{f}$ translates one and fixes the other.

If $\widetilde{f}$ fixes the leaves of both $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$ then Proposition 7.1 and Corollary 7.5 imply that we are in case (i) of the theorem.

So we have to show that we cannot be in the mixed case. Suppose that $\widetilde{f}$ fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$.

Since M is not $\mathbb{T}^3$, there are leaves of $\mathcal{W}^{cs}_{\text{bran}}$ with non-trivial fundamental group. Consider the lift L in $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ of such a leaf, with L invariant by γ in $\pi_1(M) \smallsetminus \{\text{Id}\}$. We can apply Proposition 9.3 to conclude that there is a center leaf c in L that is fixed by $\widetilde{f}$. So, in particular, $\widetilde{f}$ needs to fix a center unstable leaf containing c. Thus $\widetilde{f}$ has to also fix every leaf of $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$. □

10. Regulating pseudo-Anosov flows and translations

The rest of the paper is concerned with hyperbolic 3-manifolds. We will get positive results dealing with the non-dynamically coherent case.

That is, we want to understand the dynamics of a homeomorphism acting by translation on a branching foliation.

In order to be able to do that, we first need to build a regulating pseudo-Anosov flow transverse to the branching foliation.

The existence of such a flow is a relatively immediate consequence of the construction of the regulating flow and the fact that the branching foliation is well-approximated by foliations.

Proposition 10.1. *Let M be a hyperbolic 3-manifold and $\mathcal{F}$ a branching foliation well-approximated by foliations $\mathcal{F}_\epsilon$ such that $\mathcal{F}$ (and thus also $\mathcal{F}_\epsilon$ for small ϵ) are $\mathbb{R}$-covered and uniform. Then, there exists a transverse and regulating pseudo-Anosov flow Φ for $\mathcal{F}$.*

Proof. By [Thu, Cal00, Fen02] (see [BFFP20b, Theorem D.3]) for any ϵ, there exists a pseudo-Anosov flow Φ_ϵ transverse to and regulating for $\mathcal{F}_\epsilon$.

Now, as ϵ get small, the angle between leaves of $\mathcal{F}_\epsilon$ and leaves of $\mathcal{F}$ becomes arbitrarily small.

Then, since both $\mathcal{F}$ and $\mathcal{F}_\epsilon$ are $\mathbb{R}$-covered and uniform, for any leaf $L \in \mathcal{F}$, there exists two leaves L_1 and L_2 such that L is in between L_1 and L_2. As Φ_ϵ is regulating for $\mathcal{F}_\epsilon$, every orbit intersects both L_1 and L_2, thus L. So every orbit of Φ_ϵ intersect every leaf of $\mathcal{F}$, that is, Φ_ϵ is regulating for $\mathcal{F}$.

The fact that the flow Φ_ϵ can be chosen transverse to $\mathcal{F}$ follows from the construction of Φ_ϵ (see [Thu, Cal00, Fen02]). The flow Φ_ϵ is build by blowing down certain laminations transverse to $\mathcal{F}_\epsilon$. Moreover these laminations are transverse to any foliation that are close enough to $\mathcal{F}_\epsilon$ for a uniform angle. Since the angle between $\mathcal{F}$ and $\mathcal{F}_\epsilon$ gets arbitrarily small, Φ_ϵ will also be transverse. For a continuous family of **R**-covered foliations, this property is explicitely stated in [Cal00, Corollary 5.3.22]. □

Using the regulating pseudo-Anosov flow given by Proposition 10.1, all of [BFFP20b, Section 8] works for a branching foliation without change. Thus we obtain

Proposition 10.2. *Let M be a hyperbolic 3-manifold. Let $f\colon M \to M$ be a homeomorphism homotopic to the identity that preserves a (branching) foliation $\mathcal{F}$. Suppose that $\mathcal{F}$ is uniform and $\mathbb{R}$-covered, and that a good lift $\widetilde{f}$ of f acts as a translation on the leaf space of $\mathcal{F}$. Let Φ be a transverse regulating pseudo-Anosov flow to $\mathcal{F}$.*

Then, for every $\gamma \in \pi_1(M)$ associated with a periodic orbit of Φ, there is a compact $\hat{f}_\gamma$-invariant set T_γ in M_γ which intersects every leaf of $\hat{\mathcal{F}}_\gamma$, where $M_\gamma = \widetilde{M}\big/\langle\gamma\rangle$ and $\hat{f}_\gamma\colon M_\gamma \to M_\gamma$ is the corresponding lift of f.

Moreover, if an iterate $\hat{f}_\gamma^k$ of $\hat{f}_\gamma$ fixes a leaf L of $\hat{\mathcal{F}}_\gamma$, and γ fixes all the prongs of this orbit, then the fixed set of $\hat{f}_\gamma^k$ in L is contained in $T_\gamma \cap L$ and has negative Lefschetz index.

Almost without any change, we also obtain the corresponding version of [BFFP20b, Proposition 9.1]

Proposition 10.3. *Let f be partially hyperbolic diffeomorphism in a hyperbolic 3-manifold which preserves a branching foliation $\mathcal{W}^{cs}_{\text{bran}}$ tangent to E^{cs}. Assume that a good lift $\widetilde{f}$ of f acts as a translation on the foliation $\mathcal{W}^{cs}_{\text{bran}}$ and let Φ be a transverse regulating pseudo-Anosov flow. Then, for every $\gamma \in \pi_1(M)$ associated to the inverse periodic orbit of Φ there are $n > 0, m > 0$ such that $h = \gamma^n \circ \widetilde{f}^m$ fixes a leaf L of $\mathcal{W}^{cs}_{\text{bran}}$.*

Proof. The only difference is that we cannot say that the action of h in the leaf space is expanding since collapsing of leaves may change the behavior. However, the same proof gives the existence of an interval in the leaf space which is mapped inside itself by h^{-1} giving a fixed leaf as desired. □

Remark 10.4. Note that in the non dynamically coherent situation, the proof of [BFFP20b, Theorem B] does not give a contradiction: it could happen (and indeed happens in a situation with similar properties, see e.g., [BGHP17]) that having a fixed point in a leaf of the foliation, does not force the dynamics on the leaf space to be repelling around the leaf in terms of the action on the leaf space. This issue has previously appeared, in particular in Lemma 6.2.

Notice that if one assumes the existence of a periodic center leaf, then we can easily prove a version of [BFFP20b, Theorem B] in the non dynamically coherent setting.

Proposition 10.5. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism on a hyperbolic 3-manifold. Suppose that there exists a closed center leaf c that is periodic under f. Then f is a discretized Anosov flow.*

Proof. We start by replacing f by a power, so that f becomes homotopic to the identity.

Let $\widetilde{f}$ be a good lift of f. We will show that $\widetilde{f}$ fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ and $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$. Then, section 7 above shows that the original f (before taking a power) is dynamically coherent, hence the result follows from [BFFP20b, Theorem B].

Suppose that $\widetilde{f}$ does not fix every leaf of, say, $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$. Then Corollary 4.7 implies that the leaf space of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ is $\mathbb{R}$ and that $\widetilde{f}$ acts as a translation on it.

Let $\widetilde{c}$ be a lift of the periodic closed center leaf c. Since c is periodic and $\widetilde{f}$ acts as a translation, there exists $\gamma \in \pi_1(M)$, non-trivial such that $\gamma(\widetilde{c}) = \widetilde{f}^k(\widetilde{c})$ for some k. Now c is also closed, so there exists g (distinct from any power of γ, since they do not act in the same way on the leaf space of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$) such that $g(\widetilde{c}) = \widetilde{c}$. Thus g and γ produce a $\mathbb{Z}^2$ subgroup in $\pi_1(M)$, which is impossible since M is hyperbolic. ☐

Remark 10.6. The arguments here show that the dynamics of the transverse pseudo-Anosov flow coarsely affects the dynamics of f. In particular, if $\widetilde{f}$ is a translation with respect to a certain $\mathbb{R}$-covered branching foliation, there must be a lower bound on the topological entropy of f. It is possible that one can get a *uniform* lower bound independent of the foliation by getting that $\widetilde{f}$ must translate a certain uniform amount and this would give another way to check that a partially hyperbolic diffeomorphism in a hyperbolic 3-manifold is a discretized Anosov flows

11. Translations in hyperbolic 3-manifolds

In this section we obtain further consequences of having a partially hyperbolic diffeomorphism act as a translation in a hyperbolic 3-manifold.

We start by recalling the setting. Let $f\colon M \to M$ be a (not necessarily dynamically coherent) partially hyperbolic diffeomorphism on a hyperbolic 3-manifold. Up to replacing f by a power, we assume that it is homotopic to the identity. Up to taking a further iterate of f and a lift to a finite cover of M, we can assume that f admits branching foliations, and that the good lift $\widetilde{f}$ acts as a translation on the leaf space of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$.

Let Φ_{cs} be a transverse regulating pseudo-Anosov flow to $\mathcal{W}^{cs}_{\text{bran}}$ given by Proposition 10.1. This flow is fixed throughout the discussion.

Then Proposition 10.3 shows that, for any periodic orbit of Φ^{cs}, there exists a center stable leaf periodic by f.

11.1. Periodic center rays.

We will now produce rays in periodic center leaves which are expanding. A *ray* in L is a proper embedding of $[0, \infty)$ into L. We say that a ray is a *center ray* if it is contained in a center leaf. So a center ray c_x is the closure in L of a connected component of $c \smallsetminus \{x\}$ where c is a center curve and $x \in c$.

Let γ in $\pi_1(M)$ be associated with a periodic orbit δ_0 of the pseudo-Anosov flow Φ_{cs}. Let L be a leaf (given by Proposition 10.3) of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ fixed by $h := \gamma^n \circ \widetilde{f}^m$, with $m > 0$.

A center ray c_x is *expanding* if $h(c_x) = c_x$ and x is the unique fixed point of h in c_x and every $y \in c_x \smallsetminus \{x\}$ verifies that $h^{-n}(y) \to x$ as $n \to +\infty$.

Proposition 11.1. *Assume that a good lift $\widetilde{f}$ of f acts as a translation on the (branching) foliation $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$. Let Φ_{cs} be a regulating transverse pseudo-Anosov flow. Let γ in $\pi_1(M)$ associated with a periodic orbit δ_0 of Φ_{cs}. Let L be a leaf* *of* $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ *fixed by* $h = \gamma^n \circ \widetilde{f}^m$, *where* $m > 0$. *Assume that* γ *fixes all prongs of a lift of* δ_0 *to* $\widetilde{M}$. *Then there are at least two center rays in* L, *fixed by* h, *which are expanding.*

Remark 11.2. We should stress that we cannot guarantee to get a single center leaf with both rays expanding. For example it is very easy to construct an example such that h has Lefschetz index -1 in L, it has exactly 3 fixed center leaves in L, and only two fixed expanding rays, which are contained in distinct center leaves (see Figure 9). This situation occurs in the examples constructed in [BGHP17] in the unit tangent bundle of a surface.

We will use Proposition 11.1 and its proof to eliminate the mixed behavior in hyperbolic 3-manifolds. It should be noted that this proposition also gives some relevant information about the structure of the enigmatic double translations examples which are not ruled out by our study.

The key point is to understand how each fixed center leaf contributes to the total Lefschetz index of the map in a center-stable leaf which we can control. Since the dynamics preserves foliations and one of them has a well understood dynamical behavior (i.e., in the center stable foliation, the stable foliation is contracting) we can compute the index just by looking at the dynamics in the center foliation (see Figure 8).

As remarked above, one do have to be careful when computing the index as cancellations might happen with branching foliation (see Figure 9).

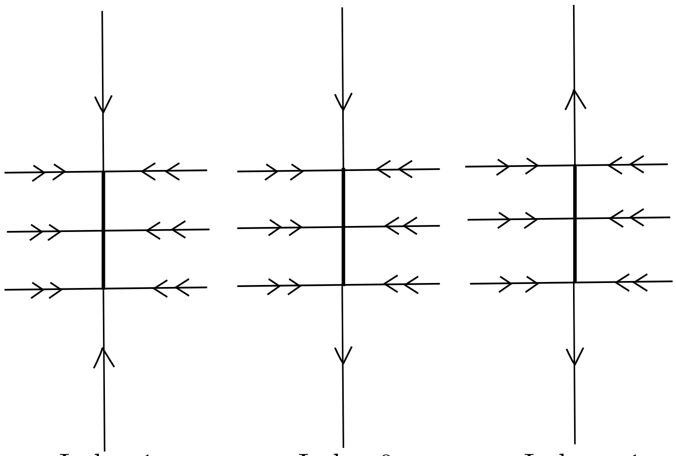


FIGURE 8. Contribution of index of a center arc depending on the center dynamics

We are now ready to give a proof of Proposition 11.1.

Proof of Proposition 11.1. By Proposition 10.2, we know that the fixed point set of h in L is contained in T_γ and has Lefschetz index $1-p$ where p is the number of stable prongs at the fixed point. In particular h has some fixed points in L.

Let $L_2 = \widetilde{f}^m(L)$. We denote by $\tau_{12}\colon L \to L_2$ the flow along $\widetilde{\Phi}^{cs}$ map.

Let $g := \gamma^n \circ \tau_{12}\colon L \to L$. The map g is a bounded distance away from h.

Claim 11.3. *Let* c_1, c_2 *be two distinct center leaves in* L *that have a non-trivial intersection. Suppose that both* c_1, c_2 *are fixed by* h, *and there exist two distinct*

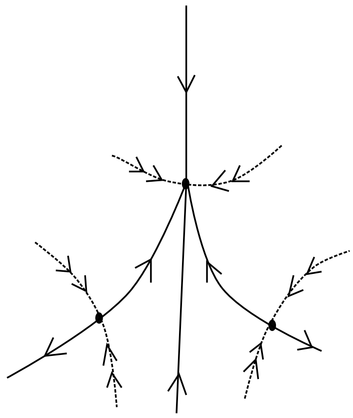


FIGURE 9. Two segments of zero index merge with a point with index 1 to produce a global -1 index.

points $z, y \in c_1 \cap c_2$ which are fixed by h. Then the center leaves c_1 and c_2 coincide on the segment between z and y.

Proof of Claim 11.3. Let $[y,z]_{c_1}$ and $[y,z]_{c_2}$ be the center segments between y and z in c_1 and c_2 respectively.

Assume for a contradiction that $[y,z]_{c_1}$ and $[y,z]_{c_2}$ are distinct. Then, up to changing y and z, we can assume that the intersection between the open intervals $(y,z)_{c_1}$ and $(y,z)_{c_2}$ is empty.

Thus, by construction, $[y,z]_{c_1}$ and $[y,z]_{c_2}$ intersect only at z and y. We let B be the bigon in L bounded by $[y,z]_{c_1}$ and $[y,z]_{c_2}$.

Note that any stable leaf that enters the bigon B must exit it (otherwise it would limit in a stable leaf entirely contained in B, which is impossible). Hence, B is "product foliated" by stable leaves. Since B is compact the length of the stable segments contained in B is bounded.

Since z, y are fixed by h it follows that B is also fixed by h. Let s be one such stable segment connecting $(z,y)_{c_1}$ to $(z,y)_{c_2}$. Then, the images of s under powers of h^{-1} stay in B but must also have unbounded length, contradiction. □

Let x be a fixed point of h. Recall from Lemma 3.14 that the set of center leaves through x in L is a closed interval. In particular h fixes the endpoints of this interval. Hence, x is contained in a center leaf c such that $h(c) = c$.

Claim 11.4. *All the fixed points of h in L are contained in the union of finitely many compact segments of center leaves in L.*

Proof of Claim 11.4. Let c be a center leaf fixed by h. Since the fixed points are contained in a compact set C (see Lemma 8.12 of [BFFP20b]), there is a minimal compact interval J in c which contains all the fixed points of h in c.

Suppose that there exists infinitely many distinct such minimal intervals J_i in center leaves c_i. Since the fixed points of h in L are in a compact set, we can choose i, j large enough, so that J_i is very close in the Hausdorff distance of L to J_j. Let z be an endpoint of J_i. Then the stable leaf $s(z)$ through z intersects the center leaf c_j. As z is fixed by h and so is c_j, contraction of the stable length implies that $z \in c_j$, thus $z \in J_j$.

Hence, both endpoints of J_i are on J_j. By Claim 11.3, it implies that $J_i \subset J_j$, and minimality of the interval J_j implies $J_j = J_i$ which is a contradiction. □

Let $\{J_i, 1 \leq i \leq i_0\}$ be a finite family of compact intervals containing all the fixed point of h, as given by Claim 11.4. Note that we do not necessarily take the minimal intervals as constructed in the proof of Claim 11.4, as we want the following properties for that family.

Claim 11.5. *We can choose the collection of intervals $\{J_i, 1 \leq i \leq i_0\}$, each in a center leaf fixed by h, satisfying the following properties:*

1. *The union $\bigcup_{1\leq i\leq i_0} J_i$ contains all the fixed points of h.*
2. *The endpoints of each interval J_i are fixed by h.*
3. *The intervals are pairwise disjoint.*

Proof of Claim 11.5. Let $c_1, \ldots, c_n$ be a minimal collection of center leaves that contains all fixed points of h in L, as given by Claim 11.4. Let J_i be the minimal compact interval containing all fixed points of h in c_i.

The family J_i then satisfies conditions (1) and (2). So we only have to show that one can split the intervals J_i further so that conditions (3) is also satisfied (while still satisfying the first two conditions).

Notice that c_i, c_j intersect if and only if J_i, J_j intersect. Thus, we can restrict our attention to each connected component of the union of the c_i's separately.

Up to renaming, assume that $\cup_{1\leq i\leq k} c_k$ is a connected component of $\cup_{1\leq i\leq n} c_k$.

Now we can consider the union of the $J_1, \ldots, J_k$ as a graph, where the vertices are are the endpoints of the segments J_i together with the points where two segments merge, and the edge are the subsegments joining the vertices. With this convention, the union of the $J_1, \ldots, J_k$ is then a tree. Otherwise there would be a bigon in L enclosed by the union, which is ruled out by Claim 11.3.

Let $\mathcal{B}$ be this tree. Our goal is to remove enough open segments from the J_i's so that no vertex of this associated tree has degree 3 or more. Consider a vertex p in $\mathcal{B}$ with degree 3 or more. Then there are two edges e_1 and e_2 abutting at p on the same side of p. We claim that e_1 cannot have points fixed by h arbitrarily close to p (except for p itself). Otherwise one would have a fixed point $y \in e_1$ such that $s(y)$ intersects e_2. Since e_2 is contained in a fixed leaf, $e_2 \cap s(y)$ is fixed by h. This implies (since h decreases stable length) that y is in e_2. Thus, by Claim 11.3, the intersection of e_1 and e_2 would contain the segment $[y, p]$, contradicting the fact that they are distinct edges.

Thus, we can remove an open interval (p, z) from, say, e_1, where z is fixed by h but (p, z) has no fixed points. In the new tree, p has index one less than before and z has index one.

Doing this recursively on each vertex of index strictly greater than 2, we will obtain, as sought, a disjoint collection of intervals that also satisfy conditions (1) and (2). □

Now we will look at the index of h on the fixed intervals J_i, $1 \leq i \leq i_0$ produced by Claim 11.5. Note that for each such interval J_i there are no other fixed points of h nearby in L. Let c be a leaf fixed by h containing J_i.

If h is contracting on c near both endpoints of J_i on the outside then the index of J_i is $+1$. This is because the stable foliation is contracting under $h = \gamma^n \circ \widetilde{f}^m$ (since $m > 0$). Hence h is contracting near J_i. If h is expanding on both sides, the index is -1. If one side is contracting and the other is expanding then the index is zero.

The global index for h can then be computed by adding the indexes of h on each of the intervals J_i, taking care of cancellations.

Let c_k, $1 \leq k \leq k_0$, be finitely many center leaves, fixed by h and containing all the J_i. We choose this collection to have the minimum possible number of leaves.

Each leaf c_k contains finitely many segments J_i, so there are exactly two infinite rays that do not contain any J_i. The contribution of c_k to the global index of h (before possible cancellations) will then be -1 if both rays are expanding, 0 if one is expanding while the other contracts and 1 if both are contracting.

Suppose for a contradiction, that there is at most one expanding ray in L. So each c_k, considered separately, has index either 0 or 1.

If there is an expanding ray, let c_k be a leaf with an expanding ray. Otherwise let c_k be any leaf. Now we need to consider how the other leaves and the possible cancellations impact the global index of h. Let c_l be a leaf that intersect c_k. If c_l shares an expanding ray with c_k, then the other ray of c_l is contracting, and eventually disjoint from the corresponding ray of c_k. The fixed set (if any) of this ray in c_l has index zero. If c_l does not share an expanding ray with c_k, then both rays of c_l are contracting. The ray that is added to the same end as the expanding ray of c_k contributes index 1. The other ray contributes index 0. In any case the index, starting at 0 or 1, does not decrease.

Now, if c_m is another leaf that is disjoint from the set above, then both rays are contracting and it contributes an index 1. So again the index does not decrease.

Thus, if there is at most one expanding ray, then the index of h is at least 0. This contradicts the fact that the index of h is $1-p$ where $p \geq 2$, and thus finishes the proof of Proposition 11.1. □

11.2. **Periodic rays and boundary dynamics.** Proposition 11.1 gave the existence of periodic rays that are coarsely expanding. Here we will show that such a ray has a well-defined ideal point on the circle at infinity of the leaf, and that it corresponds to the endpoint of a prong of the transverse regulating pseudo-Anosov flow, Φ^{cs}.

As previously, we assume that we have a center stable leaf $L \in \widetilde{\mathcal{W}}^{cs}_{\mathrm{bran}}$ such that there is a deck transformation γ for which $\gamma \circ \widetilde{f}^m(L) = L$ for some $m > 0$. We let $L_2 = \widetilde{f}^m(L)$ and define $\tau_{12} \colon L \to L_2$ the flow along $\widetilde{\Phi}^{cs}$ map. We also take as before

$$h := \gamma \circ \widetilde{f}^m \ \text{ and } \ g := \gamma \circ \tau_{12}.$$

Recall that h and g are maps of L that are a bounded distance from each other. Also g preserves the (singular) foliations $\mathcal{G}^s$ and $\mathcal{G}^u$. We again assume that if g has a fixed point x_0 in L then γ is such that g preserves each of the prongs of $\mathcal{G}^s(x_0)$ (resp. $\mathcal{G}^u(x_0)$).

The action of g on the circle at infinity $S^1(L_1)$ has an even number of fixed points, which are alternately contracting and repelling. We denote by P the set of contracting fixed points and by N the set of repelling ones. With these notations, we get the following.

Proposition 11.6. *Let $\eta \colon [0,\infty) \to L$ be a contracting fixed ray for h. Then $\lim_{t\to\infty} \eta(t)$ exists in $S^1(L)$ and it is a (unique) point in N. (Symmetrically, if η is an expanding fixed ray, its limit point belongs to P.)*

Proof. Let y in P and U a small neighborhood of y in $L \cup S^1(L)$ as in [BFFP20b, §8]. If η has a point q in $U \cap L$, then $h^n(q)$ converges to y as $n \to +\infty$, so η

could not be a contracting ray, a contradiction. So η cannot limit on any point in P. If z is in $S^1(L) \smallsetminus \{N \cup P\}$, then $h^n(z)$ converges to a point in P under forward iteration. Hence again a small neighborhood Z of z in $L \cup S^1(L)$ is sent under some iterate inside a neighborhood U as in the first part of the proof. So any point in $Z \cap L$ converges to a point in P under forward iteration. Hence η cannot limit to a point in $S^1(L) \smallsetminus \{N \cup P\}$ either. So η can only limit on points in N. Since η is properly embedded in L, the set of accumulations points of η is connected, so it has to be a single point. □

12. Mixed case in hyperbolic manifolds

In this section we show that even in the non-dynamically coherent case, the mixed behavior is impossible for hyperbolic 3-manifolds. This will be done by using the study of translations in hyperbolic 3-manifolds developed in sections 10 and 11 to provide more information on the dynamics of general partially hyperbolic diffeomorphisms.

The main result of this section is the following.

Theorem 12.1. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism homotopic to the identity on a hyperbolic 3-manifold M. Suppose that there exists a finite lift and finite power $\hat{f}$ of f that preserves two branching foliations $\mathcal{W}^{cs}_{\text{bran}}, \mathcal{W}^{cu}_{\text{bran}}$ and is such that a good lift $\widetilde{f}$ fixes a leaf of $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$. Then, f is a discretized Anosov flow.*

12.1. **The set up.** Consider a partially hyperbolic diffeomorphism f as in Theorem 12.1.

Our goal is to show that the good lift $\widetilde{f}$ of f fixes every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}, \widetilde{\mathcal{W}}^{cu}_{\text{bran}}$. Indeed, Proposition 7.1 (and Corollary 7.5) then implies that $\hat{f}$ is dynamically coherent, so we can then use [BFFP20b, Theorem B] to obtain that $\hat{f}$ is a discretized Anosov flow. In turns, thanks to Proposition 7.7, we obtain that f itself is dynamically coherent and a discretized Anosov flow.

Since Proposition 7.7 allows us to use finite lifts and powers, we assume directly that $f = \hat{f}$, that $\mathcal{W}^{cs}_{\text{bran}}$ and $\mathcal{W}^{cu}_{\text{bran}}$ are orientable and transversely orientable and that f preserves their orientations.

Since $\widetilde{f}$ is assumed to fix one leaf of $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$, Proposition 6.1 implies that every leaf of $\widetilde{\mathcal{W}}^{cu}_{\text{bran}}$ is fixed. We will prove that every leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ is fixed by $\widetilde{f}$ by contradiction. So, by Proposition 6.1, we can assume that $\mathcal{W}^{cs}_{\text{bran}}$ is $\mathbb{R}$-covered and uniform and that $\widetilde{f}$ acts as a translation on the leaf space of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$. In particular, there are no center curves fixed by $\widetilde{f}$.

Then, we can apply Proposition 5.2 to $\mathcal{W}^{cu}_{\text{bran}}$ to deduce that every periodic center leaf is coarsely expanding.

On the other hand, since $\widetilde{f}$ acts as a translation on $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$, we can use the results from sections 10 and 11. Let Φ_{cs} be a regulating pseudo-Anosov flow transverse to $\mathcal{W}^{cs}_{\text{bran}}$ given by Proposition 10.1.

The flow Φ_{cs} is a genuine pseudo-Anosov, that is it admits at least one periodic orbit which is a p-prong with $p \geq 3$ (see [BFFP20b, Proposition D.4]).

Now, we choose γ in $\pi_1(M)$, associated to this prong, and apply Proposition 10.3: Up to taking powers, we can assume that $h := \gamma \circ \widetilde{f}^k$ for some $k > 0$ fixes a leaf L of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$. Moreover, the dynamics in L resembles that of the dynamics of a p-prong, and in particular fixes every prong.

Notice that Proposition 11.1 also provides some center rays which are expanding in L for h. We will need to use some of the ideas involved in the proof of that proposition (even though the statement itself will not be used).

We summarize the discussion above in the following proposition.

Proposition 12.2. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism homotopic to the identity of a hyperbolic 3-manifold M preserving branching foliations $\mathcal{W}^{cs}_{\rm bran}, \mathcal{W}^{cu}_{\rm bran}$. Suppose that a good lift $\widetilde{f}$ fixes a leaf of $\widetilde{\mathcal{W}^{cu}_{\rm bran}}$ and acts as a translation on $\widetilde{\mathcal{W}^{cs}_{\rm bran}}$. Then, up to taking finite iterates and covers, there exists $\gamma \in \pi_1(M)$ and $k > 0$ such that a center stable leaf $L \in \widetilde{\mathcal{W}^{cs}_{\rm bran}}$ is fixed by $h := \gamma \circ \widetilde{f}^k$ and its Lefschetz index is $I_{{\rm Fix}(h)}(h) = 1 - p$ with $p \geq 3$. Moreover, every center curve fixed by h in L is coarsely expanding.*

Let γ be as in the proposition. Let L be a center stable leaf fixed by $h = \gamma \circ \widetilde{f}^k$ and $L_2 = \widetilde{f}^k(L)$. As previously, we write $\tau_{12}\colon L \to L_2$ for the map obtained by flowing from L to L_2 along $\widetilde{\Phi}^{cs}$. We set $g := \gamma \circ \tau_{12}$.

The map g acts on the compactification of L with its ideal circle $L \cup S^1(L)$ the same way as h does (see sections 10 and 11).

Let δ be the unique orbit of $\widetilde{\Phi}_{cs}$ fixed by γ and let x be the (unique) intersection of δ with L. Note that x is the unique fixed point of g. Since we assume that γ fixes the prongs of δ, then h has exactly $2p$ fixed points in $S^1(L)$. These fixed points are contracting if they correspond to an ideal point of $\mathcal{G}^u(x)$ and expanding if they are ideal points of $\mathcal{G}^s(x)$.

12.2. **Proof of Theorem 12.1.** To prove Theorem 12.1 we will first show some properties. Recall from Proposition 11.6 that every proper ray in $L \in \widetilde{\mathcal{W}^{cs}_{\rm bran}}$, fixed by h has a unique limit point in $S^1(L)$ (notice that the ray must be either expanding or contracting). We will show that the fixed rays associated to the center and stable (branching) foliations have different limit points at infinity.

Lemma 12.3. *Let s be a stable leaf in L which is fixed by h. Then the two rays of s limit to distinct ideal points of L. The same holds if c is a center leaf in L fixed by h.*

Proof. We do the proof for the center leaf c, the one for stable leaves is analogous, and a little bit easier (since there is no branching).

By hypothesis, c is fixed by h, hence it is coarsely expanding under h. It follows that there are fixed points of h in c. By Proposition 11.6 each ray of c can only limit in a point in $P \subset S^1(L)$, where, as previously, P is the set of attracting fixed points of h in $S^1(L)$. Let q_1, q_2 be the ideal points of the rays. What we have to prove is that q_1 and q_2 are distinct.

Suppose that $q_1 = q_2$. Then c bounds a unique region S in L which limits only in $q_1 \in S^1(L)$. The other complementary region of c in L limits to every point in $S^1(L)$. Let z be a fixed point of h in c. Then the stable leaf $s(z)$ of z has a ray s_1 entering S. It cannot intersect c again, and it is properly embedded in L. Hence it has to limit in q_1 as well. See Figure 10.

But now this ray is contracting for h. This contradicts Proposition 11.6 because this ray should limit in a point of N. □

Remark 12.4. The proof used strongly that periodic center leafs are coarsely expanding, in order to induce a behavior at infinity. In the examples of [BGHP17] it does happen that different stable curves land in the same ideal point at infinity in their center stable leaf.

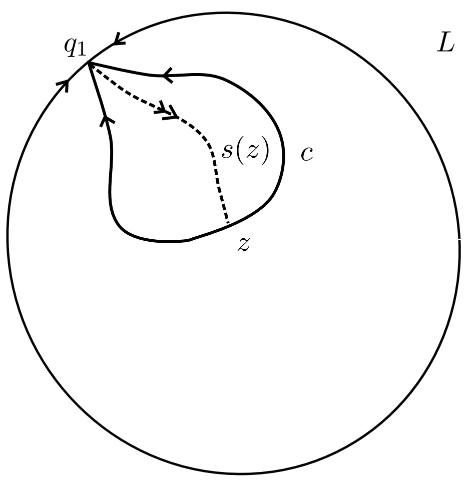


FIGURE 10. Rays have to land in different points of $S^1(L)$.

Now we show a sort of dynamical coherence for fixed center rays.

Lemma 12.5. *Suppose that c_1, c_2 are distinct center leaves in L which are fixed by h. Then c_1, c_2 cannot intersect.*

Notice that since f is not necessarily dynamically coherent, the distinct center leaves c_1, c_2 can a priori intersect each other. The proof will depend very strongly on the fact that center rays fixed by h are coarsely expanding.

Proof. Suppose that c_1, c_2 intersect. Since c_1, c_2 are both fixed by h, so is their intersection. Since h is coarsely expanding in each, then c_1, c_2 share a fixed point of h. In the the proof of Claim 11.3, we showed that c_1 and c_2 cannot form a bigon B.

It follows that there is a point x, fixed by h, which is an endpoint of all intersections of c_1 and c_2: On one side x bounds a ray e_1 of c_1 and a ray e_2 of c_2 such that e_1 and e_2 are disjoint. For a point y in e_1 near enough to x, we have that $s(y)$ must intersects c_2. Since stable lengths are contracting under powers of h, it implies that e_1 is contracting towards x near x and similarly for e_2 (see figure 11). But e_1 is coarsely expanding. Hence there must exist fixed points of h in e_1. Let $y \in e_1$ be the closest point to x which is fixed by h. Similarly, let z in e_2 closest to x fixed by h.

The leaves $s(y)$, $s(z)$ are not separated from each other in the stable leaf space in L.

Let now c be a center leaf through x, which is between c_1 and c_2 and which is the first center leaf not intersecting $s(y)$. Then $h(c) = c$. In addition c has a ray e with endpoint x and intersecting only stable leaves which intersect c_1 between x and y. It follows that this ray is contracting under h, contradicting Proposition 12.2, because this is fixed by h. ☐

Thus far, we showed that distinct center leaves in L, which are fixed by h do not intersect. Then, the proof of Claim 11.4 also implies that fixed center leaves cannot accumulate (as accumulation would imply that some fixed leaves intersect).

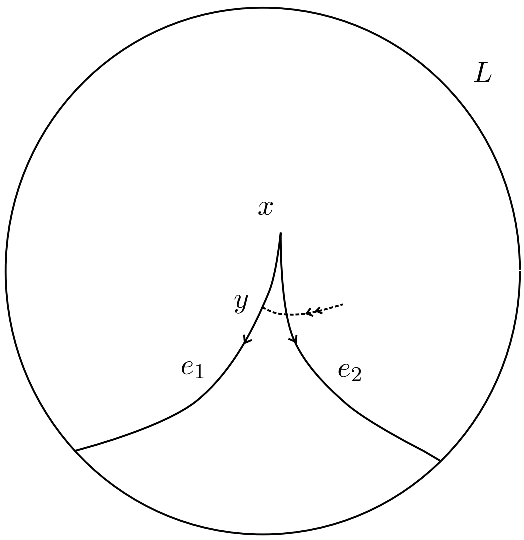


FIGURE 11. Showing the existence of fixed points below x in Lemma 12.5.

We conclude that there are finitely many center leaves in L that are fixed under h. Each such center leaf is coarsely expanding. For each such center leaf c, we consider a small enough open topological disk containing all the fixed points of h in c, and no other fixed point of h in L. Then, on such disks, the Lefschetz index of h is -1. Since the total Lefschetz number of h in L is $1-p$ it follows that:

Lemma 12.6. *There are exactly $p-1$ center leaves which are fixed by h in L.*

This together with the following lemma will allow us to make a counting argument to reach a contradiction.

Lemma 12.7. *Let c_1, c_2 be two distinct center leaves in L fixed by h. Let $y_1 \in c_1$ and $y_2 \in c_2$ be fixed points of h. Then $s(y_1)$ and $s(y_2)$ do not have common ideal points.*

Proof. Suppose, for a contradiction, that there are distinct fixed center leaves c_1, c_2 satisfying the following: There are points $y_1 \in c_1$ and $y_2 \in c_2$, fixed by h, such that $s_1 = s(y_1)$ and $s_2 = s(y_2)$ share an ideal point in $S^1(L)$.

Let q be the common ideal point of the corresponding rays of s_1 and s_2. Let e_j be the ray in s_j with endpoint y_j and ideal point q. Suppose first that no center leaf intersecting e_1 intersects e_2. Let c_0 be a center leaf intersecting e_1. Iterate c_0 by powers of h^{-1}. It pushes points in s_1 away from y_1. Since the leaves $h^{-i}(c_0)$ all intersect s_1 and none of them intersect s_2, the sequence $(h^{-i}(c_0))$ converges to a collection of center leaves as $i \to +\infty$. Then there is only one center leaf in this limit, call it c, which separates all of $h^{-i}(c_0)$ from s_2. This c is invariant under h, but it has an ideal point in q. Now q is a repelling fixed point, so c must have an attracting ray, a contradiction.

It follows that some center leaf intersecting e_1 also intersects e_2. Let c_0 be one such center leaf. Now iterate by positive powers of h. Then $(h^i(c_0))$ converges to a fixed center leaf v_1 through y_1 and a fixed center leaf v_2 through y_2. But then v_1 and c_1 are both fixed by h and both contain y_1. Lemma 12.5 implies that $c_1 = v_1$ and $c_2 = v_2$. In particular $v_1 \neq v_2$, and they are non separated from each other. In this case, consider s the unique stable leaf defined as the first leaf

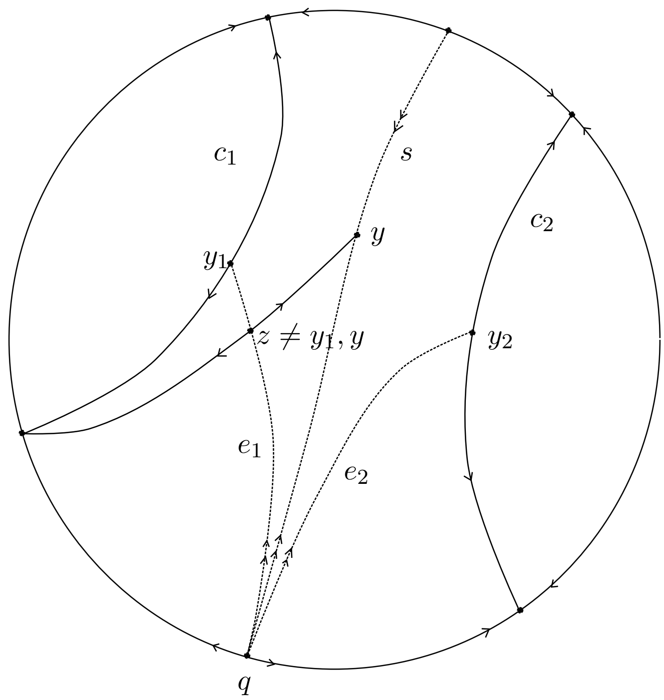


FIGURE 12. A depiction of the main objects in the proof of Lemma 12.7.

not intersecting c_1 that separates s_1 from s_2. Then, as above, h fixes s and has a fixed point y in s. But a center leaf c through y fixed by h has to intersect the interior of the ray e_1. This intersection point is the intersection of c fixed by h, and s_1 fixed by h. So this intersection point is fixed by h. But this is a contradiction, because y_1 is the only fixed point of h in s_1. So Lemma 12.7 is proven. □

We now can complete the proof of Theorem 12.1.

Proof of Theorem 12.1. By Lemma 12.6, there are $p-1$ center leaves fixed by h in L. We denote them by $c_1, \ldots, c_{p-1}$.

Each center leaf has at least one fixed point. Let y_i, $1 \leq i \leq p-1$ be a fixed point in c_i. Then, for each i, Lemma 12.3 states that $s(y_i)$ has two distinct ideal points z_i^1 and z_i^2.

Moreover, for every $i \neq j$, the ideal points of the stable leaves are distinct by Lemma 12.7. It follows that there are at least $2p-2$ distinct points in $S^1(L)$ which are repelling.

But we also know that there are exactly p points in $S^1(L)$ that are repelling under h. It follows that $2p-2 \leq p$, which implies $p=2$. However, we had that $p \geq 3$, thus obtaining a contradiction.

This finishes the proof of Theorem 12.1. □

APPENDIX A. SOME 3-MANIFOLD TOPOLOGY

Besides the 3-manifold topology presented in [BFFP20b, Appendix A] we will need an additional result important to understand certain particular deck transformations when one lifts to finite covers.

Lemma A.1. *Let M be a closed, irreducible 3-manifold with fundamental group that is not virtually nilpotent. Suppose that β is a non trivial deck transformation* *so that $d(x, \beta(x))$ is bounded above in $\widetilde{M}$. Then M is a Seifert fibered space and β represents a power of a regular fiber.*

Proof. First we assume that M is orientable. Then, the JSJ decomposition states that M has a canonical decomposition into Seifert fibered and geometrically atoroidal pieces. We lift this to a decomposition of $\widetilde{M}$ and construct a tree $\mathcal{T}$ in the following way: The vertices are the lifts of components of the torus decomposition of M, and we associate an edge if two components intersect along the lift of a torus. Such a lift of a torus is called a wall. There is a minimum separation distance between any two walls.

The deck transformation β acts on this tree. Let W be a wall. Suppose that $\beta(W)$ is distinct from W. But, as subsets of $\widetilde{M}$, the walls $W, \beta(W)$ are a finite Hausdorff distance from each other. Then $\pi(W)$, $\pi(\beta(W))$ are tori in M, and the region V in $\widetilde{M}$ between $W, \beta(W)$ projects to $\pi(V)$ which is $\mathbb{T}^2 \times [0, 1]$ in M. If this happens then M is a torus bundle over a circle. In that case, use that $\pi_1(M)$ is not virtually nilpotent, so the monodromy of the fibration is an Anosov map of $\mathbb{T}^2$. But then no β as above could satisfy the bounded distance property. It follows that $\beta(W) = W$ for any wall, and in particular $\beta(P) = P$ for any vertex of $\mathcal{T}$.

Now consider a vertex P. Suppose first that $\pi(P)$ is homotopically atoroidal. By the Geometrization Theorem, $\pi(P)$ is hyperbolic. If β restricted to P were to satisfy the bounded distance property, then it would have to be the identity on P. Hence β itself is the identity, contradiction.

Hence all the pieces of the torus decomposition of M are homotopically toroidal. Suppose now that there is one such piece $\pi(P)$ that is geometrically atoroidal (but not homotopically atoroidal). The proof of the Seifert fibered conjecture ([CJ94, Gab92]) shows that $\pi(P)$ has no boundary and $\pi(P)$ is Seifert. In other words, $M = \pi(P)$ is Seifert. So we can assume that all the pieces of the torus decomposition are geometrically toroidal. Then they are all Seifert fibered. Thus M is a graph manifold.

We will show that the torus decomposition of M is in fact trivial, proving that M is Seifert fibered. Suppose it is not true. Then the tree $\mathcal{T}$ is infinite. Let P_1, P_2, P_3 be three consecutive vertices in $\mathcal{T}$. Let W_1 be the wall between P_1 and P_2. Then $\beta(W_1)$ (as a set in $\widetilde{M}$) is a bounded distance from W_1 and sends the Seifert fibration of P in W_1 to lifts of Seifert fibers. It follows that $\beta = \delta_1^k \alpha_1$ where δ_1 represents a regular fiber in $\pi(P_1)$, and α_1 is a loop in $\pi(W_1)$. Similarly if W_2 is the wall between P_2 and P_3 then $\beta = \delta_3^i \alpha_3$ where α_3 is a loop in $\pi(W_3)$. Then α_1, α_3 are both in the boundary of $\pi(P_2)$. The loops representing $\delta_1^k \alpha_1$, $\delta_3^i \alpha_3$ are both in the boundary of $\pi(P_2)$. They represent the same element of $\pi_1(M)$ only when $k = i = 0$ and α_1, α_3 are freely homotopic. That means that P_2 is a torus times an interval, which is impossible in the torus decomposition in our situation as explained above.

It follows now that the torus decomposition of M is trivial, which implies that M is Seifert fibered. Moreover, if the base is not hyperbolic, then $\pi_1(M)$ is virtually nilpotent ([Sco83, Theorem 5.3]). But this contradicts the hypothesis of the lemma.

It follows that the base is hyperbolic. Also β induces a transformation in the universal cover of the base that is a bounded distance from the identity. This can only happen if this transformation is the identity. Therefore β represents a power of a regular Seifert fiber in M (notice that non-regular fibers induce a finite symmetry on the base, thus not the identity, and not a bounded distance from the identity).

So the Lemma is proven when M is orientable. If M is not orientable, then it has a double cover M_2 which is orientable. Now β^2 lifts to an element of $\pi_1(M_2)$ that satisfies the assumption of the lemma. So we can apply the result to M_2 and obtain that M_2 is Seifert. Thus M is doubly covered by a Seifert space, which, by a result of Tollefson [Tol78], implies that M itself is Seifert fibered. It follows that β corresponds to a power of a regular fiber. This finishes the proof of the lemma. □

Appendix B. Minimality and f-minimality

We prove that in certain situations minimality is equivalent to f-minimality. We need the following result which is of interest in itself.

Lemma B.1. *Let $\mathcal{L}_b^{cs}$ be the leaf space of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$. Let $\mathcal{B} \subset \mathcal{L}_b^{cs}$ be a closed set of leaves. Suppose that, for all $x \in \widetilde{M}$, there exists a leaf $L \in \mathcal{B}$ containing x. Then $\mathcal{B} = \mathcal{L}_b^{cs}$.*

Proof. The lemma is obvious when $\mathcal{W}^{cs}_{\text{bran}}$ is a true foliation (and one does not need to require $\mathcal{B}$ to be closed). However, when $\mathcal{W}^{cs}_{\text{bran}}$ has some branching, one could possibly have a union of leaves that cover all of $\widetilde{M}$ without using all the leaves of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$. For closed sets of leaves we show this is not possible.

Let L be a leaf of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$, x a point in L and τ an open unstable segment through x. The set of leaves of $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ intersecting τ is isomorphic to an open interval. Using the transversal orientation to $\widetilde{\mathcal{W}}^{cs}_{\text{bran}}$, we can put an order on this interval.

By our assumption, every point in τ intersects a leaf in $\mathcal{B}$. Let L' be the supremum of leaves in $\mathcal{B}$, intersecting τ and smaller than or equal to L. Since $\mathcal{B}$ is closed, we have $L' \in \mathcal{B}$. Notice that x is in both L and L'.

We claim that $L' = L$. If L is not equal to L' then they branch out. Let y be a boundary point of $L \cap L'$. Let $z \in L'$, with $z \notin L$ be close enough to y so that its unstable leaf $u(z)$ intersects L. Now take any point $w \in u(z)$ in between z and $L \cap u(z)$. Any leaf $L_1 \in \widetilde{\mathcal{W}}^{cs}_{\text{bran}}$ that contains w must contain y. Hence (because leaves do not cross), L_1 also contains x. By definition, it is above L', thus L_1 is not in $\mathcal{B}$. Since this is true for any leaf through w, it contradicts our assumption. □

Lemma B.2. *When $\mathcal{W}^{cs}_{\text{bran}}$ does not have compact leaves, then f-minimality of $\mathcal{W}^{cs}_{\text{bran}}$ is equivalent to minimality of $\mathcal{W}^{cs}_{\text{bran}}$.*

Proof. Note that minimality obviously implies f-minimality, so we only need to show the other implication.

Suppose that $\mathcal{W}^{cs}_{\text{bran}}$ is not minimal and let C be the union of a set of $\mathcal{W}^{cs}_{\text{bran}}$ leaves which is closed and not M. Let $\mathcal{W}^{cs}_\epsilon$ be an approximating foliation, with approximating map h^{cs}_ϵ sending leaves of $\mathcal{W}^{cs}_\epsilon$ to those of $\mathcal{W}^{cs}$. Then $(h^{cs}_\epsilon)^{-1}(C)$ is a set which is a union of $\mathcal{W}^{cs}_\epsilon$ leaves, which is closed and not M. In particular it contains an exceptional minimal set D. By [HH87, Theorem 4.1.3] the actual foliation $\mathcal{W}^{cs}_\epsilon$ has finitely many exceptional minimal sets $B_1, \dots, B_k$. The union B of these is not M because $D \neq M$. The set of leaves in B is a closed set of leaves denoted by $\mathcal{B}$. Then $A = h^{cs}_\epsilon(B)$ is a closed subset of M, and $\mathcal{A} = h^{cs}_\epsilon(\mathcal{B})$ is a closed set of leaves, being the image by h^{cs}_ϵ of the leaves in $\mathcal{B}$. Let $\widetilde{\mathcal{A}} = \pi^{-1}(\mathcal{A})$, we stress that this is on the leaf space level, not in terms of sets. This is a closed subset of $\mathcal{L}^{cs}_b$.

Let $A_i := h^{cs}_\epsilon(B_i)$. Every leaf of $\mathcal{W}^{cs}_{\rm bran}$ which is the image of a leaf in B_i is dense in A_i. Using this, it is easy to see that $f(A) = A$. By f-minimality it follows that $A = M$.

Since $A = M$ then $\widetilde{\mathcal{A}}$ is a closed subset of $\mathcal{L}^{cs}_b$, whose union of points in all leaves of $\widetilde{\mathcal{A}}$ is $\widetilde{M}$ as $A = M$. Lemma B.1 implies that $\widetilde{\mathcal{A}} = \mathcal{L}^{cs}_b$. Hence for each leaf E of $\mathcal{W}^{cs}_{\rm bran}$, it is the image of a leaf F in some B_i. Conversely every leaf of $\mathcal{W}^{cs}_\epsilon$ maps by h^{cs}_ϵ to a leaf of $\mathcal{W}^{cs}_{\rm bran}$.

For each leaf E of $\mathcal{W}^{cs}_{\rm bran}$, its preimage $(h^{cs}_\epsilon)^{-1}(E)$ is a closed interval of leaves of $\mathcal{W}^{cs}_\epsilon$. No leaf in the interior of the interval can be in a B_i as it is a minimal set. It follows that the complementary regions of B in M are I-bundles. These can be collapsed to generate another foliation $\mathcal{C}$. Since the B_i were minimal sets of $\mathcal{W}^{cs}_\epsilon$, then the collapsing of each of these is a minimal set of $\mathcal{C}$. Since the union is all of M, there can be only one such minimal set, so $\mathcal{W}^{cs}_\epsilon$ is minimal.

But this contradicts the fact that D is an exceptional minimal set of $\mathcal{W}^{cs}_\epsilon$. □

We state the following criteria for dynamical coherence (which in this setting is quite obvious).

Proposition B.3 (Proposition 1.6 and Remark 1.10 in [BW05])**.** *Assume that f is a partially hyperbolic diffeomorphism admitting branching foliations $\mathcal{W}^{cs}_{\rm bran}$ and $\mathcal{W}^{cu}_{\rm bran}$ and assume that*

- *no two different leaves of $\mathcal{W}^{cs}_{\rm bran}$ intersect,*
- *no two different leaves of $\mathcal{W}^{cu}_{\rm bran}$ intersect.*

Then, f is dynamically coherent.

Appendix C. The Lefschetz index

Here we define the Lefschetz index and give the main property that we used. We refer to the monograph by Franks [Fra82, Section 5] for details and other references.

For any space X and subset $A \subset X$, we denote by $H_k(X, A)$ the k-th relative homology group with coefficients in $\mathbb{Z}$.

Definition C.1. Let $V \subset \mathbb{R}^k$ be an open set and $F\colon V \subset \mathbb{R}^k \to \mathbb{R}^k$ be a continuous map such that the set of fixed point of F is $\Gamma \subset V$, a compact set. Then the *Lefschetz index* of F, denoted by $I_\Gamma(F)$ is an element in $\mathbb{Z} \cong H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\})$, defined as follows. It is the image by $(\mathrm{id} - F)_*\colon H_k(V, V - \Gamma) \to H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\})$ of the class u_Γ, where u_Γ itself is the image of the generator 1 under the composite $H_k(\mathbb{R}^k, \mathbb{R}^k - D) \to H_k(\mathbb{R}^k, \mathbb{R}^k - \Gamma) \cong H_k(V, V - \Gamma)$. Here D is a ball containing Γ.

It is easy to see that if $\Gamma = \mathrm{Fix}(F) = \Gamma_1 \cup \cdots \cup \Gamma_j$, where Γ_i are compact and disjoint then $I_\Gamma(F) = \sum_1^j I_\Gamma(F)$. Here $I_\Gamma(F)$ is the index restricted to an open set V_i of V which does not intersect the other Γ_m, see [Fra82, Theorem 5.8 (b)].

This technical definition works well with the standard examples. For a single hyperbolic fixed point q, the index at q is exactly $\mathrm{sgn}(\det(\mathrm{id} - D_qF))$ (see [Fra82, Proposition 5.7]), where det is the determinant, and sgn is the sign of the determinant. Hence in dimension 2 the index of a hyperbolic fixed point when the orientation of the bundles is preserved is -1. This can be generalized to a p-prong hyperbolic fixed point to obtain that the index is $1 - p$. This is because the index is invariant by homotopic changes. A p-prong can be easily split into

$p-1$ distinct hyperbolic points which are differentiable. In addition for any fixed set which behaves locally as a hyperbolic fixed point, the index is the same as the hyperbolic fixed point.

The main property we use is the following.

Proposition C.2 (Theorem 5.8(c) of [Fra82]). *Let P be a topological plane equipped with a metric d. Let $g, h\colon P \to P$ be two homeomorphisms. Suppose that there exists $R > 0$ such that:*

- *For every $x \in P$, one has that $d(g(x), h(x)) < R$;*
- *There is a disk D such that, for every $x \notin D$, one has that $d(x, g(x)) > 2R$.*

Then, the total index $I_{\mathrm{Fix}(g)}(g) = I_{\mathrm{Fix}(h)}(h)$.

See also [KH95, Section 8.6] for an alternate presentation of the Lefschetz index.

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