COLLAPSED ANOSOV FLOWS AND SELF ORBIT EQUIVALENCES

THOMAS BARTHELMÉ, SERGIO R. FENLEY, AND RAFAEL POTRIE

Abstract. We propose a generalization of the concept of discretized Anosov flows that covers a wide class of partially hyperbolic diffeomorphisms in 3-manifolds, and that we call collapsed Anosov flows. They are related with Anosov flows via a self orbit equivalence of the flow. We show that all the examples from [BGHP] belong to this class, and that it is an open and closed class among partially hyperbolic diffeomorphisms. We provide some equivalent definitions which may be more amenable to analysis and are useful in different situations. Conversely we describe the class of partially hyperbolic diffeomorphisms that are collapsed Anosov flows associated with certain types of Anosov flows.

1. Introduction

For about 15 years Pujals' conjecture [BWi] has served as a blueprint and motivation for the understanding and classification of partially hyperbolic diffeomorphisms in dimension 3. In most 3-manifolds, that is, those with non virtually solvable fundamental group[1], the conjecture affirmed that, up to iterates and finite lifts, a transitive partially hyperbolic diffeomorphism had to behave like the discretization of an Anosov flow: the diffeomorphism should globally fix each orbit of an associated Anosov flow, moving points along the orbits.

In the past few years, Pujals' conjecture was disproved: Examples built in [BGP, BGHP] (see also [BPP, BZ]) gave a plethora of new partially hyperbolic diffeomorphisms. All of these new examples are such that they have infinite order in the mapping class group of their supporting manifolds, contrarily to Pujals' conjecture.

Thanks to a criterion developed in [BGHP], called φ-transversality (see Definition 2.8), these new examples — as well as the older examples of [BWi] — can be described in the following way: Start with an Anosov flow ϕ_t on a manifold M. Then find a diffeomorphism φ of M that preserves the transversality of the bundles of the Anosov splitting (more precisely, such that the flow ϕ_t is φ-transversal to itself, see Definition 2.8). Finally, compose φ with a very large time of the flow ϕ_t and obtain a partially hyperbolic diffeomorphism.

Finding the good diffeomorphisms φ is generally the difficult step, but one type of map that does work (as chosen in [BWi]) is a smooth symmetry of the Anosov flow.

Key words and phrases. Anosov flows, Partially hyperbolic diffeomorphisms, 3-manifolds, Classification.

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[1] See, e.g., [HP_1] for the other case.

In this article we show that, while not obvious from the construction, all of the new examples of partially hyperbolic diffeomorphisms are related to symmetries (self orbit equivalences to be precise) of the starting flow.

More generally, the main goal of our article is to introduce, and start the study of, a new class of partially hyperbolic diffeomorphisms in dimension 3, that we call *collapsed Anosov flows*. A partially hyperbolic diffeomorphism is a collapsed Anosov flow if there exists a global collapsing map, homotopic to the identity, that semi-conjugates a self orbit equivalence of a topological Anosov flow with the diffeomorphism.

This class of diffeomorphisms has very interesting properties. In particular we show the following (formalized below as Theorems A and C).

Informal Statement. *Collapsed Anosov flows form an open and closed class of partially hyperbolic diffeomorphisms in dimension 3 that contains all known examples in manifolds with non-virtually solvable fundamental group.*

Since our goal is in part to lay down the basis for a future study of this class, we introduce four definitions: Three of them (*collapsed Anosov flow*, *strong collapsed Anosov flow* and *leaf space collapsed Anosov flow*, Definitions 2.5, 2.7 and 2.10 respectively) have to do with how restrictive one wants the semi-conjugacy to be in terms of its behavior with respect to either center curves or the branching foliations of the partially hyperbolic diffeomorphisms. The last definition (*quasi-geodesic partially hyperbolic diffeomorphisms*, Definition 2.13) is singular as it instead asks for the center foliation to be by quasigeodesics inside each center stable and center unstable leaves[2].

Under some orientability conditions, we prove equivalence between quasigeodesic partially hyperbolic diffeomorphisms, strong collapsed Anosov flows and leaf space collapsed Anosov flows (Theorems B and D). We believe that these equivalences will show themselves to be quite useful: For instance, the proof, obtained in [FP_2], that every hyperbolic 3-manifold that admits a partially hyperbolic also admits an Anosov flow relies on these equivalences.

In light of the fact that the known counter-examples to Pujals' conjecture are all collapsed Anosov flows, it is natural to ask the following (thus extending [BGP, Question 1] and making [Pot_2, Question 12] precise).

Question 1. *Let M be a 3-manifold with non virtually solvable fundamental group and $f\colon M \to M$ a (transitive) partially hyperbolic diffeomorphism. Is f a collapsed Anosov flow?*

One interest of Pujals' conjecture was to suggest that the classification of partially hyperbolic diffeomorphisms in dimension 3 could be done up to classification of Anosov flows. If the question above admits a positive answer, then that view behind Pujals' conjecture may still be true, as it seems possible to understand all the self orbit equivalences of Anosov flows without having a full classification of the flows.

While we will not suggest an answer to Question 1 in full generality, there are several contexts where we can say more:

(i) When M is hyperbolic, the answer is proven to be positive in [FP_2].
(ii) When the partially hyperbolic diffeomorphism is homotopic to the identity, the answer is likely positive (see [BFFP_1, BFFP_2]).

[2]It also requires the center stable/unstable branching foliations to be by Gromov-hyperbolic leaves.

(iii) For Seifert manifolds, current work in progress by the second and third authors also indicates a positive answer.

Another potential interest we see in collapsed Anosov flows is that it may allow to successfully decouple the dynamical study of partially hyperbolic diffeomorphisms from the question of their classification. Indeed, one may be able to obtain fine dynamical properties when restricting to particular types of collapsed Anosov flows.

This strategy has previously been successfully used in [AVW, BFT, BaG_2, DWX, FP_1, GM] for *discretized Anosov flows*, or similar concepts. Discretized Anosov flows were introduced in [BFFP_1] (although related notions appeared previously, for instance in [BWi, BoG]). One can view them as collapsed Anosov flows where the self orbit equivalence is trivial, meaning that it fixes every orbit of the flow (see §5). By [BFFP_1, BFFP_2], discretized Anosov flows represent a very large class of partially hyperbolic diffeomorphisms. An example of a dynamical consequence is [FP_1], where it is shown that discretized Anosov flows are always *accessible* unless they come from suspensions (in particular, smooth volume preserving ones are ergodic). In fact, another, albeit slightly weaker, accessibility result is obtained for some specific collapsed Anosov flows in [FP_1] (but without using that terminology), and it seems plausible that such results could be achievable for other classes of collapsed Anosov flows.

Finally, we can use the interaction of collapsed Anosov flows with self orbit equivalences of Anosov flows to classify them up to isotopy. Over the years, a deep knowledge of the orbit space of Anosov flows in dimension 3 has been attained. This in turn gives restrictions on how a self orbit equivalence can act. For instance, self orbit equivalences that are homotopic to the identity were classified in [BaG_1], showing that there are at most two types of actions in that case. Knowing restrictions about self orbit equivalences (for instance which isotopy classes can support them) directly implies restrictions on possible collapsed Anosov flows.

On the other hand, a general method to build self orbit equivalences of Anosov flow has not yet been developed. The construction methods of [BGHP] together with Theorem A gives one such method.

We illustrate what consequences this interaction gives us in a few specific cases. In particular, we give a complete description of collapsed Anosov flows up to isotopy: 1) when the manifold is the unit tangent bundle of a surface; 2) when the associated flow is the Franks–Williams example [FW]; or 3) when the collapsed Anosov flow is homotopic to the identity (see §11).

The conceptualization of the notion of collapsed Anosov flows that we introduce here has been in part motivated by [FP_2] (and to a lesser extent [BFFP_1, BFFP_2]). The indebtedness we have to these previous works does not translate, however, into their direct use in the present article. Indeed, the scope, as well as most of the techniques we use here, are different in nature from those in the aforementioned work.

2. Collapsed Anosov flows

In this paper, M will always denote a closed 3-dimensional manifold. It is possible that some notions make sense in higher dimensions but we will repeatedly use facts about foliations and Anosov flows in dimension 3 that are unknown in higher dimensions and we have not checked to which extent arguments extend (even if only in part) to higher dimensions.

In this section, we will make precise the different definitions of collapsed Anosov flows alluded to earlier and formally state the main results of this article.

First we review the notion of topological Anosov flows, in order to be able to introduce collapsed Anosov flows.

2.1. **Anosov flows and topological Anosov flows.** Recall that an *Anosov flow* is a flow $\phi_t \colon M \to M$ generated by a vector field X such that $D\phi_t$ preserves a splitting $TM = E^s \oplus \mathbb{R}X \oplus E^u$ and there exists $T > 0$ satisfying

$$\|D\phi_T v^s\| < \frac{1}{2} < 2 < \|D\phi_T v^u\|,$$

for every unit vector $v^\sigma \in E^\sigma$ $(\sigma = s, u)$.

It is well known that this property implies that the splitting is continuous. Moreover, it follows that the bundles E^s and E^u are uniquely integrable into ϕ_t-invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ tangent respectively to E^s and E^u [Ano] called the *strong stable* and *strong unstable* foliations of ϕ_t. One obtains ϕ_t-invariant foliations $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$ called the *weak stable* and *weak unstable* foliations by taking the saturation of the previous foliations by the flow. Note that these foliations are the unique foliations tangent respectively to $E^{ws} := E^s \oplus \mathbb{R}X$ and $E^{wu} := \mathbb{R}X \oplus E^u$ [Ano]. See [HPS] or [CP, §4] and references therein for more details.

The following generalizes Anosov flows: A *topological Anosov flow* is a continuous flow $\phi_t \colon M \to M$ generated by a vector field X which shares the topological properties of an Anosov flow, that is:

- It preserves two continuous one dimensional foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that if $y \in \mathcal{F}^s(x)$ then $d(\phi_t(x), \phi_t(y)) \to 0$ as $t \to \infty$ and $d(\phi_t(x), \phi_t(y)) > \varepsilon_0 > 0$ for all $t < t_0$ (t_0 depending on x, y) and the same properties but with time reversed for $\mathcal{F}^u$.
- It preserves $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$, the weak stable and unstable (two-dimensional) foliations defined as the flow saturation of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. Moreover, the foliations $\mathcal{F}^{ws}$ and $\mathcal{F}^u$ (resp. $\mathcal{F}^{wu}$ and $\mathcal{F}^s$) are uniformly topologically transverse meaning that there are local charts of uniform size at each point sending the $\mathcal{F}^{ws}$ to horizontal planes and $\mathcal{F}^u$ to vertical lines in the cube $[-1, 1]^3$ (resp. $\mathcal{F}^{wu}$ to horizontal planes and $\mathcal{F}^s$ to vertical lines).

An *orbit equivalence* between (topological) Anosov flows $\phi^1_t \colon M \to M$ and $\phi^2_t \colon N \to N$ is a homeomorphism $\beta \colon M \to N$ sending orbits of ϕ^1_t to orbits of ϕ^2_t preserving the orientation. In other words, there exists a reparametrization $\phi^2_{u(x,t)}$[3] of ϕ^2_t such that β is a conjugation, i.e., $\beta(\phi^1_t(x)) = \phi^2_{u(t,x)}(\beta(x))$, where $u(t, x)$ is monotone increasing for fixed x.

Remark 2.1. It has been recently proved that every transitive topological Anosov flow is *orbit equivalent* to an Anosov flow [Sha].

Definition 2.2. A *self orbit equivalence* of an Anosov flow ϕ_t is an orbit equivalence between ϕ_t and itself.

Self orbit equivalences homotopic to the identity have been studied in [BaG$_1$] to understand fiberwise Anosov dynamics, but in fact there are self orbit equivalences of certain Anosov flows which are not homotopic to the identity.

[3]In order for $\phi^2_{u(x,t)}$ to be a flow, the function u must satisfy the following cocycle condition: $u(x, t + s) = u(\phi^1_t(x), s) + u(x, t)$. See, e.g., [KH, §2.2].

Definition 2.3. We say that a self orbit equivalence β is *trivial* if there exists a continuous function $\tau\colon M \to \mathbb{R}$ such that $\beta(x) = \phi_{\tau(x)}(x)$. Two self orbit equivalences α, β are said to be *equivalent* (or that they belong to the same class) if $\alpha \circ \beta^{-1}$ is a trivial self orbit equivalence.

2.2. **Partially hyperbolic diffeomorphisms.** A *partially hyperbolic diffeomorphism* is a diffeomorphism $f\colon M \to M$ such that Df preserves a splitting $TM = E^s \oplus E^c \oplus E^u$ into non-trivial bundles such that there is $\ell > 0$ verifying that for every $x \in M$ and unit vectors $v^\sigma \in E^\sigma(x)$ $(\sigma = s, c, u)$ one has:

$$\|Df^\ell v^s\| < \frac{1}{2}\min\{1, \|Df^\ell v^c\|\}, \text{ and } \|Df^\ell v^u\| > 2\max\{1, \|Df^\ell v^c\|\}.$$

As in the Anosov flow case, this condition implies that the bundles are continuous. It also implies unique integrability of the bundles E^s and E^u into foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, called *strong stable* and *strong unstable* foliations respectively (see [CP]). We denote by $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$.

Remark 2.4. It follows that given an Anosov flow, its time one map is partially hyperbolic and E^c coincides with the bundle generated by the vector field tangent to the flow.

When necessary, we will denote the dependence of bundles or foliations on the maps with a subscript, e.g., E^s_f or $\mathcal{F}^s_\phi$.

2.3. **Collapsed Anosov flows and strong collapsed Anosov flows.** We are now ready to give the formal definition of a collapsed Anosov flow.

Definition 2.5 (Collapsed Anosov Flow)**.** A partially hyperbolic diffeomorphism f of a closed 3-dimensional manifold M is said to be a *collapsed Anosov flow* if there exists a topological Anosov flow ϕ_t, a continuous map $h\colon M \to M$ homotopic to the identity and a self orbit equivalence $\beta\colon M \to M$ of ϕ_t such that:

(i) The map h is differentiable along the orbits of ϕ_t and maps the vector field tangent to ϕ_t to non-zero vectors tangent to E^c.
(ii) For every $x \in M$ one has that $f \circ h(x) = h \circ \beta(x)$.

As noted earlier, *discretized Anosov flows* (as defined in [BFFP$_1$], see §5.5) are collapsed Anosov flows, where h can be taken to be the identity and β is a trivial self orbit equivalence.

Another case, discussed previously, that is easily seen to be a collapsed Anosov flow is when an Anosov flow ϕ_t commutes with a smooth map β (as in [BWi, Proposition 4.5] for instance), then $\beta \circ \phi_1$ is a collapsed Anosov flow (with self orbit equivalence $\beta \circ \phi_1$ and h the identity). However, we show (Proposition 10.10) that these examples are always "periodic" in the sense that a power of the diffeomorphism is a discretized Anosov flow (or, equivalently, a power of the self orbit equivalence is trivial).

In contrast, the examples of [BGHP] will give, thanks to Theorem A, collapsed Anosov flows associated with self orbit equivalences of infinite order.

Remark 2.6. The definition of a collapsed Anosov flow forces the center direction of f to be orientable, since we can induce an orientation from the orientation of the flow direction via h. To see this, suppose that E^c is not orientable, and suppose that α is a closed curve starting at x in M such that it reverses the local orientation of E^c. Let γ be the deck transformation associated to α. Lift x to $\widetilde{x}$ in $\widetilde{M}$. Let $\widetilde{h}$ be a lift of h which is a lift of a homotopy of h to the identity. Then $\widetilde{h}$ commutes with every deck transformation. Since h is homotopic to the identity it is degree one, so there is $\widetilde{y}$ in $\widetilde{M}$ such that $\widetilde{h}(\widetilde{y}) = \widetilde{x}$. Let η be a curve in $\widetilde{M}$ from $\widetilde{y}$ to $\gamma(\widetilde{y})$. Along $\widetilde{h}(\eta)$ the projection of the flow lines of $\widetilde{\phi}_t$ by $\widetilde{h}$ induces a non zero vector tangent to E^c. Since γ commutes with $\widetilde{h}$ the final vector is the tangent to E^c in the direction induced by γ. But γ was supposed to reverse the direction of E^c so this is a contradiction to the fact that the tangent vectors to E^c are changing continuously along the curve $\widetilde{h}(\eta)$.

Note that we require more from a collapsed Anosov flow than just being semi-conjugated to a self orbit equivalence of an Anosov flow. Indeed condition (i) of Definition 2.5 asks that the semi-conjugacy h at least preserve the center direction.

There are several reasons for this condition: First, going back to at least [HPS], an overarching idea has been that any kind of classification for partially hyperbolic diffeomorphisms should be "up to center dynamics" (where the precise meaning of this can be taken to be more strong or less strong, and somehow has to be adapted to the particular situation of study). Therefore we see condition (i) as the minimal requirement in order to keep to the spirit of this paradigm. A second, less philosophical, reason is that a collapsed Anosov flow thus defined is a natural extension of the concept of discretized Anosov flow introduced in [BFFP$_1$] (see §5.5, in particular Proposition 5.15). Finally, Definition 2.5 provides us with a model of the dynamical behavior of the partially hyperbolic diffeomorphism to compare it with. Moreover, since some features of the dynamics of a self orbit equivalence β can be readily understood, we hope that Definition 2.5 is enough to understand some of the dynamical properties of a collapsed Anosov flow, as has been the case for discretized Anosov flows.

There is an issue one quickly runs into when one wants to extract more geometrical or topological information about the partially hyperbolic diffeomorphism from the collapsed Anosov flow definition: The map h sends orbits of the Anosov flow to center curves (i.e., curves tangent to the center direction) of the diffeomorphism f. However, it is usually difficult in partially hyperbolic dynamics to extract much knowledge about the behavior of center stable and center unstable (branching) foliations from coarse information about center curves. In fact, the center curves obtained via h may, a priori, not even be inside the intersection of a center stable and center unstable leaf[4].

One can resolve this issue, while preserving the interest of the class of partially hyperbolic diffeomorphisms thus defined, by requiring that the semi-conjugacy h somehow sends the weak stable and unstable directions of the flow to the center stable and unstable directions of the diffeomorphism. This leads us to our next definition.

Definition 2.7 (Strong Collapsed Anosov Flow)**.** A partially hyperbolic diffeomorphism f of a closed 3-manifold M is called a *strong collapsed Anosov flow* if there exists a topological Anosov flow ϕ_t, a continuous map $h\colon M \to M$ homotopic to identity and a self orbit equivalence $\beta\colon M \to M$ of ϕ_t such that:

(i) The map h is differentiable along orbits of ϕ_t and maps the vector field tangent to ϕ_t to non vanishing vectors tangent to E^c.
(ii) The image by h of a leaf of $\mathcal{F}^{ws}_\phi$ (resp. $\mathcal{F}^{wu}_\phi$) is a C^1-surface tangent to E^{cs} (resp. E^{cu}).
(iii) The map h locally preserves the orientation in leaves of both foliations.

[4]It is not even known if a collapsed Anosov flow necessarily admits invariant center stable or center unstable branching foliations, as the existence result of Burago–Ivanov [BI], see §3, requires some orientability conditions.

(iv) For every $x \in M$ one has that $f \circ h(x) = h \circ \beta(x)$.

By being a C^1-surface tangent to E^{cs} we mean that if $\widetilde{h}$ is a lift of h to $\widetilde{M}$ and L is a leaf of $\widetilde{\mathcal{F}}^{ws}_\phi$ then $\widetilde{h}(L)$ is a C^1, properly embedded plane in $\widetilde{M}$ tangent to E^{cs}.

Under these conditions, we can make precise what we mean by h locally preserves the orientation: the map h sends a segment I of an orbit on a leaf L in $\mathcal{F}^{ws}_\phi$, oriented by the direction of the flow, to a segment $h(I)$ of a center curve that locally splits $h(L)$ in two. We thus ask for the left side of I to be send by h to the left side of $h(I)$ (note that there may be some collapsing).

Clearly, a strong collapsed Anosov flow is a collapsed Anosov flow, but we do not know whether those definitions are distinct or equivalent.

Notice that a strong collapsed Anosov flow automatically admits a pair of invariant center stable and center unstable branching foliations (see §3 for the precise definition) by looking at the image under h of the weak foliations of the Anosov flow. Definition 2.5 on the other hand does not directly imply the existence of such branching foliations. But even if one assumes that a collapsed Anosov flow has branching foliations (or even true foliations), it is not clear that it is enough to make it a strong collapsed Anosov flow. Part of the issue arising here is that, in general, the center direction of a a partially hyperbolic diffeomorphism is not uniquely integrable (even when it integrates to a foliation, see [HHU_2]).

Let us mention here, that we obtain some results about unique integrability of the center direction in §10.2.

2.4. **First result and examples.** In [BGHP] a notion of transversality was introduced that allows to produce new examples of partially hyperbolic diffeomorphisms. This encompasses results proved in previous papers [BPP, BGP, BZ].

Definition 2.8. Let $\phi_t\colon M \to M$ be an Anosov flow generated by a vector field X in a closed 3-manifold and preserving a splitting $TM = E^s \oplus \mathbb{R}X \oplus E^u$ and $\varphi\colon M \to M$ a diffeomorphism. We say that ϕ_t is *φ-transverse to itself* if $D\varphi(E^u)$ is transverse to $E^s \oplus \mathbb{R}X$ and $D\varphi^{-1}(E^s)$ is transverse to $\mathbb{R}X \oplus E^u$.

Note that this notion makes sense more generally when considering any partially hyperbolic diffeomorphism instead of the Anosov flow ϕ_t, see [BGHP].

Using this notion, [BGHP] proves:

Proposition 2.9 (Proposition 2.4 [BGHP]). *If an Anosov flow ϕ_t is φ-transverse to itself, then there exists $T > 0$ such that , for all $t > T$, the map $f_t := \phi_t \circ \varphi \circ \phi_t$ is*[5] *partially hyperbolic.*

Not only does [BGHP] give that criterion for building partially hyperbolic diffeomorphisms, but it also gives many examples (using results of [BPP, BGP, BZ]) of maps φ and Anosov flows ϕ_t that are φ-transverse to themselves.

But while great at providing examples, this criterion fails, at least directly, to give any direct understanding of the structure that these maps may enjoy. Indeed, it is a priori not obvious, and it may even seem contradictory, how these examples may act: On the one hand, many of them are not homotopic to the identity, while when t is large, the dynamics seems to be governed by the Anosov flow ϕ_t.

Our first main result gives an understanding of how the behaviors of φ and ϕ_t must play together and makes clear the structure of these examples.

[5] Note that we wrote it this way for convenience, since f_t is smoothly conjugate to $\phi_{2t} \circ \varphi$ and to $\varphi \circ \phi_{2t}$.

Theorem A. *Let $\phi_t\colon M \to M$ be an Anosov flow on a closed 3-manifold and $\varphi\colon M \to M$ a diffeomorphisms such that ϕ_t is φ-transversal to itself. Then, there exists $t_0 > 0$ such that for all $t > t_0$ the diffeomorphism $f_t = \phi_t \circ \varphi \circ \phi_t$ is a strong collapsed Anosov flow of the flow ϕ_t.*

With the help of Theorem A one can prove that all the partially hyperbolic diffeomorphisms built in [BGHP] are collapsed Anosov flows.[6] This not only gives a wealth of examples, but also shows that all known constructions of partially hyperbolic diffeomorphisms on 3-manifolds with non virtually solvable fundamental group are collapsed Anosov flows.

In the examples that we advertised earlier, i.e., the discretized Anosov flows and the examples of [BWi], the map h of Definition 2.7 could always be taken to be a homeomorphism (in fact, the identity). Now, some of the collapsed Anosov flows obtained through Theorem A show why we cannot always ask for the collapsing map h to be a homeomorphism: Indeed, if h is injective, then the image by h of the weak stable and weak unstable foliations of the Anosov flow ϕ_t are center stable and center unstable foliations of the strong collapsed Anosov flow f. In particular, f must be *dynamically coherent*[7]. Since some examples built in [BGHP] are shown to be non dynamically coherent, the associated map h must be non-injective.

It is an interesting question to try to determine when the map h can be a homeomorphism, or equivalently when a strong collapsed Anosov flow may be dynamically coherent. Some examples build in [BPP, BZ] are dynamically coherent and associated with a non-periodic self orbit equivalence. But the associated Anosov flow is *non transitive.*

So far, the only collapsed Anosov flows associated with a *transitive* Anosov flow that are known to be dynamically coherent are such that the self orbit equivalent is periodic (i.e., such that a power is a trivial self orbit equivalence).

2.5. Leaf space collapsed Anosov flows.

Although not explicit, the definition of a strong collapsed Anosov flow implies the existence of center stable $\mathcal{W}^{cs}$ and center unstable $\mathcal{W}^{cu}$ branching foliations (we defer their precise definitions to §3) that are invariant under f. By taking the intersection of these branching foliations (in an appropriate way), one gets an invariant 1-dimensional *center* branching foliation $\mathcal{W}^c$.

It is possible to generalize the definition of a leaf space of a true foliation to the branching case (see §3 or [BFFP$_2$]), and we thus obtain the center leaf space $\mathcal{L}^c$, on which any lift $\widetilde{f}$ of f to the universal cover acts naturally.

For a collapsed Anosov flow which preserves branching foliations, this center leaf space $\mathcal{L}^c$ should be the same as the orbit space of an Anosov flow, and the action of $\widetilde{f}$ should correspond to the action of a lift of a self orbit equivalence. This idea is made precise in the next definition of a *leaf space collapsed Anosov flow.*

For a topological Anosov flow $\phi_t\colon M \to M$ we denote by $\mathcal{O}_\phi$ the orbit space of the flow $\widetilde{\phi}_t$ which is the lift of ϕ_t to $\widetilde{M}$. We recall that $\mathcal{O}_\phi$ is homeomorphic to $\mathbb{R}^2$ and $\pi_1(M)$ acts naturally on $\mathcal{O}_\phi$, see [Bar$_1$, Fen$_1$].

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[6] To be precise, one proves that all examples *à la* [BGHP], understood as any example obtained via Proposition 2.9, are collapsed Anosov flows by applying Theorems A and C together. See Remark 10.4.

[7] A partially hyperbolic diffeomorphism f is called *dynamically coherent* if it preserves a pair of foliations tangent to respectively $E^s \oplus E^c$ and $E^c \oplus E^u$.

Definition 2.10 (Leaf space collapsed Anosov flow)**.** We say that a partially hyperbolic diffeomorphism f of a closed 3-manifold is a *leaf space collapsed Anosov flow* if it preserves center stable and center unstable branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ and there exists a topological Anosov flow ϕ_t and a homeomorphism $H\colon \mathcal{O}_\phi \to \mathcal{L}^c$ which is $\pi_1(M)$-equivariant.

That H is $\pi_1(M)$-equivariant means that if $\gamma \in \pi_1(M)$ is a deck transformation, then, $H(\gamma o) = \gamma H(o)$, for any $o \in \mathcal{O}_\phi$.

Remark 2.11*.* We emphasize that the map $H\colon \mathcal{O}_\phi \to \mathcal{L}^c$ in the definition above is a *homeomorphism*, and not just a surjective continuous map as $h\colon M \to M$ was in Definitions 2.5 and 2.7. We can require this because, although distinct center leaves may merge, they always represent different points in the center leaf space $\mathcal{L}^c$.

Remark 2.12*.* Note that Definition 2.10 does not involve a self orbit equivalence explicitly. However, it is easy to note that there is a self orbit equivalence class associated to a leaf space collapsed Anosov flow since the action of a lift $\widetilde{f}$ of f to $\widetilde{M}$ induces a permutation of leaves of $\mathcal{L}^c$ which via H induces a permutation of orbits of ϕ_t. The fact that from this one can actually construct a self orbit equivalence follows from a standard averaging argument, see for instance [Bar_1, Theorem 3.4].

The homeomorphism H of Definition 2.10 identifies the center leaf space of a leaf space collapsed Anosov flow f with the orbit space of an Anosov flow ϕ_t. The difficulty to go from there to a strong collapsed Anosov flow (Definition 2.7) is to build a map h on the manifold from the map H which is only on the orbit/center leaf space. This is done (in §9) using a standard averaging argument (although made harder by the existence of branching).

There is however a wrinkle to smooth out before this: The map H of Definition 2.10 is not explicitly required to behave well with respect to the center stable and unstable (branching) foliations. That is, H is not assumed to identify the weak (un)stable leaf space of ϕ_t with the center (un)stable leaf space of f. However, thanks to the fact that pairwise transverse foliations invariant by an Anosov flow are unique (see Proposition 5.5), H will automatically identify the weak (un)stable leaf space of the Anosov flows with the center (un)stable leaf space of the diffeomorphism (Proposition 5.6).

Thus we obtain the following equivalence:

Theorem B. *If a diffeomorphism f is a strong collapsed Anosov flow then it is a leaf space collapsed Anosov flow. Moreover, if E^s or E^u are orientable, the converse also holds.*

In order to prove Theorem A, we will prove that the examples are leaf space collapsed Anosov flows and use Theorem B (some additional work allows to bypass the orientability condition in Theorem B).

2.6. **The space of collapsed Anosov flows.** Based on a result of [HPS], that we expand upon in Appendix B, we are able to obtain a global stability result for collapsed Anosov flows.

Theorem C. *The space of collapsed Anosov flows for a given Anosov flow ϕ_t and self orbit equivalence class β is open and closed among partially hyperbolic diffeomorphisms on 3-manifolds. Similarly, the space of leaf space collapsed Anosov flows is open and closed among partially hyperbolic diffeomorphisms.*

Similar statements for other classes of systems have been obtained in [Pot$_1$, FPS]. This result has also been announced for discretized Anosov flows in *any* dimension in [Mar].

In terms of classification, Theorem C gives us that any partially hyperbolic diffeomorphism in a connected component of a collapsed Anosov flow (in the space of partially hyperbolic diffeomorphisms) is also a collapsed Anosov flow, for the same flow and the same self orbit equivalence class. In particular, two leaf space collapsed Anosov flows in the same connected component have homeomorphic center leaf spaces and act equivariantly on them.

However, to stay even closer to the spirit of the first efforts at a classification of partially hyperbolic diffeomorphisms, as in [HPS] or Pujals' conjecture [BWi], we may want to ask more: One may hope that inside a connected component, not only are the center leaf spaces homeomorphic, but so is the structure of *branching of center leaves*. More precisely, suppose that f_1 and f_2 are two leaf space collapsed Anosov flows associated with an Anosov flow ϕ_t and the same self orbit equivalence class β. Then $\mathcal{L}^c_1$ the center leaf space of f_1 is homeomorphic to $\mathcal{L}^c_2$, via the composition $H_2 \circ H_1^{-1}$, where the H_i are as in Definition 2.10. However it may a priori happen that two center leaves $c_1, c_1' \in \mathcal{L}^c_1$ merge (i.e., have a non empty intersection in $\widetilde{M}$), while their images by $H_2 \circ H_1^{-1}$ do not.

We show (in §5.5) that this issue does not arise for discretized Anosov flows (or, as a consequence, for collapsed Anosov flows for which the self orbit equivalence class is periodic), but, in general, we do not know whether the branching structure is completely determined by the Anosov flow and the self orbit equivalence class.

Question 2. *Is the branching locus*[8] *of a collapsed Anosov flow determined by the Anosov flow ϕ_t and the self orbit equivalence class (or, at least, is the branching locus constant in a connected component of partially hyperbolic diffeomorphisms)?*

Related questions can be found in [HPS, §7].

A first step towards Question 2 could be to prove that, if a collapsed Anosov flow is dynamically *incoherent*, then all collapsed Anosov flows in its connected component are also dynamically incoherent. This is true in certain manifolds, or classes of partially hyperbolic diffeomorphisms (e.g., hyperbolic manifolds [BFFP$_2$, Theorem B], or Seifert manifolds when the action on the base is pseudo-Anosov [BFFP$_3$]). One natural, seemingly simple but far from well-understood, class of examples where this is not known is for partially hyperbolic diffeomorphisms in Seifert manifolds which act as a Dehn-twist on the base.

2.7. **Quasigeodesic partially hyperbolic diffeomorphisms.** The last definition we introduce describes a class of partially hyperbolic diffeomorphism that are, in some sense, geometrically well-behaved.

As before, we consider $f\colon M \to M$ a partially hyperbolic diffeomorphism which preserves branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent respectively to E^{cs} and E^{cu} (cf. §3).

We say that a curve in a leaf L of the lifted branching foliation $\widetilde{\mathcal{W}^{cs}}$ (or $\widetilde{\mathcal{W}^{cu}}$) is a *quasigeodesic* if it admits a parametrization $\eta\colon \mathbb{R} \to L$ such that $d_L(\eta(t), \eta(s)) \geqslant$

[8]To be precise, consider $\mathcal{O}_\phi$ to be the orbit space of the lift $\widetilde{\phi}_t$ of ϕ_t to the universal cover. We can define the *branching locus* as a function $B\colon \mathcal{O}_\phi \times \mathcal{O}_\phi \to \{0,1\}$ such that if $B(o_1, o_2) = 1$ then the corresponding center leaves intersect in $\widetilde{M}$. One could define more refined notions taking into account how many connected components of intersection they have, or the direction on which center curves branch, etc.... All these things could a priori be determined by the data of the flow and the self orbit equivalence class and be independent of the partially hyperbolic diffeomorphism that realizes this as a collapsed Anosov flow.

$c|t-s|-c'$ for some $c \in (0,1), c' > 0$ where d_L is the path metric induced on L by the pullback metric from $\widetilde{M}$. A family of curves is *uniformly quasigeodesic* if the constants c, c' can be chosen independently of the curve. Following common usage in the field, we say that a curve α in a leaf L of $\mathcal{W}^{cs}$ or $\mathcal{W}^{cu}$ is a quasigeodesic, if a lift $\tilde{\alpha}$ to a leaf $\widetilde{L}$ in $\widetilde{M}$ is a quasigeodesic.

Definition 2.13 (Quasigeodesic partially hyperbolic diffeomorphism)**.** Consider $f\colon M \to M$ a partially hyperbolic diffeomorphism. We say that f is *quasigeodesic* if it preserves center stable and center unstable branching foliations with Gromov-hyperbolic leaves such that the center curves are uniform quasigeodesics inside center stable and center unstable leaves.

Note that this definition is independent of the choice of Riemannian metric on M (cf. Proposition A.5).

For true foliations, deciding whether their leaves are Gromov-hyperbolic can be done via Candel's uniformization theorem (see [CC, §I.12.6]). While Candel's theorem does not directly apply to branching foliations, it may be used when these branching foliations are *well-approximated* by true foliations (see Appendix A.3), as occurs for example in the existence theorem of Burago–Ivanov (Theorem 3.3). In particular, one can prove that, when the branching foliations are *minimal* and the manifold has fundamental group with exponential growth, then the leaves are Gromov-hyperbolic [FP_1, §5.1]. Notice that the weak stable and weak unstable foliations of Anosov flows always have Gromov-hyperbolic leaves [Fen_1].

It turns out that quasigeodesic partially hyperbolic diffeomorphism and leaf space collapsed Anosov flows are one and the same class (at least under some orientability conditions), thereby giving a nice geometrical description of (strong) collapsed Anosov flows.

Theorem D. *A leaf space collapsed Anosov flow is a quasigeodesic partially hyperbolic diffeomorphism. Moreover, if the bundles E^s and E^u are orientable, the converse holds.*

Although we do not use it to prove Theorem A, this characterization can be used to prove that some partially hyperbolic diffeomorphism are collapsed Anosov flows, as is done in [FP_2]. (In fact, [FP_2] motivated some of the results in this article, including Theorem D.)

The geometric description we obtain for collapsed Anosov flows is in fact more precise than this: We show that the center leaves of a quasigeodesic partially hyperbolic diffeomorphism must form a *quasigeodesic fan* inside each center stable or unstable leaf, as is the case for orbits of Anosov flows (see Theorem 6.11). In addition, we prove that the branching of center leaves, if it exists, is fairly well-behaved, see Lemma 10.5.

A parallel can be made between the cases studied here and the classification of partially hyperbolic diffeomorphisms on 3-manifolds with (virtually) nilpotent fundamental group: On these manifolds, while the branching foliations are not Gromov hyperbolic, the center leaves may be quasigeodesics inside their branching center (un)stable leaves. Determining when this is the case turned out to be a successful strategy for the classification, see [HP_1, HP_2].

Remark 2.14. Both Theorem C and Theorem D, giving the equivalence between strong collapsed Anosov flows, leaf space collapsed Anosov flows and quasi-geodesic partially hyperbolic diffeomorphism require some orientability conditions for one of their directions. The knowledgeable reader might surmise that this is linked to the theorem of Burago–Ivanov (Theorem 3.3), giving the existence of branching foliations under some orientability conditions of the bundles. This is only partly true: each of the Definitions 2.7, 2.10 and 2.13 assumes already the existence of branching foliations, but what we do need for some arguments from Burago–Ivanov Theorem is that these branching foliations are well approximated by true foliations.

While we think it likely that both Theorem C and Theorem D would hold without the orientability assumptions, we are not able to prove this at this time.

In particular, one step that would be very helpful to solve this problem, would be to prove uniqueness of the invariant branching foliations tangent to the center stable and center unstable bundles for partially hyperbolic diffeomorphisms (see Question 4).

The uniqueness question, which has a very wide scope of potential applications, is quite open in general. Here we prove it for the examples of Theorem A in Proposition 10.2.

2.8. **Realization of self orbit equivalences.** One way of looking at the definition of collapsed Anosov flows is as a partially hyperbolic realization of a self orbit equivalence of an Anosov flow.

Quite clearly, not every self orbit equivalence of an Anosov flow can be a partially hyperbolic diffeomorphism: Just consider a trivial self orbit equivalence $\phi_{h(\cdot)}$ of an Anosov flow $\phi_t \colon M \to M$ where $h \colon M \to \mathbb{R}$ is such that $h(x_0) = 0$ for some $x_0 \in M$, which therefore cannot be partially hyperbolic. However, if we consider the *equivalence class* of a trivial self orbit equivalence, then that class has an element that can be represented as a partially hyperbolic diffeomorphism.

Therefore, the following natural problem presents itself.

Question 3. *Is every self orbit equivalence class of an Anosov flow realized by a collapsed Anosov flow?*

Notice that a positive answer would in particular imply that there exists examples of partially hyperbolic diffeomorphisms in hyperbolic 3-manifolds which are not discretized Anosov flows, even up to finite powers, see [BFFP$_2$, Theorem B].

While not complete enough to fully answer Question 3, the constructions of [BGHP] lead, via Theorem A, to the realization of many classes of self orbit equivalences. In fact, for some Anosov flows, their construction is enough to realize (virtually) all self orbit equivalence classes.

On the other hand, a basic understanding of the self orbit equivalences of Anosov flows, such as the one obtained in [BaG$_1$] for those homotopic to the identity, directly leads to restrictions on possible collapsed Anosov flows (up to isotopy).

We chose to illustrate both of these principles on three specific, but important, examples.

In Theorem 11.1, we show how [BaG$_1$, Theorem 1.1] translates to strong collapsed Anosov flows that are homotopic to identity.

In Theorem 11.2 and Theorem 11.5, we show that, on the unit tangent bundle of a hyperbolic surface and when considering the Franks–Williams example (or some generalizations of it) the answer to Question 3 is (virtually) positive and we describe all possible collapsed Anosov flows up to isotopy.

2.9. **Organization of the paper.** In §3, we recall the definition of branching foliation and the existence theorem of Burago–Ivanov. We also state a more precise existence theorem for true foliations that approximate branching foliations.

This precision can be extracted from the original proof of Burago–Ivanov and we explain how to do that in Appendix A.

In §4, we prove Theorem C. To prove it, we first recall some results that can be extracted from [HPS], as explained in Appendix B.

In §5, we prove some general facts on topological Anosov flows and show that discretized Anosov flows (in the sense of [BFFP_1, BFFP_2]) are (strong) collapsed Anosov flows.

In §6, we prove (Theorem 6.11) that the center leaves of a quasigeodesic partially hyperbolic diffeomorphism must make a *quasigeodesic fan* in each center (un)stable leaf. To prove this, we study general subfoliations by quasigeodesic leaves of a foliation and obtain some results that apply in the general case.

In §7, we prove Theorem D.

In §8 and §9, we prove both directions of Theorem B.

In §10, we prove Theorem A. We also (see §10.2) prove a result about the uniqueness of center stable and center unstable branching foliations, in the setting of the examples of Theorem A, as well as a (branching) version of unique integrability of their center curves (Proposition 10.6).

Finally, in §11, we prove some classification results about collapsed Anosov flows and self orbit equivalences.

3. Branching foliations and leaf spaces

In this section we review the notion of branching foliations introduced in [BI] and their leaf spaces. Under some orientability assumptions, partially hyperbolic diffeomorphisms always preserve branching foliations which are well approximated by foliations, so it makes sense to consider partially hyperbolic diffeomorphisms preserving some branching foliations. We assume basic familiarity with foliations in 3-manifolds, see, e.g., [BFFP_1, Appendix B] and references therein.

Given a plane field E in a 3-manifold M we call *complete surface tangent to* E a C^1-immersion $\varphi\colon U \to M$ from a simply connected domain $U \subset \mathbb{R}^2$ into a 3-manifold M which is complete for the pull-back metric and such that $D_p\varphi(\mathbb{R}^2) = E(\varphi(p))$ at every $p \in U$.

Definition 3.1 (Branching foliation)**.** A branching foliation $\mathcal{F}$ of a 3-manifold M tangent to E is a collection of complete surfaces tangent to E such that:

(i) every point $x \in M$ belongs to (the image of) some surface,
(ii) the surfaces pairwise do not topologically cross (see below),
(iii) the family is complete in the pointed compact open topology (see below),
(iv) it is minimal in the sense that one cannot remove any surface from the collection and still satisfy properties (i) to (iii).

The condition of no topological crossing is quite subtle, since the crossing may take place far in the manifold (it cannot be defined locally, and it is part of the reason surfaces are defined to be with simply connected domain). Following [BI, Section 4], given two complete surfaces $\varphi\colon U \to M$ and $\psi\colon V \to M$ tangent to E we say that they *topologically cross* if there is a curve $\gamma\colon (0,1) \to U$ a C^1-immersion $\Psi\colon V \times (-\varepsilon,\varepsilon) \to M$ such that $\Psi(\cdot,0) = \psi$ and a map $\tilde{\gamma}\colon (0,1) \to V \times (-\varepsilon,\varepsilon)$ whose image intersects both $V \times (0,\varepsilon)$ and $V \times (-\varepsilon,0)$ such that $\varphi \circ \gamma = \Psi \circ \tilde{\gamma}$. This notion is well defined and symmetric on the surfaces.

Remark 3.2. The key difference between branching foliations defined above and the *branched laminations* introduced in [HPS, §6.B] is this added assumption (ii) of no topological crossing between surfaces. Of course, this added notion only makes sense in the codimension one setting.

The completeness of the family stated in the definition of branching foliation should be understood in the compact open topology, meaning that if there is a sequence $\varphi_n : U_n \to M$ of complete surfaces tangent to E in the family and one has that $\varphi_n(p_n) \to x$ for some points $p_n \in U_n$, then there is a surface $\varphi : U \to M$ in the family such that for a point $p \in U$ it verifies that the map φ in an arbitrarily large ball around p is C^1-close to some reparametrization (see next paragraph) of the maps φ_n in a large ball around p_n (see also [BI, Lemma 7.1]).

Note that condition (iv) above is not stated explicitly in [BI], but can be easily deduced by choosing one leaf in each equivalence class (up to topological reparametrization). There is a big ambiguity on the choice of the parametrizations and since we want to focus on their images, we want to avoid it. For that, we will say that two complete surfaces $\varphi : U \to M$ and $\psi : V \to M$ tangent to E are the *same up to reparametrization* if there is a homeomorphism $h : U \to V$ such that $\varphi = \psi \circ h$.

It is standard in the literature to abuse notation and talk about leaf of a branching foliation $\mathcal{F}$ to refer either to the complete surface $\varphi : U \to M$ up to reparametrization, or to its image. In this article, we will try to avoid this for clarity. In fact, some of our results a posteriori help justifying this classical abuse (see the end of Remark 3.4).

In [BI, Theorem 7.2] it is shown that branching foliations can be approximated arbitrarily well by true foliations. The statement of [BI, Theorem 7.2] does not state explicitly some properties of the approximation that we will need. We explain in Proposition A.1 of the appendix how the following statement indeed follows from the proof in [BI, Theorem 7.2].

Theorem 3.3. *Let $\mathcal{F}$ be a branching foliation tangent to a transversely orientable distribution E on a closed 3-manifold M. Then, for every $\varepsilon > 0$ there exists a foliation $\mathcal{F}_\varepsilon$ and a continuous map $h_\varepsilon : M \to M$ such that the following conditions hold:*

(i) the angle between E and $T\mathcal{F}_\varepsilon$ at every point is smaller than ε,
(ii) for every surface $\varphi : U \to M$ in $\mathcal{F}$ there is a unique leaf L of the foliation $\mathcal{F}_\varepsilon$ such that h_ε is a local C^1 diffeomorphism from L to the surface: That is, for every $x \in L$ there is a neighborhood V of x in L and an open subset $W \subset U$ such that $\varphi^{-1} \circ h_\varepsilon : V \to W$ is a diffeomorphism,
(iii) $d(h_\varepsilon(x), x) < \varepsilon$ for every $x \in M$.

The uniqueness of the correspondence between leaves of the true and branching foliations, given by item (ii) above, allows to simplify the definition of the leaf spaces of the center stable, center unstable and center (branching) foliations given in [BFFP$_2$, Section 3].

Let $f : M \to M$ be a partially hyperbolic diffeomorphism and assume that f preserves branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent respectively to $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$. That f preserves the branching foliation means that each surface of the collection is mapped, up to reparametrization, to another surface in the collection.

Remark 3.4. There is no closed contractible curve everywhere transverse to E^{cs} (cf. [BI, Lemma 2.3]) since by Novikov theorem this would imply the existence of a Reeb component for some of the approximating foliations, and thus a closed curve tangent to E^u which is impossible. This also has the important consequence that the approximating foliation $\mathcal{F}^{cs}_\varepsilon$ given by Theorem 3.3, for small ε, is *Reebless*. The same holds for E^{cu} and $\mathcal{W}^{cu}$ and $\mathcal{F}^{cu}_\varepsilon$. Notice that once that one knows that the branching foliation is Reebless, one can simplify a bit its treatment, in particular, when lifting to the universal cover, there is no ambiguity in identifying surfaces up to reparametrization with their images.

Remark 3.4 and Palmeira's theorem imply that the universal cover $\widetilde{M}$ of M is homeomorphic to $\mathbb{R}^3$ and that the leaf space of the lifted foliations $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ and $\widetilde{\mathcal{F}^{cu}_\varepsilon}$ of $\mathcal{F}^{cs}_\varepsilon$ and $\mathcal{F}^{cu}_\varepsilon$ respectively are 1-dimensional simply connected (possibly non-Hausdorff) manifolds. Theorem 3.3 implies that all of these leaf spaces $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ for $\varepsilon > 0$ are independent of $\varepsilon > 0$ and are naturally bijective with the leaf space of $\widetilde{\mathcal{W}^{cs}}$. It allows one to put a topology on the leaf space $\mathcal{L}^{cs} = \widetilde{M}/_{\widetilde{\mathcal{W}^{cs}}}$ which is the same as the topology on $\widetilde{M}/_{\widetilde{\mathcal{F}^{cs}_\varepsilon}}$ independently of $\varepsilon > 0$. In the same way we define a topology on $\mathcal{L}^{cu} = \widetilde{M}/_{\widetilde{\mathcal{W}^{cu}}}$. It also allows to define the action of $\tilde{f}$, a lift of f on these spaces, since $\tilde{f}$ preserves the lifted branching foliations $\widetilde{\mathcal{W}^{cs}}$ and $\widetilde{\mathcal{W}^{cu}}$.

The same holds for every deck transformation $\gamma \in \pi_1(M)$ that acts on these leaf spaces canonically. Using these identifications there is a canonical action on the leaf spaces of $\widetilde{\mathcal{F}^{cs}_\varepsilon}, \widetilde{\mathcal{F}^{cu}_\varepsilon}$ by either lifts of f or deck transformations.

We obtain also a way to define a leaf space $\mathcal{L}^c$ for the center branching foliation. A center leaf in $\widetilde{M}$ is a connected component of the intersection of a leaf L of $\widetilde{\mathcal{W}^{cs}}$ and a leaf U of $\widetilde{\mathcal{W}^{cu}}$. The center leaf space is this set with the natural topology induced from the quotient of the subset of the Cartesian product two original leaf spaces. Another way to see this is using the identification of leaf spaces of $\widetilde{\mathcal{W}^{cs}}, \widetilde{\mathcal{F}^{cs}_\varepsilon}$; and $\widetilde{\mathcal{W}^{cu}}, \widetilde{\mathcal{F}^{cu}_\varepsilon}$ to define the center leaf space as the leaf space of the foliation $\widetilde{\mathcal{F}^{c}_\varepsilon}$ obtained as the intersection of the foliations $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ and $\widetilde{\mathcal{F}^{cu}_\varepsilon}$. This is again well defined independently of ε and there is a well defined action of $\pi_1(M)$ on this leaf space as well as an action of $\tilde{f}$ any lift of f to $\widetilde{M}$.

Remark 3.5. The notions of leaf spaces of $\widetilde{\mathcal{W}^{cs}}$ and $\widetilde{\mathcal{W}^{cu}}$ coincide with the ones studied in [BFFP$_2$, Section 3] where we did not rely on the approximating foliations. The definition of the center leaf space $\mathcal{L}^c$ taken here may however differ slightly[9] from the one defined in [BFFP$_2$, Section 3] which is a quotient of this definition: In [BFFP$_2$, Section 3] if two connected components c_1 of $L_1 \cap U_1$ and c_2 of $L_2 \cap U_2$ (L_i in $\widetilde{\mathcal{W}^{cs}}$, U_i in $\widetilde{\mathcal{W}^{cu}}$) are the same set in $\widetilde{M}$, then they produced a single center leaf. Here we do not identify them. So the center leaf space defined in [BFFP$_2$] is a quotient of the one we define here.

In the cases we will be interested in, there will be a nice topological structure in the leaf space $\mathcal{L}^c$ which will be homeomorphic to $\mathbb{R}^2$. Notice that in this setting, the foliations $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ induced in $\mathcal{L}^c$ by $\widetilde{\mathcal{F}^{cs}_\varepsilon}, \widetilde{\mathcal{F}^{cu}_\varepsilon}$ are (topologically) transverse and invariant under the action of $\pi_1(M)$ and $\tilde{f}$.

The assumption that a partially hyperbolic diffeomorphism of a 3-manifold preserves branching foliations is justified, since it always holds up to finite cover and iterate as the following fundamental result of [BI] shows.

Theorem 3.6 (Burago–Ivanov)**.** *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism with splitting $TM = E^s \oplus E^c \oplus E^u$ such that the bundles are oriented*

[9]We do not know whether there exists examples where the two definitions are actually different, but, at least formally, they are not the same.

and Df preserve their orientation. Then, there are f-invariant branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent respectively to $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$.

To be precise, the invariance by f of the branching foliations means the following: If (φ, U) is a leaf of $\mathcal{W}^{cs}$ then $(f \circ \varphi, U)$ is also a leaf of $\mathcal{W}^{cs}$ modulo reparametrization.

Notice that the branching foliations constructed in [BI] are invariant under *every* diffeomorphism that preserves the bundles E^{cs} and E^{cu} and preserves orientations of the bundles E^c, E^s and E^u. Some other consequences of their construction is explored in Appendix A. One goal being to better understand the uniqueness properties these foliations may have.

Notation 3.7. Given a branching foliation $\mathcal{F}$ on M we will denote by (φ, U) the leaves of $\mathcal{F}$ to refer to the surface $\varphi\colon U \to M$. If $f\colon M \to M$ is a diffeomorphism that preserves the branching foliation $\mathcal{F}$ we will denote by $f(\varphi, U)$ to the leaf $(\psi, V) \in \mathcal{F}$ which is a reparametrization of $(f \circ \varphi, U)$.

4. The space of collapsed Anosov flows

We want to show Theorem C. First we recall some results from [HPS] that we need.

4.1. **Graph transform method.** The structural stability results of Hirsch, Pugh and Shub [HPS] provide conditions implying that perturbations of a partially hyperbolic diffeomorphisms preserving a foliation tangent to the center direction are *leaf conjugate* to the original one. Their classical stability result (see [HPS, §7]) requires the center bundle to be integrable plus a technical condition called *plaque expansivity*. We refer the reader to [HPS] for the precise definitions of these notions since we do not use them here.

In [HPS, §6] the authors develop a more general theory that permits leaves to merge (see also [CP, Theorem 4.26]). The more general theory allows one to remove the technical conditions at the expense of not knowing if centers remain disjoint after perturbation. Since in our case this is what usually happens, this is precisely what we need.

We will use [HPS, Theorem 6.8] (which is part of [HPS, Theorem 6.1]). We need a uniform version of the result whose proof is exactly the same[10]. We state it in dimension 3 for simplicity, but it holds in any dimension.

We first need some definitions from [HPS]. A C^1-*leaf immersion* is a C^1-immersion, $\imath\colon V \to M$, of a manifold V (which is typically a disjoint union of possibly uncountably many connected complete manifolds) to M whose image is a closed set in M. For a diffeomorphism $g\colon M \to M$, a C^1-leaf immersion $\imath\colon V \to M$ is said to be g-*invariant* if there exists a C^1-diffeomorphism $\imath_* g\colon V \to V$ verifying $\imath \circ \imath_* g = g \circ \imath$. Two C^1-leaf immersions $\imath, \imath'$ from V to M are said to be C^1-*close* if they are uniformly C^1-close, meaning that there exists $\varepsilon > 0$ such that for every $x \in V$ we have $d(\imath(x), \imath'(x)) < \varepsilon$ and[11] $\|D_x\imath - D_x\imath'\| < \varepsilon$.

[10] The only difference in the statement is that it implies the size of the neighborhood to be independent of $f \in \mathcal{U}$, where $\mathcal{U}$ is as in Theorem 4.1 . This is implied directly by the proof in [HPS] as the construction of the C^1-neighborhood of a map depends only on the C^1-size of f as well as the geometry of the bundles E^s, E^c and E^u. These quantities are certainly constant in a neighborhood of a given f_0.

[11] To make sense of the difference of derivatives, one can for instance, embed M in some $\mathbb{R}^k$ with large k.

Theorem 4.1. *Let $f_0\colon M \to M$ be a partially hyperbolic diffeomorphism. There exists $\mathcal{U}$ a C^1-neighborhood of f_0 such that if $g, g' \in \mathcal{U}$ and $\imath_g\colon V \to M$ is a g-invariant C^1-leaf immersion tangent to E^c_g, then there exists $\imath_{g'}\colon V \to M$ a g'-invariant C^1-leaf immersion tangent to $E^c_{g'}$ and C^1-close to $\imath_g$ and a homeomorphism $\tau\colon V \to V$ which is C^0-close to the identity*[12] *verifying that $(\imath_g)_* g(x) = (\imath_{g'})_* g'(\tau(x))$ for every $x \in V$.*

We will also need a version of the result above for branching foliations[13].

If $g\colon M \to M$ and $g'\colon M \to M$ are partially hyperbolic diffeomorphisms preserving respectively branching foliations $\mathcal{W}^{cs}_g$ and $\mathcal{W}^{cs}_{g'}$ tangent to E^{cs}_g and $E^{cs}_{g'}$. We say that $\mathcal{W}^{cs}_g$ and $\mathcal{W}^{cs}_{g'}$ are *ε-equivalent* if:

(i) There exists a $\pi_1(M)$-invariant homeomorphism H from $\mathcal{L}^{cs}_g$ to $\mathcal{L}^{cs}_{g'}$, the leaf spaces of $\widetilde{\mathcal{W}^{cs}_g}$ and $\widetilde{\mathcal{W}^{cs}_{g'}}$ in $\widetilde{M}$ respectively.

(ii) There are lifts $\tilde{g}$ and $\tilde{g}'$ of g and g' respectively such that the actions on $\mathcal{L}^{cs}_g$ and $\mathcal{L}^{cs}_{g'}$ are conjugate via H, that is, $H \circ \tilde{g} = \tilde{g}' \circ H$.

(iii) Given $L = (\varphi, U) \in \mathcal{L}^{cs}_g$ a leaf of $\widetilde{\mathcal{W}^{cs}_g}$ we have that the leaf $H(L) = (\psi, V)$ of $\widetilde{\mathcal{W}^{cs}_{g'}}$ is uniformly ε-C^1-close to L. This means that there exists a diffeomorphism $\eta\colon U \to V$ such that the maps φ and $\psi \circ \eta$ are uniformly ε-close as well as their derivatives.

We can now state the result we will need.

Theorem 4.2. *Let $f_0\colon M \to M$ be a partially hyperbolic diffeomorphism of a closed 3-manifold M. There exists $\mathcal{U}$ an open neighborhood of f_0 in the C^1 topology and $\varepsilon > 0$ with the property that every $g \in \mathcal{U}$ is partially hyperbolic and if $\mathcal{W}^{cs}_g$ is a branching foliation tangent to E^{cs}_g and invariant under g, then, for every $g' \in \mathcal{U}$ there is a branching foliations $\mathcal{W}^{cs}_{g'}$, invariant under g' and ε-equivalent to $\mathcal{W}^{cs}_g$.*

The proof of both Theorem 4.1 and Theorem 4.2 are the same as the ones given in [HPS] with some simplifications (due to the fact that we are in small dimension and the lamination is normally expanded and not normally hyperbolic). For the convenience of the reader, we will include a short sketch of the proof of Theorem 4.2 in Appendix B (part of the justification for this appendix is the fact that [HPS, §6] proves many other results and what we need is not always easily separated from what we do not need). The sketch will also serve to show how the uniform estimates follow from the same arguments (and how the non-crossing condition is automatically satisfied). This will also serve to explain how Theorem 4.1 follows from [HPS].

4.2. **Proof of Theorem C.** Recall (cf. Remark 2.12) that a leaf space collapsed Anosov flow induces naturally a self orbit equivalence class (i.e., a self orbit equivalence up to trivial self orbit equivalences).

Proposition 4.3. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism. Then, there exists a neighborhood $\mathcal{U}$ of f such that if there is $g \in \mathcal{U}$ which is a leaf space collapsed Anosov flow associated to a topological Anosov flow $\phi_t\colon M \to M$*

[12] In particular, it preserves connected components.

[13] See [HPS, §6.B] for the related notion of branched lamination (which is also intimately related to the statement of Theorem 4.1) which differs from the notion of branching foliations we use in this paper. The latter has to do with codimension one phenomena and features the non-topological crossing condition that makes no sense in the setting of [HPS, §6.B].

and a self orbit equivalence class $[\beta]$*. Then, every* $g' \in \mathcal{U}$ *is a leaf space collapsed Anosov with respect to* ϕ_t *and* $[\beta]$*.*

Proof. Let $\mathcal{U}$ be a neighborhood given by Theorem 4.2. Then, for any $g' \in \mathcal{U}$, we obtain a pair of branching foliations $\mathcal{W}^{cs}_{g'}$ and $\mathcal{W}^{cu}_{g'}$ with the same dynamics as g in their leaf spaces.

Let $\widetilde{\mathcal{W}^{cs}_{g'}}$ and $\widetilde{\mathcal{W}^{cu}_{g'}}$ be the lifted foliations to the universal cover. For each leaf F of $\widetilde{\mathcal{W}^{cs}_{g}}$ there is a unique leaf $F' = H(F)$ of $\widetilde{\mathcal{W}^{cs}_{g'}}$ which is ε close to it, and vice versa.

Let c be a center leaf of g. It is a connected component of the intersection of a leaf F of $\widetilde{\mathcal{W}^{cs}_{g}}$ and L of $\widetilde{\mathcal{W}^{cu}_{g}}$. Hence, there is a unique component c' of the intersection of $H(F)$ and $H(L)$ which is ε-close to c. So the center leaf spaces of g and g' are equivariantly homeomorphic. So one gets Definition 2.10 for g' which implies that g' is a leaf space collapsed Anosov flow with respect to the flow ϕ_t. Moreover, H conjugates the respective actions of lifts of g and g' on their center leaf spaces. Hence, their corresponding self orbit equivalence are equivalent (cf. Remark 2.12). $\square$

This proposition implies Theorem C for leaf space collapsed Anosov flows. The open property is immediate. In order to see that it is a closed property consider $f_n\colon M \to M$ leaf space collapsed Anosov flows converging to a partially hyperbolic diffeomorphism $f\colon M \to M$. If we apply Proposition 4.3 to f we get a neighborhood $\mathcal{U}$ such that if $g \in \mathcal{U}$ is leaf space collapsed Anosov flow, then every $g' \in \mathcal{U}$ is leaf space collapsed Anosov flow. Since $f_n \to f$ it follows that for large n we have that $f_n \in \mathcal{U}$ and so we can apply the proposition with $g = f_n$ and $g' = f$.

An analogous proof detailed below will give Theorem C for collapsed Anosov flows using Theorem 4.1 instead of Theorem 4.2. This case is much more involved because we need to construct a map in the manifold and not just on the leaf space level.

Proposition 4.4. *The space of collapsed Anosov flows is open and closed among partially hyperbolic diffeomorphisms.*

Proof. Let $f_0\colon M \to M$ be a partially hyperbolic diffeomorphism. We will show that there is a neighborhood $\mathcal{U}$ of f_0 verifying that if there is $g \in \mathcal{U}$ which is a collapsed Anosov flow, then every $f \in \mathcal{U}$ is a collapsed Anosov flow. This shows that being collapsed Anosov flow is an open and closed property among partially hyperbolic diffeomorphisms as explained above.

For such a $f_0\colon M \to M$ we will take $\mathcal{U}$ to be the neighborhood given by Theorem 4.1 and assume that there is $g \in \mathcal{U}$ which is a collapsed Anosov flow. That is, there exists an Anosov flow $\phi_t\colon M \to M$, a continuous map $h\colon M \to M$ homotopic to the identity as in Definition 2.5 and a self orbit equivalence β such that $g \circ h = h \circ \beta$. We want to construct, for $g' \in \mathcal{U}$ a map $h'\colon M \to M$ and a self orbit equivalence β' of ϕ_t which verify Definition 2.5.

First, we will consider a leaf immersion $\imath_g\colon V \to M$ induced by h and ϕ_t. This is defined as follows: Consider V to be the disjoint union of orbits of ϕ_t, each one with the smooth structure induced by the length of the curves in M. Note that even if V is a disjoint union, we can think of points in V as points of M so we can apply both h and β to these leaves. We define $\imath_g(x) = h(x)$. This is a well defined C^1-leaf immersion since leaves can be lifted to the universal cover where the lift of g acts and induces a map from V to V which is exactly β. In particular, we get that $(\imath_g)_* g = \beta$.

We now consider $g' \in \mathcal{U}$ and Theorem 4.1 gives us a C^1-leaf immersion $\imath_{g'}: V \to M$ and a homeomorphism $\tau: V \to V$ which is globally C^0-close to the identity such that $(\imath_g)_* g(x) = (\imath_{g'})_* g'(\tau(x))$. We need to construct h' and β' using this map.

Note that for $\varphi: V \to V$ a C^1-diffeomorphism we get that $\imath_{g'} \circ \varphi$ is also a C^1-leaf immersion with the same properties, so we need to show that there is a choice of φ which makes $h': M \to M$ continuous when defined as $h'(x) = \imath_{g'} \circ \varphi(x)$ where we identify V with M as a set. The subtlety here is that even though V and M are identified as sets, their topologies are completely different. In particular V has many more open sets.

To obtain h', β' we will take advantage of the fact that $\imath_g$ was defined using h which is continuous and that $\imath_{g'}$ and $\imath_g$ are uniformly C^1-close. First some local considerations. The curves $i_g(\alpha)$ where α is a component of V are all integral curves of E^c_g and likewise those of $i_{g'}$ are integral curves of $E^c_{g'}$. In a fixed small scale one can choose local coordinates (x, y, z) so that the curves $i_g(\alpha)$ are all ε_0 C^1-close to vertical curves, with fixed ε_0. The same happens for $i_{g'}(\alpha)$. Hence in a local box, for a fixed point z in $i_{g'}(\alpha)$ there is a unique point denoted by $\eta(z)$ in the corresponding local sheet of $i_g(\alpha)$ which is the closest point to z. This defines a function η. Switching the roles, this implies that this function is locally injective. Finally this function has derivative which is non zero everywhere.

So, given $x \in V$ we consider I_x the ε-neighborhood around x with the metric of V (induced by M), in particular this is contained in the same component of V. Consider $\ell_x = \imath_g(I_x)$. Take $\varphi_0(x)$ to be the preimage by $\imath_g$ of the closest point in ℓ_x to $\imath_{g'}(x)$. The map $\varphi_0: V \to V$ is continuous and close to the identity. By integrating in I_x and using the orientation, one can make it to be a C^1-diffeomorphism φ of V. We claim that $\imath_{g'} \circ \varphi^{-1}$ works. Consider $x_n \to x$ a converging sequence in M. It follows that $\ell_{x_n} \to \ell_x$ uniformly in the C^1-topology. We get that $\imath_{g'}(\varphi^{-1}(x_n))$ maps to the closest point in average to ℓ_{x_n} in $\imath_{g'}(I_{x_n})$ and this point varies continuously. This shows that $\imath_{g'} \circ \varphi^{-1}$ is continuous seen as a map from M to M.

Once we got the leaf immersion $\imath_{g'} \circ \varphi^{-1}$, that we will now rename to be denoted by $\imath_{g'}$ to simplify notation, we can also define $\beta': M \to M$ viewing M as the disjoint union of the components of V. We set $\beta'(x) = (\imath_{g'})_* g'(x) \in V \cong M$. The map β' is bijective since it is bijective in each component of V and maps components to components bijectively. We need to check that β' is continuous with the topology of M (which is weaker than the one of V). But the continuity of β' is a direct consequence of the continuity of g' which forces the maps $(\imath_{g'})_* g'$ in different components of V to be close when the components are close in M.

The equation $g' \circ h' = h' \circ \beta'$ is automatically verified. □

Remark 4.5. We can also show that being a quasigeodesic partially hyperbolic diffeomorphism is an open and closed property: If f is a quasigeodesic partially hyperbolic diffeomorphism, in a finite cover, an iterate of f is a leaf space collapsed Anosov flow (cf. Theorem D). Suppose that $f_n \to f$ is a sequence of quasigeodesic collapsed Anosov flows converging to a partially hyperbolic diffeomorphism f. Using the neighborhood $\mathcal{U}$ of f given by Theorem 4.2, it follows that there are f-invariant branching foliations tangent to E^{cs}, E^{cu}. Let g be a lift of a finite iterate f^i of f to a finite cover M_1 of M so that the lifted bundles E^c, E^s, E^u in M_1 are orientable and g preserves the orientations. Let g_n be the lifts to M_1 of f^i_n which converge to g. Since g_n converges to g and g preserves orientations of the bundles then same happens for g_n for n big enough. We assume it is true for all n. It now follows from Theorem D that the g_n are leaf space collapsed

Anosov flows. By Theorem C it follows that g is a leaf space collapsed Anosov flow. Hence using Theorem D again, it follows that g is a quasigeodesic collapsed Anosov flow. Since f itself preserves branching foliations, it now follows that f is a quasigeodesic collapsed Anosov flow, because the foliations of g are lifts of foliations of f. This proves that being a quasigeodesic collapsed Anosov flow is a closed property among partially hyperbolic diffeomorphisms. The open property is proved analogously.

As mentioned in §1, we may wonder whether a collapsed Anosov flow is automatically a strong, or leaf space, collapsed Anosov flow. Notice that, if not, then Theorem C implies that there is at least one entire connected component of partially hyperbolic diffeomorphisms on which all maps are collapsed Anosov flows, but none are leaf space collapsed Anosov flows.

To try to decide whether all collapsed Anosov flows are leaf space collapsed Anosov flows, one tool that would greatly help is if the following was true:

Question 4. *Suppose that the bundles E^c, E^s, E^u are orientable. Is the invariant branching foliation of a collapsed Anosov flow tangent to the center stable (resp. the center unstable) bundle unique?*

Notice that this question also naturally arises in the existence theorem of Burago–Ivanov (Theorem 3.3), as their construction yields two, a priori distinct, center (un)stable branching foliations (see also Appendix A).

But the scope of potential use, if Question 4 were to be true, is much greater: When studying partially hyperbolic diffeomorphisms in dimension 3, if one wants to use branching foliations (which so far has been the main tool to understand partially hyperbolic diffeomorphisms geometrically or topologically), then one has to use the existence result of Burago–Ivanov. Now that result comes with an orientability condition, thus forcing one to take a finite lift and finite power to ensure the existence of such foliations. Knowing uniqueness of such foliations would then allow to prove that they can project to the original manifold. Hence, one may hope to obtain geometrical consequences for the original map as well as for its lifts and powers.

5. Some results about topological Anosov flows

5.1. Foliations of Anosov flows.

Let $\phi_t\colon M \to M$ be a topological Anosov flow on a closed 3-manifold M. We study here the ϕ_t-invariant foliations saturated by orbits. We say that a foliation $\mathcal{F}$ is *ϕ_t-saturated* if for every leaf $L \in \mathcal{F}$ and $x \in L$ we have that $\phi_t(x) \in L$ for all $t \in \mathbb{R}$.

Proposition 5.1. *Let $\mathcal{F}$ be a foliation by surfaces which is saturated by orbits of ϕ_t and such that $\mathcal{F}^{ws}_\phi \neq \mathcal{F}$. Then there is an attractor of ϕ_t on which $\mathcal{F} = \mathcal{F}^{wu}_\phi$.*

Proof. We use the spectral decomposition of Anosov flows, see [FH, §5.3], which also works for topological Anosov flows using essentially the same arguments. This implies that the set of points in M whose ω-limit set is contained in an attractor of the Anosov flow is open and dense. Note that the set of points on which $\mathcal{F}^{ws}_\phi \neq \mathcal{F}$ is open, therefore, there is an open set U of points whose ω-limit set is contained in an attractor $A \subset M$ of the flow ϕ_t and such that $\mathcal{F}^{ws}_\phi \neq \mathcal{F}$.

In particular, there is a point $x \in U$ which belongs to the stable manifold $\mathcal{F}^{ws}_\phi$ of a periodic orbit o. Notice that since $\mathcal{F}$ is ϕ_t-saturated, this implies that the closure of the local leaf of $\mathcal{F}(x)$ contains the weak unstable manifold $\mathcal{F}^{wu}_\phi$ of o, because $\mathcal{F}^{ws}_\phi(x) \pitchfork \mathcal{F}(x)$.

Claim 5.2. *Let S be a surface topologically transverse to $\mathcal{F}^{ws}_\phi(o)$ where o is a periodic point of ϕ. Then, $\phi_t(S)$ as $t \to \infty$ contains $\mathcal{F}^{wu}_\phi(o)$.*

Proof. The fact that the surface S is topologically transverse to $\mathcal{F}^{ws}_\phi(o)$ means that (up to iterating forward) there is a small transversal D to the flow through o on which the trace of S contains a curve transverse to the trace of $\mathcal{F}^{ws}_\phi(o)$ with D. The Poincaré first return map to D is conjugate to a fixed saddle on $o \cap D$ and its forward iterates then make S converge to the trace of $\mathcal{F}^{wu}_\phi(o)$ by forward iteration. $\square$

Since the foliation is continuous, this implies that the leaf $\mathcal{F}^{wu}_\phi(o)$ is contained in $\mathcal{F}$. Since this leaf is dense in the attractor A it follows that $\mathcal{F}$ coincides with $\mathcal{F}^{wu}_\phi$ in A as announced. $\square$

A direct corollary is:

Corollary 5.3. *Let ϕ_t be a transitive topological Anosov flow, then, there are exactly two ϕ_t-saturated foliations, which are $\mathcal{F}^{ws}_\phi$ and $\mathcal{F}^{wu}_\phi$.*

Proof. Note that if ϕ_t is transitive, then M is the unique attractor and repeller. $\square$

Remark 5.4. If ϕ_t is not transitive, then Corollary 5.3 does not hold. Indeed, it is possible to construct several ϕ_t saturated foliations which coincide with the weak-stable and unstable foliations in subsets of the non-wandering set, but which are different from both of these foliations in the wandering region. Indeed, to make a concrete example, consider the Franks–Williams [FW] Anosov flow $\phi_t \colon M \to M$ with an attractor A and a repeller R such that every orbit not in $A \cup R$ intersects a C^1 smooth torus T transverse to ϕ_t and choose a foliation $\mathcal{G}$ of T which is transverse[14] to both $\mathcal{F}^{ws}_\phi \cap T$ and $\mathcal{F}^{wu}_\phi \cap T$. If one considers the orbit of $\mathcal{G}$ by ϕ_t one gets a ϕ_t-saturated foliation on $M \setminus (A \cup R)$ that can be completed to a ϕ_t-saturated foliation by taking the foliation $\mathcal{F}^{wu}_\phi$ in A and $\mathcal{F}^{ws}_\phi$ in R. Notice that one can construct uncountably many such foliations. Other examples can be constructed along the same lines using the zoo of examples from [BBY].

Notice however that while non-transitive (topological) Anosov flows may have several flow saturated foliations, one cannot choose them to be pairwise transverse as we will show:

Proposition 5.5. *Let ϕ_t be a topological Anosov flow and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two topologically transverse ϕ_t-saturated foliations. Then, up to relabeling, one has that $\mathcal{F}_1 = \mathcal{F}^{ws}_\phi$ and $\mathcal{F}_2 = \mathcal{F}^{wu}_\phi$.*

Proof. The proof is very similar to that of Proposition 5.1. In the transitive case the result follows directly from Corollary 5.3, so we will assume that ϕ_t is non-transitive.

Consider a point $x \in M$ such that its forward orbit accumulates in an attractor A and its backward orbit in a repeller R. Then, we claim that in x the foliations must coincide with $\mathcal{F}^{ws}_\phi$ and $\mathcal{F}^{wu}_\phi$. If this were not the case, then, say $\mathcal{F}_1$ does not coincide with either of them in a neighborhood of x. Assume that $\mathcal{F}_2$ does not coincide with $\mathcal{F}^{ws}_\phi$ in a neighborhood of x (if it does not coincide with $\mathcal{F}^{wu}_\phi$ one makes a symmetric argument). Then, it follows by the argument in Proposition 5.1 that both $\mathcal{F}_1$ and $\mathcal{F}_2$ must coincide with $\mathcal{F}^{wu}_\phi$ in A, so they cannot be transverse.

[14]Notice that these foliations are indeed C^1, so one can take any foliation generated by a vector field between the two tangent spaces.

Now, notice that points whose forward orbit accumulates in an attractor and backward orbit in a repeller are an open and dense subset of M(cf. [FH, §5.3]). This also works exactly the same for topological Anosov flows. So this completes the proof. $\square$

5.2. **Leaf space collapsed Anosov flows respect weak foliations.** Here we will consider Definition 2.10, where ϕ_t is the Anosov flow in question.

Recall that we denote by $\mathcal{O}^{ws}_\phi$ and $\mathcal{O}^{wu}_\phi$ the one-dimensional foliations of $\mathcal{O}_\phi$ induced respectively by $\widetilde{\mathcal{F}^{ws}_\phi}$ and $\widetilde{\mathcal{F}^{wu}_\phi}$, the weak stable and weak unstable foliations of $\tilde{\phi}_t$ which are precisely the lifts of the foliations $\mathcal{F}^{ws}_\phi$ and $\mathcal{F}^{wu}_\phi$ to $\widetilde{M}$. We also denote by $\mathcal{O}^{cs}_f$ and $\mathcal{O}^{cu}_f$ the foliations induced in $\mathcal{L}^c$ by the center stable and center unstable branching foliations.

Proposition 5.6. *If f is a leaf space collapsed Anosov flow (Definition 2.10) then up to taking ϕ_{-t} the map H maps $\mathcal{O}^{ws}_\phi$ to $\mathcal{O}^{cs}_f$ and $\mathcal{O}^{wu}_\phi$ to $\mathcal{O}^{cu}_f$.*

Proof. Assume that f verifies Definition 2.10. Here H is the map $H\colon \mathcal{O}_\phi \to \mathcal{L}^c$ which is $\pi_1(M)$-invariant. Consider the preimage under H of the center stable and center unstable foliations $\mathcal{O}^{cs}_f$ and $\mathcal{O}^{cu}_f$ in $\mathcal{L}^c$. These clearly project to foliations in M by the $\pi_1(M)$-invariance, and provide different foliations which are ϕ_t saturated.

It follows from Proposition 5.5 that one must be $\mathcal{F}^{ws}_\phi$ and the other $\mathcal{F}^{wu}_\phi$. Thus, up to changing the flow ϕ_t to the flow η_t defined by $\eta_t = \phi_{-t}$, the homeomorphism H must map the foliations $\mathcal{O}^{ws}_\phi$ and $\mathcal{O}^{wu}_\phi$ to $\mathcal{O}^{cs}_f$ and $\mathcal{O}^{cu}_f$ respectively. $\square$

5.3. **Expansive flows and topological Anosov flows.** We first define the notion of expansive flow:

Definition 5.7. A non singular flow $\phi_t\colon M \to M$ is *expansive* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in M$ and $\sigma\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\sigma(0) = 0$ such that $d(\phi_t(x), \phi_{\sigma(t)}(y)) \leqslant \delta$ for every $t \in \mathbb{R}$ then $y = \phi_s(x)$ for some $|s| < \varepsilon$.

Remark 5.8. The use of ε in the definition of expansivity is to account for the recurrence of the flow in M itself and so that orbits that auto-accumulate also separate. If one knows that the flow $\phi_t\colon M \to M$ has properly embedded orbits[15] then to establish expansivity it is enough to show that there is some δ such that different orbits cannot be Hausdorff distance less than δ form each other. In such cases we will call δ an *expansivity constant* for ϕ_t. We refer the reader to [BWa] for more on expansive flows.

The following is a direct consequence of [IM, Theorem 1.5] or [Pat, Lemma 7]:

Theorem 5.9. *Let $\phi_t\colon M \to M$ be an expansive flow preserving a foliation. Then ϕ_t is a topological Anosov flow.*

Proof. The results [IM, Theorem 1.5] or [Pat, Lemma 7] show that an expansive flow preserves transverse *singular* foliations (one stable and one unstable) whose singularities consist of periodic orbits whose local structure is of a p-prong with $p \geqslant 3$.

We claim that prong singularities of singular stable and unstable foliations are incompatible with preserving a foliation. Suppose that ϕ_t preserves a foliation $\mathcal{F}$ and ϕ_t has a singular p-prong orbit α. Let L be the leaf of $\mathcal{F}$ through α. The

[15]This cannot happen if M is compact, but will sometimes be easy to know for instance, when lifting the flow to the universal cover.

arguments in the proof of Proposition 5.1 show that on each side of α the leaf L has to agree with either a prong of a stable leaf of ϕ_t of α or an unstable prong. Now look transversely. Since there are at least six prongs of ϕ_t at α (at least 3 stable and at least 3 unstable), then locally transversally one component of $M - L$ intersects at least one stable and one unstable prong of α. A nearby leaf L' of $\mathcal{F}$ intersecting that component will intersect a stable and an unstable prong of α. Flowing forward along ϕ_t preserves L', brings it closer to α along the stable of α and farther from α along the unstable of α. This forces L' to topologically cross itself, which is impossible for a foliation.

This contradiction shows that the flow cannot have any singularities, and therefore it is a topological Anosov flow. $\square$

Remark 5.10. Notice that the previous Theorem does not require the foliation invariant by the flow to coincide with the weak stable and unstable foliations of the topological Anosov flow (cf. Remark 5.4). If ϕ_t preserves two transverse foliations then these must coincide with the weak stable and unstable foliations of the topological Anosov flow thanks to Proposition 5.5.

5.4. **The quasigeodesic property.** In this section, we prove Gromov hyperbolicity of the leaves of the weak foliations of a topological Anosov flow, and the quasigeodesic property of its orbits along the leaves. This was proved in [Fen_1, §5] for Anosov flows and we adapt his proof to the topological setting.

Before properly stating the result, we comment on the Gromov hyperbolicity of leaves of $\widetilde{\mathcal{F}^{ws}_\phi}$ and $\widetilde{\mathcal{F}^{wu}_\phi}$: Since ϕ_t is only a topological Anosov, these leaves could just be topological, and not even contain rectifiable curves. This would be problematic in order to induce a structure of path metric space. However, it is possible to change the smooth structure so that individual leaves are C^1 (see [Cal_1]). We thus do that and fix a chosen metric.

Proposition 5.11. *Let $\phi_t\colon M \to M$ be a topological Anosov flow. Then, leaves of $\widetilde{\mathcal{F}^{ws}_\phi}$ and $\widetilde{\mathcal{F}^{wu}_\phi}$ are Gromov hyperbolic and orbits of $\widetilde{\phi}_t$ are uniform quasigeodesics in each leaf.*

Proof. We give a detailed outline of the proof since in [Fen_1] the assumption of being an Anosov flow (rather than a topological Anosov flow) is used. We do the proof for $\mathcal{F}^{ws}$, the one for $\mathcal{F}^{ws}$ is completely analogous. First, using results of Calegari [Cal_1] one can assume that leaves of $\mathcal{F}^{ws}_\phi$ are C^1.

For a set X we denote by $\mathrm{cc}_x(X)$ the connected component of X containing x. We choose a finite covering of M by the interior of local product structure boxes B_i with the following property:

- Each B_i is of the form $\bigcup_{t\in[-\varepsilon,\varepsilon]}\phi_t(D_i)$ and D_i is a square transverse to ϕ_t with four boundaries contained respectively in leaves of $\mathcal{F}^{ws}_\phi$ and $\mathcal{F}^{wu}_\phi$.
- For every pair of points $x, y \in B_i$ it follows that $\mathrm{cc}_x(\mathcal{F}^{ws}_\phi(x)\cap B_i)$ intersects $\mathrm{cc}_y(\mathcal{F}^{wu}_\phi(y))$ in a unique point and $\mathrm{cc}_x(\mathcal{F}^{wu}_\phi(x)\cap B_i)$ intersects $\mathrm{cc}_y(\mathcal{F}^{ss}_\phi(y))$ in a unique point.

Let a_0 be the maximum of the diameters of the B_i's. In addition there is $a_1 > 0$ so that any set of diameter $< a_1$ is contained in at least one B_i.

In the universal cover, we get a locally finite covering $\{\tilde{B}^j_i\}$ where each $\tilde{B}^j_i$ projects homeomorphically to B_i by the universal covering projection. Let a_3 be the minimum area of any local sheet of $\mathcal{F}^{ws}$ contained in some box B_i.

Stable segments are only continuous, hence not necessarily rectifiable. We define a "coarse length" of stable segments as follows. First notice that a stable segment in $\widetilde{M}$ can only intersect a box $\tilde{B}_i^j$ in a single component. So we define the "coarse length" of a stable segment s as the minimum number of boxes $\tilde{B}_i^j$ that can cover it. Since ϕ is a topological Anosov flow, there is $t_0 > 0$ such that for every stable segment I of coarse length between $9a_0$ and $10a_0$ we have that $\widetilde{\phi}_t(I)$ has diameter less than a_1 for every $t \geqslant t_0$ and hence it is contained in some box $\tilde{B}_i^j$. In other words flowing time t_0 divides the coarse length of a stable segment by roughly a factor of 10. Hence for long stable segments, its diameter is a bounded additive error from $2\log_{10} a_2$, where a_2 is its original coarse length.

To show that leaves of $\mathcal{F}^{ws}_\phi$ are Gromov hyperbolic it is enough to show that disks inside the leaves of $\widetilde{\mathcal{F}^{ws}_\phi}$ have exponential area with respect to the radius with uniform constants. We have just proved this because a stable segment of diameter d intersects at least $C10^{d/2}$ distinct $\tilde{B}_i^j$, where C is a global constant. A fixed percentage of the union of these $\tilde{B}_i^j$ have to be disjoint, yielding exponential area in terms of d.

This proves uniform Gromov hyperbolicity of leaves of $\mathcal{F}^{ws}$.

Now we prove that orbits are uniformly quasigeodesics inside the leaves of $\widetilde{\mathcal{F}^{ws}_\phi}$. We need to show that there exists $c > 1$ such that for every $L \in \widetilde{\mathcal{F}^{ws}_\phi}$ and $x \in L$ we have that $d_L(x, \widetilde{\phi}_t(x)) \geqslant c^{-1}t - c$. By lifting to a finite cover assume that $\mathcal{F}^{wu}$ is transversely orientable. This does not affect the quasigeodesic property. We will use a different metric in the leaves of $\mathcal{F}^{ws}$. First we reparametrize the flow so that it is a parametrization by unit speed. The new flow is still an expansive flow, so it still has strong stable and unstable foliations, even though these probably have changed. For any leaf L of $\mathcal{F}$ and x in L choose a basis of $T_x\mathcal{F}^{ws}$ as follows. Recall that leaves of $\mathcal{F}^{ws}$ are C^1. One basis vector is $v_x = T_x\phi$. Continuously choose a unit normal w_x to $T_x\phi$ in $T_x\mathcal{F}^{ws}$. Continuity is in M. We now define a metric in $T_x\mathcal{F}^{ws}$ as follows: For any $v \in T_x\mathcal{F}^{ws}$, there exists unique reals b and c such that $v = bv_x + cw_x$, and we set

$$|v| = |b| + |c|.$$

This definition gives a Finsler metric on each leaf.

Given a leaf E of $\widetilde{\mathcal{F}}^{ws}$, we define the distance in E to be the path metric generated by the Finsler metric $|\cdot|$ above lifted to $\widetilde{M}$.

Given any x, y in the same flow line of $\widetilde{\phi}$ in E let $y = \widetilde{\phi}_t(x)$ with $t > 0$. There is a projection from E to the flowline ℓ of x by projecting along the strong stable segments. For any piece wise smooth curve with endpoints in ℓ, the projection above decreases its length. This is because of the definition of the infinitesimal metric. Hence the flowline is a length minimizing geodesic in this metric. Since the metrics in E are quasicomparable, then flow lines are uniform quasigeodesics in their $\widetilde{\mathcal{F}^{ws}}$ leaves.

This finishes the proof of the proposition. ☐

Remark 5.12. In fact, the proof shows that inside each leaf of $\widetilde{\mathcal{F}^{ws}}$ the orbits of the flow form a *quasigeodesic fan* (cf. Definition 6.5).

5.5. **Discretized Anosov flows revisited.** Here we show that discretized Anosov flows defined in [BFFP$_1$] fit well with all the definitions of collapsed Anosov flows.

Definition 5.13. A partially hyperbolic diffeomorphism $f\colon M \to M$ is a *discretized Anosov flow* if there exists a topological Anosov flow $\phi_t\colon M \to M$ and a continuous function $\tau\colon M \to \mathbb{R}$ such that $f(x) = \phi_{\tau(x)}(x)$ for every $x \in M$.

In [BFFP$_1$, Appendix G] we asked for the function τ to be positive, but this is unnecessary:

Proposition 5.14. *If f is a discretized Anosov flow, then the function $\tau\colon M \to \mathbb{R}$ cannot vanish.*

Proof. In [BFFP$_1$, Proposition G.2], we proved that if f is a discretized Anosov flow, then f must be dynamically coherent. The argument presented in [BFFP$_1$, Proposition G.2] assumed that the map τ was positive, but we will show below how they can easily be modified in order not to use this assumption. Once we know that f is dynamically coherent, we will directly deduce that τ cannot vanish.

First, we prove, that the vector field X tangent to the flow ϕ_t needs to be in the center bundle of f:

If X is tangent to E^s at some point, then as in [BFFP$_1$, Proposition G.2], this implies that there is an interval along the flow direction totally tangent to E^s. Since E^s is uniquely integrable, then this interval in the flow direction is contained in a stable leaf. Now, the function τ is bounded, so the length along a flow line from x to $f(x)$ is bounded. Iterating negatively by f increases the stable length exponentially, so first we can assume that there is x in M such that the interval in the center leaf from x to $f(x)$ is contained in a stable leaf. Then again applying negative powers of f, produces a contradiction to τ being bounded. It follows that X is never tangent to E^s. The symmetric argument implies that it is never tangent to E^u either. Then, as in the proof of [BFFP$_1$, Proposition G.2] one proves, that X is always tangent to E^c and that f is dynamically coherent.

Since f is dynamically coherent, we can consider the good lift $\widetilde{f}$ obtained via the lift of the natural homotopy along the flowlines of the lifted topological Anosov flow. This lift $\widetilde{f}$ cannot have fixed points (see, e.g., [BWi, Corollary 3.11] or [BFFP$_1$, Lemma 3.13]), thus τ cannot vanish. □

The following relates the notion of discretized Anosov flows and collapsed Anosov flows.

Proposition 5.15. *If f is a discretized Anosov flow, then it verifies Definition 2.7 with h being a homeomorphism and β being a trivial self orbit equivalence. Conversely, if f verifies Definition 2.7 with β a trivial self orbit equivalence, then f is a discretized Anosov flow.*

Proof. To prove the direct assertion let us just take h to be the identity. In [BFFP$_1$, Proposition G.2] it is shown that the center stable and center unstable foliations of f correspond to the weak stable and weak unstable foliations of the topological Anosov flow (in particular, these weak foliations whose leaves are a priori only C^0 have C^1-leaves). Then $\beta(x) = \phi_{\tau(x)}(x)$ which proves the result.

For the converse statement, notice first that since f verifies Definition 2.7 then the image under h of leaves of $\mathcal{F}^{ws}_\phi$ provides a branching foliation tangent to E^{cs}, and likewise for $\mathcal{F}^{wu}_\phi$. Finally the image of any flow line is a curve tangent to E^c, providing a branching center foliation. Consider a good lift $\widetilde{f}$ corresponding to lifting β to a homotopy along the flow lines, and using a lift of h lifting a homotopy to the identity. The equation $f \circ h(x) = h \circ \beta(x)$ then implies that $\widetilde{f}$ preserves every center leaf in $\widetilde{M}$. Again the argument of [BWi, Corollary 3.11] (or [BFFP$_1$, Lemma 3.13]) implies that the lift to the universal cover cannot have fixed points. Moreover, when lifted to the universal cover, one has the same situation as in the *doubly invariant case*, as studied in [BFFP$_2$, §7.2]. This proves that the branching foliations are actual foliations, proving dynamical coherence of f. Now this immediately implies that f is a discretized Anosov flow (see also [BFFP$_1$, §6]). □

With what we proved so far, we can show that when the self orbit equivalence is trivial, then the two notions of collapsed Anosov flows (Definition 2.5) and strong collapsed Anosov flows (Definition 2.7) coincide.

Proposition 5.16. *Let f be a collapsed Anosov flows such that the associated self orbit equivalence β is trivial, then f is a strong collapsed Anosov flows.*

Proof. Since β is trivial, the image of the flowline foliation by h is an f-invariant branching foliation, whose leaves are tangent to E^c. This is a center branching foliation in this case. Moreover, a good lift $\widetilde{f}$ leaves invariant every center leaf.

As in [BFFP$_2$, Lemma 7.3] we know that $\widetilde{f}$ moves point a bounded distance in each center. An argument similar to [BFFP$_2$, Lemma 7.4] allows to show that center curves are disjoint or coincide. This implies that f verifies Definition 5.13. □

In view of the above, we can even wonder whether it would be sufficient for a definition of collapsed Anosov flow to only require the partially hyperbolic diffeomorphism to be semi-conjugate to a self orbit equivalence:

Question 5. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism such that there exists a (topological) Anosov flow $\phi_t\colon M \to M$, a self orbit equivalence $\beta\colon M \to M$ and a map $h\colon M \to M$ continuous and homotopic to the identity such that $f \circ h = h \circ \beta$. Is f a collapsed Anosov flow?*

6. Quasigeodesic behavior inside foliations

In this section we study some properties of one dimensional foliations which subfoliate a two dimensional foliation with Gromov hyperbolic leaves. Then, we restrict to the partially hyperbolic setting and show Theorem 6.11 that is the key step to obtain Theorem D which will be shown in the next section.

6.1. **One dimensional foliations inside two dimensional foliations.**

Let $\mathcal{F}$ be a foliation on a 3-manifold. In this section, we will assume that there is a metric on M that makes every leaf of $\mathcal{F}$ negatively curved. Then we can even assume the metric on each leaf is constant curvature -1 by Candel's uniformization theorem. This assumption is verified whenever the foliation does not have a transverse invariant measure of zero Euler characteristic (by Candel's uniformization theorem, see [CC, §I.12.6] or [Cal$_3$, §8] for a precise statement).

Consider a one dimensional foliation $\mathcal{G}$ which subfoliates $\mathcal{F}$ (i.e., leaves of $\mathcal{F}$ are saturated by leaves of $\mathcal{G}$). We suppose here that $\mathcal{G}$ has differentiable leaves.

Definition 6.1. The foliation $\mathcal{G}$ is a *uniform quasigeodesic* foliation of $\mathcal{F}$ if every leaf $\ell \in \mathcal{G}$ is a quasigeodesic in its corresponding leaf $L \in \mathcal{F}$ with uniform constants.

Let us make precise what we mean by uniform constants in the above definition:

Call $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ be the lifts of $\mathcal{F}$ and $\mathcal{G}$ respectively to the universal cover. Let ℓ be a leaf of $\widetilde{\mathcal{G}}$ in a leaf L of $\widetilde{\mathcal{F}}$. Then ℓ is a C-*quasigeodesic* if there is a constant $C > 1$ such that for every $x, y \in \ell$ we have that $d_\ell(x,y) < C d_L(x,y) + C$. Here d_L denotes the distance in L given by a Riemannian metric and d_ℓ denotes the distance in ℓ induced by restricting the Riemannian metric to ℓ.

In Definition 6.1, we require that there exists a constant $C > 1$ such that, for any $L \in \mathcal{F}$ and any $\ell \in \mathcal{G}$, the leaf ℓ is a C-quasigeodesic. Note that, by compactness of M, this definition does not depend on the choice of metric (see Proposition A.5).

Remark 6.2. One can in fact prove, by adapting the proof of [Cal$_3$, Lemma 10.20], that if $\mathcal{G}$ subfoliates $\mathcal{F}$ with quasigeodesic leaves, then it is automatically a *uniform* quasigeodesic foliation.

Notice then that the usual Morse Lemma (cf. [BH, §III.H.1]) implies that given a leaf $L \in \widetilde{\mathcal{F}}$ and $\ell \in \widetilde{\mathcal{G}}$ in L, there exists a unique geodesic $\hat{\ell}$ in L which is bounded Hausdorff distance in L away from ℓ and determines two distinct point ℓ^+, ℓ^- on $S^1(L)$ the circle at infinity (or Gromov boundary) of L called the endpoints (or ideal points) of ℓ or $\hat{\ell}$.

Remark 6.3 (Uniform bound). There is a positive constant $k > 0$ such that for any ℓ leaf of $\widetilde{\mathcal{G}}$ in a leaf L of $\widetilde{\mathcal{F}}$, if $\hat{\ell}$ is the geodesic in L with ideal points ℓ^-, ℓ^+ then the Hausdorff distance in L, $d_H(\ell, \hat{\ell}) < k$. In addition if x, y in ℓ, if $\hat{\ell}_{x,y}$ is the geodesic segment in L from x to y and $\ell_{x,y}$ the compact segment in ℓ from x to y, then $d_H(\ell_{x,y}, \hat{\ell}_{x,y}) < k$. See [BH, Theorem III.H.1.7].

Let $\widetilde{\mathcal{G}}|_L$ be the foliation $\widetilde{\mathcal{G}}$ when restricted to L. As is the case for foliations by geodesics [Cal$_2$, Construction 5.5.4], we are able to show that foliations by quasigeodesics of a hyperbolic plane are quite restrictive:

Proposition 6.4. *Given $L \in \widetilde{\mathcal{F}}$ we have that the leaf space $\mathcal{L}_{\widetilde{\mathcal{G}},L} = L/_{\widetilde{\mathcal{G}}}$ of the foliation $\widetilde{\mathcal{G}}$ is homeomorphic to $\mathbb{R}$ and either there is a point $p \in S^1(L)$ such that every leaf of $\widetilde{\mathcal{G}}|_L$ has p as one of its endpoints or there are exactly two points in $S^1(L)$ invariant under every isometry of L preserving the foliation $\widetilde{\mathcal{G}}|_L$. If r_n is a sequence of rays*[16] *in leaves of $\widetilde{\mathcal{G}}$ converging to a ray r, then the ideal points of r_n in $S^1(L)$ converge to the ideal point of r.*

Proof. We first show that the leaf space $L/_{\widetilde{\mathcal{G}}}$ is Hausdorff. Suppose this is not true and there are ℓ_n leaves in $\widetilde{\mathcal{G}}$ converging to two distinct leaves ℓ, ℓ' of $\widetilde{\mathcal{G}}$.

Let x, y be points in ℓ, ℓ' respectively. Then there are x_n, y_n in ℓ_n converging to x, y respectively. Hence $d_L(x_n, y_n)$ is bounded. We claim that $d_{\ell_n}(x_n, y_n)$ goes to infinity. Otherwise up to subsequence we would have $d_{\ell_n}(x_n, y_n) \leqslant a_0$. But using the local product structure of foliations, we would deduce that y is in ℓ, a contradiction.

Hence $d_{\ell_n}(x_n, y_n)$ must converge to infinity, but, since $d_L(x_n, y_n)$ is bounded, this contradicts the uniform quasigeodesic behavior. Therefore $L/_{\widetilde{\mathcal{G}}}$ is Hausdorff, and hence homeomorphic to $\mathbb{R}$.

We now show that the ideal points of rays of leaves of $\widetilde{\mathcal{G}}$ in $S^1(L)$ vary continuously. Let x_n be a sequence in L converging to x in L, and ℓ_n, ℓ the leaves of $\widetilde{\mathcal{G}}$ through x_n and x respectively.

Let r_n be rays in ℓ_n starting in x_n converging to a ray r in ℓ starting in x. Let y_n, y be the ideal points of r_n, r respectively. We want to show that y_n converges to y.

Suppose this is not the case, we assume up to subsequence that y_n converges to $z \neq y$. Since r_n converges to r and r has ideal point y, then for n large r_n has a point t_n very close to y in the compactification $L \cup S^1(L)$. The ray r_n also has a point z_n very close to z in $L \cup S^1(L)$. We assume that $t_n \to y$, $z_n \to z$. The compact segment s_n of l_n from t_n to z_n is at most k distant in L from the geodesic segment connecting them. Since t_n is very close to y and z_n is very close

[16] A *ray* of a leaf ℓ of $\widetilde{\mathcal{G}}$ in a leaf $L \in \widetilde{\mathcal{F}}$ is the closure of a connected component of $\ell \smallsetminus \{x\}$ for some $x \in \ell$. Each ray has a well defined ideal point $r_\infty \in S^1(L)$ which coincides with the corresponding ideal point of ℓ.

to z, then all of these geodesic segments intersect a fixed compact set of L, and similarly for the segments s_n of r_n. Up to another subsequence s_n converges to a leaf ℓ' of $\widetilde{\mathcal{G}}$. This leaf is not ℓ, contradicting that $L/_{\widetilde{\mathcal{G}}}$ is Hausdorff. This proves the continuity property.

Identify the leaf space $L/_{\widetilde{\mathcal{G}}}$ with $\mathbb{R}$, with parametrization $\ell_t, t \in \mathbb{R}$ and consider a sequence ℓ_{t_n}, $t_n \to +\infty$. Notice first that the endpoints $(\ell_{t_n})^{\pm}$ determine a weakly nested sequence of intervals in $S^1(L)$ which needs to shrink as $n \to \infty$. If they do not shrink they limit to a geodesic g in L. But recall that the ℓ_n are at distance at most k corresponding geodesics (cf. Remark 6.3): If the endpoints are trapped by the endpoints of g, then the leaves are trapped by a neighborhood of size k of g and cannot escape in L, contradiction.

Hence we get two points of $S^1(L)$ one for $+\infty$ and one for $-\infty$. If these two points coincide then we get that every leaf of $\widetilde{\mathcal{G}}$ must have that limit point as a limit point. Otherwise, we get the other condition. Obviously any isometry of L leaving the foliation invariant has to preserve this pair of ideal points. $\square$

We now define some structures related to what follows from the previous proposition.

Definition 6.5. We say that a leaf $L \in \widetilde{\mathcal{F}}$ is a *weak quasigeodesic fan* for the foliation $\widetilde{\mathcal{G}}$ if there is a point $p \in S^1(L)$ such that every leaf of $\widetilde{\mathcal{G}}|_L$ has p as one of its limit points. In this case we call p the *funnel point* of $\widetilde{\mathcal{G}}|_L$. The leaf $L \in \widetilde{\mathcal{F}}$ is a *quasigeodesic fan* if moreover given a point $q \in S^1(L) \smallsetminus \{p\}$ there is a unique leaf of $\widetilde{\mathcal{G}}|_L$ whose endpoints are p and q. We say that a leaf A of $\mathcal{F}$ is a quasigeodesic fan or a weak quasigeodesic fan if a lift L of it to $\widetilde{M}$ is a quasigeodesic fan or a weak quasigeodesic fan respectively.

Lemma 6.6. *If $\mathcal{G}$ is a uniform quasigeodesic subfoliation of $\mathcal{F}$ then, for every leaf $L \in \widetilde{\mathcal{F}}$ we have that there are at most two points in $S^1(L)$ which are not endpoints of any of the curves in $\widetilde{\mathcal{G}}|_L$.*

Moreover, if a leaf $L \in \widetilde{\mathcal{F}}$ is a weak quasigeodesic fan, then every points of $S^1(L)$ is the endpoint of a center leaf. See Figure 1.

Proof. We use the notation of Proposition 6.4. We justify this for the case that the limit points of $\ell_{t_n}, n \to \infty$ and $\ell_{t_n}, n \to -\infty$ are distinct points x, y in $S^1(L)$. Let I, J be the complementary intervals of x, y in $S^1(L)$. First note that not all leaves can have both endpoints in the closure of J. Consider the curve α which is k distant from the geodesic connecting x, y and in the side limiting on I. By the property of k, then any leaf of $\widetilde{\mathcal{G}}$ is entirely contained in the component of α that limits on J. This is impossible. Hence there are leaves with one ideal point in I, and similarly for J. Let ℓ be a leaf of $\widetilde{\mathcal{G}}|_L$ with one endpoint in I. Clearly the other endpoint of ℓ cannot be in I. We proved in Proposition 6.4 that the endpoints of corresponding rays of ℓ_t vary continuously when t varies. Since they limit to x, y when $t \to \pm\infty$ it follows that the set of limit points of rays of leaves of $\widetilde{\mathcal{G}}|_L$ contains I. This shows that the case $x = y$ occurs exactly when $\widetilde{\mathcal{G}}|_L$ is a weak quasigeodesic fan in L. When x, y are distinct we may have that neither is an ideal point of a leaf of $\widetilde{\mathcal{G}}$, one of them is, or both of them are. $\square$

From Proposition 6.4 we deduce:

Corollary 6.7. *If $\mathcal{F}$ a foliation by hyperbolic leaves admits a uniformly quasigeodesic subfoliation $\mathcal{G}$ then every leaf of $\mathcal{F}$ has cyclic fundamental group (thus a leaf is either a plane, an annulus or a Möbius band).*

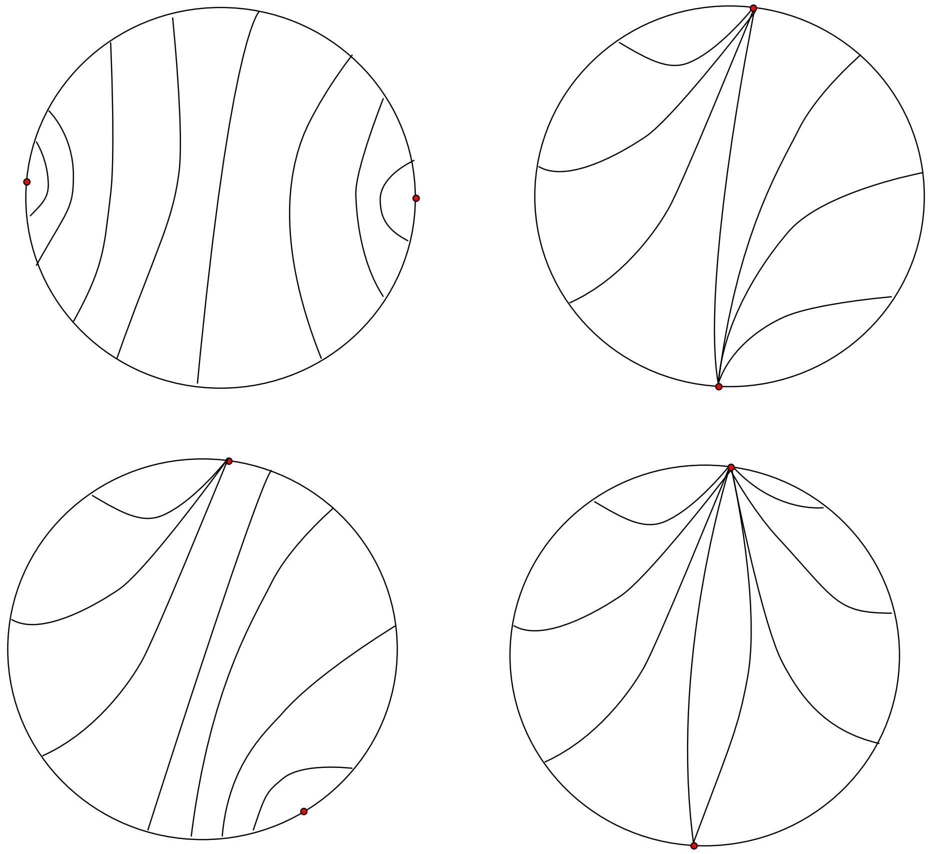

FIGURE 1. Some quasigeodesic foliations of the disk which are not quasigeodesic fans (the bottom right one is a weak quasigeodesic fan).

Proof. Deck transformations of M act as isometries, so if a deck transformation fixes some leaf $L \in \widetilde{\mathcal{F}}$ then it is an isometry which preserves $\widetilde{\mathcal{G}}|_L$. Note that it must be a hyperbolic isometry since there is a uniform injectivity radius of leaves of $\mathcal{F}$. Hyperbolic isometries that fix a given point or a pair of points at infinity commute, so this concludes. $\square$

Our first goal is to show that there are weak quasigeodesic fan leaves of $\mathcal{F}$, and that the collection of such leaves forms a sublamination of $\mathcal{F}$. This is the analogue of [Cal$_2$, Lemma 5.3.6] (see also [Cal$_2$, Lemma 5.5.5]) which is done for the case of geodesic subfoliations in leaves of $\mathcal{F}$.

We first need a technical result that produces some weak quasigeodesic fan leaves from certain configurations and will be used several times.

Lemma 6.8. *Suppose that x_n is a sequence in $\widetilde{M}$ such that there are disks D_n in the leaves $L_n \in \widetilde{\mathcal{F}}$ centered at x_n with radius converging to infinity and satisfying the following: There are disks E_n in L_n of bounded diameter, such that the distance in L_n from E_n to D_n goes to infinity and such that any leaf of $\widetilde{\mathcal{G}}|_{L_n}$ intersecting D_n also intersects E_n. Then, given a sequence of deck transformations $\gamma_{n_j} \in \pi_1(M)$ such that $\gamma_{n_j} x_{n_j} \to x$ for some subsequence $n_j \to \infty$ we have that the leaf through x is a weak quasigeodesic fan.*

Proof. We can assume without loss of generality that $x_n \to x$ up to changing by deck transformations and taking a subsequence.

Call $a_1 > 0$ an upper bound of the diameters of the E_n. Assume by contradiction that the leaf L of $\widetilde{\mathcal{F}}$ through x is not a weak quasigeodesic fan. Then there is a pair of leaves ℓ, ℓ' of $\widetilde{\mathcal{G}}|_L$ which do not share any ideal point in $S^1(L)$. These curves are at at most k distant in L from the corresponding geodesics because of the uniform bound, see Remark 6.3.

Since ℓ, ℓ' do not share ideal points, then two properties follow:

(i) there are points y, y' in ℓ, ℓ' respectively attaining the minimum distance a_0 between points in ℓ, ℓ',

(ii) there is $t > 0$ such that if $z \in \ell$, and $z' \in \ell'$ then if both $d_L(z, y)$ and $d_L(z', y')$ are larger than t then

$$d_L(z, \ell'), d_L(z', \ell) > 10(a_0 + a_1 + k + 4).$$

The points x_n converge to x in L. The distance from x to y, y' in L is finite, so up to changing x_n in L_n by a bounded distance (and choosing a subdisk of D_n with radius still going to infinity with n) we may assume that x_n converges to y. Since the foliations $\widetilde{\mathcal{G}}|_{L_n}$ converge to $\widetilde{\mathcal{G}}_L$ we see that the foliations in the disk of radius $100(t + a_0 + 1)$ (recall that t, a_0 are fixed) around x_n converge to the foliation in a disk of radius $100(t + a_0 + 1)$ around y in L. In L on both sides of y, y' the leaves ℓ, ℓ' spread more than $10(a_0 + a_1 + k + 4)$ from each other. So we see this in some of the leaves of $\widetilde{\mathcal{G}}|_{L_n}$ as well. This is within fixed distance t. But the property of E_n means that these leaves come back within a_1 of each after a distance larger than t if n is big enough.

We now use that these curves are uniform quasigeodesics. Recall their properties:

(i) they are within $a_0 + 1$ from each other near y, y';

(ii) they are within within $a_1 + 1$ from each other when they both intersect E_n.

This implies that the geodesic segments connecting these pairs of points are within $(a_0 + a_1 + 2)$ throughout. By the uniform quasigeodesic property the segments in leaves of $\widetilde{\mathcal{G}}|_{L_n}$ are within $(a_0 + a_1 + 2) + 2k$ from each other throughout. But we proved that they have points where the curves are more than $10(a_0+a_1+2+2k+2)$ apart from each other in between.

This contradiction shows that the limit leaf is a weak quasigeodesic fan and finishes the proof of the lemma. □

We can now show:

Proposition 6.9. *The set of leaves $L \in \widetilde{\mathcal{F}}$ which are weak quasigeodesic fans for $\widetilde{\mathcal{G}}$ is non empty, closed and $\pi_1(M)$-invariant. Hence it induces a sublamination of $\mathcal{F}$ in M.*

Proof. The $\pi_1(M)$ invariance property is obvious.

We first show that the set of weak quasigeodesic fan leaves is non empty. Let L be a leaf of $\widetilde{\mathcal{F}}$. We will construct sets D_n, E_n in L satisfying the hypothesis of the previous lemma. Let ℓ_1 be a leaf of $\widetilde{\mathcal{G}}|_L$. Let I be the closed interval of leaves of $\widetilde{\mathcal{G}}|_L$ all of which share both endpoints with ℓ_1. This could be a degenerate interval, that is, ℓ_1 itself. Let ℓ be a boundary leaf of I. Now consider a leaf ℓ' sufficiently near ℓ intersecting a transversal τ from x in ℓ to x' in ℓ'. In addition assume that ℓ' is not in I. Let E be a disk containing τ. Let $E_n = E$ of fixed diameter.

Since ℓ' is not in I, it has at least one ideal point z' which is not an ideal point of ℓ. Let r' be the ray of ℓ' starting in x' and with ideal point z'. Let r be the ray of ℓ starting in x and going in the same direction as r'.

Recall that the leaf space of $\widetilde{\mathcal{G}}|_L$ is the reals $\mathbb{R}$. Let V be the complementary region of $r \cup \tau \cup r'$ which only contains rays of leaves of $\widetilde{\mathcal{G}}|_L$ which intersect τ. Hence every leaf of $\widetilde{\mathcal{G}}|_L$ intersecting V also intersects the fixed set E.

Finally since r' and r do not have the same ideal points and are quasigeodesics we can find D_n a set of diameter greater than n contained in V and such that the distance in L from D_n to E is greater than n. Taking $L_n = L$ for any n, we can apply the previous lemma and get leaves of $\mathcal{F}$ which are weak quasigeodesic fans. This proves the first assertion of the proposition.

Now we prove that the set of leaves that are not weak quasigeodesic fan is open.

Let L be a leaf that is not a weak quasigeodesic fan. Then there are leaves ℓ, ℓ' which do not share any ideal points. As in the previous lemma:

(i) there are points $y \in \ell, y' \in \ell'$ realizing the minimum distance a_0 between them,
(ii) for any $a_2 > 0$, there is $t > 0$ such that if distance along ℓ from y to z is greater than t then $d_L(z, \ell') > a_2$ and vice versa for points in ℓ'.

Hence once a_1, a_2, t are fixed we obtain for any leaf F sufficiently near L that we have leaves ℓ_F, ℓ'_F in $\widetilde{\mathcal{G}}|_F$ satisfying this property in F. Specifically this does not hold for *every* point z in ℓ_F with distance in ℓ_F from a fixed point is $> t$, but for some points. We choose $a_2 > a_0 + 100k$. Fix this pair of leaves ℓ_F, ℓ'_F.

Now suppose that F is a weak quasigeodesic fan. We will obtain a contradiction. For any two leaves ζ, ζ' in $\widetilde{\mathcal{G}}|_F$ they have a common endpoint in some direction. If they share both endpoints then they are within $2k$ of each other. So if $a_2 > 2k$ then the pair ℓ_F, ℓ'_F cannot be ζ, ζ'. Since $a_2 > 2k$, it follows that ℓ_F, ℓ'_F cannot be ζ, ζ'.

Next, suppose that ζ, ζ' share one but not both ideal points. The corresponding geodesics $\hat{\zeta}, \hat{\zeta}'$ of F to ζ, ζ' are asymptotic, but disjoint. By negative curvature in the direction where they are asymptotic, the distance in F between points y_t in $\hat{\zeta}$ converging to the common ideal point and $\hat{\zeta}'$ is always decreasing, modulo a bounded error, and converging to zero. Since ζ, ζ' are k distant from $\hat{\zeta}, \hat{\zeta}'$ respectively, then the distance in F between points y_t in ζ converging to the common ideal point and ζ' in F is roughly decreasing modulo an error of at most $4k$. But the leaves ℓ, ℓ' have points very distant ($> a_2 > a_0 + 100k$ from the other leaf), then follow along to points roughly a_0 distant, then again some points very distant ($> a_2$). Therefore ℓ, ℓ' cannot be ζ, ζ'.

This contradicts the existence of leaves ℓ, ℓ' in F, which have to be some pair ζ, ζ'. This contradiction finishes the proof that the set of non weak quasigeodesic fans is open. This finishes the proof of the proposition. $\square$

6.2. **Branching foliations.** Now consider two transverse branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ in M (the names are given for obvious reasons) which determine a one dimensional branching foliation $\mathcal{W}^c$ by intersection. We consider $\widetilde{\mathcal{W}^{cs}}$, $\widetilde{\mathcal{W}^{cu}}$ the lifts to the universal cover. We assume that $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ are transversely orientable. Let $\mathcal{F}^{cs}_\varepsilon, \mathcal{F}^{cu}_\varepsilon$ be the approximating foliations from $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ given by Theorem 3.3 for some small $\varepsilon > 0$. Let and $\widetilde{\mathcal{F}^{cs}_\varepsilon}, \widetilde{\mathcal{F}^{cu}_\varepsilon}$ their lifts to $\widetilde{M}$. These determine a foliation $\widetilde{\mathcal{F}^c_\varepsilon}$ which subfoliates both. The foliation $\widetilde{\mathcal{F}^c_\varepsilon}$ also projects to a one dimensional foliation $\mathcal{F}^c_\varepsilon$ which subfoliates both $\mathcal{F}^{cs}_\varepsilon, \mathcal{F}^{cu}_\varepsilon$. Since $\mathcal{F}^{cs}_\varepsilon, \mathcal{F}^{cu}_\varepsilon$ have C^1 smooth leaves, then leaves of $\mathcal{F}^c_\varepsilon$ are C^1.

We can then copy the notions above to define:

Definition 6.10. We say that $\mathcal{W}^c$ is by *uniform quasigeodesics* in $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ if leaves of $\mathcal{F}^{cs}_\varepsilon, \mathcal{F}^{cu}_\varepsilon$ are Gromov hyperbolic and $\widetilde{\mathcal{F}^c_\varepsilon}$ is a foliation by uniform quasigeodesics in both $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ and $\widetilde{\mathcal{F}^{cu}_\varepsilon}$ as in Definition 6.1 . Similarly we can define as in Definition 6.5 leaves of $\widetilde{\mathcal{W}^{cs}}$ or $\widetilde{\mathcal{W}^{cu}}$ (or leaves of $\mathcal{W}^{cs}, \mathcal{W}^{cu}$) being *(weak)-quasigeodesic fans* by the identification between the leaves of $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ and $\widetilde{\mathcal{W}^{cs}}$ (resp. $\widetilde{\mathcal{F}^{cu}_\varepsilon}$ and $\widetilde{\mathcal{W}^{cu}}$).

The leaves of $\widetilde{\mathcal{W}^{cs}}, \widetilde{\mathcal{W}^{cu}}$ have their intrinsic geometry induced from the Riemannian geometry of $\widetilde{M}$. These leaves are quasi-isometric to the corresponding leaves of $\widetilde{\mathcal{F}}^{cs}_\varepsilon, \widetilde{\mathcal{F}}^{cu}_\varepsilon$. In particular the notions above are independent of ε.

6.3. **The partially hyperbolic setting.** Here we state the main result of this section:

Theorem 6.11. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism preserving branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ such that the foliation $\mathcal{W}^c$ is by uniform quasigeodesics (cf. Definition 6.10). Then, the center leaves of $\mathcal{W}^c$ form a quasigeodesic fan in each $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ leaf.*

We will split the proof of Theorem 6.11 into two parts. Proposition 6.15 shows that every leaf of $\widetilde{\mathcal{W}^{cs}}$ must be a weak quasigeodesic fan and Proposition 6.20 shows that different centers in a leaf of $\widetilde{\mathcal{W}^{cs}}$ do not have the same pair of points at infinity. Both proposition follow the same strategy, first we construct an invariant lamination of *good leaves* where the property we want holds, and then we apply Proposition 6.12 below to show that every leaf is a good leaf.

6.4. **A general result about invariant laminations.** Here we give a general result that will be used repeatedly in what follows and might be interesting in itself. The result is stated for $\mathcal{W}^{cs}$, but obviously works for $\mathcal{W}^{cu}$ as well.

Proposition 6.12. *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism preserving a branching foliation $\mathcal{W}^{cs}$ tangent to E^{cs} and $\widetilde{f}$ a lift to $\widetilde{M}$. Suppose that M is orientable. Let $\mathcal{P} \subset \mathcal{L}^{cs}$ be a closed $\pi_1(M)$- and $\widetilde{f}$-invariant subset of the leaf space of $\widetilde{\mathcal{W}^{cs}}$. Assume that the stabilizer of each leaf of $\mathcal{L}^{cs}$ is at most infinite cyclic. Then, for every $\mathcal{N}$ connected component of $\mathcal{L}^{cs} \smallsetminus \mathcal{P}$ there is $\gamma \in \pi_1(M)$, a leaf $L \in \mathcal{N}$ and a leaf $L' \in \mathcal{N}$ such that $\gamma L = L$ but $\gamma L' \neq L'$.*

Proof. Assume by contradiction that every leaf in $\mathcal{N}$ is invariant by the same deck transformations. Then we can assume by taking finite covers and iterates that all the bundles are orientable and f preserves orientation. Consider the approximating foliation $\mathcal{F}^{cs}_\varepsilon$, with lift $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ and leaf space $\mathcal{L}^{cs}_\varepsilon$, which is canonically equivariantly homeomorphic to $\mathcal{L}^{cs}$. Let $\mathcal{P}_\varepsilon$ be the closed set corresponding to $\mathcal{P}$ and $\mathcal{N}_\varepsilon$ the open set corresponding to $\mathcal{N}$. Then the set of leaves in $\mathcal{N}_\varepsilon$ projects to an open $\mathcal{F}^{cs}_\varepsilon$ foliated set U in M. The hypothesis mean that every leaf in $\mathcal{N}_\varepsilon$ is invariant by the same deck transformations. In particular the foliation $\mathcal{F}^{cs}_\varepsilon$ restricted to U has trivial holonomy (the germ of holonomy of every closed curve in a leaf of $\mathcal{F}^{cs}_\varepsilon$ in U is trivial).

Now use [CC, Theorem I.9.2.1] applied to $\mathcal{F}^{cs}_\varepsilon$ in U. This implies that the leaf space of $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ in $\mathcal{N}_\varepsilon$ is homeomorphic to $\mathbb{R}$. In particular the same is true for the leaf space of $\widetilde{\mathcal{W}^{cs}}$ in $\mathcal{N}$.

Our assumption is that either every leaf in $\mathcal{N}$ has trivial stabilizer, or that every leaf of $\mathcal{N}$ has exactly the same stabilizer which is $\mathbb{Z}$. Denote by $G < \pi_1(M)$ the subgroup of deck transformations fixing $\mathcal{N}$. The group G is the same group which fixes $\mathcal{N}_\varepsilon$. We need the following property:

Claim 6.13. *Up to deck transformations $\mathcal{N}$ is $\widetilde{f}$ periodic.*

Proof. The projection V of $\mathcal{N}$ to M may not be open if $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ are not foliations and rather branching foliations. Nevertheless V is not a single leaf and has non empty interior, hence contains an open unstable segment τ. Let x in τ. Iterating positively by f one gets a limit point of the sequence $f^n(x)$. Since $\mathcal{P}$ is $\widetilde{f}$ and $\pi_1(M)$-invariant then for some fixed n, $f^n(V)$ and V intersect in their interiors. Hence $\widetilde{f}^n(\mathcal{N})$ is a deck translate of $\mathcal{N}$. □

By the claim, after taking an iterate we can assume that G is f_*-invariant. We will take such an iterate. We will need some arguments from standard 3-manifold topology. If the stabilizer of leaves in $\mathcal{N}_\varepsilon$ is always trivial, then as it acts freely on $\mathbb{R}$ it follows that G is abelian. By [Hem, Theorem 9.13] we get that G can be either 0, $\mathbb{Z}$, $\mathbb{Z}^2$ or $\mathbb{Z}^3$. Suppose on the other hand that the subgroup stabilizing every leaf of $\mathcal{N}_\varepsilon$ is infinite cyclic, generated by γ. It is very easy to see that for any α in G then $\alpha\gamma\alpha^{-1} = \gamma^{\pm}$, hence $\langle\gamma\rangle$ is a normal subgroup. In addition $G/\langle\gamma\rangle$ acts freely on $\mathbb{R}$ hence it is abelian. Since $\alpha\gamma\alpha^{-1} = \gamma^{\pm}$ it follows that G has a subgroup of index 2 which is abelian. Again by [Hem, Theorem 9.13] this subgroup G' of index 2 can only be $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3$. Notice that $f_*(\gamma) = \gamma^{\pm}$ so f_* preserves G'. So in any case f_* preserves an abelian subgroup G' of index at most 2, which can only be $0, \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3$. We need the following:

Claim 6.14. *The action of f_* in G' does not have eigenvalues of modulus larger than* 1.

Proof. Let U be the open set in M which is the projection of the leaves in $\mathcal{N}_\varepsilon$. The completion $\hat{U}$ of U has an octopus decomposition (cf. [CC, Proposition I.5.2.14]) with a thin part T and a core K such that K is compact and $\hat{U}$ retracts onto K hence $\pi_1(\hat{U})$ is finitely generated.

By the orientability conditions it follows that the boundaries of K are tori. Since leaves are properly embedded in $\widetilde{M}$ it follows that either K is a solid torus or that all boundary components of K are π_1-injective in $\pi_1(M)$ and hence $\pi_1(\hat{U})$ injects in $\pi_1(M)$ (note that the image is exactly G).

We proved before that G' can be only $0, \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3$.

The claim is trivial if G' is either 0 or $\mathbb{Z}$. If $G' = \mathbb{Z}^3$, using that M is prime we can apply [Hem, Theorem 9.11] to deduce that M has virtually abelian fundamental group, and then the result follows from [BI].

Finally, if $G' = \mathbb{Z}^2$ then [Hem, Theorem 10.5] implies that K is $\mathbb{T}^2 \times [0,1]$ up to double cover. This case was dealt with in [HHU$_1$]. We just provide a couple of details: in this case up to finite index G' is generated by α, β, where α corresponds to a curve in $K \cap T$. The curve α has to be preserved up to finite order. This implies the result. □

We now complete the proof of Proposition 6.12. The contradiction will be given by a volume versus length argument that will imply that the action of f_* on G' must have an eigenvalue of modulus larger than one. More precisely, [HPS, Proposition 5.2] implies that if there is an open f-invariant set $X \subset M$ such that the inclusion $\imath\colon X \subset M$ verifies that $\imath_*(\pi_1(X))$ is abelian and there is a strong unstable manifold inside X which is at distance $\geqslant \varepsilon$ from the boundary of X, then f_* must have an eigenvalue of modulus larger than 1 in $\imath_*(\pi_1(X))$. The same proof applies if $i_*(\pi_1(X))$ has a subgroup of index 2 which is abelian and preserved by f_*. Notice that

$$\imath_*(\pi_1(X)) \subset \imath_*(\pi_1(\hat{U})) = \imath_*(G).$$

We will apply the result from [HPS] to the interior X of the projection of the closure of N to M. This is an open f-invariant set (after taking the iterate we considered before). Let x be a point in ∂X. Let $\ell_x \in [0, \infty]$ be the length of the open unstable segment inside X whose boundaries are in ∂X and one of them is x. This interval is possibly trivial giving $\ell_x = 0$ or a complete ray giving $\ell_x = \infty$. It follows that the function ℓ_x of x cannot be bounded in ∂X. Otherwise it would have a maximum and backward iteration would give a contradiction. This allows to construct an unstable curve completely contained in the interior of X. Moreover, the closure of such unstable leaf must be at positive distance of ∂X because of local product structure. This completes the proof. $\square$

6.5. **Funnel leaves.** Here we show:

Proposition 6.15. *In the setting of Theorem 6.11 we have that every leaf of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ is a weak quasigeodesic fan for $\mathcal{F}^c$.*

Recall that we proved in Proposition 6.9 that the set P of leaves of $\widetilde{\mathcal{W}^{cs}}$ which are weak quasigeodesic fans is non empty, $\pi_1(M)$ invariant, $\widetilde{f}$ invariant, and closed. We did that in the (non branching) foliations setting, but subsection 6.2 implies the result in the branching foliations setting as well. Let Λ be the projection of the leaves in P to M. This is a closed, f-invariant set of $\mathcal{W}^{cs}$ leaves, that is a sublamination of $\mathcal{W}^{cs}$. We want to show that these are all the leaves of $\mathcal{W}^{cs}$.

Notice that, if we assumed that the branching foliations are f-minimal (see [BFFP$_1$, BFFP$_2$]), which happens for instance when f is transitive, then (by definition of f-minimality) Λ would automatically consist of all the leaves of $\mathcal{W}^{cs}$.

So the rest of this section will deal with the general case, and the reader only interested in the transitive case can skip this section.

In order to prove that Λ covers all the leaves of $\mathcal{W}^{cs}$, we will first consider a slightly larger lamination such that the leaves in the complementary region are all planes. This will allow us to apply Proposition 6.12. First we show that annular leaves which are not in Λ can only accumulate on Λ.

Lemma 6.16. *Let A be an annular leaf of $\mathcal{W}^{cs}$ in the complement of Λ. Then A only limits on points in Λ, that is, the closure of A is contained in $\Lambda \cup A$.*

Proof. Let A be an annular leaf of $\mathcal{W}^{cs}$ in the complement of Λ. Let γ be a generator of $\pi_1(A)$. Let L be a lift of A to $\widetilde{M}$ invariant by γ. There are two possibilities depending on the action of γ:

Claim 6.17. *If γ does not act freely on the center leaf space in L (i.e., if there is a center curve which is invariant under γ) then A limits only on points of Λ.*

Proof of Claim 6.17. The set of center leaves in L invariant by γ is a non empty, bounded and closed interval I. Let c, c' be the boundary leaves of this interval.

Recall from the proof of Corollary 6.7 that the action of γ on $S^1(L)$ fixes only the ideal points z_1, z_2 of c, c' and acts as a translation on any complementary interval. Let e be a center leaf in L not in I. Hence it has one ideal point, call it t which is not an ideal point of c, with t contained in the interval component J of

$S^1(L) \setminus \{z_1, z_2\}$. If the other ideal point t' of e is also not an ideal point of c, then t' is also in J. Recall that γ acts as a translation on J. There are two options: if $\gamma(t), \gamma(t')$ link with t', t' in $S^1(L)$ then e and $\gamma(e)$ intersect transversely, which is a contradiction. If $\gamma(t), \gamma(t')$ do not link with t', t', then this shows that the leaf space of the center foliation in L is non Hausdorff. This contradicts Proposition 6.4.

It follows that every such center leaf e shares an ideal point with c, c'. Suppose it is z_1. In addition if e' is another leaf with an ideal point in J, then the other ideal point of e' is also z_1.

Let now x_i be points in A converging to a point x in M. In particular, they are getting further and further apart from a fixed closed core curve α in A. Up to taking a subsequence, we can assume that they are all in the same complementary component of α in A. Lift x_i to points y_i in L. The distance in L, call it d_i, from y_i to c is going to infinity. Let D_i be disks in L around y_i of radius $d_i/2$. Any center curve intersecting D_i has one ideal point in J, hence also an ideal point which is z_1. Hence any two of these eventually get $3k$ from each other. So we can find a set E_i of diameter less than $3k$ intersecting all of these leaves. Now Lemma 6.8 implies that the leaf through the limit of x_i is a weak quasigeodesic fan. In particular it is in Λ. □

Claim 6.18. *If γ acts freely on the center leaf space in L then A limits only on leaves of Λ.*

Proof of Claim 6.18. As in the previous claim, we choose x_i in A with lifts y_i in L. Choose α a simple closed core curve in A, and let $\tilde{\alpha}$ be its lift to L. Choose x_i in a fixed component of $A \setminus \alpha$, hence y_i are in a fixed component of $L \setminus \tilde{\alpha}$, whose limit set is $J \cup \{z_1, z_2\}$. Let c_i be a center leaf through y_i.

The center leaf space in L is $\mathbb{R}$, hence for any c center in L, then $c, \gamma(c)$ are connected by a transversal. It follows that we can choose α transverse to the center foliation in A, and $\tilde{\alpha}$ intersects all centers in L.

We need an additional fact: if e is any center leaf in L, then no ideal point of e is z_1 or z_2. Suppose not, say z_1 is an ideal point of e. Let z be the other ideal point. If $z = z_2$ then by iterating by γ then one obtains a center leaf invariant by γ which was treated in the previous claim.

If z is not z_2, then iterate by positive or negative powers of γ so that z converges to z_2. This would produce a center leaf fixed by γ – again this would mean we are in the setting of the previous claim.

So no e has ideal point z_1 or z_2. Let w_i be the ideal point of c_i which is in J. Since $\tilde{\alpha}$ intersects all centers in L it follows that the other ideal point of c_i is not in J and hence it is in I. Since w_i is not z_1 or z_2 then $\gamma(w_i)$ and $\gamma^{-1}(w_i)$ are different from w_i. Since the distance from w_i to $\tilde{\alpha}$ is going to infinity, it follows that the distance d_i from w_i to $\gamma(c_i)$ and $\gamma^{-1}(c_i)$ is going to infinity as well. Let D_i be the disk in L centered at y_i with radius $\min\{d_i/2, d_L(y_i, \tilde{\alpha})\}$. Any center leaf intersecting D_i intersects $\tilde{\alpha}$ between $s_i = \gamma^{-1}(c_i) \cap \tilde{\alpha}$ and $s_i' = \gamma(c_i) \cap \tilde{\alpha}$. This set has diameter bounded by twice the length of α. So we can choose this set contained in E_i of bounded diameter.

With these properties, Lemma 6.8 implies that any leaf containing a limit of the x_i is in Λ. □

Claims 6.17 and 6.18 together complete the proof of Lemma 6.16. □

We also need the following technical result.

Lemma 6.19. *Consider the set $P_0 \subset \mathcal{L}^{cs}$ consisting of leaves invariant under some non trivial deck transformation. Then, the set $P \cup P_0$ is a closed set of leaves of $\mathcal{L}^{cs}$ (that is, a sublamination) which is $\widetilde{f}$- and $\pi_1(M)$-invariant.*

Moreover, if $P \cup P_0 = \mathcal{L}^{cs}$ then there are two possibilities:

(i) *either $P = \mathcal{L}^{cs}$, or,*
(ii) *in each connected component of $\mathcal{L}^{cs} \smallsetminus P$ all leaves are invariant by the same deck transformations.*

Proof. Recall that Λ is be the projection of P to M. We will also use the approximating foliation $\mathcal{F}^{cs}_{\varepsilon}$ of $\mathcal{W}^{cs}$ (up to finite lift this is no problem, see Theorem 3.3) and denote by Λ_ε the lamination in $\mathcal{F}^{cs}_{\varepsilon}$ induced by the blown up leaves of Λ. Since leaves with non trivial fundamental group are clearly f-invariant, it is enough to show the rest of the claims in the Lemma for the foliation $\mathcal{F}^{cs}_{\varepsilon}$ and the lamination Λ_ε.

Consider the completion $\hat{U}$ of a connected component U of $M \smallsetminus \Lambda_\varepsilon$ and its octopus decomposition (cf. [CC, Proposition I.5.2.14]) with a thin part T and a core K so that K is compact and $T = T_1 \cup \ldots \cup T_m$ where each T_i (an arm) is an I-bundle.

Lemma 6.16 implies that every annulus leaf B of $\mathcal{F}^{cs}_{\varepsilon}$ in $M \smallsetminus \Lambda_\varepsilon$ accumulates only in Λ_ε. Suppose that it is contained in the component U as above. Recall that $\hat{U} = K \cup T$. We choose K big enough so that each component $K \cap T$ (which is also an annulus) is transverse to $\mathcal{F}^{cs}_{\varepsilon}$. Then except for a compact subannulus in B, the rest of B is contained in T. In particular since the foliation restricted to each component of T is a foliated I-bundle, it follows that $B \cap K$ is a compact annulus K_B. Using [CC, Theorem I.6.1.1] we know that the set of leaves of $\mathcal{F}^{cs}_{\varepsilon}$ restricted to K which are compact is a compact set. Notice that the intersection of a leaf B of $\mathcal{F}^{cs}_{\varepsilon}$ in U with K is compact if and only if B is an annulus (the other option is B is a plane). Hence the set of annuli leaves in $\mathcal{F}^{cs}_{\varepsilon}|_K$ is a compact set.

This shows the first statement of Lemma 6.19 holds: $P \cup P_0$ is a sublamination of $\mathcal{W}^{cs}$ which is f invariant.

We need to show that if the set of annulus leaves in U is all of U then every leaf in U is invariant under the same deck transformation. This follows from the fact (cf. Corollary 6.7) that every leaf is either a plane or an annulus. Since K is compact, there is a finite set $\{\gamma_1, \ldots, \gamma_k\}$ in $\pi_1(K)$ such that leaves which are not planes in U must be fixed by some of the γ_i. This is because any such annulus leaf is incompressible in K, and distinct leaves are disjoint. Hence there are finitely many of these which are pairwise not isotopic [Hem]. If they are isotopic then they correspond to the same deck transformation. Note that since every leaf is a plane or an annulus, we deduce that if all leaves in K are fixed by some of the γ_i, this gives a partition of K by disjoint compact sets, which implies that it is a unique compact set as we wanted to show.

Finally if the set $P \cup P_0$ is not all of the leaves of $\mathcal{W}^{cs}$, then the preimage $\mathcal{P}_0$ in the leaf space $\mathcal{L}^{cs}$ is not all of $\mathcal{L}^{cs}$. Let $\mathcal{N}$ be a complementary component of $\mathcal{P}_0$. By Claim 6.13 up to deck transformations $\mathcal{N}$ is $\widetilde{f}$ periodic. All leaves of $\widetilde{\mathcal{W}^{cs}}$ in $\mathcal{N}$ have trivial stabilizer. Proposition 6.12 shows that this is impossible. We conclude that this case cannot happen so we have only the possibilities (i) and (ii).

This finishes the proof of Lemma 6.19. □

We now can prove Proposition 6.15:

Proof of Proposition 6.15. According to the first part of Lemma 6.19, the family $P \cup P_0$ is a $\widetilde{f}$- and $\pi_1(M)$-invariant lamination. Hence we can apply Proposition 6.12 to it. Since no leaves outside of $P \cup P_0$ is fixed by any deck transformation, Proposition 6.12 implies that $P \cup P_0 = \mathcal{L}^{cs}$. Now the second part of Lemma 6.19, yields that $P = \mathcal{L}^{cs}$.

This shows that every leaf is a weak quasigeodesic fan. □

6.6. **Unique centers for given limit points.** Here we show the following, that together with Proposition 6.15 completes the proof of Theorem 6.11:

Proposition 6.20. *If every leaf of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ are weak quasigeodesic fan, then they all are quasigeodesic fans.*

Remark 6.21. Notice that in order to obtain, in Proposition 6.15, that each leaf of $\mathcal{W}^{cs}$ is a *weak* quasigeodesic fan, we only needed to use the fact that the center leaves were uniform quasigeodesic in $\mathcal{W}^{cs}$ (and vice versa for $\mathcal{W}^{cu}$). To get here that it is actually a quasigeodesic fan, we need to use the fact that center leaves are uniform quasigeodesic in *both* $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$.

We are going to prove Proposition 6.20 by contradiction, dealing with leaves of $\mathcal{W}^{cu}$, the case of $\mathcal{W}^{cs}$ being symmetric.

By Lemma 6.6, for any leaf V of $\widetilde{\mathcal{W}^{cu}}$ with funnel point $p \in S^1(V)$ and every point q in $S^1(V) \smallsetminus \{p\}$, there is a center leaf in V with ideal point q. By contradiction, we will assume that there is a leaf V_0 of $\widetilde{\mathcal{W}^{cu}}$ which has more than one center curve with the same pair of limit points $p, q \in S^1(V_0)$. Since the leaf V_0 is a weak quasigeodesic fan, the set of center leaves that have p and q as limit points forms a non trivial closed interval. Let I be the interior of the interval of leaves of $\widetilde{\mathcal{W}^{cs}}$ which intersects V_0 in some of those centers. We think of I as an open interval of $\mathcal{L}^{cs}$.

Remark 6.22. The funnel direction in leaves of $\widetilde{\mathcal{W}^{cs}}$ varies continuously. This is because the funnel direction in a leaf L of $\widetilde{\mathcal{W}^{cs}}$ is the one where leaves are eventually within $2k$ of each other and in the other direction some of them diverge a lot. So near L one sees in the same direction center leaves which are within $2k+1$ of each other for a long distance, while in the opposite direction they diverge substantially from each other. This means that the same direction as that in L is the funnel direction of the nearby leaves.

For any L in I the funnel direction in L defines a direction in the center $L \cap V_0$. Since these vary continuously with L, it follows that up to switching p and q, the stable funnel direction for any L in I is the direction in $L \cap V_0$ with ideal point p. This implies that for any L in I, the rays in the funnel direction of $L \cap V_0$ are eventually $2k+1$ from each other. We let

$$Q = \bigcup_{n \in \mathbb{Z}} \bigcup_{\gamma \in \pi_1(M)} \widetilde{f}^n(\gamma I).$$

This is a non empty, open $\widetilde{f}$ and $\pi_1(M)$-invariant subset of $\mathcal{L}^{cs}$ and we consider $P = \mathcal{L}^{cs} \smallsetminus Q$. Let Λ be the lamination in M obtained by projecting the leaves in P to M. We want to show that P is everything, and therefore get a contradiction, since I and hence Q is not empty. For this, we will again apply Proposition 6.12 to a lamination Λ^* that contains Λ; to construct it we need some preliminary results.

We will use the approximating foliation setting. Let Λ_ε be the sublamination of $\mathcal{F}^{cs}_\varepsilon$ associated with Λ and let U be a connected component of $M \smallsetminus \Lambda_\varepsilon$. We will need the following technical property:

Claim 6.23. *Let L_1, L_2 be two leaves in Q. Then, there is a constant $K = K(L_1, L_2) > 0$ such that for every pair of center leaves $c_i \in L_i$, for $i = 1, 2$, we have that there is a ray r_1 of c_1 and a ray r_2 of c_2 both in the funnel directions of L_1 and L_2 respectively, such that the Hausdorff distance $d_H(r_1, r_2)$ in $\widetilde{M}$ is less than K.*

Proof. We can cover an interval joining L_1 and L_2 by finitely many translates and iterates of I. Each translate is a deck translate of an $\widetilde{f}$ iterate of I. Deck translates do not change the geometry. The map $\widetilde{f}$ has bounded derivatives so distorts distances by a bounded multiplicative amount. Hence it is enough to prove this for leaves in I. Let then L_1, L_2 in I and r_1, r_2 rays of centers c_i in L_i such that r_i is in the funnel direction in L_i. Then in L_i the center c_i has in the funnel direction the same ideal point as $V_0 \cap L_i$. Hence r_i has a subray with Hausdorff distance in L_i less than $2k + 1$ from a subray of $V_0 \cap L_i$ in the funnel direction of L_i. Then in V_0, the centers $V_0 \cap L_1$, $V_0 \cap L_2$ have subrays which are less than $2k + 1$ in Hausdorff distance in V_0 from each other. Then the rays r_i have subrays less than $6k + 3$ Hausdorff distant from each other in $\widetilde{M}$. This gives the desired bound. □

The main property we need is the following:

Lemma 6.24. *Let B be an annular leaf of $\mathcal{F}^{cs}_\varepsilon$ in U. Then B only limits on points in Λ_ε. In particular this shows that Q cannot be all of $\mathcal{L}^{cs}$.*

Proof. We will give a similar proof to that of Lemma 6.16.

Let A be the leaf in $\mathcal{W}^{cs}$ corresponding to B under the map h^{cs} given by Theorem 3.3. Since B is an annulus, so is A and we call again γ a generator of $\pi_1(A)$.

As in Lemma 6.16, there are two options for γ, it either acts freely on the center leaf space in L (the lift of A to $\widetilde{M}$ fixed by γ), or it does not.

Thanks to Proposition 6.15, every center leaf share one ideal point (the funnel point), which is therefore a fixed point of γ. We explained before (cf. Corollary 6.7) that by compactness of M, γ cannot act parabolically on $S^1(L)$, so it must fix two points on $S^1(L)$. Hence, γ fixes a center curve in L, which project to a closed center curve in A.

Let e be the corresponding closed center curve in B. Let $\widetilde{U}$ be a lift of U to $\widetilde{M}$. Suppose that B limits in a point in U. Hence there are infinitely many lifts L_i of B contained in $\widetilde{U}$ and limit to L leaf in $\widetilde{U}$. Each such lift L_i contains a lift c_i of e. The leaf L is contained in an image $\gamma \widetilde{f}^n(I)$, so there a unique K as in the claim above that works for any pair E_1, E_2 in $\gamma \widetilde{f}^n(I)$. The claim also works for the approximating foliations, doing the intersections of leaves of $\widetilde{\mathcal{F}^{cs}_\varepsilon}, \widetilde{\mathcal{F}^{cu}_\varepsilon}$. Hence for any i, j then c_i, c_j have rays a fixed bounded distance K from each other in the funnel direction in L_i, L_j. But every c_i is a lift of a fixed closed curve e. As the bound is the same, we get a contradiction, since the lifts of e form a uniformly discrete set in $\widetilde{M}$. This contradiction proves the first assertion of the lemma.

To prove the second assertion suppose that $Q = \mathcal{L}^{cs}$. First recall that $\mathcal{W}^{cs}$ has an annular leaf A. Otherwise all leaves of $\mathcal{W}^{cs}$ are planes. By a result of Rosenberg [Ros] it implies that M is the 3-torus, so $\pi_1(M)$ is abelian, which we are assuming is not the case. Hence $\mathcal{W}^{cs}$ has an annular leaf A. Since it is non compact it limits somewhere. If $Q = \mathcal{L}^{cs}$ the argument to prove the first assertion leads to a contradiction. This shows that Q is not $\mathcal{L}^{cs}$. $\square$

End of the proof of Proposition 6.20. From Lemma 6.24, we deduce that Λ is not empty. Let Λ' be the union of the annular leaves of $\mathcal{W}^{cs}$. For any annular leaf A not in Λ, the previous lemma shows that it limits only on Λ. This is the technical property that is needed to deduce that $\Lambda \cup \Lambda'$ is a sublamination of $\mathcal{W}^{cs}$ (as in the proof of the first assertion of Lemma 6.19).

Hence we can finish in exactly the same way as Proposition 6.15: $\Lambda \cup \Lambda'$ is lamination to which Proposition 6.12 applies which yields that $\Lambda \cup \Lambda'$ is all of $\mathcal{W}^{cs}$. Now the second part of Lemma 6.16 also applies here so Λ is itself all of $\mathcal{W}^{cs}$. This contradicts the fact that $Q = \mathcal{W}^{cs} \smallsetminus \Lambda$ is non empty, thus ends the proof of Proposition 6.20. $\square$

7. A criterion. Proof of Theorem D

In this section we prove Theorem D. We start by proving the converse direction, in §7.1 and §7.2, and prove the direct direction in §7.3.

In particular, we consider $f\colon M \to M$ to be a partially hyperbolic diffeomorphism preserving branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ whose leaves are Gromov hyperbolic with the induced metric. We assume that centers in each leaf of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ are uniform quasigeodesics so that Theorem 6.11 applies.

To show that being quasigeodesic partially hyperbolic diffeomorphism implies leaf space collapsed Anosov flow we will assume that the bundles E^s, E^c and E^u are orientable. (Note that orientability of E^c is a consequence of the definition and Theorem 6.11.)

7.1. Constructing an expansive flow.

Let $f\colon M \to M$ be a quasigeodesic partially hyperbolic diffeomorphism. We assume that the bundles E^s, E^c and E^u are orientable.

Let $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ be the center stable and unstable branching foliations given by Definition 2.13. Since the bundles are assumed to be orientable, we can apply Theorem 3.3 to obtain approximating foliations $\mathcal{F}^{cs}_\varepsilon$ and $\mathcal{F}^{cu}_\varepsilon$ with maps h^{cs} and h^{cu}. The intersection of $\mathcal{F}^{cs}_\varepsilon$ and $\mathcal{F}^{cu}_\varepsilon$ gives rise to an orientable foliation $\mathcal{F}^c_\varepsilon$ tangent to a vector field X^c.

Note that Theorem 6.11 shows that in each leaf of $\mathcal{F}^{cs}_\varepsilon$ (resp. $\mathcal{F}^{cu}_\varepsilon$) we have that the foliation $\mathcal{F}^c_\varepsilon$ is made of uniform quasigeodesics and that no two of them share both points at infinity. (In fact, Theorem 6.11 implies that inside each leaf of $\mathcal{F}^{cs}_\varepsilon$ (resp. $\mathcal{F}^{cu}_\varepsilon$) the foliation $\mathcal{F}^c_\varepsilon$ is a quasigeodesic fan, but we will not need this in the following.)

Proposition 7.1. *The flow $\phi^c_t\colon M \to M$ generated by X^c is expansive and preserves the transverse foliations $\mathcal{F}^{cs}_\varepsilon$ and $\mathcal{F}^{cu}_\varepsilon$.*

Proof. Recall (see §6.2) that since f is a quasigeodesic partially hyperbolic diffeomorphism, the leaves of the approximating foliations $\mathcal{F}^{cs}_\varepsilon$ and $\mathcal{F}^{cu}_\varepsilon$ are also Gromov hyperbolic (one can even choose these to be by hyperbolic surfaces [Cal_3, Chapter 8]). By hypothesis, the orbits of the flow ϕ^c_t are quasigeodesics in the leaves of each of the foliations.

There is $\delta_0 > 0$ such that every leaf of $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ and $\widetilde{\mathcal{F}^{cu}_\varepsilon}$ is properly embedded in its δ_0-neighborhood in $\widetilde{M}$ (see, e.g., [Cal_3]).

By that we mean that:

(i) any set of diameter less than δ_0 is contained in a foliated chart of each of these foliations; and

(ii) if p is in L leaf of $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ or $\widetilde{\mathcal{F}^{cu}_\varepsilon}$ then the ball of radius δ_0 around p in $\widetilde{M}$ intersects L only in the local sheet of L through p.

Now choose $\delta < \delta_0$ so that if two points x, y in $\widetilde{M}$ are less than δ apart then $\widetilde{\mathcal{F}^{cs}_\varepsilon}(x)$ intersects $\widetilde{\mathcal{F}^{cu}_\varepsilon}(y)$ in a point less than δ_0 from both of them and similarly for $\widetilde{\mathcal{F}^{cs}_\varepsilon}(y) \cap \widetilde{\mathcal{F}^{cu}_\varepsilon}(x)$. We will show that δ serves as an expansivity constant for the flow $\widetilde{\phi^c_t}$, and this implies that the flow ϕ^c_t is expansive too. Since there is no recurrence for the flow in $\widetilde{M}$ the definition of expansivity is equivalent to showing that different orbits cannot remain bounded Hausdorff distance apart, cf. Remark 5.8.

Assume by contradiction that two different orbits o_1 and o_2 of $\widetilde{\phi^c_t}$ in $\widetilde{M}$ are at Hausdorff distance less than δ. These orbits correspond to leaves of the intersected foliation $\widetilde{\mathcal{F}^c_\varepsilon}$ between $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ and $\widetilde{\mathcal{F}^{cu}_\varepsilon}$. Suppose first that they are in the same leaf of L of $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ (or $\widetilde{\mathcal{F}^{cu}_\varepsilon}$). Since they are not the same orbit, they cannot have both ideal points the same in $S^1(L)$, by Theorem 6.11. Hence they diverge from each other infinitely in L in some direction. By the choice of δ_0 they diverge from each other at least δ_0 (and hence at least δ) in $\widetilde{M}$ as well in that direction. Suppose now that o_1, o_2 are not the same leaf of $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ or $\widetilde{\mathcal{F}^{cu}_\varepsilon}$. Let then o_3 be the intersection of $\widetilde{\mathcal{F}^{cs}_\varepsilon}(o_1) \cap \widetilde{\mathcal{F}^{cu}_\varepsilon}(o_2)$. Then o_3 is distinct from both o_1, o_2. Since o_1, o_2 are always less than δ apart then o_3 is less than δ_0 apart from either o_1 or o_2. Since o_3, o_1 are in the same $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ leaf the first argument shows that this is a contradiction, that is o_1, o_3 have to diverge from each other more than δ_0. This shows that δ works as an expansive constant for the flow.

It is obvious that the flow preserves the described foliations. This finishes the proof of the proposition. □

7.2. Deducing that the map is a collapsed Anosov flow.

We can now show:

Proposition 7.2. *The flow ϕ^c_t is a topological Anosov flow and f is a leaf space collapsed Anosov flow with respect to ϕ^c_t.*

Proof. Notice first that by Proposition 7.1 and Theorem 5.9 we know that the flow ϕ^c_t is a topological Anosov flow. Moreover, by Proposition 5.5 we know that the foliations $\mathcal{F}^{cs}_\varepsilon$ and $\mathcal{F}^{cu}_\varepsilon$ correspond to the weak stable and unstable foliations respectively (maybe up to changing orientation of the vector field X^c).

Using the maps h^{cs} and h^{cu} given by Theorem 3.3 in the universal cover one can construct a $\pi_1(M)$-invariant homeomorphism H from the orbit space of $\widetilde{\phi^c_t}$ and the center leaf space of f as follows: A center leaf in $\widetilde{M}$ is a component c of the intersection of a leaf L of $\widetilde{\mathcal{W}^{cs}}$ and a leaf G of $\widetilde{\mathcal{W}^{cu}}$. There are unique leaves

$$L' \in \widetilde{\mathcal{F}^{cs}_\varepsilon}, \quad G' \in \widetilde{\mathcal{F}^{cu}_\varepsilon} \text{ so that } \widetilde{h}_{cs}(L') = L, \quad \widetilde{h}_{cu}(G') = G.$$

There is a unique component α of the intersection of L' and G' (that is, an orbit of $\widetilde{\phi}_t$) which is ε close to c. The map H is the one that sends this orbit α to c.

This completes the proof. □

7.3. The quasigeodesic property.

Here we show:

Proposition 7.3. *Let $f\colon M \to M$ be a leaf space collapsed Anosov flow. Then, the $\mathcal{W}^{cs}$-foliation is by Gromov hyperbolic leaves and the center foliation inside each leaf of $\mathcal{W}^{cs}$ is a quasigeodesic fan.*

Proof. We do the proof for $\mathcal{W}^{cs}$, the same proof works for $\mathcal{W}^{cu}$.

Up to taking a finite cover and a lift of an iterate of f we may assume that E^s, E^c, E^u are orientable and f preserves the lifted foliation. Since the quasi-geodesic properties are verified in the universal cover, this does not change the result. In addition f is still a leaf space collapsed Anosov flow in the cover. Let ϕ_t be the Anosov flow associated to f. Let $H\colon \mathcal{O}_\phi \to \mathcal{L}^c$ be the associated homeomorphism between orbit space of $\widetilde{\phi}_t$ and center leaf space in $\widetilde{M}$. Proposition 5.6 implies that H maps $\mathcal{O}^{ws}_\phi$ to $\mathcal{O}^{cs}_f$ and $\mathcal{O}^{wu}_\phi$ to $\mathcal{O}^{cu}_f$.

Using Theorem 3.3, we can approximate $\mathcal{W}^{cs}$, $\mathcal{W}^{cu}$ by actual foliations $\mathcal{F}^{cs}_\varepsilon, \mathcal{F}^{cu}_\varepsilon$. The intersection of $\mathcal{F}^{cs}_\varepsilon, \mathcal{F}^{cu}_\varepsilon$ is a one-dimensional foliation $\mathcal{G}$ in M, with lift $\widetilde{\mathcal{G}}$. Given any flow line α of $\widetilde{\phi}_t$ it is the intersection of a stable leaf L_0 with an unstable leaf U_0. Under H these leaves L_0, U_0 map to leaves L_1 of $\widetilde{\mathcal{W}^{cs}}$ and U_1 of $\widetilde{\mathcal{W}^{cu}}$ respectively. Thanks to item (ii) of Theorem 3.3, the leaves L_1, U_1 are ε near respectively from unique leaves L of $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ and U of $\widetilde{\mathcal{F}^{cu}_\varepsilon}$.

Therefore α is associated with a unique leaf of $\widetilde{\mathcal{G}}$ and vice versa. This association is a homeomorphism from the orbit space $\mathcal{O}_\phi$ to the leaf space of $\widetilde{\mathcal{G}}$. This homeomorphism is clearly $\pi_1(M)$ equivariant.

By the result of Haefliger, Ghys and Barbot [Bar$_2$, Prop.1.36], it follows that there is a homeomorphism η from M to M sending the flow foliation of ϕ_t to the foliation $\mathcal{G}$. We can then orient the foliation $\mathcal{G}$ using this homeomorphism. Hence this foliation becomes the flow foliation of a flow ψ_t. Since the flow ϕ_t is expansive then the flow ψ_t is also expansive. By Theorem 5.9 it follows that ψ_t is a topological Anosov flow. By the equivalence of the flow foliations of ϕ_t and ψ_t it now follows that the stable foliation of ψ_t is $\mathcal{F}^{cs}_\varepsilon$. By Proposition 5.11 it follows that the foliation $\mathcal{F}^{cs}_\varepsilon$ is by Gromov hyperbolic leaves and the flow lines in leaves of $\mathcal{F}^{cs}_\varepsilon$ are uniform quasigeodesics.

This implies that the leaves of $\mathcal{W}^{cs}$ are Gromov hyperbolic and the center leaves in leaves of $\widetilde{\mathcal{W}^{cs}}$ are uniform quasigeodesics.

This finishes the proof of Proposition 7.3. □

This finishes the proof of Theorem D.

8. Strong implies leaf space collapsed Anosov flow

In this section we show that Definition 2.7 implies Definition 2.10. The main point is to construct the branching foliations from the map h provided by Definition 2.7. The rest of the conditions will be rather direct.

Proposition 8.1. *If f is a strong collapsed Anosov flow, then it is a leaf space collapsed Anosov flow.*

We first show the following lemma:

Lemma 8.2. *Let f be a strong collapsed Anosov flow (Definition 2.7), then there are f-invariant branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to E^{cs} and E^{cu} respectively such that the image of each of the leaves of $\mathcal{W}^{cs}$ (resp. $\mathcal{W}^{cu}$) coincides with $h(\mathcal{F}^{ws}_\phi(x))$ (resp. $h(\mathcal{F}^{wu}_\phi(x))$) for some $x \in M$.*

Proof. The statement is symmetric, so we show it for E^{cs}.

Using a result of Calegari [Cal$_1$] we can assume that leaves of $\mathcal{F}^{ws}_\phi$ are C^1. Take the pull back of the ambient metric. For each leaf $L \in \mathcal{F}^{ws}_\phi$ we define a continuous local homeomorphism $\varphi_L\colon U_L \to M$ where U_L is the universal cover of the leaf $L \subset M$ with this intrinsic metric. Note that U_L is a complete metric space, homeomorphic to $\mathbb{R}^2$.

By assumption, the image by h of L is a C^1-surface tangent to E^{cs}. This means that when lifted to the universal cover, there is $\hat{\varphi}_L \colon U_L \to \widetilde{M}$ a C^1-proper embedding tangent to E^{cs} such that its image coincides with $\tilde{h} \circ \varphi_L \colon U_L \to \widetilde{M}$. Moreover, since h is homotopic to identity, one gets that the image of $\hat{\varphi}_L$ is complete with the metric induced by the embedding. This fact, as well as the non-topological crossings is ensured by the local preservation of the orientation that makes backtracking impossible.

Finally, to show the minimality condition, we need to show that the image of two different leaves of $\widetilde{\mathcal{F}^{ws}_\phi}$ by $\tilde{h}$ are different. Suppose then that L_1, L_2 are distinct leaves of $\widetilde{\mathcal{F}^{ws}_\phi}$ which are mapped to the same surface by $\tilde{h}$. Suppose first that L_1, L_2 intersect a common unstable leaf F. In particular, the set of leaves separating L_1 from L_2 is an interval. If for some leaf L in this interval we have $\tilde{h}(L) \neq \tilde{h}(L_1)$, then since there is no topological crossing between leaves, it also follows that $\tilde{h}(L_1) \neq \tilde{h}(L_2)$, which contradicts our assumption. Thus $\tilde{h}(L) = \tilde{h}(L_1)$. For any such L, the intersection $L \cap F$ is a single flow line α_L. The above shows that $\tilde{h}(\alpha_L)$ is contained in $\tilde{h}(L_1)$. Therefore the region in F made up of the flow lines between α_{L_1} and α_{L_2} is mapped into $\tilde{h}(L_2)$. Therefore this is mapped into a region tangent to E^{cs}. This contradicts the fact that F is mapped to a surface tangent to E^{cu} because h is close to the identity and the region between α_{L_1} and α_{L_2} contains arbitrarily large disks.

For general leaves L_1, L_2 the region between them is the connected component of $\widetilde{M} - (L_1 \cup L_2)$ which limits on both of them. If $\tilde{h}(L_1) = \tilde{h}(L_2)$ then for any leaf L between L_1, L_2 then $\tilde{h}(L) = \tilde{h}(L_1)$. There is a leaf L which is between L_1 and L_2 and which intersects a common unstable F with L_1. By the first case $\tilde{h}(L_1) \neq \tilde{h}(L)$. This leads to a contradiction.

This finishes the proof of the lemma. □

Proof of Proposition 8.1. We just need to show that there is a homeomorphism between the leaf spaces. Thanks to Lemma 8.2, this is the one induced by h. □

9. Leaf space implies strong collapsed Anosov flow

In this section we will show that Definition 2.10 implies Definition 2.7 under some orientability assumption. Together with Proposition 8.1 it completes the proof of Theorem B.

Proposition 9.1. *If f is a leaf space collapsed Anosov flow and E^{cs} is transversally orientable, then it is a strong collapsed Anosov flow.*

The strategy is quite simple, we wish to map each orbit of the Anosov flow to the corresponding center curve given by Definition 2.10. The difficulty in implementing the strategy has to do with the fact that we only have a map at the level of leaf spaces, so we first need to construct an actual map of the manifold which realizes this equivalence, for this, we first construct a specific realization of the (topological) Anosov flow that allows us to get this map in a natural way. Once this is done, a standard averaging argument achieves the local injectivity along orbits of the flow.

9.1. Constructing a convenient realization of the Anosov flow.

Let f be a leaf space collapsed Anosov flow with E^{cs} transversally orientable. We consider $\phi_t \colon M \to M$ the topological Anosov flow and $H \colon \mathcal{O}_\phi \to \mathcal{L}^c$ given by Definition 2.10.

We will start by applying Theorem 3.3 to $\mathcal{W}^{cs}$ to get an approximating foliation $\mathcal{F}^{cs}_\varepsilon$. We denote by $\widetilde{\mathcal{W}^{cs}}$ and $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ the lifts to $\widetilde{M}$. As explained in §A.3 we can consider a metric on M that makes leaves of $\mathcal{F}^{cs}_\varepsilon$ negatively curved. In Proposition 7.3 we proved that the center leaves inside each leaf of $\widetilde{\mathcal{W}^{cs}}$ form a quasigeodesic fan. Then we can pull them back to each leaf of $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ to get a funnel point $p(L) \in S^1(L)$ in each leaf $L \in \mathcal{F}^{cs}_\varepsilon$.

We consider the flow $\widetilde{\psi}_t \colon \widetilde{M} \to \widetilde{M}$ defined as follows: For a point $x \in L \in \widetilde{\mathcal{F}^{cs}_\varepsilon}$ we consider $\widetilde{\psi}_t(x)$ to be moving along the geodesic through x with endpoint the funnel point of L at unit speed. This definition is clearly $\pi_1(M)$-invariant, and this flow descends to M and we denote it by ψ_t.

In Proposition 7.1 we proved the following:

Proposition 9.2. *The flow ψ_t is topologically Anosov and orbit equivalent to ϕ_t by an orbit equivalence homotopic to the identity.*

9.2. **Averaging to construct the map.** We will now construct a map $h_0 \colon M \to M$ which maps orbits of ψ_t (cf. Proposition 9.2) to curves tangent to the center. Later we will modify this map and construct the self orbit equivalence to verify Definition 2.7. Denote by $H_0 \colon \mathcal{O}_\psi \to \mathcal{L}^c$ the $\pi_1(M)$-invariant homeomorphism between leaf spaces. Recall that Proposition 5.6 implies that H_0 maps the weak-stable/unstable foliations of ψ_t to the center stable and unstable branching foliations of f.

Construction of a map: For a fixed small $\varepsilon > 0$, we denote by $h^{cs} \colon M \to M$ the collapsing map from $\mathcal{F}^{cs}_\varepsilon$ to $\mathcal{W}^{cs}$ given by Theorem 3.3.

Pick a point $x \in \widetilde{M}$ and let ℓ_x be a center leaf in $\widetilde{M}$ which is the center leaf $H_0(o_x)$ where o_x is the orbit of x by $\widetilde{\psi}$. Note that o_x is a geodesic in a negatively curved surface, and we can push the metric in $L_x := \widetilde{\mathcal{F}^{ws}_\phi}(x) = \widetilde{\mathcal{F}^{cs}_\varepsilon}(x)$ to $\widetilde{h}^{cs}(L_x)$ which is a leaf of $\widetilde{\mathcal{W}^{cs}}$. We can push the metric because h^{cs} is a local diffeomorphism between respective leaves of $\mathcal{F}^{cs}_\varepsilon$ and $\mathcal{W}^{cs}$, and this lifts to diffeomorphisms between respective leaves of $\widetilde{\mathcal{F}^{cs}_\varepsilon}$ and $\widetilde{\mathcal{W}^{cs}}$. With this metric, o_x is a geodesic in L_x and ℓ_x is a quasigeodesic in L_x with the same endpoints.

We can then define a map $p_x \colon \ell_x \to \widetilde{h}^{cs}(o_x)$ by orthogonal projection in L_x. Since L_x is negatively curved the orthogonal projection is a uniquely defined function and it is continuous.

Lemma 9.3. *The map p_x is proper, in particular extends continuously (as the identity) to the compactification of ℓ_x and $\widetilde{h}^{cs}(o_x)$.*

Proof. This follows directly from the fact that ℓ_x is a quasigeodesic with the same endpoints as the geodesic $\widetilde{h}^{cs}(o_x)$ with respect to the chosen metric. $\square$

In principle, the map p_x can fail to be injective, so one cannot define an inverse. But there is a standard procedure of averaging going back at least to [Ful] (see also [HP$_1$, Section 8] for discussion) which allows to find a natural way to invert p_x.

We can define from p_x a map $\hat{p}_x \colon \ell_x \to \mathbb{R}$ by identifying $\widetilde{h}^{cs}(o_x)$ with $\mathbb{R}$ via the map $b_x \colon \widetilde{h}^{cs}(o_x) \to \mathbb{R}$ such that $b_x(\widetilde{h}^{cs}(\psi_t(x)) = t$.

For $y, z \in \ell_x$ we denote by $[y, z]$ the segment of ℓ_x between y and z. For any $t \in \mathbb{R}$ we denote by $y + t$ the point in ℓ_x at oriented distance t from y. Lemma 9.3 then implies that if we choose an appropriate orientation along E^c we have that the map $\hat{p}_x$ verifies that for every $y \in \ell_x$ we have $\lim_{t \to \pm\infty} \hat{p}_x(y + t) = \pm\infty$.

Let $p^T_x \colon \ell_x \to \mathbb{R}$ be the map defined by

$$p_x^T(y) = \int_{[y,y+T]} \hat{p}_x(z)dz.$$

Lemma 9.3 implies that for $T > 0$ large enough we have that not only p_x^T is C^1 along ℓ_x but also its derivative does not vanish. Indeed, since $\hat{p}_x$ is continuous, we have

$$p_x^T(y+t) - p_x^T(y) = \int_{[y+T,y+T+t]} \hat{p}_x(z)dz - \int_{[y,y+t]} \hat{p}_x(z)dz$$
$$\sim t(\hat{p}_x(y+T) - \hat{p}_x(y)).$$

Said otherwise, we deduce that if $u_x(t) = p_x^T(y+t)$ for some $y \in \ell_x$ then $u_x'(t) > 0$ everywhere. It follows that we can define an inverse map $q_x \colon \mathbb{R} \to \ell_x$ which is a C^1-diffeomorphism preserving orientation which is the inverse of p_x^T. We collect some properties of q_x in the following statement:

Lemma 9.4. *The map $x \mapsto q_x$ vary continuously in the C^1-topology in compact parts and is $\pi_1(M)$-invariant in the sense that for $\gamma \in \pi_1(M)$ we have that $q_{\gamma x}(t) = \gamma q_x(t)$. Moreover, there is a C^1 increasing diffeomorphism $u_x \colon \mathbb{R} \to \mathbb{R}$ such that $u_x(0) = 0$ and if $x_t = \psi_t(x)$ then $q_{x_t}(0) = q_x(u_x(t))$.*

Proof. All the objects we considered depend on continuous and $\pi_1(M)$-invariant choices. The last property just follows from the way we defined q_x and the fact that q_{x_t} also has ℓ_x as target since $\ell_x = \ell_{x_t}$. $\square$

Now we can define the map $h \colon M \to M$. For $x \in \widetilde{M}$ we define $\widetilde{h}(x)$ to be $q_x(0) \in \ell_x$, since this is continuous and $\pi_1(M)$-invariant it induces a continuous map h in M homotopic to the identity.

Verifying the properties. We will now verify the sought properties of h.

Lemma 9.5. *The map $h \colon M \to M$ is smooth along the orbits of ψ_t and the derivative maps the vector field to a (positively oriented) non-zero vector tangent to E^c. That is, h verifies condition (i) in Definition 2.7.*

Proof. Fix an orbit o_x of $\widetilde{\psi}_t$ and we get that by definition for every $y \in o_x$ we have that $\ell_x = \ell_y$. Therefore, the map h will map o_x to ℓ_x. By Lemma 9.4 we deduce that the image by h of the vector field is a positively oriented vector in E^c. $\square$

We can now proceed to prove Proposition 9.1.

Proof of Proposition 9.1. Since the partially hyperbolic diffeomorphism is a leaf space collapsed Anosov flow it preserves branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$. The fact that h maps weak stable into surfaces tangent to E^{cs} is direct from its construction since it maps leaves of $\widetilde{\mathcal{F}^{ws}_\psi}$ to surfaces tangent to E^{cs}. Now, by Proposition 5.6 we also get that the weak unstable maps to surfaces tangent to E^{cu}. The lift $\tilde{h}$ of h to $\widetilde{M}$ maps every weak stable/unstable by construction into a properly embedded surface in $\widetilde{M}$ respecting the orientation.

We now need to construct the self orbit equivalence $\beta \colon M \to M$ which makes the commutation $f \circ h = h \circ \beta$ work. For this, given $x \in M$ consider $y = f \circ h(x)$. Note that y may belong to several center curves. But since h is injective along orbits of the flow ψ_t it makes sense to consider its inverse restricted to the center curve $\ell_y := f(\ell_x)$ which is also the image of an orbit of ψ_t. Then, one can define $\beta(x) = (h|_{\ell_y})^{-1}(y)$.

One gets that $\beta(x)$ is continuous by construction and continuity of h (as well as continuity of the maps q_x, cf. Lemma 9.4). Moreover, β is injective since it is injective along orbits as well as maps different orbits to different orbits. Finally, β is surjective since the equation $f \circ h = h \circ \beta$ implies that β has degree one as a map. This implies that β is a homeomorphism which clearly preserves orbits and its orientation thus a self orbit equivalence for ψ_t. □

The averaging method gives several ways on which a given collapsed Anosov flow can be realized (different choices of h that affect the choice of β). The following remarks should also be taken into account if one wants to formulate uniqueness properties for collapsed Anosov flows.

Remark 9.6. Let $f\colon M \to M$ be a collapsed Anosov flow with respect to a topological Anosov flow $\phi_t\colon M \to M$ and the self orbit equivalence $\beta\colon M \to M$. In particular, there exists $h\colon M \to M$ homotopic to the identity such that $f \circ h = h \circ \beta$. Assume that $\alpha\colon M \to M$ is another self orbit equivalence of ϕ_t. Then, it follows that taking $\hat{h} = h \circ \alpha$ and $\hat{\beta} = \alpha^{-1} \circ \beta \circ \alpha$ we get that $f \circ \hat{h} = \hat{h} \circ \hat{\beta}$.

Thus, if α is homotopic to the identity, then f is also a collapsed Anosov flow associated with the Anosov flow ϕ_t via the collapsing map $\hat{h}$ and the self orbit equivalence $\hat{\beta}$.

Remark 9.7. Let $f\colon M \to M$ be a collapsed Anosov flow with respect to a topological Anosov flow $\phi_t\colon M \to M$ and self orbit equivalence $\beta\colon M \to M$ and let $\psi_t\colon M \to M$ be a flow conjugate to ϕ_t by a homeomorphism $g\colon M \to M$, that is, $\psi_t = g^{-1} \circ \phi_t \circ g$. Then, if $h\colon M \to M$ is the map homotopic to the identity such that $f \circ h = h \circ \beta$ then one has that if $\hat{h} = h \circ g$ and $\hat{\beta} = g^{-1} \circ \beta \circ g$ then $\hat{\beta}$ is a self orbit equivalence of ψ_t and $f \circ \hat{h} = \hat{h} \circ \hat{\beta}$.

Thus, if g is homotopic to the identity, then f is also a collapsed Anosov flow associated with the Anosov flow ψ_t via the collapsing map $\hat{h}$ and the self orbit equivalence $\hat{\beta}$.

10. Examples

10.1. **Proof of Theorem A.** In order to prove Theorem A, we first collect some facts that are easily extracted from [BGHP].

Proposition 10.1. *Let $\phi_s\colon M \to M$ be an Anosov flow generated by a vector field X and $\varphi\colon M \to M$ a diffeomorphism such that ϕ_s is φ-transverse to itself. Then, there exists $t_0 > 0$ and a function $\delta\colon [t_0, \infty) \to \mathbb{R}_{>0}$ with $\delta(t) \to 0$ as $t \to \infty$ such that for every $t > t_0$ one has that the diffeomorphism $f_t = \phi_t \circ \varphi \circ \phi_t$ verifies:*

- *(i) f_t is partially hyperbolic and the bundles E^s_t, E^c_t and E^u_t of f_t make an angle less than $\delta(t)$ with the bundles E^s_ϕ, $\mathbb{R}X$ and E^u_ϕ respectively;*
- *(ii) for every immersed curve $c\colon \mathbb{R} \to M$ everywhere tangent to E^c_t there exists $x \in M$ and a homeomorphism $u\colon \mathbb{R} \to \mathbb{R}$ such that $d(c(u(s)), \phi_s(x)) < \delta(t)$ for every $s \in \mathbb{R}$, moreover, the point x is unique in that if y verifies the same, then $y = \phi_s(x)$ for some $s \in \mathbb{R}$;*
- *(iii) for every $x \in M$ there is an immersed curve $c\colon \mathbb{R} \to M$ everywhere tangent to E^c_t and a homeomorphism $u\colon \mathbb{R} \to \mathbb{R}$ such that $d(c(u(s)), \phi_s(x)) < \delta(t)$ for every $s \in \mathbb{R}$.*

Proof. Item (i) is a direct consequence of [BGHP, Proposition 2.4] and [BGHP, Remark 2.6].

Item (ii) follows from the standard shadowing lemma for Anosov flows (see e.g, [BGHP, Theorem 5.3]) and item (iii) from its global version (cf. [BGHP, Theorem

5.5]). Note that [BGHP, Theorem 5.5] is stated for flows, so to do this we apply the trick in [BGHP, Proposition 5.11], we lift to a finite cover, take an iterate, so that we can apply Theorem 3.6 to get branching foliations. Using Theorem 3.3 we construct a flow whose orbits are arbitrarily close to curves tangent to the center, so we can apply [BGHP, Theorem 5.5] to this flow to get the center curve which then projects to M. □

The strong information we get with the previous proposition allows us to show the following result which answers positively Question 4 in the setting of the examples of [BGHP].

Proposition 10.2. *Let $\phi_s\colon M \to M$ be an Anosov flow generated by a vector field X and $\varphi\colon M \to M$ a diffeomorphism such that ϕ_s is φ-transverse to itself. Let $f_t = \phi_t \circ \varphi \circ \phi_t$. Assume that the invariant bundles of f_t are orientable and that t is sufficiently large. Then, f_t is a leaf space collapsed Anosov flow and there is a unique f_t-invariant branching foliation tangent to E^{cs}_t and a unique f_t-invariant branching foliation tangent to E^{cu}_t.*

Proof. By taking $g = f_t^k$ we can assume that g preserves the orientation of the bundles. So, by Theorem 3.6, there are g-invariant branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent respectively to E^{cs}_t and E^{cu}_t. Note that if we show that these are the unique g-invariant branching foliations for g, then it will follow that $f_t(\mathcal{W}^{cs})$ and $f_t(\mathcal{W}^{cu})$ are g-invariant branching foliations, so we deduce that $f_t(\mathcal{W}^{cs}) = \mathcal{W}^{cs}$ and $f_t(\mathcal{W}^{cu}) = \mathcal{W}^{cu}$.

To show the uniqueness we first show that g is a leaf space collapsed Anosov flow with respect to ϕ_t using the following claim based on Proposition 10.1. We choose t large enough so that the value of δ given by Proposition 10.1 is much smaller than the local product structure size (note that the bundles E^s_t, E^c_t and E^u_t make uniform angles boundedly away from zero as t increases). We also take $100\delta \leqslant \alpha$ where α is the expansivity constant for the Anosov flow ϕ_t.

Claim 10.3. *Given a pair of g-invariant branching foliations $\mathcal{W}^{cs}_1, \mathcal{W}^{cu}_1$ and a center curve $c = L \cap F$ in $\widetilde{M}$ obtained by intersecting a leaf $L \in \widetilde{\mathcal{W}^{cs}_1}$ with a leaf $F \in \widetilde{\mathcal{W}^{cu}_1}$ we have that there is a unique orbit o_x of $\widetilde{\phi}_t$ which is at Hausdorff distance less than δ from c. Conversely, for every orbit o_x of $\widetilde{\phi}_t$ there is a unique pair of leaves $L_x \in \widetilde{\mathcal{W}^{cs}_1}$ and $F_x \in \widetilde{\mathcal{W}^{cu}_1}$ such that the curve $c_x = L_x \cap F_x$ is at distance less than δ from o_x.*

Proof of Claim 10.3. The first statement follows directly from Proposition 10.1 (ii). To get the second statement, we use Proposition 10.1 (iii). Note that this gives that for every orbit o_x of $\widetilde{\phi}_t$ we can find at least one pair of leaves $L_x \in \widetilde{\mathcal{W}^{cs}_1}$ and $F_x \in \widetilde{\mathcal{W}^{cu}_1}$ so that $c_x = L_x \cap F_x$ is δ close to o_x.

We need to show uniqueness of c_x. So we let c_x and c'_x be center leaves δ-close to the same orbit o_x of $\widetilde{\phi}_t$. We write $c_x = L_x \cap F_x$ and $c'_x = L'_x \cap F'_x$ with $L_x, L'_x \in \widetilde{\mathcal{W}^{cs}_1}$ and $F_x, F'_x \in \widetilde{\mathcal{W}^{cu}_1}$. We want to show that $L_x \cap F'_x = L'_x \cap F_x = c_x$ which implies that $c_x = c'_x$. We argue for $L_x \cap F'_x$ since the other is similar. Assuming that $L_x \cap F'_x = c''_x \neq c_x$ we get that all center curves of L_x in between form an interval since leaves of $\widetilde{\mathcal{W}^{cu}_1}$ do not cross. Therefore, we can find centers which will have backward iterates c_1, c_2 (since they are intersected by strong stable leaves) whose distance in $\widetilde{M}$ will be in between 4δ and $\alpha/4$. Let o_1, o_2 be orbits of $\widetilde{\phi}_s$ which are δ near c_1, c_2 respectively, by Proposition 10.1 (ii). In particular o_1, o_2 are within $\alpha/4 + 2\delta$ from each other, which is less than α. Since α is the expansivity constant of the flow ϕ_s it follows that o_1, o_2 are the same orbit.

Hence c_1, c_2 are in fact within 2δ from each other, contrary to construction. This contradiction means that c_x is unique given o_x. $\square$

Now, we have showed that there is a bijection between the orbit space of the Anosov flow and the leaf space of g which can be easily seen to be a $\pi_1(M)$-invariant homeomorphism. Indeed, the invariance under the action of $\pi_1(M)$ comes from the fact that Proposition 10.1 (iii) is done in M and the continuity follows from the fact that the leaves of the foliations vary continuously in compact sets. This shows that g is a leaf space collapsed Anosov flow (and therefore is also quasigeodesic partially hyperbolic by Theorem D).

To complete the proof we must show that the (g-invariant) branching foliations tangent to E^{cs}_t and E^{cu}_t respectively are unique. We deal with E^{cs}_t. Assume there is a pair of g-invariant branching foliations $\mathcal{W}^{cs}_1$ and $\mathcal{W}^{cs}_2$ tangent to E^{cs}_t and let $\mathcal{W}^{cu}$ be one g-invariant branching foliation tangent to E^{cu}_t.

Note that applying the claim to the pairs $(\mathcal{W}^{cs}_1, \mathcal{W}^{cu})$ and $(\mathcal{W}^{cs}_2, \mathcal{W}^{cu})$ we get the structure of a leaf space collapsed Anosov flow for g in two different ways. If $\mathcal{W}^{cs}_1$ and $\mathcal{W}^{cs}_2$ are not equal the following happens: there is a leaf L_1 of $\widetilde{\mathcal{W}^{cs}_1}$ which is not a leaf of $\widetilde{\mathcal{W}^{cs}_2}$. We work with the maps between the leaf spaces and the orbit space of $\tilde{\phi}_s$, because of the leaf space collapsed Anosov flow structure. By Proposition 5.6, L is associated with a weak stable leaf E of $\tilde{\phi}_s$. By the same proposition the weak stable E is associated with a leaf L_2 of $\widetilde{\mathcal{W}^{cs}_2}$. Since L_1 is not a leaf of $\widetilde{\mathcal{W}^{cs}_2}$ there is a center leaf c_1 in L_1 such that the corresponding center leaf c_2 under these identifications is not contained in L_1. In other words c_1, c_2 are distinct curves tangent to the center bundle, but associated with the same orbit o_x of $\widetilde{\phi_s}$.

If that is the case, still we know by Proposition 10.1 (ii) that c_1, c_2 are at distance less than 2δ from each other. Now consider $f(c_1), f(c_2)$. If δ is sufficiently small then $f(c_1), f(c_2)$ are less than $\alpha/2$ from each other, so by Proposition 10.1 (ii), the corresponding flowlines from $f(c_1), f(c_2)$ are within $\alpha/2 + 2\delta$ from each other, and, as in the proof of the claim, these orbits are the same. Iterating, this happens for all forward and backward iterates.

But this is impossible unless $c_1 = c_2$ since the unstable lengths increase by forward iteration and stable lengths increase under backward iteration, both beyond the local product structure boxes size, which is much bigger than δ.

This finishes the proof of Proposition 10.2. $\square$

Now we can use the previous proposition to deduce Theorem A.

Proof of Theorem A. Consider $t > 0$ large enough so that both Proposition 10.1 and Proposition 10.2 holds.

We can choose a finite normal cover $P : \hat{M} \to M$ such that the lifts of all bundles are orientable. An iterate of f_t lifts to $\hat{M}$ and we can consider a lift g of a possibly further iterate so that g preserves the orientation of the bundles. Applying Proposition 10.2 to g we get that g is a leaf space collapsed Anosov flow and that it admits a unique pair of g-invariant branching foliations $\mathcal{W}^{cs}_0$ and $\mathcal{W}^{cu}_0$ tangent to E^{cs} and E^{cu} respectively.

As explained in Remark A.3 using the uniqueness of branching foliations, we obtain that $\mathcal{W}^{cs}_0$ (and $\mathcal{W}^{cu}_0$) must coincide with the uppermost and lowermost branching foliations constructed in [BI]. More specifically the uppermost center stable foliation is the same as the lowermost center stable foliation. This implies that these branching foliations project to M since given γ a deck transformation of $\hat{M}$ with respect to the cover P we get that it preserves the bundles, so it verifies that $\gamma\mathcal{W}_0^{cs}$ and $\gamma\mathcal{W}_0^{cu}$ are branching foliations tangent to E^{cs} and E^{cu} and depending on how γ acts on the orientation it preserves the uppermost branching foliation or it maps it into the lowermost one. Since these are equal by Proposition 10.2 we deduce $\gamma\mathcal{W}_0^{cs} = \mathcal{W}_0^{cs}$ and $\gamma\mathcal{W}_0^{cu} = \mathcal{W}_0^{cu}$.

Now denote by $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ the projection of these branching foliations to M. Let $\mathcal{B}$ be the lift of $f_t(\mathcal{W}^{cs})$ to $\hat{M}$. Since f_t^k lifts to g in $\hat{M}$, it follows that $f_t^k(\mathcal{W}^{cs}) = \mathcal{W}^{cs}$. The foliation $g(\mathcal{B})$ projects to $f_t^k \circ f_t(\mathcal{W}^{cs})$, which is then equal to $f_t(\mathcal{W}^{cs})$, so $g(\mathcal{B}) = \mathcal{B}$. Then the uniqueness of branching foliations in $\hat{M}$ implies that $\mathcal{B} = \mathcal{W}_0^{cs}$, and we finally conclude that $f_t(\mathcal{W}^{cs}) = \mathcal{W}^{cs}$.

Hence f_t preserves branching foliations $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ and f_t is also a leaf space collapsed Anosov flow. By Theorem D we get that f_t is also a quasigeodesic partially hyperbolic diffeomorphism.

To show that f_t is a strong collapsed Anosov flow, we point out to the proof of Theorem B in §9.

Theorem A states that f_t is also a strong collapsed Anosov flow. However, since we do not assume that the bundles are orientable, we cannot use Theorem B directly to deduce this. Instead, we redo and adapt some of the steps of the proof Theorem B in §9 to these particular examples.

In §9 we constructed a map which sent an orbit of an Anosov flow ψ_t which was orbit equivalent to the original Anosov flow $\hat{\phi}_t$ to a curve tangent to E^c. This worked fine under orientability assumptions, so we get a map $\hat{h}\colon \hat{M} \to \hat{M}$ with these properties. Our goal is to show that we can project that map to M.

We consider an orbit equivalence $\hat{k}\colon \hat{M} \to \hat{M}$ from the flow $\hat{\phi}_t\colon \hat{M} \to \hat{M}$ (the lift of ϕ_t to $\hat{M}$) to the flow ψ_t constructed in §9. We let $\hat{h}_0 = \hat{h} \circ \hat{k}^{-1}$ which maps orbits of $\hat{\phi}_t$ to curves tangent to the centers. If we consider a deck transformation γ with respect to $P\colon \hat{M} \to M$ and an orbit o_x of $\hat{\phi}_t$ we claim that

$$\hat{h}_0(\gamma o_x) = \gamma \hat{h}_0(o_x).$$

Indeed, by construction, for any orbit o, $\hat{h}_0(o)$ is the unique curve tangent to E^c which is δ near o. Now $\gamma\hat{h}_0(o_x)$ is a curve tangent to E^c which is δ near the orbit $\gamma(o_x)$. Hence, the above formula must hold.

Using this we can prove that we can make a quotient map of $\hat{h}_0$ to M. Given a center leaf c in M we say that c is closed if given a lift $\tilde{c}$ in $\widetilde{M}$, there is a non trivial deck transformation α such that $\alpha(\tilde{c}) = \tilde{c}$. We have already proved that f_t is a leaf space collapsed Anosov flow, which implies that c is closed if and only if it is associated with a closed orbit of ϕ_s. Let $\gamma_1, \dots, \gamma_n$ be the deck transformations of the cover $\hat{M} \to M$. Given y a point in a non periodic orbit of ϕ_s, let $x_1, \dots, x_n$ be the lifts of y to $\hat{M}$, which are related by the $\{\gamma_i\}$. We consider the center leaves in M or $\hat{M}$ which are not closed, or equivalently the non periodic orbits of ϕ_s or $\hat{\phi}_s$. So given y, there are finitely many x_i. For each x_i, we compute $\hat{h}_0(x_i)$, which by the formula above projects by P to the same center leaf in M. In this center leaf there is an induced metric given by length along the centers. This metric induces an identification with $\mathbb{R}$. Using this identification, we can compute the average of $P(\hat{h}_0(x_i))$ for $1 \leqslant i \leqslant n$. Let $h_0(y)$ be this average. Note that we have used that the center leaf is not closed, as otherwise it is more complicated to take averages.

Now we use the following properties: There are finitely many γ_i, the length along center leaves varies continuously, and $\hat{h}_0$ is continuous on the non periodic center leaves. These properties imply that this function extends to a continuous function in all of M.

We now obtained the collapsing function h_0 sending orbits of ϕ_s to curves tangent to E^c in M. Finally we need to construct the self orbit equivalence β to satisfy $f_t \circ h_0 = h_0 \circ \beta$. The construction is now exactly as in the end of the proof of Proposition 9.1 since no orientation is needed then. This shows that f_t is a strong collapsed Anosov flow. □

Remark 10.4. Notice that, in the proof of Theorem A (more precisely, in Proposition 10.2), the time t_1 we require to have so that f_t is leaf space collapsed Anosov flow for all $t > t_1$ may be greater than the time t_0 required so that f_t is a partially hyperbolic diffeomorphism for all $t > t_0$.

Hence, Theorem A does not directly say that *all* the examples *à la* [BGHP] (meaning all examples proven to be partially hyperbolic using Proposition 2.9) are (leaf space) collapsed Anosov flows.

However, since f_t is partially hyperbolic for all $t > t_0$ and leaf space collapsed Anosov flow for all $t > t_1 \geqslant t_0$, Theorem C implies that all f_t, $t > t_0$ are indeed leaf space collapsed Anosov flows.

10.2. **Uniqueness of curves tangent to the center bundle.** In this section we show that under some uniqueness properties of the branching foliations like the ones obtained in Proposition 10.2 we can deduce a stronger form of uniqueness of integrability of the center bundle. This also motivates Question 4 as a way to understand finer geometric properties of the center bundle beyond the fact that it can help to remove orientability assumptions in our results.

We first prove a general fact about quasigeodesic partially hyperbolic diffeomorphisms that may be of interest and which essentially states that the center direction inside center stable (or center unstable) leaves is a semi-flow (i.e., it can only branch in one direction).

Lemma 10.5. *Suppose that f is a quasigeodesic partially hyperbolic diffeomorphism with branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$. Given L a leaf of $\widetilde{\mathcal{W}^{cs}}$ suppose that two center leaves c_1, c_2 in L intersect in x. Then c_1, c_2 coincide in the ray from x to the funnel point in L. The symmetric statement holds for leaves in $\widetilde{\mathcal{W}^{cu}}$.*

Proof. Suppose this is not the case. There are two options:

(i) There are y, z in the ray of c_1 to the funnel point, so that both belong to the intersection of c_1, c_2 but no point in the segment of c_1 between them is in c_2. This is called a finite bigon; or

(ii) There is y in $c_1 \cap c_2$ so that the ray in c_1 from y to the funnel point is disjoint from c_2. This is called an infinite bigon.

We first show that option (i) cannot happen. Let B be the bigon formed by the segments in $c_1 \cup c_2$ bounded by y, z. Let ℓ_i be the segment in c_i from y to z. Consider the negative iterate by f of B: Since the stable lengths converge to infinity, the diameters of $f^{-n}(B)$ goes to infinity as $n \to +\infty$. The curves $f^{-n}(\ell_i)$ are uniform quasigeodesics arcs with same pair of endpoints, hence they are a uniform bounded distance from each other. Consider points midway in $f^{-n}(B)$: up to subsequences and deck transformations the two boundary center rays converge to distinct center leaves in the same center stable leaf, and which have the same ideal points. This is disallowed by Proposition 6.20.

A similar argument rules out option (ii) by considering the infinite bigon B and taking points at increasing distance from the point where they intersect in the direction where they converge to the same point. The same argument gives two center leaves which have the same ideal points.

This proves the lemma. □

We can use this to get a precise description of curves tangent to E^c assuming uniqueness of branching foliations.

Proposition 10.6. *Suppose that f is a quasigeodesic partially hyperbolic diffeomorphism such that all the bundles are orientable and f preserves the orientations. Suppose that there is a unique pair of center stable and center unstable branching foliations that are invariant by f. Then any curve in M which is tangent to E^c is the intersection of a center stable and a center unstable leaf.*

Proof. Let $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ be branching foliations given by Theorem 3.6. As explained in Proposition A.2, two natural f-invariant branching foliations tangent to E^{cs} are constructed in [BI]: The lowermost one in the positive center direction, and the uppermost one. By hypothesis, these two branching foliations must coincide.

Orient the center bundle to be positive in the center stable funnel direction. Now suppose that c is a curve in $\widetilde{M}$ tangent to E^c. Let x be a point in c. Consider a ray r in c starting at x and in the positive direction. Suppose that x is in a center stable leaf U.

Claim 10.7. *The ray r is contained in U.*

Proof of Claim 10.7. Consider U_1, U_2 the uppermost and lowermost center stable leaves of $\widetilde{\mathcal{W}^{cu}}$ through x. Since we assumed that center stable and center unstable branching foliations are unique, $\mathcal{W}^{cu}$ is both the lowermost and uppermost branching foliation of [BI] (see Appendix A). In particular, this implies that U_2 is the lowermost local center unstable surface from x in the positive center direction as constructed by Burago–Ivanov in [BI]. Similarly U_1 is the uppermost local center unstable surface through x. If one does a local saturation S of c through stable leaves, then [BI, Lemma 3.1] shows that S is a C^1 surface tangent to E^{cu}. In particular S is locally between U_1 and U_2. Let L be a center stable leaf containing c. Let $c_i = U_i \cap c$. Now c_i are center leaves in L both through x. Lemma 10.5 shows that the rays of c_i starting at x and in the positive direction coincide. In particular U_1, U_2 coincide locally near x and so does U. Hence r is locally contained in U.

This situation has a uniformity: there is fixed $\varepsilon_0 > 0$ so that one can always get a segment of length ε_0 in r contained in U. This generates point x_1 in r at least ε_0 along r from x. Notice that x_1 is in every center unstable leaf in $[U_1, U_2]$. Now restart with x_1. Get U_1^1, U_2^1 the uppermost and lowermost leaves of $\widetilde{\mathcal{W}^{cs}}$ through x_1. Notice that the intervals $[U_1, U_2] \subset [U_1^1, U_2^1]$. Apply the same argument for a length $\geqslant \varepsilon_0$ along r to get second segment in r now contained in every leaf in $[U_1^1, U_2^1]$ and hence in U. Then iterate, obtaining points x_j in r escaping in r. This proves the claim. □

Now we prove that there is a $\widetilde{\mathcal{W}^{cu}}$ leaf that contains all of c. Let $p_0 = x$. For each i, we choose a point p_i in c that is a distance along c at least 1 from p_{i-1}. We choose the sequence so that the p_i escapes in the direction opposite to the funnel. This direction is opposite to where the points x_j were. Let U_i a $\widetilde{\mathcal{W}^{cu}}$ leaf with p_i in U_i. Let $r_i = [p_i, +\infty)$ be the ray of c starting in p_i and going in the direction of the funnel. By Claim 10.7 the entire ray r_i is contained in U_i. All U_i contain p_0. The set of $\widetilde{\mathcal{W}^{cu}}$ leaves through p_0 is a compact interval. Up to subsequence assume U_i converges to a leaf V as $i \to \infty$. Then since all U_i for $i >= j$ contain p_j then V contains p_j. Hence V contains all the p_i's. By the claim then V contains the entire curve c.

By the same arguments c is contained in a $\widetilde{\mathcal{W}^{cs}}$ leaf E. This finishes the proof of the proposition. □

Remark 10.8. Note that in the case of the examples worked out in Theorem A we are able to get that for large enough $t > 0$ the diffeomorphism f_t when lifted to a finite cover satisfies the hypothesis of Proposition 10.2 and so Proposition 10.6 can be applied to obtain that every curve tangent to E^c is obtained (in $\widetilde{M}$) as the intersection of a center stable and a center unstable leaf of the branching foliations. This is a form of unique integrability of the center bundle, even if different center curves may merge. Note in particular that if f_t is dynamically coherent, this implies that E^c is uniquely integrable as a bundle. In particular notice the difference: one can prove that f_t is partially hyperbolic for all $t \geqslant t_1$, but to get the unique integrability of the center bundle as above one needs $t \geqslant t_0$, where in theory $t_0 > t_1$.

We also note that the property of not having unique f-invariant center stable or center unstable branching foliations is an open property among partially hyperbolic diffeomorphisms thanks to Theorem 4.2. The closed property may fail because in the limit different branching foliations may collapse to a single branching foliation.

However as a direct consequence of Theorem C we get the following: in the connected component of partially hyperbolic diffeomorphisms containing some f_t we have that f_t has to be a collapsed Anosov flow with respect to the same flow and same self orbit equivalence of the flow (same in terms of the action on the orbit spaces), for every pair of branching foliations it may have[17].

Remark 10.9. The previous remark applies very well to the case of partially hyperbolic diffeomorphisms in the connected component of the time one map of an Anosov flow. Here, by Theorem C the whole connected component of partially hyperbolic diffeomorphisms consists of discretized Anosov flows. This uses the last part of the previous remark as well as Proposition 5.15. Moreover, since the Anosov flow is generated by a C^1 vector field, the center direction of its time one map is uniquely integrable. It follows that in the whole connected component of partially hyperbolic diffeomorphisms, if there were more than one pair of branching foliations, these should correspond to discretized Anosov flows – again by the last part of the previous remark. But in [BFFP$_2$, Lemma 7.6] using that they are discretized Anosov flows, we showed that this implies that there is a unique pair of branching foliations. As a consequence we obtain that the center direction is uniquely integrable (since it integrates to a foliation) in the whole connected component of partially hyperbolic diffeomorphisms containing the time one map of an Anosov flow[18].

10.3. **C^1 self orbit equivalences and collapsed Anosov flows.** Thanks to the concept of φ-transversality of [BGHP] and Theorem A, we can readily obtain many collapsed Anosov flows: Finding a map φ for which a flow is φ-transverse to itself is generally not easy (see [BGHP]). But one instance when it *is* easy is when one has a map β, which is a (at least) C^1 self orbit equivalence of a smooth Anosov flow ϕ. Indeed, since β preserves the weak stable and unstable directions and preserves the flow direction, the flow is trivially β-transverse to itself (see Definition 2.8).

[17] Technically to get this one needs to show that having branching foliations for which f is not a collapsed Anosov flow is also an open and closed property, but this follows directly from the same Theorem 4.2.

[18] Or, maybe more generally, a discretized Anosov flow for which the center direction is uniquely integrable.

Hence, for such a β, the map $\phi_t \circ \beta \circ \phi_t$ is a collapsed Anosov flow of ϕ thanks to Theorem A, and it is clearly dynamically coherent as it preserves the weak stable and weak unstable foliations of ϕ.

The first such examples were constructed in [BWi], but these examples are such that a power is a discretized Anosov flow.

One can wonder whether different smooth self orbit equivalences could lead to genuinely new collapsed Anosov flows (that is, ones such that no power is a discretized Anosov flow). It turns out that, at least when the Anosov flow is transitive, this is not the case, as we observe a form of smooth rigidity:

Proposition 10.10. *Let β be a C^1 self orbit equivalence of a smooth (at least C^1) transitive Anosov flow. Then there exists k such that β^k is a trivial self orbit equivalence. (Moreover there is an upper bound for k that only depends on the flow and the manifold.)*

Proof. In the proof of [Bar_2, Proposition 6.6], Barbot shows that if a map $\widetilde{\beta}_{\mathcal{O}}$ on the orbit space of a smooth Anosov flow ϕ is a π_1-equivariant, C^1 diffeomorphism, then there exist a time-change ψ of ϕ such that β is a conjugation of ψ with itself, where $\beta\colon M \to M$ is a C^1-map such that its lift to the orbit space is $\widetilde{\beta}_{\mathcal{O}}$ (or $\widetilde{\beta}_{\mathcal{O}}^2$ if $\widetilde{\beta}_{\mathcal{O}}$ reverses the direction of the flow). (The flow ψ is build on the projectivized bundle of the orbit space, see also [BF]).

In other words, β is in the centralizer of ψ. By [BaG_2, Lemma 1.4], the centralizer of ψ quotiented out by the elements of the centralizer that act as the identity on the orbit space is finite. Hence, there exists k, which can be chosen depending only on the flow and the manifold, such that β^k is trivial. □

11. Some classification results

In this section we will present some relatively direct results giving settings where one can use self orbit equivalences to classify all collapsed Anosov flows or vice-versa. The three settings we will describe are: Collapsed Anosov flows that are homotopic to the identity, Collapsed Anosov flows on T^1S, the unit tangent bundle of a hyperbolic surface, and Collapsed Anosov flows associated with the Franks–Williams example.

Those are not the only cases where one can obtain such a complete understanding, but they are among the easiest and nicely showcase the type of tools one has to prove such results.

We emphasize that the result below gives a complete picture of self orbit equivalences of certain Anosov flows, but only a classification up to isotopy for collapsed Anosov flows, as we do not yet know how different two collapsed Anosov flows associated with the same self orbit equivalence can be.

11.1. The homotopic to the identity case.

In [BaG_1], self orbit equivalences of transitive Anosov flows that are homotopic to the identity were completely classified. Thus, we can translate [BaG_1, Theorem 1.1], using Proposition 5.15, in terms of collapsed Anosov flow to obtain the following.

Theorem 11.1. *If f is a strong collapsed Anosov flow homotopic to the identity associated to a transitive Anosov flow ϕ^t, then f is either a discretized Anosov flow or a* double translation *in the sense of* [BFFP_2].

Moreover, if the associated Anosov flow ϕ_t is either not $\mathbb{R}$-covered, or has non transversely-orientable weak foliations, then f must be a discretized Anosov flow.

Note that it is still unknown whether double translations exists or not outside of Seifert manifolds, but in [FP_2] the second and third authors show that any double translation on a hyperbolic manifold must be a collapsed Anosov flow associated with the "one-step up" self orbit equivalence of an $\mathbb{R}$-covered Anosov flow.

Proof. Let f be a strong collapsed Anosov flow that is homotopic to the identity, associated with a transitive Anosov flow ϕ_t. Let h and β be the associated collapsing map and self orbit equivalence. Since $f \circ h = h \circ \beta$ and that both f and h are homotopic to the identity, we deduce that β is also homotopic to the identity. Thus we can apply [BaG$_1$, Theorem 1.1].

If the flow ϕ_t is not $\mathbb{R}$-covered or has non transversely-orientable weak foliations, then item (1) and (3), respectively, of [BaG$_1$, Theorem 1.1] implies that β is trivial, thus f is a discretized Anosov flow thanks to Proposition 5.15.

If ϕ_t is $\mathbb{R}$-covered, then item (4) of [BaG$_1$, Theorem 1.1] gives that either β is trivial, which gives that f is a discretized Anosov flow, or that β is a power of the "one-step up" self orbit equivalence η. We will not recall what η is exactly, just that a good lift of it acts as a translation on both leaf spaces of ϕ_t. Let $\widetilde{f}$ be a lift of f to the universal cover obtained from lifting an homotopy to the identity. Since f is a strong collapsed Anosov flow, it admits center stable and center unstable branching foliations that are the images by h of the weak stable and weak unstable foliations of ϕ_t. Hence, a lift $\widetilde{h}$ realizes a semi-conjugacy between the action of $\widetilde{\beta}$ and the action of $\widetilde{f}$ on the respective leaves spaces. Since $\widetilde{\beta}$ acts as a translation, so does $\widetilde{f}$. So f is a double translation in the sense of [BFFP$_2$]. □

11.2. **Unit tangent bundle of surfaces.** When considering unit tangent bundle of surfaces, it is also possible to give a complete picture of collapsed Anosov flows, at least up to isotopy.

Theorem 11.2. *Let T^1S be the unit tangent bundle of a hyperbolic surface S.*

Let f be a collapsed Anosov flow on T^1S with associated flow ϕ. Then the isotopy class of f is in a lift of $\mathrm{MCG}(S)$ to $\mathrm{MCG}(T^1S)$. More precisely, let g^t be the geodesic flow on T^1S for a fixed hyperbolic metric and $e\colon T^1S \to T^1S$ an orbit equivalence between ϕ and g^t. Let $\widehat{\mathrm{MCG}(S)} \subset \mathrm{MCG}(T^1S)$ be the lift of $\mathrm{MCG}(S)$ given by taking the derivative. Then the isotopy class of f is inside $i_{[e]}\widehat{\mathrm{MCG}(S)}$, the conjugation of $\widehat{\mathrm{MCG}(S)}$ by the isotopy class of e.

Moreover, any isotopy class in $i_{[e]}\widehat{\mathrm{MCG}(S)}$ admits a collapsed Anosov flow.

The same statements hold for self orbit equivalences of ϕ.

Remark 11.3. Note that one can choose the orbit equivalence e above such that it induces a map homotopic to the identity on S.

Remark 11.4. In [BGHP], it is shown that the isotopy classes of partially hyperbolic diffeomorphisms on T^1S do not form a subgroup of $\mathrm{MCG}(T^1S)$. However, as we see here, this lack of a group structure is only because there are many Anosov flows on T^1S that are orbit equivalent to the geodesic flow, but not via an orbit equivalence that is homotopic to identity. Indeed, once an Anosov flow ϕ is fixed, the isotopy classes of collapsed Anosov flow associated with ϕ form a subgroup of $\mathrm{MCG}(T^1S)$.

Proof. Let ϕ be an Anosov flow on T^1M and $e\colon M \to M$ an homeomorphism such that $e^{-1} \circ \phi^t \circ e$ is a time-change of g^t.

In [BGHP, Theorem 1.2], it was shown that any isotopy class in $\widehat{\mathrm{MCG}(S)}$ admits a partially hyperbolic diffeomorphism which is, according to Theorem A, a collapsed Anosov flow associated with g^t. Hence, for any class in $\widehat{\mathrm{MCG}(S)}$, there exists a self orbit equivalence β of g^t. Thus, $e \circ \beta \circ e^{-1}$ is a self orbit equivalence of ϕ^t and such self orbit equivalences will cover all of $i_{[e]}\widehat{\mathrm{MCG}(S)}$.

We can also build a collapsed Anosov flow using the same method, but we would need to require smoothness of e which does not a priori hold. So instead, we let $\bar{e}$ be a diffeomorphism in the same isotopy class as e and define $\bar{\phi}^t = \bar{e} \circ g^t \circ \bar{e}^{-1}$. Then, for any collapsed Anosov flow f associated with g^t, the map $\bar{e} \circ f \circ \bar{e}^{-1}$ is a collapsed Anosov flow of $\bar{\phi}$ and the isotopy classes of such collapsed Anosov flow cover all of $i_{[e]}\widehat{\mathrm{MCG}(S)}$.

The other direction can be proven for instance as in [Mat]: We can show that the isotopy class of a self orbit equivalence of ϕ is necessarily in $i_{[e]}\widehat{\mathrm{MCG}(S)}$. This will imply that the isotopy class of a collapsed Anosov flow must also necessarily be in $i_{[e]}\widehat{\mathrm{MCG}(S)}$.

If β is a self orbit equivalence of ϕ, then up to conjugation by e, we can assume that β is a self orbit equivalence of g^t, and we have to show that $[\beta] \in \widehat{\mathrm{MCG}(S)}$. This follows as in the proof of [Mat, Proposition 3.6] (see also [BGHP, Theorem 3.6] or [BF]). □

11.3. **Collapsed Anosov flows of the Franks–Williams example.** The Franks–Williams [FW] example is the first, most famous and simplest non-transitive Anosov flow on a 3-manifolds. We denote the Franks–Williams flow by ϕ_{FW} and by M_{FW} the manifold supporting that flow. Note that ϕ_{FW} is the only non-transitive Anosov flow up to orbit equivalence on M_{FW} (see [YY]). We will not recall the construction of ϕ_{FW} (see [FW] or, e.g., [BBY]), but instead list the properties that we will use:

(i) The manifold M_{FW} decomposes into two atoroidal pieces separated by a torus T transverse to ϕ_{FW} (which is unique up to isotopy along the flow lines). In particular, by Mostow's rigidity theorem, the mapping class group of M_{FW} is up to finite index generated by Dehn twists along the transverse tori[19].
(ii) The stable and unstable foliations restrict to two transverse foliations on the transverse torus with four closed leaves (two stable and two unstable leaves) and Reeb components in between. We denote by α the element of $\pi_1(T)$ representing the closed leaves;
(iii) Each periodic orbit of ϕ_{FW} is unique in its free homotopy class, except for the four periodic orbits (two in each atoroidal pieces) associated with the closed leaves of T which are pairwise freely homotopic.

Theorem 11.5. *Up to finite power, any self orbit equivalence or collapsed Anosov flow of ϕ_{FW} is in the isotopy class of a Dehn twist of T in the direction of α. Moreover, up to a finite power, two self orbit equivalences of ϕ_{FW} in the same isotopy class are equivalent.*

Conversely, any such isotopy classes can be realized by a collapsed Anosov flow or self orbit equivalence of ϕ_{FW}.

Remark 11.6. One can show that two self orbit equivalences in the same isotopy class are equivalent without taking a finite power, but the proof is easiest when allowing finite powers and we leave the more precise statement for a future general study of self orbit equivalences.

Remark 11.7. A cosmetic adaptation of the following proof allows to more generally classify collapsed Anosov flows and self orbit equivalences of Anosov flows

[19] see e.g., [BGHP] for the definition of a Dehn twist on a torus in a 3-manifold

that are obtained in the following way: Create any number of hyperbolic plugs (in the sense of [BBY]) by doing a derived from Anosov construction on finitely many orbits of a suspension of an Anosov diffeomorphism of the torus. Glue the hyperbolic plugs together in any of the ways allowed to get a (transitive or non-transitive) Anosov flow (see [BBY]).

Such Anosov flows will satisfy a version of each of the items (i), (ii), and (iii) above. That is, the JSJ decomposition of the manifold has only atoroidal pieces, each torus is transverse to the flow with two closed center leaves and Reeb components for each of the weak foliations restricted to the torus, and every periodic orbits aside from finitely many will be alone in their free homotopy class.[20]

Proof. We start by proving the converse part of the theorem: Since $\alpha \in \pi_1(T)$ represents the free homotopy class of the closed leaves of the weak stable and weak unstable foliations restricted to T, by [BGHP, Theorem 1.3], the isotopy class of any Dehn twist in the direction of α admits a partially hyperbolic diffeomorphism. This diffeomorphism is a collapsed Anosov flow by Theorem A.

Now, suppose that β is a self orbit equivalence of ϕ_{FW}. Then up to finite power it fixes both pieces of the torus decomposition of M_{FW}, and, up to isotopy along flow lines, it also fixes T.

By Mostow's rigidity theorem, the mapping class group restricted to each atoroidal piece is finite. Hence, a further power, say k, of β will be isotopic to identity in each pieces.

So β^k must send each periodic orbit to one freely homotopic to it. By construction of the Franks–Williams example (see item (iii) above), β^{2k} will then fix every periodic orbit of ϕ_{FW}. In particular, the isotopy class of β^{2k} must preserve the conjugacy class of α, the element of $\pi_1(T)$ that is freely homotopic to the exceptional periodic orbits of ϕ_{FW}. Therefore, the isotopy class of β^{2k} must be generated by the Dehn twist on T in the direction of α.

So all we have left to do is show that if two self orbit equivalences are in the same isotopy class, then they are equivalent. Equivalently, it suffices to show that if β is homotopic to the identity, then it fixes every orbit of ϕ^t.

Let $\widetilde{\beta}$ be a lift of β to the universal cover obtained by lifting the homotopy to identity. Recall that by item (iii) above, $\widetilde{\beta}$ must fix all the lifts of periodic orbits, except possibly the lift of the four exceptional periodic orbits. Moreover, $\widetilde{\beta}^2$ must preserve each half-leaves of lifted periodic orbits. Hence, any orbit obtained as an intersection of a weak stable and weak unstable leaf of a periodic orbit is fixed by $\widetilde{\beta}^2$. That set is dense in $\widetilde{M}$ [Fra]. Therefore, by continuity, $\widetilde{\beta}^2$ acts as the identity on the orbit space of ϕ_{FW}, which ends the proof. □

Appendix A. Branching foliations and prefoliations revisited

In this section we obtain more information about branching foliations. We will assume some familiarity with the constructions in [BI] and repeatedly refer to statements or proofs in that paper.

A.1. **Uniqueness of approximating leaves.** The constructions of Burago and Ivanov have a lot of inherent redundancy. What we mean is that there are a lot of surfaces $S\colon dom(S) \to M$ with the same image. Since these are not embeddings

[20]To show that a periodic orbit γ crossing one transverse torus T is also alone in its free homotopy class, remark that, otherwise, it would have to be freely homotopic to the *inverse* of another periodic γ' (see, e.g., [Fen_2]), but that would imply that γ' has to cross T in the opposite direction as γ, contradicting the transversality of T.

one has to be more careful with the meaning of "same image". We follow Burago–Ivanov and say that two surfaces S_1, S_2 are equivalent if there is a homeomorphism $g\colon dom(S_1) \to dom(S_2)$ such that $S_1 = S_2 \circ g$. More generally if this works for subsets of the domains we say this is a change of parameter of the subsurfaces. This is another reason to consider $dom(F)$ to be a plane.

What we call by "leaves" of the branching foliation, are the equivalence classes of these identifications. With this understanding one can prove:

Proposition A.1. *Let $\mathcal{A}$ be a branching foliation. Let $\mathcal{B}_\varepsilon$ be the approximating foliations constructed by Burago–Ivanov. There is a one to one correspondence between the leaves of $\mathcal{A}$ and the leaves of $\mathcal{B}_\varepsilon$ for any $\epsilon > 0$.*

Proof. We will use the notations and terminology of the proof of [BI, Theorem 7.2]. They construct a "push off" function F which pushes different branching leaves through a point apart. Then given any $\alpha > 0$ they construct a foliation $\mathcal{A}_{\alpha F}$ such that as $\alpha \to 0$ the tangent planes to the leaves of $\mathcal{A}_{\alpha F}$ converge to the bundle E. So $\mathcal{B}_\varepsilon$ is $\mathcal{A}_{\eta(\varepsilon)F}$ for some function η which converges to 0 as ε converges to 0.

We review the important points to construct F. They consider a smooth vector field W which is almost perpendicular to the bundle E. Let ϕ be the flow generated by W. They consider a finite cover $\{U_i\}$ of M by foliated boxes of W and with coordinates (x_i, y_i, z_i) such that E is almost horizontal (x, y) directions) and W is almost vertical (z direction) in each U_i.

In each U_i they consider $\mathcal{A}_i$ the set of pairs (S, x) where S is an element of $\mathcal{A}$, $x \in dom(S)$ and $S(x) \in U_i$. They define a non strict total order $\geqslant_i$ on $\mathcal{A}_i$ as follows: choose $A_1, A_2 \in \mathcal{A}_i$, $A_1 = (S_1, x_1), A_2 = (S_2, x_2)$. There is an intrinsic ball $D = B_r(x_1) \subset dom(S_1)$ such that a piece of A_2 is the graph of a C^1 function $f\colon D \to R$ as follows: the surface $S_1^f\colon D \to M$ given by

$$S_1^f(x) \;=\; \phi^{f(x)}(S_1(x)), \quad x \in D$$

coincides, up to a change of parameter sending x_1 to x_2, with a region in S_2. Let r be the maximum radius of such a ball (possibly $r = \infty$). Since the surfaces have no topological crossing, the function f does not change sign. We set $A_2 \geqslant_i A_1$ if $f \geqslant 0$ and $A_1 \geqslant_i A_2$ if $f \leqslant 0$.

Burago and Ivanov remark that it is possible that both inequalities $A_1 \geqslant_i A_2$ and $A_2 \geqslant_i A_1$ hold. This means that S_1 and S_2 coincide up to a parameter change, which sends x_1 to x_2, in which case they write $A_1 \cong A_2$.

We remark that we identified surfaces of $\mathcal{A}$ if they have the same image up to parameter change. Under this identification $\geqslant_i$ is a total order in $\mathcal{A}_i$, which is denoted by $>_i$. So the set of equivalence classes of $\mathcal{A}_i$ is the same as $\mathcal{A}_i$.

Then [BI, Lemma 7.2] shows that $(\mathcal{A}_i, >_i)$ is order isomorphic to an open interval and they pick a homeomorphism $\theta_i\colon \mathcal{A}_i \to (0,1)$. The important point to understand here is that θ_i is different for different branching leaves B_1, B_2: even if the leaves B_1, B_2 pass through a common point y in U_i, and even if they coincide in a pass through U_i. But if B_1, B_2 are not the same leaf globally, then $\theta_i(B_1) \neq \theta_i(B_2)$. That is θ_i differentiates different branching leaves, even if locally (which can be a big set) they are the same image.

They use the functions θ_i to define functions F_i which are meant to "push" leaves of $\mathcal{A}$ inside the foliated boxes U_i. The push off is done along flow lines of ϕ. The functions F_i are summed to produce a function $F = 1/k \sum_{i=1}^k F_i$. Given $\alpha > 0$, Burago–Ivanov push leaves of $\mathcal{A}$ using the function αF and they show the pushed off leaves form an actual foliation (that is, with no branching). The map

h in the statement of the Burago–Ivanov theorem, which sends leaves of $\mathcal{A}_{\alpha F}$ to leaves of $\mathcal{A}$ is just the opposite of the push off map: the map h slides points back along flowlines of ϕ.

Now we come to the property we want to prove. Suppose that two leaves B, C of $\mathcal{A}_{\alpha F}$ project to the same leaf G of $\mathcal{A}$. But if it is the same leaf G, then by the discussion above all the functions θ_i are specified. Hence the functions F_i are specified along G and there is only one push off leaf in $\mathcal{A}_{\alpha F}$ associated to G. This shows that B, C are the same leaf of $\mathcal{A}_{\alpha F}$.

This finishes the proof of the Proposition. □

A.2. **Properties of some branching foliations.** Now we go back to the specific branching foliations associated with partially hyperbolic diffeomorphisms, as constructed by Burago and Ivanov.

The next property we want to consider is the local "highest" and "lowest" leaves from a point. The construction of Burago and Ivanov of the branching foliations for partially hyperbolic diffeomorphisms starts as follows. Consider a point p in M. They fix a smooth disk D through p and transversal to the stable foliation. The disk is small to be contained in foliated boxes of all the bundles E^c, E^s, E^u. The E^{cs} bundle intersects the tangent bundle to D in a one dimensional bundle, call it G. They consider all C^1 curves in D tangent to G. Among all these tangent curves passing through p there is a lowest curve in the forward direction. We refer to [BI, §5]. The local saturation of this is a C^1 surface (see [BI, Proposition 3.1]). Locally it is the "lowest" surface tangent to E^{cs} through p in the positive direction.

We prove the following:

Proposition A.2. *One can do the construction of the branching foliations of* [BI] *such that through every point p there is a branching leaf which is the lowest locally in the positive E^c direction. More specifically there is a fixed size $\delta > 0$, so for every p in M the locally lowest forward surface for p containing a half disk of radius at least δ centered at p is in a leaf of the branching foliation. In addition for the same foliation for every p there is also a branching leaf which is the highest locally in the negative E^c direction.*

Proof. In fact we prove that the branching foliations that Burago and Ivanov construct satisfy the conclusions of the Lemma.. The main result we need is [BI, Proposition 4.13]. This result concerns "partial" branching foliations, which satisfy only the non topological crossing condition of branching foliations. Proposition 4.13 extends this partial foliation in a particular way. This result is proved in [BI, §6] using results in dimension 2 developed in [BI, §5].

In particular since there may be many leaves through a given point, one has to keep track of which leaves are "above" other leaves. They introduce a total order in the set of leaves through a point p and this has to be preserved when one moves along paths common to both leaves being considered. To introduce a new leaf, they have to specify where it should be located with respect to already existing order in the set of leaves through p. A location is given by what they call a "section" of the leaves through p, which corresponds to a cut in the ordering of all the already existing leaves through p. The constructed surfaces are called "upper enveloping surfaces", see [BI, Definition 6.1]. They show that for any section at p one can construct a new partial branched surface through p that fits exactly in that section and that does not cross topologically any of the already existing surfaces.

The beginning step of the induction process is with the empty set. Through every point the section is empty. In this case (empty section) the upper enveloping surface (in the positive E^c direction) through a point p is locally the lowest surface through p. This is because the surface has to be what is called an upper enveloping surface. These surfaces are locally obtained as stable saturations of curves tangent to the E^c bundle, which are called upper envelope curves see [BI, page 558]. The upper envelope is the supremum of descending curves, see [BI, page 558] and the definition of descending curves, cf. [BI, Definitions 5.2 and 5.4]. In particular in the initial step there are no surfaces so the sections are the empty sections. In this case [BI, Definition 5.2, item (2)] says that the initial step is the lowest forward integral curve from the point.

The local stable saturation of the lowest forward integral curve is the lowest local surface in the forward E^c direction, tangent to E^{cs} and through the point p. In addition this surface is a "patch": the edges of the surface are separated by at least a fixed size $\delta > 0$, see [BI, Definition 4.7].

This proves the first assertion of the Lemma.

To prove the second assertion of the Lemma one has to go in the negative direction of E^c. In the construction of the branching foliations in [BI] they go alternatively forward and backward, constructing patches of surfaces starting at the points.

So the initial step puts in the lowest surface tangent to E^{cs} through any p in M and going in the forward direction. In the second step the orientations are reversed, so going forward now corresponds to going backwards in the original E^c direction, and lowest is highest in the original partially constructed branching foliation.

This step is done after we already have some partial surfaces and sections through points. So given a point p consider the empty section of all surfaces through p. Then [BI, Lemma 6.11] shows that there is a forward envelope surface with p in the boundary and the section is the empty section at p. Since the section at p is empty, then the initial step is labeled by the empty set again (see [BI, Definition 5.2, item (3)] for descending curves). This means that locally this is the lowest forward surface through p. But recall that we switched orientations, so forward means backwards from p in the original orientation, and lowest means highest in the original orientation.

This proves the second property of the Lemma and finishes the proof. □

Remark A.3. In the same way we could have switched the orientation of E^u in the beginning – but not of E^c. Doing the construction in [BI] produces a branching foliation containing the highest local surface tangent to E^{cs} in the forward center direction and the lowest local surface tangent to E^{cs} in the backwards direction. In particular, we see that every curve tangent to E^c must be locally contained between both branching foliations. The reason for this is that if c is any such local center curve in the forward direction through the point, then its local stable saturation is a C^1 surface through the point and tangent to E^{cs}. As proved in [BI] this surface is locally "above" the lowest surface through the point.

Remark A.4. In general we cannot have both lowest and highest local surfaces (of fixed size) and in both forward and backwards directions for all p in M as part of leaves of the foliation. Here is an example one dimension lower in the plane. Consider the differential equation $\frac{dy}{dx} = 3y^{2/3}$. It generates a vector field in the direction $(1, 3y^{2/3})$. This vector field is not uniquely integrable along the x axis. General solutions are made up of pieces of curves $y = (x+c)^3$ or segments in the x axis. Outside the x axis this is uniquely integrable producing segments of curves $y = (x + c)^3$.

Consider first the curves that are highest forward. For any point p in the plane the highest forward curve through that points is contained in the curve $y = (x + c)^3$ through p. Since the requirement is that one has to have a fixed sized $\delta > 0$ of highest forward for every point, then if p is below the x axis, but sufficiently close to the x axis, the δ size highest forward curve through p is a part of the cubic which crosses the x axis. But the highest backward curve of every point in the x is the ray of the x axis ending positively at that point. One has to have at least a size δ for every point in the x axis. These two sets of curves cross topologically, so cannot be part of the same branching foliation.

A.3. **Smooth approximation and Candel metrics.** The following states that the coarse nature of leaves of the branching foliations with the metric induced by the manifold is good enough. We refer the reader to [BH, §III.H] for the basic notions about Gromov hyperbolic metric spaces.

Proposition A.5. *Let $\mathcal{F}$ be a branching foliation well approximated by foliations $\mathcal{F}_\varepsilon$ such that $\mathcal{F}_\varepsilon$ are by hyperbolic leaves. Then, for every Riemannian metric in M the pullback of the metric to the leaves of $\mathcal{F}$ makes them Gromov hyperbolic.*

Proof. For this, we choose ε small enough so that we have nice local product structure neighborhoods and take a (continuous) Riemannian metric on M by considering the Candel metric (cf. [Can]) on $\mathcal{F}_\varepsilon$ on $T\mathcal{F}_\varepsilon$ and taking a fixed vector field transverse to an ε-cone around $T\mathcal{F}$ to complete an orthonormal basis. For this metric, it is possible to verify in these local product structure neighborhoods the CAT(κ) condition for some $\kappa < 0$ which is a local condition (see [BH, §II.2]).

Now, since being Gromov hyperbolic is invariant under quasi-isometries and M is compact, we have that changing the metric does not change the fact that leaves are Gromov hyperbolic (see [BH, §III.H]). □

Note that in [Thu, §4] it is claimed that one can choose a smooth metric in M which makes every leaf of $\mathcal{F}_\varepsilon$ to have curvature arbitrarily close to -1. and For smooth foliations this is proved in [AY, Theorem B] and attributed to Ghys (see also [AY, Remark 6.2]). In our case, leaves of $\mathcal{F}$ may be just C^1, so it is more delicate to talk about curvature but still we only look at coarse geometric properties, so our statement suffices.

Remark A.6. This implies that there is a well defined notion of complete geodesics in leaves, and that through each tangent vector $v \in T_xL$ in a leaf $L \in \mathcal{F}$ there is a unique geodesic in the leaf through x with velocity v. In particular, one can compactify each leaf with a circle and consider a visual metric in this circle in a natural way. See also [BH, §III.H.3] for definitions valid for general metric spaces.

Appendix B. Graph transform method

Here we revisit the results in [HPS] to get Theorem 4.2. Then we comment on Theorem 4.1 which is similar.

Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism of a closed 3-manifold M. By considering a different Riemannian metric, we can assume that the bundles E^s, E^c and E^u are almost pairwise orthogonal and that expansion, contraction and domination is seen in one iterate (see [CP, §2]). We can choose a neighborhood $\mathcal{U}$ of f so that every $g \in \mathcal{U}$ is partially hyperbolic and the invariant bundles of g have the same property with respect to the same Riemannian metric.

We can also choose E a smooth one-dimensional subbundle of TM which is transverse (and almost orthogonal) to E^{cs}_g for every $g \in \mathcal{U}$. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ we have that the exponential mapping is a smooth embedding from $E(\varepsilon)$ to M, meaning that for every $x \in M$, if we consider $E(x, \varepsilon)$ to be the ε-neighborhood of 0 in the space $E(x) \subset T_x M$ then the exponential map $\exp_x : E(x, \varepsilon) \to M$ is an embedding with derivative close to 1.

These are the choices of $\mathcal{U}$ and ε that one needs to make, and if one follows the proof in [HPS, Pages 94-107] one can see that Theorem 4.2 follows. For the convenience of the reader, we will indicate the main points, particularly because our setting is simpler.

Proof of Theorem 4.2. Consider $g \in \mathcal{U}$ admitting an invariant branching foliation $\mathcal{W}^{cs}_g$ tangent to E^{cs}. To avoid confusions, we will use Notation 3.7.

We can consider that the collection of immersions $(\varphi, U) \in \mathcal{W}^{cs}_g$ as a unique immersion $\imath : V \to M$ where V is an uncountable union of complete simply connected surfaces, each connected component corresponding to a leaf of $\mathcal{W}^{cs}_g$. The immersion $\imath$ is clearly a C^1-*leaf immersion* which is normally expanded with respect to g ([HPS, §6]), that is:

(i) the connected components of V with the metric induced by $\imath$ by pullback are complete,
(ii) there is a map $\imath_* g : V \to V$ such that $g \circ \imath = \imath_* g \circ \imath$,
(iii) for every $x \in V$ we have that $D_x \imath T_x V = E^{cs}(\imath(x))$.

The only point which needs some justification is *(ii)* but this follows rather easily by considering the lift of $\mathcal{W}^{cs}_g$ to the universal cover where leaves are properly embedded planes and therefore it is easy to induce a map from leaf to leaf even when these may not be injectively immersed in M.

As in [HPS, (6.2)] we can define a *plaquation* of $\imath$ consisting of embeddings $\{\rho : D \to V\}_{\rho \in \mathcal{P}}$ of the unit disk $D = \{v \in \mathbb{R}^2 \ : \ \|v\| \leqslant 1\}$ such that the interiors of $\rho(B)$ as $\rho \in \mathcal{P}$ cover V and such that the family $\{\imath \circ \rho\}_{\rho \in \mathcal{P}}$ is precompact in $\mathrm{Emb}^1(D, M)$.

Claim B.1. *We can choose the plaquation $\mathcal{P}$ with the following additional properties:*

(i) *For every $x \in V$ there is a plaque $\rho \in \mathcal{P}$ centered at x, this means, $\rho(0) = x$.*
(ii) *If one considers the vector bundle E_ρ induced by E over the image of ρ (which is a trivial bundle), we have that the exponential map* $\exp : E_\rho(\varepsilon) \to M$ *is an embedding for every $\varepsilon < \varepsilon_0$.*

Proof. For the second item, notice that since $\exp_x : E(x, \varepsilon) \to M$ is an embedding tangent to $E(x)$ at x there is $\delta > 0$ such that if a disk is tangent to a subbundle making a definite angle with E and the disk has maximal radius smaller than δ then the exponential map will be an embedding from the bundle E restricted to the disk for vectors of norm less than ε. Now, we can choose a covering of V by disks around every point which are mapped by $\imath$ into disks of maximal radius smaller than δ but minimal radius larger than $\delta/10$. This family will be tangent to E^{cs}_g which makes a uniform angle to E independently of $g \in \mathcal{U}$ and it will be clearly pre-compact in the space of embeddings (see [HPS, (6.2)]). $\square$

Now, for each $g' \in \mathcal{U}$ we want to construct using a graph transform argument a C^1-leaf immersion $\imath_{g'}: V \to M$ producing a branching[21] foliation $\mathcal{W}^{cs}_{g'}$ with the same dynamics as the one of g on $\mathcal{W}^{cs}_g$.

To describe the strategy, let (φ, U) be a leaf of $\mathcal{W}^{cs}_g$, we want to construct a new surface $(\varphi_{g'}, U)$ which will be part of the branching foliation $\mathcal{W}^{cs}_{g'}$. The surface $(\varphi_{g'}, U)$ will be defined as $\lim_n (g')^{-n}(g^n(\varphi, U))$. We need to explain what we mean by this, and this is why the plaquations play a role in the proof. As in [HPS] we will work directly with $\imath$ and construct $\imath_{g'}$ since it allows to treat all leaves of $\mathcal{W}^{cs}_g$ simultaneously.

Let us construct what we mean by the graph transform. Consider $\mathcal{P}$ to be the plaquation of V as in Claim B.1. Since in $\mathcal{U}$ the derivative along E^{cs} is uniformly bounded we can define another plaquation $\hat{\mathcal{P}}$ which consists on the restrictions of the plaques $\rho \in \mathcal{P}$ to $\hat{D} = \{v \in \mathbb{R}^2 : \|v\| < \hat{\delta}\}$ where $\hat{\delta}$ is chosen so that the image of the plaque $\hat{\rho} \in \hat{\mathcal{P}}$ centered at $x \in M$ by g is contained in the interior of the plaque $\rho' \in \mathcal{P}$ centered at $\imath_* g(x)$. Notice that all this is uniform in $\mathcal{U}$ since it only depends on the norms of the derivatives. We will denote by ρ_x and $\hat{\rho}_x$ the plaques from $\mathcal{P}$ and $\hat{\mathcal{P}}$ respectively which are centered at x.

Denote E_x (resp. $\hat{E}_x$) to be the vector bundle over D (resp. $\hat{D}$) induced by E via the map $\imath \circ \rho_x$ (resp. $\imath \circ \hat{\rho}_x$). As before, for $z \in D$ (resp. $z \in \hat{D}$) we denote by $E_x(z, \delta)$ (resp. $\hat{E}_x(z, \delta)$) to be the interval of length 2δ centered at 0 on the fiber of E_x (resp. $\hat{E}_x$) over z.

Given a section ξ of the bundle $E_x(\varepsilon)$ or $\hat{E}_x(\varepsilon)$ (that is, a continuous map from D to $E_x(\varepsilon)$ such that $\xi(z) \in E_x(z, \varepsilon)$) we can define its graph as $\mathrm{graph}(\xi) \subset M$ to be the image under the exponential map of the image of ξ. By the choice of ε this is a topologically embedded disk, moreover, if ξ has some additional regularity (Lipschitz, or C^1) then the disk is embedded in this regularity.

By our choices of D and $\hat{D}$ one can check:

Claim B.2. *Let ξ be a section of the bundle $E_x(\varepsilon)$ then, there is a well defined section $(g')_*\xi$ of the bundle $\hat{E}_y$ where $y = (\imath_* g)^{-1}(x)$ such that the image by g' of* $\mathrm{graph}((g')_*\xi)$ *is contained in* $\mathrm{graph}(\xi)$.

Using this, we will construct the graph transform of a coherent family of sections $\{\xi_x\}_{x \in V}$ by gluing together enough images under $(g')_*$ of plaques. There will be a unique fixed point of this graph transform which will provide the new branching foliation for g' with the desired properties.

We say that a family of sections $\{\xi_x\}_{x \in V}$ such that each ξ_x is a section of $E_x(\varepsilon)$ is a *coherent family of sections of* $\mathcal{P}$ if whenever the images of ρ_x and ρ_y intersect it follows that $\mathrm{graph}(\xi_x)$ and $\mathrm{graph}(\xi_y)$ intersect in the image under the exponential map of the restriction of the section ξ_x to $\rho_x^{-1}(\rho_y(D) \cap \rho_x(D))$ (notice that this is the same to say that they intersect in the image under the exponential map of the restriction of the section ξ_y to $\rho_y^{-1}(\rho_y(D) \cap \rho_x(D))$). Similarly, one can define a coherent family of sections of $\hat{\mathcal{P}}$.

Given a coherent family of sections $\{\xi_x\}_{x \in V}$ of $\mathcal{P}$ one can define a coherent family of sections of $\hat{\mathcal{P}}$ by restriction. Similarly, since every plaque of $\mathcal{P}$ is covered by plaques of $\hat{\mathcal{P}}$, the coherent property allows to obtain, from a coherent family of sections $\{\hat{\xi}_x\}_{x \in V}$ of $\hat{\mathcal{P}}$ a coherent family of sections $\{\xi_x\}_{x \in V}$ by gluing the sections in a cover of the image of $\rho_x(D)$ by $\{\hat{\rho}_{y_i}(\hat{D})\}_i$; this is independent of the choice of the covering.

[21] The non topological crossing condition is not discussed in [HPS] since they work in higher codimension, but will follow rather directly from the construction in our case.

We thus get:

Claim B.3. *Given a coherent family of sections* $\{\xi_x\}_{x\in V}$ *of* $\mathcal{P}$ *one can define a new coherent family of sections* $\{\zeta_x\}_{x\in V} = (g')_\sharp\{\xi_x\}_{x\in V}$ *by gluing the coherent family of sections* $\{(g')_*\xi_x\}_{x\in V}$ *over* $\hat{\mathcal{P}}$.

The map $(g')_\sharp$ is what is called a *graph transform* and it is a standard argument (see, e.g., [HPS, §4 and §5] or [CP, §4]) to show that one can metricize the space of Lipschitz sections with bounded Lipschitz constant to get that $(g')_\sharp$ is a contraction and therefore has a unique fixed point. This fixed point can be showed to consist on sections whose graphs are tangent to $E^{cs}_{g'}$ (and therefore it is C^1). Moreover, the uniqueness of the fixed point is stronger, as every $(g')_\sharp$ invariant family of coherent sections must coincide with this fixed point which follows by the fact that Dg' expands uniformly the direction generated by E.

This produces a new C^1-leaf immersion $\imath_{g'}$ whose leaves are tangent to $E^{cs}_{g'}$ and which are permuted in the same way as g permutes the leaves of $\imath$. The dynamics inside each leaf differs by something that is smaller than the size of the plaques[22].

We need to check that leaves do not topologically cross which follows quite directly since by uniqueness one obtains the branching foliation by iterating by $(g')^n_\sharp$ the original branching foliation (which corresponds to the family of trivial sections corresponding to $\imath$). Since iterates preserve the local orientation, the limit cannot create crossings. □

Similar argument allow to get Theorem 4.1 (this is indeed totally contained in [HPS, §6]).

Comments on the proof of Theorem 4.1. The setup of the proof of Theorem 4.1 is very similar to the one in Theorem 4.2 except that instead of normal expansion one has normal hyperbolicity (and naturally one cannot talk about topological crossings in higher codimension, but this is not so relevant for the proof).

Let us comment on this difference. We emphasize again that this is done in [HPS, §6], and the only difference is the uniformity of the constants that is not precisely stated there, so we will only sketch the argument very briefly to try to convince the reader that the arguments do not require more than a control on the C^1-size of the partially hyperbolic map and the angles between bundles (to be able to construct the plaques and set up the graph transform operator).

In particular, one needs to first use the stable manifold theorem to construct stable manifolds and unstable manifolds through each plaque; this is done with standard graph transform methods (see [HPS, Theorem 6.1(a)]). This gives families of two dimensional plaques that now are respectively normally expanded and contracted. One can apply the same arguments as in Theorem 4.2 to these families and obtain continuations of these plaques (which will now be coherent only in the center direction). Intersecting these plaques one obtains the desired result. See [HPS, §6], in particular [HPS, pages 94-100] for more details on how the constants are chosen. □

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[22]This is how the notion of plaque expansivity arises naturally.

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Queen's University, Kingston, ON
Email address: thomas.barthelme@queensu.ca
URL: sites.google.com/site/thomasbarthelme

Florida State University, Tallahassee, FL 32306
Email address: fenley@math.fsu.edu

Centro de Matemática, Universidad de la República, Uruguay
Email address: rpotrie@cmat.edu.uy
URL: http://www.cmat.edu.uy/~rpotrie/