CHARACTERISTIC CLASSES OF SINGULAR VARIETIES APRIL 23-27, BANACH CENTER, WARSAW

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Preface. These five lectures aim to explain an algebro-geometric approach to the study of different notions of Chern classes for singular varieties, with emphasis on results leading to concrete computations.

The notes are organized so that every page deals with essentially one topic (a device which I am borrowing from Marvin Minsky's *"The Society of Mind"*). Every one of the five lectures consists of five pages.

My main goal in the lectures was not to summarize the history or to give a complete, detailed treatment of the subject; five lectures would not suffice for this purpose, and I doubt I would be able to accomplish it in any amount of time anyway. My goal was simply to provide enough information so that interested listeners could start working out examples on their own. As these notes are little more than a transcript of my lectures, they are bound to suffer from the same limitations. In particular, I am certainly not quoting here all the sources that should be quoted; I offer my apologies to any author that may feel his or her contribution has been neglected.

The lectures were given in the mini-school with the same title organized by Professors Pragacz and Weber at the Banach Center. Jörg Schürmann gave a parallel cycle of lectures at the same mini-school, on the same topic but from a rather different viewpoint. I believe everybody involved found the counterpoint provided by the accostment of the two approaches very refreshing. I warmly thank Piotr Pragacz and Andrzej Weber for giving us the opportunity to present this beautiful subject.

Contents

1. Lecture I

1.1. **Cardinality of finite sets vs. Euler characteristic vs. Chern-Schwartz-MacPherson classes.** Let $\mathcal{F}in$ denote the category of finite sets. I want to consider a functor $\mathcal{C}$ from $\mathcal{F}in$ to abelian groups, defined as follows: for S a finite set, $\mathcal{C}(S)$ denotes the group of functions $S \to \mathbb{Z}$.

Note: we could see $\mathcal{C}(S)$ as the group of linear combinations $\sum_V m_V 1\!\!1_V$, where V runs over the subsets of S, $m_V \in \mathbb{Z}$, and $1\!\!1_V$ is the constant 1 on V and 0 in the complement of V. We could even select V to be the singletons $\{s\}$, with $s \in S$, if we wanted.

How do we make $\mathcal{C}$ into a functor? For $f : S \to T$ a map of finite sets, we have to decide what $\mathcal{C}(f)$ does; and for this it is enough to decide what function $T \to \mathbb{Z}$

$$\mathcal{C}(f)(1\!\!1_V)$$

should be, for every subset $V \subset S$; and for this, we have to decide the value of

$$\mathcal{C}(f)(1\!\!1_V)(t)$$

for $t \in T$. Here is the definition:

$$\mathcal{C}(f)(1\!\!1_V)(t) = \#(f^{-1}(t) \cap V)$$

where # denotes 'number of elements'. Exercise: this makes $\mathcal{C}$ into a functor.

This trivial observation is the source of equally trivial, but rather interesting properties of the counting function. Note that $\mathcal{C}(\{p\}) = \mathbb{Z}$, and for the constant map $\kappa : S \to \{p\}$,

$$\mathcal{C}(\kappa)(1\!\!1_S) = \#S \quad .$$

So if S_1, S_2 are two subsets of S and $S = S_1 \cup S_2$, thinking about the covariance for

$$S_1 \amalg S_2 \to S = S_1 \cup S_2 \to \{p\}$$

tells us that

$$\#(S_1 \cup S_2) = \#S_1 + \#S_2 - \#(S_1 \cap S_2) \quad ;$$

and, more generally, the 'inclusion-exclusion' counting principle follows.

A much more remarkable observation is that the *topological Euler characteristic* satisfies the same properties. If S admits a structure of CW complex, define $\chi(S)$ to be the number of vertices, minus the number of edges, plus the number of faces, ... Then whenever S_1, S_2, S all admit such a structure one verifies immediately that

$$\chi(S_1 \cup S_2) = \chi(S_1) + \chi(S_2) - \chi(S_1 \cap S_2) \quad ;$$

and, more generally, an inclusion-exclusion principle for χ holds. So we could think of the Euler characteristic as a 'counting' function.

The main character in these lectures will be 'the next step' in this philosophy: the *Chern-Schwartz-MacPherson* class of a variety V, $c_{\mathrm{SM}}(V)$, will be an even fancier analog of 'counting', in the sense that it will satisfy the same 'inclusion-exclusion' principle. In fact, the Euler characteristic will be part of the information carried by the CSM class: for V a compact complex algebraic variety, $\chi(V)$ will be the degree $\int c_{\mathrm{SM}}(V)$ of $c_{\mathrm{SM}}(V)$, that is, the degree of the zero-dimensional part of $c_{\mathrm{SM}}(V)$. The class $c_{\mathrm{SM}}(V)$ will live in a homology theory for V.

My emphasis will be: how do we *concretely* compute such classes?

But maybe the first question should be: what does 'computing' mean?

1.2. **Computer demonstration.** Objection: we have not defined $c_{\rm SM}(V)$ yet, so it is unfair to ask how to compute it. This is correct. However, I have defined the topological Euler characteristic, so it is fair to ask: can we compute it now? Isn't the definition itself, as 'vertices minus sides plus faces…', already a 'computation'?

That depends. How would one use this in practice to compute the Euler characteristic of the subscheme of $\mathbb{P}^3$ given by the ideal

$$(x^2+y^2+z^2, xy-zw) \quad ?$$

The point is that what it means to 'compute' something strictly depends on what information one starts from. Of course if I start from a description of V from which a triangulation is obtained easily, then the Euler characteristic can be computed just as easily. As an algebraic geometer, however, I may have to be able to start off from the raw information of a scheme; for example, from a defining homogeneous ideal in projective space. And then? how do I 'compute' a CW-complex realization of the support of a scheme starting from its ideal? In this sense, $\chi(V) = \#\text{vertices} - \#\text{edges} + \dots$ is not a 'computation': if I already knew so much about V as to be able to count vertices, edges, etc. then I would not gain much insight about V by applying this formula.

By contrast, *here* is what a computation is:

```
themis{aluffi}1: Macaulay2
Macaulay 2, version 0.9
--Copyright 1993-2001, D. R. Grayson and M. E. Stillman
--Singular-Factory 1.3b, copyright 1993-2001, G.-M. Greuel, et al.
--Singular-Libfac 0.3.2, copyright 1996-2001, M. Messollen

i1 : load "CSM.m2"
--loaded CSM.m2

i2 : QQ[x,y,z,w];

i3 : time CSM ideal(x^2+y^2+z^2,x*y-z*w)
                                   3     2
Chern-Schwartz-MacPherson class : H  + 4H
     -- used 49.73 seconds
```

this tells me that the Chern-Schwartz-MacPherson class of that scheme is $4H^2 + H^3 = 4[\mathbb{P}^1] + [\mathbb{P}^0]$ (once it is pushed forward into projective space); hence its Euler characteristic is 1.

I will have accomplished my goal in these lectures if I manage to explain how this computation is performed. As a preview of the philosophy behind the whole approach, we will 'divide and conquer': split the information in the Chern-Schwartz-MacPherson class into the sum of an 'easy' term (this will be what I will call the 'Chern-Fulton' class), and an 'interesting' one (usually called 'Milnor class') accounting specifically for the singularities of the scheme.

'Computing' the Milnor class will be the most substantial part of the work. To give an idea of how difficult it may be, here is a rather loose question:

Is there a natural scheme structure on the singularities of a given variety V, which determines the Milnor class of V?

To my knowledge, this is completely open! But it is understood rather well for hypersurfaces of nonsingular varieties, so that will be my focus in most of my lectures.

1.3. **Chern-Schwartz-MacPherson classes: definition.** Back to our 'counting' analogy. For V a variety, let $\mathcal{C}(V)$ denote the abelian group of finite linear combinations $\sum m_W 1\!\!1_W$, where W are (closed) subvarieties of V, $m_W \in \mathbb{Z}$, and $1\!\!1_W$ denotes the function that is 1 along W, and 0 in the complement of W. Elements of $\mathcal{C}(V)$ are called 'constructible functions' on V. How to make $\mathcal{C}$ into a functor?

For $f : V_1 \to V_2$ a *proper* function (so that the image of a closed subvariety is a closed subvariety), it suffices to define $\mathcal{C}(f)(1\!\!1_W)$ for a subvariety W of V_1; so we have to prescribe $\mathcal{C}(f)(1\!\!1_W)(p)$ for $p \in V_2$. We set

$$\mathcal{C}(f)(1\!\!1_W)(p) = \chi(f^{-1}(p) \cap W) \quad ,$$

in complete analogy with the counting case. Exercise: this makes $\mathcal{C}$ into a functor.

Now observe that there are *other* functors from the category of varieties, proper maps to abelian groups. I will denote by $\mathcal{A}$ the *Chow group* functor. As a quick reminder, $\mathcal{A}(V)$ can be obtained from $\mathcal{C}(V)$ by setting to zero $\sum m_W 1\!\!1_W$ if there is a subvariety U of V and a rational function φ on U such that the divisor of φ equals $\sum m_W[W]$. The equivalence corresponding to the subgroup generated by these constructible functions is called 'rational equivalence'; $\mathcal{A}(V)$ is the abelian group of 'cycles modulo rational equivalence'. It is a functor for proper maps under a seemingly less interesting prescription: for $f : V_1 \to V_2$ proper, simply set $\mathcal{A}(f)([1\!\!1_W]) = d[1\!\!1_{f(W)}]$, where d is the degree of $f|_W$. The Chow group $\mathcal{A}(V)$ should be thought of as a 'homology'; indeed, there is a natural transformation $\mathcal{A} \to H_*$; in fact, $\mathcal{A}(V) = H_*(V)$ in many interesting case, e.g., $V = \mathbb{P}^n$.

By construction there is a map $\mathcal{C}(V) \to \mathcal{A}(V)$; but the *functors* $\mathcal{C}$ and $\mathcal{A}$ do not have so much to do with each other—this is easily seen *not* to be a natural transformation.

As a side remark, note however that even this naive recipe does define a natural transformation on the associated graded functors $G\mathcal{C} \rightsquigarrow G\mathcal{A}$ (where the G is taken with respect to the evident filtration by dimension); the objection is that this does not lift to a natural transformation $\mathcal{C} \rightsquigarrow \mathcal{A}$ in the most obvious way.

Does it lift at all? If it does, does it lift in some particularly interesting way? Let us assume that there is a lift, that is, a homomorphism $c_* : \mathcal{C}(V) \to \mathcal{A}(V)$ for all V, satisfying covariance. What can we say *a priori* about it?

Just as we did when we were playing with finite sets, consider the constant map $\kappa : V \to \{p\}$. The covariance diagram would then say that

$$\int c_*(1\!\!1_V) = \chi(V) \quad .$$

So whatever $c_*(1\!\!1_V)$ is, its degree must be the Euler characteristic of V. This should sound reminiscent of something... if for example V is nonsingular, what class canonically defined on V has the property that its degree is $\chi(V)$?

Answer: $c(TV) \cap [V]$ —in words, the 'total homology Chern class of the tangent bundle of V'. By one of the many descriptions of Chern classes, $\int c(TV) \cap [V]$ measures the number of zeros of a tangent vector field on V, counted with multiplicities; this is $\chi(V)$, by the Poincaré-Hopf theorem.

So we could make an educated guess: maybe a natural transformation c_* does exist, with the further amazing property that $c_*(1\!\!1_V)$ equals $c(TV) \cap [V]$ whenever V is nonsingular.

This was conjectured by Pierre Deligne and Alexandre Grothendieck, and an explicit construction of c_* was given by Robert MacPherson ([Mac74]).

Definition. $c_{\mathrm{SM}}(V) := c_*(1\!\!1_V)$ is the 'Chern-Schwartz-MacPherson class' of V.

1.4. **Other classes: quick tour.** Thus, Chern-Schwartz-MacPherson classes are 'characteristic classes for singular varieties', in the sense that they are defined for all varieties and agree with the ordinary characteristic classes for nonsingular ones.

The definition of Chern-Schwartz-MacPherson classes I just gave is only partly useful for computations, and does not hint at the subtleties of MacPherson's construction. Neither does it say anything about the subtleties of the alternative construction given by Marie-Hélène Schwartz, which in fact predated MacPherson's contribution (but to my knowledge does not address the functorial set-up); see [BS81] and [Bra00].

MacPherson's construction is obtained by taking linear combinations of another notion of 'characteristic class' for singular varieties, also introduced by MacPherson, and usually named 'Chern-Mather class'.

To define these, assume the given variety V is embedded in a nonsingular variety M. We can map every *smooth* $v \in V$ to its tangent space T_vV, seen as a subspace of T_vM; this gives a rational map $V \dashrightarrow Grass(\dim V, TM)$. The closure $\widetilde{V}$ of the image of this map is called the *Nash blow-up* of V. It comes equipped with a projection ν to V, and with a tautological bundle $\mathcal{T}$ inherited from the Grassmann bundle. Since $\mathcal{T}$ 'agrees with' TV where the latter is defined, it seems a very sensible idea to define a characteristic class by

$$c_{\mathrm{Ma}}(V) = \nu_*(c(\mathcal{T}) \cap [\widetilde{V}]) \quad .$$

(Exercise: this is independent of the ambient variety M.) This is the Chern-Mather class. It is clear that if V is nonsingular to begin with, again $c_{\mathrm{Ma}}(V) = c(TV) \cap [V]$. Still, $c_{\mathrm{Ma}}(V) \neq c_{\mathrm{SM}}(V)$ in general. MacPherson's natural transformation can be defined in terms of Chern-Mather classes; we'll come back to this later.

There are other sensible ways to define 'characteristic classes' for singular varieties—in fact, for arbitrary schemes. A seemingly very distant approach leads to classes known as *(Chern-)Fulton* and *(Chern-)Fulton-Johnson* classes. Both of these are defined as $c(TM) \cap S(V, M)$, where again we are embedding V in a nonsingular ambient M, and $S(V, M)$ is a class capturing information about the embedding.

—For Fulton classes, $S(V, M)$ is the *Segre class* of V in M. These will be very important in what follows, so I'll talk about them separately.

—For Fulton-Johnson classes, $S(V, M)$ is the Segre class of the *conormal sheaf* of V in M.

If $\mathcal{I}$ is the ideal of V in M, the conormal sheaf of V in M is the coherent sheaf $\mathcal{I}/\mathcal{I}^2$; that is, $\mathcal{I}$ restricted to V (I mean: tensored by $\mathcal{O}_V = \mathcal{O}_M/\mathcal{I}$). To think about the Segre class of a coherent sheaf $\mathcal{F}$ on V, consider the corresponding 'linear fiber space' $\mathrm{Proj}(\mathrm{Sym}\mathcal{F}) \xrightarrow{p} V$; this comes with an invertible sheaf $\mathcal{O}(1)$, and we can set

$$s(\mathcal{F}) = p_* c(\mathcal{O}(-1))^{-1} \cap [\mathrm{Proj}(\mathrm{Sym}\mathcal{F})] \quad .$$

So Fulton-Johnson classes capture, up to restricting to V and standard intersection-theoretic manouvers, the Symmetric algebra of $\mathcal{I}$.

What do Fulton classes capture? *Exactly the same kind of information,* but for the *Rees* algebra rather than the *Symmetric* algebra. This is not a minor difference in general, but it is completely invisible if $\mathcal{I}$ is (locally) a complete intersection.

Important example. If V is a hypersurface in a nonsingular variety M (or more generally a local complete intersection) then both Fulton and Fulton-Johnson classes yield $c(TM)c(N_MV)^{-1} \cap [V]$. If V is nonsingular, this is automatically $c(TV) \cap [V]$.

The main point here is that, at least when V is a local complete intersection, these classes are 'easier' than the functorial Chern-Schwartz-MacPherson classes.

So you may hit upon the idea of trying to relate the two.

1.5. **Preliminaries: Segre classes.** Segre classes will show up often (they already have), and it will be important to have a certain technical mastery of them. If V is a proper closed subscheme of M, the *Segre class* of V in M is the element of $\mathcal{A}V$ (the Chow group of V characterized by the following two requirements:

- if $V \subset M$ is a regular embedding, then $s(V,M) = c(N_V M)^{-1} \cap [V]$;
- if $\pi : M' \to M$ is a proper birational map, and $p : \pi^{-1}(V) \to V$ is the restriction of π, then $p_* s(\pi^{-1}V, M') = s(V,M)$.

These are enough to define $s(V,M)$ in any circumstance, since by the second item we can replace M with the blow-up of M along V, and V by the exceptional divisor in the blow-up; this is regularly embedded, so the first item clinches the class.

In fact this argument reduces the computation of a Segre class $s(V,M)$ to the computation when V is a *hypersurface* of M, that is, it is locally given by one equation. The shorthand for the Segre class is then

$$s(V,M) = \frac{[V]}{1+V} \quad ,$$

by which one means the sum $(1 - V + V^2 - V^3 + \dots) \cap [V] = V - V^2 + V^3 - \dots$.

Often one can use this in reverse.

Example. Consider the second Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. The hyperplane H in $\mathbb{P}^5$ restricts to twice the hyperplane h in $\mathbb{P}^2$. Standard sequences lead to the following computation of the Segre class of $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$:

$$s(v_2(\mathbb{P}^2), \mathbb{P}^5) = \frac{(1+h)^3}{(1+2h)^6} \cap [\mathbb{P}^2] = [\mathbb{P}^2] - 9[\mathbb{P}^1] + 51[\mathbb{P}^0] \quad .$$

Now let $\widetilde{\mathbb{P}}^5$ be the blow-up of $\mathbb{P}^5$ along the Veronese surface, and let E be the exceptional divisor. The birational invariance of Segre classes then says that $s(E, \widetilde{\mathbb{P}}^5)$ must push-forward to $s(v_2(\mathbb{P}^2), \mathbb{P}^5)$; that is,

$$E - E^2 + E^3 - E^4 + E^5 \mapsto [\mathbb{P}^2] - 9[\mathbb{P}^1] + 51[\mathbb{P}^0] \quad .$$

This is enough information to determine the Chow *ring* of the blow-up. Interpreting $\mathbb{P}^5$ as the $\mathbb{P}^5$ of conics, then a standard game in enumerative geometry would compute the number of *conics tangent to five conics in general position* as

$$(6H - 2E)^5 = 7776 - 2880 \cdot 4 + 480 \cdot 2 \cdot 9 - 32 \cdot 51 = 3264 \quad .$$

(Exercise: make sense out of this!) In fact, Segre classes provide a systematic framework for enumerative geometry computation; but this is of relatively little utility, as Segre classes are in general extremely hard to compute.

Why? Because blow-ups are hard to compute. If $\mathcal{I}$ is the ideal of V in M, 'computing' the blow-up of M along V amounts to realizing

$$\text{Proj}(\text{Rees}\mathcal{I}) = \text{Proj}(\mathcal{O} \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \dots) \quad :$$

this isn't easy.

One way to do this in practice is to see V as the vanishing of a section of a vector bundle on M: $s : M \to E$; then it is not hard to show that the blow-up of M along V is the closure of the image of M in $\mathbb{P}(E)$ by the induced rational section $\overline{s} : M \dashrightarrow \mathbb{P}(E)$. Then the line bundle of the exceptional divisor is the restriction of the tautological $\mathcal{O}(-1)$ from $\mathbb{P}(E)$. We will use this remark later on.

For applications of Segre classes to enumerative geometry, see [Ful84], Chapter 9 (and elsewhere).

2. Lecture II

2.1. **(Chern-)Fulton classes vs. (Chern-)Fulton-Johnson classes.** I would like to look at a simple example to explore the difference between the Chern-Fulton and Chern-Fulton-Johnson business.

Consider the planar triple point S with ideal $I = (x^2, xy, y^2)$ in the affine plane $\mathbb{A}^2$. For such an object we can be most explicit.

First of all, I is dominated by $k[x,y]^{\oplus 3}$:

$$k[x,y]\langle s,t,u\rangle \to I \to 0 \quad ,$$

sending $s \mapsto x^2$, $t \mapsto xy$, $u \mapsto y^2$. Tensoring by $k[x,y]/I$:

$$(k[x,y]/I)\langle s,t,u\rangle \to I/I^2 \to 0 \quad ,$$

with the same prescriptions for the map. The kernel is computed to be $(xt-ys, xu-yt)$. Thus $\mathrm{Proj}(\mathrm{Sym} I/I^2)$ is defined by the ideal $(xt-ys, xu-yt)$ in the product $S \times \mathbb{P}^2$. Now if I believe Macaulay2, taking primary decomposition here gives me $(x,y)\cap$ an embedded component. That is, $\mathrm{Proj}(\mathrm{Sym} I/I^2)$ is the *reduced* $\mathbb{P}^2$ inside the (nonreduced) $S \times \mathbb{P}^2$. The Fulton-Johnson class is then 1 times the class of the point supporting S.

What about Chern-Fulton? To obtain the Segre class of S in $\mathbb{A}^2$, blow-up $\mathbb{A}^2$ along S. We can fit the blow-up in $\mathbb{A}^2 \times \mathbb{P}^2$, and a bit of patience gives its ideal:

$$(xt-ys, xu-yt, su-t^2) \quad .$$

Note the extra generator! That is the difference between the Rees algebra of I, computed here, and the Symmetric algebra, which is what comes up in the Fulton-Johnson computation. The exceptional divisor of the blow-up is the intersection of the blow-up with $S \times \mathbb{P}^2$ (the same $S \times \mathbb{P}^2$ as before!), and taking the primary decomposition again shows it is $S\times$a conic in $\mathbb{P}^2$.

It's completely different: it was a $\mathbb{P}^2$ before, it is a curve now. Its degree is $2\times 2 = 4$, so the Chern-Fulton class is 4 times the class of the point.

Remark. Taking away the extra generator xy leaves the multiple point (x^2, y^2), a complete intersection. It is a fact that this does not change the Chern-Fulton class (because xy is integral over this ideal); on the other hand, Chern-Fulton and Chern-Fulton-Johnson agree for complete intersections, so they must both give 4 times the class of the point in this case. Exercise: verify this explicitly.

One moral to be learned from such examples is that classes such as Chern-Fulton or Chern-Fulton-Johnson are extremely sensitive to the *scheme structure.* This is important for me, since I am aiming to develop a tool that can do computations starting from an arbitrary ideal. On the other hand, no scheme considerations have entered the discussion of Chern-Schwartz-MacPherson or Chern-Mather classes.

To cover such cases, I will simply declare that the Chern-Schwartz-MacPherson class of a possibly non-reduced scheme is the class of its *support.* This turns out to be computationally convenient, but it is not arising from any 'functoriality' considerations.

2.2. **Segre classes again: inclusion-exclusion.** In view of the considerations that took us here, we may now want to ask: are classes such as Chern-Fulton or Chern-Fulton-Johnson classes also a fancy form of 'counting'? That is, do they satisfy 'inclusion-exclusion'?

Since the difference between these classes and Segre classes is a common factor ($c(TM)\cap$), the question for Fulton classes is equivalent to: do Segre classes satisfy inclusion-exclusion?

Example. Take V = the union of two distinct lines L_1, L_2 in the projective plane. Then $s(V,\mathbb{P}^2) = \frac{1}{(1+2H)} \cap [V] = [L_1] + [L_2] - 4[pt]$, with hopefully evident notations. On the other hand, $s(L_i,\mathbb{P}^2) = [L_i] - [pt]$. Thus

$$s(L_1,\mathbb{P}^2) + s(L_2,\mathbb{P}^2) - s(V,\mathbb{P}^2) = 2[pt] \neq s(L_1 \cap L_2,\mathbb{P}^2) \quad .$$

In other words, inclusion-exclusion fails miserably for Segre classes, on the very first example one may try.

Is there a way out? Actually yes, [Alu03c]; it is completely trivial, and I didn't notice it for many years. The remark is that variations on the definition of Segre classes do tend to satisfy inclusion-exclusion.

This follows immediately from 8th grade algebra. The simplest case, which is also the only one I need in what follows, goes like this:

$$\begin{aligned}&\frac{R_1+E}{1+R_1+E}+\frac{R_2+E}{1+R_2+E}-\frac{R_1+R_2+E}{1+R_1+R_2+E}-\frac{E}{1+E}\\ &=\frac{R_1R_2(2+R_1+R_2+2E)}{(1+R_1+E)(1+R_2+E)(1+R_1+R_2+E)(1+E)}\end{aligned}$$

How can this possibly say something useful? Assume X_1, X_2 are effective Cartier divisors in an ambient scheme M, and $Y = X_1 \cap X_2$ (scheme-theoretically, of course). Blow-up M along Y, and let E be the exceptional divisor. Then, by our definition of Segre classes, $s(Y,M) = p_*\frac{[E]}{1+E}$, where p is the blow-up map. On the other hand, p^*X_i will consist of E and of a proper transform R_i ('residual'). Note that $R_1R_2 = 0$, since the proper transforms do not meet (this is why I am stressing that the intersection must be 'scheme-theoretic'). Further, by the projection formula, $p_*\frac{[R_i+E]}{1+R_i+E} = \frac{[X_i]}{1+X_i} = s(X_i,M)$. That is, 8th grade algebra says

$$s(Y,M) = s(X_1,M) + s(X_2,M) - p_*\frac{[R_1]+[R_2]+[E]}{1+R_1+R_2+E} \quad :$$

something very close to inclusion-exclusion. The funny term

$$p_*\frac{[R_1]+[R_2]+[E]}{1+R_1+R_2+E}$$

is where one would expect 'the class of the union'; this would instead equal

$$p_*\frac{[R_1]+[R_2]+2[E]}{1+R_1+R_2+2E} \quad .$$

My point of view is that inclusion-exclusion *does* works for Segre classes once one takes care to correct them adequately for (something like) multiple contributions of subsets. The case I just illustrated (which generalizes nicely to arbitrarily many X_i of any kind...) is one way to correct the classes. We will run into another one later on.

2.3. **Remark on notations.** Before massaging the formula we just obtained into something yielding any readable information, I must introduce two simple notational devices.

These concern rational equivalence classes in an ambient scheme M, which I will now assume to be pure-dimensional. The notations are best written by indexing the classes by *co*dimension: so I will write a class $A \in \mathcal{A}N$ as $A = \sum_{i\ge 0} a^i$, denoting by a^i the piece of A of codimension i. Then I will let

$$A^\vee = \sum_{i\ge 0} (-1)^i a^i \quad ,$$

and I will sloppily call this the 'dual' of A. Note that the notation hides the ambient space, which may sometime lead to confusion. The rationale for the notation is simple: if E is a vector bundle on M, or more generally a class in the K-theory of vector bundles, then with this notation one simply has

$$(c(E)\cap A)^\vee = c(E^\vee)\cap A^\vee \quad .$$

The second piece of notation is similar, but a bit more interesting. Let $\mathcal{L}$ be a line bundle on M. I will let

$$A\otimes\mathcal{L} = \sum_{i\ge 0} \frac{a^i}{c(\mathcal{L})^i} \quad .$$

This also hides the ambient M; when necessary, I subscript the tensor: $\otimes_M$. But in these lectures all tensors will be in the ambient variety.

The rationale for the second notation is similar to the first: if E is a class in K-theory *and of rank 0,* then one can check that

$$(c(E)\cap A)\otimes\mathcal{L} = c(E\otimes\mathcal{L})\cap(A\otimes\mathcal{L}) \quad .$$

Watch out for the 'rank 0' part!

Of course the notation $A\otimes\mathcal{L}$ suggests an 'action', and this is easy to verify: if $\mathcal{M}$ is another line bundle, one checks that

$$(A\otimes\mathcal{L})\otimes\mathcal{M} = A\otimes(\mathcal{L}\otimes\mathcal{M}) \quad .$$

Also, $(A\otimes\mathcal{L})^\vee = A^\vee\otimes\mathcal{L}^\vee$.

All these observations are simple algebra of summations and binomial coefficients; they are useful insofar as they compress complicated formulas into simpler ones by avoiding $\sum$'s and elementary combinatorics.

Here is an example of such manipulations, which I will need in a moment.

$$\frac{-E}{1+X-E} = \left(\frac{E}{1+E-X}\right)^\vee = (E\otimes\mathcal{O}(E-X))^\vee = ((E\otimes\mathcal{O}(E))\otimes\mathcal{O}(-X))^\vee$$
$$= \left(\frac{E}{1+E}\otimes\mathcal{O}(-X)\right)^\vee = \left(\frac{E}{1+E}\right)^\vee\otimes\mathcal{O}(X) \quad .$$

2.4. **Key example.** Let's go back to the 'inclusion-exclusion' formula we obtained a moment ago:

$$s(Y,M) = s(X_1,M) + s(X_2,M) - p_* \frac{[R_1+R_2+E]}{1+R_1+R_2+E} \quad .$$

Here we had $Y = X_1 \cap X_2$, where X_1, X_2 are hypersurfaces; and E is the exceptional divisor of the blow-up along Y, R_i the proper transform of X_i. Assuming that the ambient M is nonsingular, and capping through by $c(TM)$, gives

$$c_{\mathrm{F}}(Y) = c_{\mathrm{F}}(X_1) + c_{\mathrm{F}}(X_2) - c(TM) \cap p_* \frac{[R_1+R_2+E]}{1+R_1+R_2+E} \quad ,$$

where c_{F} denotes Chern-Fulton class.

Now assume X_1, X_2, and Y are nonsingular. Then $c_{\mathrm{F}} = c_{\mathrm{SM}}$, since both equal the classes of the tangent bundle. That is:

$$c_{\mathrm{SM}}(Y) = c_{\mathrm{SM}}(X_1) + c_{\mathrm{SM}}(X_2) - c(TM) \cap p_* \frac{[R_1+R_2+E]}{1+R_1+R_2+E}$$

in this extremely special case. On the other hand, c_{SM} satisfies inclusion-exclusion on the nose:

$$c_{\mathrm{SM}}(Y) = c_{\mathrm{SM}}(X_1) + c_{\mathrm{SM}}(X_2) - c_{\mathrm{SM}}(X_1 \cup X_2) \quad .$$

The conclusion is that if X_1, X_2 are transversal nonsingular hypersurfaces in a nonsingular ambient variety M, and $X = X_1 \cup X_2$, then

$$c_{\mathrm{SM}}(X) = c(TM) \cap p_* \frac{[R_1+R_2+E]}{1+R_1+R_2+E} \quad .$$

Let's work on the funny piece $\frac{[R_1+R_2+E]}{1+R_1+R_2+E}$ in this formula. First, $R_i + E$ stands for (the pull-back of) X_i; so we can rewrite this as $\frac{[R_1+R_2+E]}{1+R_1+R_2+E} = \frac{[X-E]}{1+X-E}$. Next, another bit of 8th grade algebra:

$$\frac{[X-E]}{1+X-E} = \frac{X}{1+X} + \frac{1}{1+X} \cdot \frac{-E}{1+X-E}$$

Using the example from the previous section:

$$\frac{-E}{1+X-E} = \left(\frac{E}{1+E}\right)^{\vee} \otimes \mathcal{O}(X)$$

Put everything together and use the projection formula:

$$\begin{aligned} c_{\mathrm{SM}}(X) &= c(TM) \cap p_* \left(\frac{X}{1+X} + \frac{1}{1+X} \cdot \left(\left(\frac{E}{1+E}\right)^{\vee} \otimes \mathcal{O}(X) \right) \right) \\ &= c(TM) \cap \left(s(X,M) + \frac{1}{1+X} \cdot (s(Y,M)^{\vee} \otimes \mathcal{O}(X)) \right) \quad . \end{aligned}$$

Now we are back in M: everything relating to the blow-up has been absorbed into terms in the original ambient space. We may also note that $c(TM) \cap s(X,M) = c_{\mathrm{F}}(X)$, so that what we are really saying is that

$$c_{\mathrm{SM}}(X) = c_{\mathrm{F}}(X) + c(TM) \cap \left(\frac{1}{c(\mathcal{O}(X))} \cdot (s(Y,M)^{\vee} \otimes \mathcal{O}(X)) \right) \quad .$$

We have proved that this holds if X is the union of two nonsingular hypersurfaces in a nonsingular variety M, meeting transversally along Y.

2.5. **Main theorem.** That is a reasonably pretty formula, but how do we interpret it in more 'intrinsic' terms? 'What is' $Y = X_1 \cap X_2$, in terms of $X = X_1 \cup X_2$, when X_1 and X_2 are nonsingular and transversal?

Working locally, suppose (F_i) is the ideal of X_i. Then X has ideal (F_1F_2). The 'singularity subscheme' of a hypersurface with ideal (F) is the scheme defined by the ideal (F, dF), where dF is shorthand for the partial derivatives of F. It is clear that this scheme is supported on the singular *locus* of X; the specified ideal gives it a scheme structure (it is easy to see that this structure patches on affine overlaps).

For our X, this ideal would be

$$(F_1F_2, F_1dF_2 + F_2dF_1) \quad .$$

It is clear that this is supported on $Y = X_1 \cap X_2$, since X_1 and X_2 are nonsingular. But in fact we are asking X_1 and X_2 to be *transversal:* thus dF_1 and dF_2 are independent at every point of Y. Hence the ideal of the singularity subscheme is $(F_1F_2, F_1, F_2) = (F_1, F_2)$: that is, precisely the ideal of $X_1 \cap X_2 = Y$. In other words, Y *is the singularity subscheme of* X in this case.

Therefore, we can rephrase the formula we have obtained above: if X is a (very special) hypersurface in a nonsingular ambient variety M, and Y is the singularity subscheme of X, then

$$c_{\mathrm{SM}}(X) = c_{\mathrm{F}}(X) + c(TM) \cap \left(\frac{1}{c(\mathcal{O}(X)} \cdot (s(Y,M)^\vee \otimes \mathcal{O}(X)) \right) \quad .$$

Theorem 2.1. *This formula holds for* every *hypersurface in a nonsingular variety.*

This is the main result of the lectures: everything else I can say is simply a variation or restatement or application of this theorem.

Many proofs are known of this statement, and I will review some of them in the next lecture. The first proof, going back to 1994 ([Alu94]), proved this formula over $\mathbb{C}$ and in a weak, 'numerical' sense. The formula is in fact true in the Chow group of X, and over any algebraically closed field of characteristic 0 (the theory of CSM classes extends to this context, by work of Gary Kennedy, [Ken90]).

The 1994 proof is instructive in the sense that it clarifies what kind of information the difference $c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X)$ carries. Read the term of dimension 0:

$$\int c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X) = \chi(X) - \chi(X_g) \quad ,$$

where $\int c_{\mathrm{SM}}(X) = \chi(X)$ as we have seen, and X_g stands for a general, nonsingular hypersurface in the same rational equivalence class as X (should there be one). It is well-known (see for example [Ful84], p.245-246) that if the singularities of X are isolated then this difference is (up to sign) the sum of the Milnor numbers of X. Adam Parusiǹski ([Par88]) defines an invariant of (not necessarily isolated) hypersurface singularities by taking it to be this difference in general. On the other hand, I had obtained a formula for Parusiǹski's number in terms of the Segre class of the singularity subscheme of X ([Alu95]). A bit of work using the funny $\otimes$ notations shows then that the Theorem holds for the dimension 0 component of the class. The numerical form can be obtained by reasoning in terms of general hyperplane sections.

Because of this history, it makes sense to call the difference $c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X)$ the 'Milnor class' of X (up to sign, depending on the author). I do not think Milnor has been informed…

3. Lecture III

3.1. **Main theorem: Proof I.** I would like to review some of the approaches developed in order to prove the formula for the Milnor class of a hypersurface X in a nonsingular variety M:

$$c_{\rm SM}(X) - c_{\rm F}(X) = c(TM)\cap\left(c(\mathcal{O}(X))^{-1}\cap(s(Y,M)^\vee\otimes\mathcal{O}(X))\right) \quad .$$

The first one ([Alu99a]) is rather technical, but has the advantage of working over an arbitrary algebraically closed field of characteristic 0, and for arbitrary hypersurfaces (allowing multiplicities for the components).

The idea is the following. It would of course be enough to show that the class defined by

$$c_?(X) = c_{\rm F}(X) + c(TM)\cap\left(c(\mathcal{O}(X))^{-1}\cap(s(Y,M)^\vee\otimes\mathcal{O}(X))\right)$$

is covariant in the same sense as the CSM class. This would be very nice, but I have never quite managed to do it directly.

Using resolution of singularity, however, it is enough to prove a weak form of covariance, and this can be done. Specifically:

- if X is a divisor with normal crossings and (possibly multiple) nonsingular components, then $c_?(X) = c_{\rm SM}(X)$;
- if $\pi : \widetilde{M} \to M$ is a blow-up along a nonsingular subvariety of the singular locus of X, then $\pi_* c_?(\pi^{-1}X) = c_?(X) + \pi_* c_{\rm SM}(\pi^{-1}X) - c_{\rm SM}(X)$.

Indeed, by resolution of singularities one can reduce to the normal crossing case after a number of blow-ups; then $c_?$ and $c_{\rm SM}$ agree there, by the first item; and as they map down each step of the resolution, $c_?$ and $c_{\rm SM}$ change in the same way, so they must agree for X. Note that this forces us to work with nonreduced objects!

The proofs of the two listed properties are rather technical. The first one in the reduced case boils down to the following: if $X = X_1\cup\cdots\cup X_r$ is a reduced divisor with normal crossings and nonsingular components, with singularity subscheme Y, then

$$s(Y,M) = \left(\left(1 - \frac{c(\mathcal{L}^\vee)}{c(\mathcal{L}_1^\vee)\cdots c(\mathcal{L}_r^\vee)}\right)\cap[M]\right)\otimes\mathcal{L}$$

where $\mathcal{L} = \mathcal{O}(X)$ and $\mathcal{L}_i = \mathcal{O}(X_i)$. Proof: induction, and properties of $\otimes$.

The other item is more interesting. If $\widetilde{M}$ is obtained by blowing up M along a nonsingular Z of codimension d, one is reduced to showing that

$$\pi_* c_?(\pi^{-1}(X)) = c_?(X) + (d-1)c(TZ)\cap[Z] \quad .$$

This should be much easier than it is! After a number of manipulations, one is led to transferring the question to within the bundle $\mathbb{P}(\pi^*\mathcal{P}^1_M\mathcal{L}\oplus\mathcal{P}^1_{\widetilde{M}}\mathcal{L})$, where $\mathcal{P}^1$ denotes 'principal parts'. I will return to these later on, so I won't say much about them here. Suffice it to say that there are two classes in this bundle, related to the blow-up of M along the singularity subscheme Y of X, and to the blow-up of $\widetilde{M}$ along the singularity subscheme of $\pi^{-1}(X)$. The statement is translated in a suitable triviality of the *difference* between these two classes.

Once one phrases the problem in this manner, one sees right away what is the most natural tool to attack it: it's the graph construction—maybe not surprisingly, since this was the key tool in the original paper by MacPherson. Here it must be applied to the graph of the 'differential' map $\pi^*\mathcal{P}^1_M\mathcal{L}\to\mathcal{P}^1_{\widetilde{M}}\mathcal{L}$. Lots and lots of technical details later, the needed relation is proved.

3.2. **Characteristic cycles.** The proof I just surveyed was obtained in 1995-6, but only found the glory of the printed page in 1999: the reason is simply that referees did not like it at all. They pointed out that a different viewpoint would probably yield a much shorter and more insightful proof.

That turned out to be correct. The key is to rephrase the whole question in terms of *characteristic cycles,* a translation of the constructible function framework that goes back to Claude Sabbah and yields a very powerful approach. The subject is discussed at length in Lecture 3 of Jörg Schürmann's lecture cycle.

Sabbah summarizes the situation very well, in the following quote from [Sab85]: *la théorie des classes de Chern de* [Mac74] *se ramène à une théorie de Chow sur T^*M, qui ne fait intervenir que des classes fondamentales.* The functor of constructible functions is replaced with a functor of *Lagrangian cycles* of T^*M (or its projectivization $\mathbb{P}(T^*M)$); then the key operations on constructible functions become more geometrically transparent, and this affords a general clarification of the theory. I can recommend [Ken90] for a good treatment of $c_{\rm SM}$ classes from this point of view.

I'll summarize the situation here. Let M be a nonsingular variety. If $V \subset M$ is nonsingular, then we have a sequence

$$0 \to T^*_V M \to T^*M|_V \to T^*V \to 0$$

where $T^*_V M$ denotes the *conormal bundle* (or 'space') of V; we view this as a subvariety of the total space of T^*M. If V is singular, do this on its nonsingular part, then close it up in T^*M to obtain its conormal space $T^*_V M$. Linear combinations of cycles $[T^*_V M]$ (these are the *classes fondamentales* in Sabbah's quote) form an abelian group $L(M)$ (L stands for 'Lagrangian').

Now go back to the *Nash blow-up* $\nu : \widetilde{V} \to V$, with tautological bundle $\mathcal{T}$. For each $p \in V$ we can define a number as follows:

$$\mathrm{Eu}_V(p) = \int c(\mathcal{T}) \cap s(\nu^{-1}(p), \widetilde{V}) \quad .$$

This is the 'local Euler obstruction', a constructible function originally defined (in a different way) by MacPherson in his paper. As it happens, these functions span $\mathcal{C}(M)$, so we may use them to define a homomorphism Ch from $\mathcal{C}(V)$ to $\mathcal{L}(V)$: require that $\mathrm{Ch}(\mathrm{Eu}_V) = (-1)^{\dim V}[T^*_V M]$. The cycle $\mathrm{Ch}(\varphi)$ corresponding to a constructible function φ is called its *characteristic cycle.* In particular, every subvariety V of M has a characteristic cycle $\mathrm{Ch}(1\!\!1_V)$: this is a certain combination of $T^*_V M$ and of conormal spaces to subvarieties of V, according to the singularities of V.

Now all the ingredients are there. The original definition given by MacPherson for the natural transformation c_* is a combination of Chern-Mather classes, with coefficients determined by local Euler obstructions. A relatively straightforward computation shows that Chern-Mather classes can be computed in terms of conormal spaces. The homomorphism $\mathcal{C} \rightsquigarrow L$ is concocted so as to be compatible with this set-up. All in all, we get an explicit expression for c_*:

$$c_*(\varphi) = (-1)^{\dim M - 1} c(TM) \cap \pi_* \left(c(\mathcal{O}(1))^{-1} \cap [\mathbb{P}\mathrm{Ch}(\varphi)] \right) \quad ,$$

where π is the projection $\mathbb{P}(T^*M) \to M$ ([PP01], p.67).

The operation on the right-hand-side may look somewhat unnatural, but is on the contrary the most direct way to deal with classes in a projective bundle; we may come to this later. I call the whole operation (maybe up to some sign) *casting the shadow* of $\mathrm{Ch}(\varphi)$. Thus, the Chern-Schwartz-MacPherson class of V is nothing but the shadow of its characteristic cycle.

3.3. **Main theorem: proof II.** In [PP01], Adam Parusiǹski and Piotr Pragacz give a proof (in fact, two proofs) of a theorem which implies the formula for the difference $c_{\rm SM}(X) - c_{\rm F}(X)$ for X a hypersurface of a nonsingular variety M, at least in the reduced case.

The proof is a clever interplay of certain constructible functions, which are very close to the geometry of the hypersurface, and characteristic cycles in the sense of the previous section. I'll have to pass in silence the story concerning the interesting constructible functions, but I want to expose the point of contact with [Alu99a].

The formula in [Alu99a] (writing $\mathcal{L} = \mathcal{O}(X)$):

$$c_{\rm SM}(X) = c_{\rm F}(X) + c(TM) \cap \left(c(\mathcal{L})^{-1} \cap (s(Y, M)^\vee \otimes \mathcal{L})\right)$$

writes the 'hard' part of $c_{\rm SM}$ in terms of the Segre class of the singularity subscheme Y of X. As we have seen previously, an equivalent formulation is in terms of the exceptional divisor E of the blow-up $\widetilde{M}$ of M along Y; in fact, tracing the different terms yields a rather pretty formula (Theorem I.3 in [Alu99a] works this out):

$$c_{\rm SM}(X) = c(TM) \cap \pi_*\left(\frac{[X' - E]}{1 + X' - E}\right) \quad ,$$

where π is the blow-up map, and I am writing X' for the inverse image $\pi^{-1}(X)$.

Now I mentioned in one of the first sections that there is an efficient way to realize a blow-up, when the center of the blow-up is the zero-scheme of a section of a vector bundle. This is our situation: Y can be viewed as the zero-scheme of a section of the bundle $\mathcal{P}^1_M\mathcal{L}$ which made its appearence above. This bundle fits a nice exact sequence:

$$0 \to \Omega^1_M \otimes \mathcal{L} \to \mathcal{P}^1_M\mathcal{L} \to \mathcal{L} \to 0 \quad ;$$

the section is obtained by combining the differential of a defining equation F for X (in the Ω^1 part) with F itself (in the cokernel). The ideal of this section is generated by F and dF, hence it gives Y.

By the general blow-up story, then, $\widetilde{M}$ lives in $\mathbb{P}(\mathcal{P}^1_M\mathcal{L})$. *Further,* the part over X lives in the subbundle $\mathbb{P}(\Omega^1_M \otimes \mathcal{L})$, since the cokernel part of the section is 0 along X. *Further still,* $\mathbb{P}(\Omega^1_M \otimes \mathcal{L}) = \mathbb{P}(T^*M)$ since tensoring by a line-bundle just changes the meaning of $\mathcal{O}(1)$.

Summarizing, we see that the cycle $[X' - E]$ lives naturally in $\mathbb{P}(T^*M)$, which is the home of the characteristic cycle of X. Once one unravels notations, the operation getting $c_{\rm SM}(X)$ from $[X' - E]$ turns out to match precisely the one getting c_* from the characteristic cycle.

In the end, and modulo another bit of 8th grade algebra, one sees that *the main formula is equivalent to the statement that $[X' - E]$ be the characteristic cycle of X:* if X is a (reduced) hypersurface in a nonsingular variety M, then $\mathrm{Ch}(1\!\!1_X) = [X' - E]$.

This framework offers a different way to prove the main formula. We have to verify that $\mathrm{Ch}^{-1}([X' - E])$ is the constant function 1 on X. Unraveling local Euler obstructions and using minimal general knowledge about Segre classes reduces the statement to the equality

$$\int \frac{s(\pi^{-1}(p), X' - E)}{1 + X' - E} = 1 \qquad \forall p \in X$$

This can be checked directly, by a multiplicity computation. The details are in [Alu00], a spin-off of [PP01].

3.4. **Differential forms with logarithmic poles.** Yet another viewpoint on the main formula comes from aiming to understand the class of the *complement* of X in M rather than the class of X itself: after all,

$$c_*(1\!\!1_{M\setminus X}) = c_*(1\!\!1_M) - c_*(1\!\!1_X) = (c(TM)\cap[M]) - c_{\rm SM}(X) \quad ,$$

so the information can be used to recover $c_{\rm SM}(X)$. One price to be paid here is that one gets $c_{\rm SM}(X)$ as a class in the ambient variety M rather than on X itself; this is common to other approaches, and there does not seem to be a way around it.

The observation is now that it is in fact *straightforward* to compute $c_*(1\!\!1_{M\setminus X})$: it turns out that this can be realized in a way somewhat similar to the computation of Chern-Mather classes. That is, there is a blow-up $\widetilde{M}$ of M and a vector bundle on $\widetilde{M}$ whose (honest) Chern classes push forward to $c_*(1\!\!1_{M\setminus X})$. Further, this can be done even if X is not a hypersurface!

For this, blow-up X and resolve singularities, so as to obtain a variety $\pi : \widetilde{M} \to M$ such that $X' = \pi^{-1}(X)$ is a divisor with normal crossing and nonsingular components. Denote by X'' the support of X'. Then there is an interesting *locally free* sheaf on $\widetilde{M}$, denoted $\Omega^1_{\widetilde{M}}(\log X'')$: over an open set U where X'' has ideal $(x_1 \cdot \dots \cdot x_r)$ (where $x_1, \dots, x_n$ are local parameters), sections of $\Omega^1_{\widetilde{M}}(\log X'')$ can be written

$$\alpha_1 \frac{dx_1}{x_1} + \dots + \alpha_r \frac{dx_r}{x_r} + \alpha_{r+1} dx_{r+1} + \dots + \alpha_n dx_n \quad :$$

that is, they consist of differential forms 'with logarithmic poles' along the components of X''.

Claim. $c_*(1\!\!1_{M\setminus X}) = \pi_*(c(\Omega^1_{\widetilde{M}}(\log X'')^\vee) \cap [\widetilde{M}])$.

This is surprisingly easy, actually: put $\Omega^1_{\widetilde{M}}(\log X'')$ in a sequence, then observe that the resulting class satisfies enough 'inclusion-exclusion' as to be forced to agree with the CSM class. Details are in [Alu99b]. The same remark was made by Mark Goresky and William Pardon (a small lemma in [GP02]).

This is another case in which we have 'computed' something, but we are essentially no wiser than before. A certain amount of detective work should be performed before we can extract the information packaged in the Claim. I will summarize in the next section what is involved in this work in the case of hypersurfaces. To my knowledge, no one has tried the same for higher codimension varieties, although the Claim given above works just as well.

I should add that it is not clear that the Claim may not be computationally useful as is: I am told that embedded resolution of singularities has in fact been implemented, and it should not be hard to compute the Chern class of an explicit vector bundle and push-forward. This is another tempting project, waiting for a willing soul to pursue it.

3.5. **Main theorem: proof III.** How can we use differentials with logarithmic poles to effectively compute $c_{\rm SM}(X)$ for a hypersurface? 'Effectively' means: without resolving singularities.

The question is how to identify a bundle *on* M, and some operation on this bundle that will yield the same classes as the projection of the $\Omega(\log)$ sheaf as seen above.

The operation that works is again something reminiscent of Chern-Mather classes of a variety V. Recall that these are defined as the projection of the classes of a tautological bundle $\mathcal{T}$ from the Nash blow-up $\widetilde{V} \overset{\nu}{\to} V$. This bundle agrees with $\nu^* TV$ when the latter is defined. In fact, the *co*tangent *sheaf* Ω^1_V is always defined, and by the same token must agree with $\mathcal{T}^\vee$ wherever ν is an isomorphism. We can go one step further: there is in fact an onto morphism

$$\nu^*\Omega_V \to \mathcal{T}^\vee \to 0 \quad ,$$

whose kernel is torsion on $\widetilde{V}$. Thus, we can think (up to duals) of the Chern-Mather operation as something done on the sheaf of differential forms of V to 'make it locally free modulo torsion'; and the Chern-Mather classes are obtained by mod-ing out the torsion and taking ordinary Chern classes of what is left.

That can be done for every coherent sheaf $\mathcal{F}$ on V. The resulting class is called the *Chern-Mather class* $c_{\rm Ma}(\mathcal{F})$, and has been studied by Marie-Hélène Schwartz a while back and Michał Kwieciński more recently ([Sch82] and [Kwi94]).

So the plan is to show that the push-forward of the classes of the $\Omega(\log)$ sheaf actually computes the Chern-Mather class of a sheaf that can be effectively described in M. This works out, but is as usual rather technical. The sheaf on M is very natural, given the data of a hypersurface X: I have already pointed out that X determines a section

$$0 \to \mathcal{O}_M \to \mathcal{P}^1_M\mathcal{L}$$

where $\mathcal{L} = \mathcal{O}(X)$; tensoring by $\mathcal{L}^\vee$ we have an injection

$$0 \to \mathcal{L}^\vee \to \mathcal{L}^\vee \otimes \mathcal{P}^1_M\mathcal{L} \quad ;$$

define $\overline{\Omega}_X$ to be the sheaf (on M!) obtained as the cokernel of this map.

Claim. $c_{\rm Ma}(\overline{\Omega}_X) = \pi_*(c(\Omega^1_{\widetilde{M}}(\log X'')) \cap [\widetilde{M}])$.

The obvious approach to proving this would be to construct a surjection from $\pi^*\overline{\Omega}_X$ to the $\Omega(\log)$ sheaf. I was not able to do this, but I was able to construct a locally free sheaf onto which $\pi^*\overline{\Omega}_X$ maps: after blowing up further if necessary in order to assume that $Y' := \pi^{-1}(Y)$ is a Cartier divisor, one can show that there is a surjection

$$\pi^*\overline{\Omega}_X \to \frac{\mathcal{L}^\vee \otimes \pi^*\mathcal{P}^1_M\mathcal{L}}{\mathcal{L}^\vee \otimes \mathcal{O}(Y')} \to 0 \quad ,$$

so one can aim to showing that

$$c\left(\frac{\mathcal{L}^\vee \otimes \pi^*\mathcal{P}^1_M\mathcal{L}}{\mathcal{L}^\vee \otimes \mathcal{O}(Y')}\right) = c(\Omega^1_{\widetilde{M}}(\log X'')) \quad .$$

This does work out—the key to it, after some standard set-up work, is again the graph construction.

At the same time, the classes on the left-hand-side can be computed 'formally'; when one does this, the main theorem comes to light. So this gives another proof of our main statement for hypersurfaces.

Can the same trick be used in higher codimension? I don't know!

4. Lecture IV

4.1. **The higher codimension puzzle.** Once more, here is the main theorem of these lectures: if X is a hypersurface in a nonsingular variety M, then

$$c_{\rm SM}(X) = c_{\rm F}(X) + c(TM) \cap \left(c(\mathcal{O}(X))^{-1} \cap (s(Y,M)^\vee \otimes \mathcal{O}(X))\right) \quad .$$

The next natural question is: what about higher codimension? can one remove the hypothesis that X be a hypersurface?

In a very weak sense the answer is: yes, cf. the approach via differential forms with logarithmic poles: the main formula there works for arbitrary X. But this doesn't teach us much, since we have almost no access to the resolution necessary in order to apply that formula. One would want to have a statement that remains within M, just as the main formula given above.

The first natural step to take is to go from hypersurfaces to complete intersections, or local complete intersections if one is brave. For the alternative viewpoints on Milnor classes mentioned in the previous lecture this has been carried out, first by Jean-Paul Brasselet et al., [BLSS02]; and now it is part of Jörg Schürmann powerful theory. Schürmann's work generalizes the approach by Parusiński and Pragacz in the sense that it concentrates on the 'interesting constructible functions' mentioned in the previous lecture; this is one of the main topics of Lecture 5 in Schürmann's cycle. One way to rephrase that approach is to view the Milnor class as arising from a constructible function corresponding to vanishing cycles; as Schürmann explains, this can be done for arbitrary complete intersections.

From this perspective, our 'Main Theorem' shows that *for hypersurfaces* the vanishing cycle information is essentially captured by the Segre class of the singularity subscheme. This statement cries out for a generalization, but one is still lacking.

Thus, a puzzle remains for higher codimensions, even in the cases covered by these approaches. For this, I must go back to a question I posed in one of the first sections:

Is there a natural scheme structure on the singularities of a given variety V, which determines the Milnor class of V?

The Milnor class is, up to sign, simply the difference $c_{\rm SM}(X) - c_{\rm F}(X)$ (recall that for local complete intersection Fulton and Fulton-Johnson classes agree, so this convention is compatible with the one used by other authors in that context). The main formula answers this puzzle for hypersurfaces: the Milnor class of a hypersurface X is determined by the *singularity subscheme* of X, that is, the subscheme of X defined by the partials of a defining equation.

It is very tempting to guess that a similar statement should hold for more general varieties. Candidates for immediate generalizations of the 'singularity subscheme' are not hard to find: for example, we could take the base scheme of the rational map from V to its Nash blow-up. Still, I do not know of any formula that will start from this scheme, perform an intersection-theoretic operation analogous to taking the Segre class and manipulating the result, and would thereby yield the Milnor class of V.

Some comments on what may be behind such an operation will have to wait until the last lecture. In this lecture I will leave the main philosophical question aside, and use a brute-force apprach to obtain the information for higher codimension without doing any new work at all.

4.2. **Segre classes: Inclusion-exclusion again.** Brute force means: go back to inclusion-exclusion considerations. I have already pointed out that a variation on the definition of Segre class provides a notion that satisfies a certain kind of inclusion-exclusion.

There happens to be another variation on the theme of a Segre class that satisfies inclusion-exclusion. We can in fact force inclusion-exclusion down the throat of Segre classes, by recasting the main formula (for hypersurfaces) as part of the definition.

Explicitly, define the *SM-Segre class* of a hypersurface X in a nonsingular variety M, with singularity subscheme Y, to be

$$s^\circ(X,M) := s(X,M) + c(\mathcal{O}(X))^{-1} \cap (s(Y,M)^\vee \otimes \mathcal{O}(X)) \quad ;$$

and for a proper subscheme Z of M, say

$$Z = X_1 \cap \cdots \cap X_r \quad ,$$

with X_i hypersurfaces, set

$$s^\circ(Z,M) := \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1<\cdots<i_s} s^\circ(X_{i_1} \cup \cdots \cup X_{i_s}, M) \quad .$$

Here $X_{i_1} \cup \cdots \cup X_{i_s}$ is any hypersurfaces supported on the union.

There are a few points to be made here. The first is a formal exercise: check that $s^\circ(Z,M)$ satisfies inclusion-exclusion (so $s^\circ(Z_1 \cap Z_2, M) = s^\circ(Z_1,M) + s^\circ(Z_2,M) - s^\circ(Z_1 \cup Z_2, M)$, for a suitable $Z_1 \cup Z_2$). The second point is more striking: this definition should look *extremely unlikely,* due to the amazing range of choices involved in it. What if I change the X_i to another collection of hypersurfaces cutting out Z? (for example, what if I throw in a few inessential generators of the ideal of Z?) what if at each stage in the $\sum$ I choose some other hypersurface supported on $X_{i_1} \cup \cdots \cup X_{i_s}$?

What magic property of Segre classes ensures that this definition does not depend on these choices?

I do not know! It must be something powerful indeed, but I have no clue as to its true nature.

So how do I know that the definition of $s^\circ(Z,M)$ really does not depend on the choices leading to it? Simply because

$$c_{\rm SM}(Z) = c(TM) \cap s^\circ(Z,M) \quad .$$

Exercise: realize that this is completely obvious, modulo the main result. (The impatient reader may look up [Alu03b], Theorem 3.1.)

One last, equally obvious, point: if I can compute $s^\circ(X,M)$ for hypersurfaces, then I can compute it for everything. There is nothing to this, but it has an important implication: I have almost bypassed the higher codimension puzzle. The SM-Segre class $s^\circ(Z,M)$ stands to $c_{\rm SM}(Z)$ in essentially the same way as the ordinary $s(Z,M)$ stands to $c_{\rm F}(Z)$.

The qualifiers 'almost…', 'essentially…' are there for an important reason: as defined, the class $s^\circ(Z,M)$ lives *in* M*,* not *in* Z*.* After the fact I know that there is a class on Z which agrees with $s^\circ(Z,M)$ after push-forward to M, but in terms of computations I won't be able to squeeze the class down to its proper place. In fact, this seems to be another puzzle about the definition that must go back to some mysterious and powerful property of Segre classes.

In any case, these considerations force me to shift my aim a little, and focus on computing the *push-forward of* $c_{\rm SM}(Z)$ *to the ambient variety* M*.* This seems to be the most natural question at this point.

4.3. **Putting together the algorithm.** At this point the algorithm behind the `Macaulay2` code I showed in action in the first lecture should have few secrets left. *Suppose you can compute Segre classes* in some ambient space M; then formal manipulations (8th grade algebra again) will allow you to compute $s^\circ(Z, M)$. For this, you will need to

- obtain a set of hypersurfaces cutting out Z (for example, a homogeneous ideal for Z if $M = \mathbb{P}^n$);
- for each subset S of this set, compute the singularity subscheme Y_S of the union X_S of the hypersurfaces in S;
- compute the ordinary Segre class of Y, and use it to obtain $s^\circ(X_S, M)$;
- put all the information together and obtain $s^\circ(Z, M)$.

Obtaining $c_{\mathrm{SM}}(Z)$ from $s^\circ(Z, M)$ is trivial, as pointed out in the previous section.

In short: *if you are able to compute ordinary Segre classes, then you can compute Chern-Schwartz-MacPherson classes.* This means that after all this work we are back to the very beginning: how to compute Segre classes? as I mentioned in the first lecture, this is in general a very hard problem.

Since the actual implementation of the algorithm works in $M = \mathbb{P}^n$, it is worth rewriting the main formula for hypersurface in this case. So let X be a hypersurface in $\mathbb{P}^n$; then $\mathcal{O}(X) = \mathcal{O}(d+1)$, where $d = \deg X - 1$ (the shift makes some other formulas a little more pleasant). If $F \in k[z_0, \dots, z_n]$ is a homogeneous polynomial defining X, then the singularity subscheme Y of X is defined by taking the ideal $(\frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_n})$ (by Euler's formula, this contains F). So we may see Y as a zero of a section

$$\mathbb{P}^n \to \mathcal{O}(d)^{n+1} \quad :$$

again our trick to realize a blow-up tells us what to do—the blow-up of $\mathbb{P}^n$ along Y is the closure of the graph of the induced rational map

$$\mathbb{P}^n \dashrightarrow \mathbb{P}^n \quad .$$

Let $\Gamma \subset \mathbb{P}^n \times \mathbb{P}^n$ be this closure. Denote by H, K respectively the pull-backs of the hyperplane class from the first, resp. second factor. As Γ is a class of dimension n, it is determined by the $(n+1)$ coefficients in

$$[\Gamma] = g_0 K^n + g_1 H K^{n-1} + \cdots + g_n H^n \quad .$$

Claim. The push-forward of $c_{\mathrm{SM}}(X)$ to $\mathbb{P}^n$ is

$$(1+H)^{n+1} - \sum_{d=0}^{n} g_d \, (-H)^d (1+H)^{n-d}$$

as an element of $\mathbb{Z}[H]/(H^{n+1}) = \mathcal{A}\mathbb{P}^n$.

This is of course no more and no less than the main theorem again. The switch from $s(Y, M)$ to the information about Γ is something that has to do with Segre classes, so I'll talk about it separately. Everything else is 8th grade algebra, and is explained more in detail in [Alu03a].

4.4. **Segre classes in projective space: concrete computations.** We are back to Segre classes, and how to compute them in practice. We have already seen where the main information will be, but it only takes a moment to repeat the set-up more in general.

So let Z be a proper subscheme of $\mathbb{P}^n$, with homogeneous ideal $(f_0,\dots,f_r)$. We may assume all generators have the same degree: indeed, otherwise we can multiply a generator of lower degree by powers of all homogeneous coordinates and bring its degree up.

This realizes Z as the zero-scheme of a section of $\mathcal{O}(d)^{r+1}$, for some d. Once more, the blow-up of $\mathbb{P}^n$ along Z will simply be the closure Γ of the graph of the rational map

$$\mathbb{P}^n \dashrightarrow \mathbb{P}(\mathcal{O}(d)^{r+1}) = \mathbb{P}^n \times \mathbb{P}^r \to \mathbb{P}^r$$

defined by the f_i's.

Incidentally, it is not conceptually hard (but computationally *very* demanding) to compute equations for Γ. The key step is performed by elimination theory, a slow but well understood process. It is easy to instruct Macaulay2 to do this.

As before, the class $[\Gamma]$ is determined by the $(n+1)$ numbers g_i for which

$$[\Gamma] = g_0K^r + g_1K^{r-1}H + g_2K^{r-2}H^2 + \cdots \quad :$$

these can be viewed as obtained by intersecting Γ with general copies of the hyperplane K, then pushing forward to $\mathbb{P}^n$ and computing degrees. Again, a tool like Macaulay2 can perform these operations.

How do we reconstruct the Segre class $s(Z,\mathbb{P}^n)$ (or rather its push-forward to $\mathbb{P}^n$) from these numbers? Assemble them together into a class $G = g_0 + g_1H + g_2^2H^2 + \dots$ in $\mathcal{A}\mathbb{P}^n$.

Claim. The push-forward of $s(Z,\mathbb{P}^n)$ is $1 - c(\mathcal{O}(dH))^{-1} \cap (G \otimes \mathcal{O}(dH))$.

To see this, write $s(Z,\mathbb{P}^n)$ as the push forward of $\frac{[E]}{1+E}$ from Γ, where E is the exceptional divisor in the blow-up. The class of E is the hyperplane class in $\mathbb{P}^n\times\mathbb{P}^r = \mathbb{P}(\mathcal{O}(dH)^{r+1})$: chasing classes around shows that we need the push-forward of

$$\frac{[dH-K]}{1+dH-K} \quad .$$

And this game should now look familiar. Omitting evident notations (using the projection formula, etc.):

$$\begin{aligned}\frac{dH-K}{1+dH-K} = 1 - \frac{1}{1+dH-K} &= 1 - \frac{1}{1+dH}\cdot\frac{1+dH}{1+dH-K}\\ &= 1 - c(\mathcal{O}(dH))^{-1}\left(\frac{1}{1-K}\otimes\mathcal{O}(dH)\right)\\ &= 1 - c(\mathcal{O}(dH))^{-1}\cdot(G\otimes\mathcal{O}(dH))\end{aligned}$$

since $\frac{1}{1-K} = (1+K+K^2+\dots)$ does precisely what it should (intersect Γ with higher and higher powers of K to get the components of G).

It is not hard to instruct Macaulay2 to perform these operations (see [Alu03a]) although computation time takes off exponentially, making any serious application impractical at the moment.

4.5. **'Almost unrelated' example: enumerative geometry.** The most immediate application of an algorithm computing Segre classes would be enumerative geometry. This is not immediately related to characteristic classes of singular variety (but see the next lecture), but it is quite a bit of fun. So here is a simple example which is actually within reach of the Macaulay2 routine. I am taking this almost verbatim from [Alu03a].

The enumerative question has to do with configurations of point on $\mathbb{P}^1$, and is a baby version of a much harder question in $\mathbb{P}^2$. The harder question is: how many translations of a given curve in $\mathbb{P}^2$ contain 8 general points?

Why 8? because 'translation' means 'in the same PGL(3) orbit', and PGL(3) has dimension 8; so it is natural to expect that the orbit should have dimension 8, and that 8 conditions should determine a finite number.

Why is this interesting? For example, for a general plane quartic the answer is the degree of the rational map from a general $\mathbb{P}^6$ inside the $\mathbb{P}^{14}$ of quartics to $\mathcal{M}_3$. More generally, the question should be seen as an 'isotrivial' variation on the Gromov-Witten invariants theme.

It is a little harder to defend why the analogous question should be interesting in dimension 1; but it is in fact of some interest in representation theory, and it is a good warm-up case for the harder question. In $\mathbb{P}^1$, the question becomes: how many PGL(2) translates of a given configuration of points (allowing and counting multiplicities) contain 3 points in general position?

The question translates easily into an intersection question in the $\mathbb{P}^3$ of 2×2 matrices: if $(F(s:t))$ is the ideal defining the 'configuration of points', then the condition that a translate contains the point p is written $F(\varphi(p)) = 0$, where φ varies in $\mathbb{P}^3$. This is a certain surface, and we are after 'honest' intersections of three general such surfaces.

The matter of 'honesty' has to do with the fact that if φ is a matrix whose image is a zero of F, then $F(\varphi(p)) = 0$ no matter what p is. That is, the intersection of any number of surfaces of the type we are considering contains a certain 'base scheme' S, of which it is not hard to compute the ideal in any given case. The intersections we want to compute are those that are *not* contained in S.

In fact, we know what the 'total intersection number' of three surfaces should be: if d is the degree of F, then the surfaces have degree d, so by Bézout's theorem three of them meet at d^3 points, counting multiplicities. The question is, what is the contribution of S to this number?

Fulton-MacPherson's intersection theory give a neat answer: the contribution is

$$\int (1 + dH)^3 \cap s(S, \mathbb{P}^3) \quad .$$

Thus if we can compute the Segre class of S in $\mathbb{P}^3$ then we can answer our enumerative question. I tried this for the specific $F = s(s+3t)^2(s+5t)(s+16t)$, letting the characteristic of the ground field vary. For example, this collapses to two points over $\mathbb{F}_2$, it's three points with multiplicities $3, 1, 1$ over $\mathbb{F}_3$, etc. My Macaulay2 routine can compute these Segre classes in (very) small characteristic, and the result matches precisely the general formula I had obtained with Carel Faber a long time ago ([AF93]).

I must add that this is really an extremely small example. We would need computers millions of times larger and faster than the ones we have, in order to solve serious enumerative questions this way.

5. Lecture V

5.1. **Multiplicity of discriminants.** I would now like to talk about a few items that are somewhat loosely related with characteristic classes of singular varieties. They are all rather concrete questions, so they should serve as 'after-the-fact motivation'.

I'll begin with a very concrete computation. Consider the space of plane curves of degree d: a projective space $\mathbb{P}^N$, with $N = \frac{d(d+3)}{2}$. In this projective space there is a *discriminant* hypersurface D, consisting of all *singular* plane curves. If X is a singular curve, then I can think of it as a point of D. What is the multiplicity of the hypersurface D at the point X?

It has been known for a long time that if X has no multiple components then the multiplicity is simply the sum of the Milnor numbers of X. So this has the potential of having to do with Milnor classes, and characteristic classes of singular variety…

It is in fact very easy to give a formula for this multiplicity. The hypersurface D can be realized as the projection from $\mathbb{P}^2 \times \mathbb{P}^N$ of a locus D' of codimension 3; and in fact equations for this locus are nothing but the 3 partial derivatives of the generic homogeneous polynomial of degree d in the homogeneous coordinates $\mathbb{P}^2$. Denoting by H, K respectively the hyperplane classes in $\mathbb{P}^2$ and $\mathbb{P}^N$, each derivative is a polynomial of class $(d-1)H + K$, so the normal bundle of D' has class $(1 + (d-1)H + K)^3$.

Now we want the multiplicity $m_X D$ of X in D. Segre classes compute multiplicities: $s(X, D) = m_X D[X]$. Segre classes are birational invariants, so $s(X, D) = \pi_* s(\pi^{-1}X, D')$, where $\pi : D' \to D$ is the projection. Capping with the classes of the ambient gives the intrinsic Fulton class, hence: $c(TD') \cap s(\pi^{-1}X, D') = c_{\mathrm{F}}(\pi^{-1}X) = c(T\mathbb{P}^2 \times \mathbb{P}^N) \cap s(\pi^{-1}X, \mathbb{P}^2 \times \mathbb{P}^N)$ or in other words

$$s(\pi^{-1}X, D') = (1 + (d-1)H + K)^3 \cap s(\pi^{-1}X, \mathbb{P}^2 \times \mathbb{P}^N) \quad .$$

There is more: $\pi^{-1}X$ is manifestly the *singularity subscheme* Y of X, and it sits in a fiber $\mathbb{P}^2$ so its Segre class does not depend on the $\mathbb{P}^N$ factor (another consequence of the fact that Fulton classes are intrinsic), and further K does nothing to it. Thus

$$s(\pi^{-1}X, D') = (1 + (d-1)H)^3 \cap s(Y, \mathbb{P}^2) \quad .$$

This should start looking familiar. Of course $(1+(d-1)H)^3$ is nothing but the Chern class of the trivial extension of $T^*\mathbb{P}^2$ tensored by $\mathcal{L} = \mathcal{O}(d)$:

$$s(\pi^{-1}X, D') = c(\mathcal{L})c(T^*M \otimes \mathcal{L}) \cap s(Y, M)$$

where I have written $M = \mathbb{P}^2$. The multiplicity of the discriminant is the degree of this class.

Claim. This holds in general, for a hypersurface X in a nonsingular variety M, with singularity subscheme Y.

The proof isn't hard—just trace the computation in the general case; details can be found in [AC93]. What does this have to do with Milnor classes? The point I want to make is that $s(\pi^{-1}X, D') = s(Y, D')$ 'is' the Milnor class! More precisely:

Claim. $c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X) = \left(\dfrac{s(Y, D')}{c(\mathcal{L})^{\dim M}} \right)^\vee \otimes \mathcal{L}$.

Proof: Exercise. Use the above formula for $s(\pi^{-1}X, D')$ and the properties of $^\vee$ and $\otimes$ to reduce to the main formula.

'Undoing this', it is very easy to provide a formula for the multiplicity of a discriminant in terms of c_{SM} and c_{F}. It seems that discriminants naturally embody a lot of information about characteristic classes of what they parametrize.

5.2. **Constraints on singular loci of hypersurfaces.** One class that has surfaced in the considerations in the previous section:

$$c(T^*M \otimes \mathcal{L}) \cap s(Y, M)$$

turns out to pack a good amount of information. After the fact this is easy to believe, since it *is* the Milnor class up to 8th grade algebra and notation juggling. But when I run into it at first I was not aware of the connection with characteristic classes, and I studied it independently of such considerations, cf. [Alu95], calling it the 'μ-class of Y with respect to $\mathcal{L}$', $\mu_{\mathcal{L}}(Y)$. It is in fact possible to figure out the exact dependence of this class with respect to $\mathcal{L}$, but this is not important here.

It should come to no surprise that the Milnor class can be written in terms of the μ class; the precise formula is

$$c_{\rm SM}(X) - c_{\rm F}(X) = c(\mathcal{L})^{\dim X} \cap (\mu_{\mathcal{L}}(Y)^\vee \otimes \mathcal{L})$$

(exercise). That also means that almost everything I say here could be directly transcribed in terms of Milnor classes.

The subtlest property of the μ-class is that its notation makes sense: that is, that the ambient variety M does not matter. In other words, if you manage to realize Y as the singularity subscheme of a hypersurface with bundle $\mathcal{L}$ (restricted to Y) in an*other* nonsingular ambient M, and compute the μ-class there, you will get the same class. Note that this isn't so clear, even after the connection with Milnor classes has been established: who knows what has happened to X in the process?

But it is true. One very special case of this fact leads to very nice applications: what if Y (the singularity subscheme of X) is itself *nonsingular*? Then I can pathologically consider it as the singularity subscheme of the 'hypersurface' with equation $0 = 0$ on Y itself, and the main property of μ-classes then tells me

$$c(T^*M \otimes \mathcal{L}) \cap s(Y, M) = \mu_{\mathcal{L}}(Y) = c(T^*Y \otimes \mathcal{L}) \cap [Y]$$

(since $s(Y, Y) = [Y]$).

One particularly pretty consequence of this formula is that it poses constraints on 'what' nonsingular Y can be singularity subschemes of hypersurfaces in a given M.

Example. The twisted cubic (with the reduced structure) cannot be the singularity subscheme of a hypersurface in $M = \mathbb{P}^n$.

Indeed, the twisted cubic is abstractly $\mathbb{P}^1$, embedded as a curve of degree 3 in $\mathbb{P}^n$. If it were the singularity subscheme of a hypersurface of degree d, then we would have:

$$c(T^*\mathbb{P}^n \otimes \mathcal{O}(X)) \cap s(\mathbb{P}^1, \mathbb{P}^n) = \mu_{\mathcal{O}(3d)}(\mathbb{P}^1) = c(T^*\mathbb{P}^1 \otimes \mathcal{O}(3d)) \cap [\mathbb{P}^1]$$

that is, calling h the hyperplane on $\mathbb{P}^1$:

$$\frac{(1+3(d-1)h)^{n+1}}{1+3dh} \frac{(1+h)^2}{(1+3h)^{n+1}} \cap [\mathbb{P}^1] = \frac{(1+(3d-1)h)^2}{1+3dh} \cap [\mathbb{P}^1]$$

and finally (take degree 1 terms):

$$(3d-6)n - 4 = 3d - 2$$

which is nonsense (read modulo 3).

Many such examples can be found. The moral is that it is *very unlikely* for a singularity subscheme to be nonsingular. A 'general' example (whatever this may mean) must have embedded components somewhere. The topologist's viewpoint on this should be that the 'general' singular hypersurface must have a 'complicated' Whitney stratification. I don't know if there are explicit results in this direction.

5.3. **Enumerative geometry again.** I brought up enumerative geometry already a couple of times, in ways that were only tangentially related to characteristic classes. But characteristic classes can actually be used to attack some enumerative problems, at least in comparatively simple instances.

Many enumerative problems can be interpreted as asking what is the degree of a specific locus in a discriminant. For example, the number of nonsingular plane cubics tangent to 9 lines (that is: my Ph.D. thesis) can be computed as the degree of the closure of the locus of plane *sextic* curves with exactly 9 cusps. Even the celebrated numbers of rational plane curves of a given degree containing the right number of general points is manifestly the degree of a geometrically significant locus in a discriminant.

One difficulty is that these problems are completely unrelated: what connection can there be between the degree of one locus and the degree of another locus, even within the same hypersurface in a variety? (The discriminant hypersurface in this case.) Well, there is a way out of this bind: what if we asked for the $c_{\rm SM}$ class instead?

So I would like to pose a variation on the enumerative problem. For every 'geometrically significant' locus X in a discriminant (for example: for every closure of a PGL(3) orbit in the discriminant of plane curves of a given degree), aim to computing $c_{\rm SM}(X)$, as a class in the ambient projective space.

This is at least as difficult as computing degrees, since the very first term in $c_{\rm SM}(X)$ will carry the degree information. *On the other hand,* the information of $c_{\rm SM}$ classes *is* structured: because every class corresponding to a known constructible function on the discriminant will give a relation between these $c_{\rm SM}$ classes.

The problem is now that there are not that many choice of 'geometrically significant' constructible functions on a discriminant D. Two possibilities are $1\!\!1_D$ and ${\rm Eu}_D$, so this is a natural place to start: what can one say about $c_{\rm SM}(D)$ and $c_{\rm Ma}(D)$, and can one use this knowledge to get any enumerative information?

If one can play this game at all, it is because the Nash blow-up of a discriminant is easily accessible. We saw this already in one previous section, where I computed multiplicities: the variety D' I used there was nothing but the Nash blow-up. The tautological bundle can also be computed with ease, and this gives access to the function ${\rm Eu}_D$:

Claim. If $X \in D$, then ${\rm Eu}_D(X) = \int \mu_{\mathcal{O}(X)}(Y)$.
Phrased otherwise, this says that the local Euler obstruction of a discriminant at a hypersurface X is nothing but Parusiński's Milnor number of X. (This holds provided D is a hypersurface.) Now studying the situation in low codimension says that ${\rm Eu}_D = 1\!\!1_D + 1\!\!1_{\overline{C}} + 1\!\!1_{\overline{G}} + \cdots$, where C is the locus of 'cuspidal' hypersurfaces, and G is the locus of 'binodal' hypersurfaces. One could go further down the list of singularities, although this gets combinatorially messy.

But this is enough for some applications. It says that

$$c_{\rm SM}(D) = c_{\rm Ma}(D) - [\overline{C}] - [\overline{G}] + \text{higher codimension terms.}$$

Playing the same game with other classes gives more such relations. When enough relations are known, one can start solving the relations, getting information about the loci C, G, etc. This way one can recover classical formulas for cuspidal and binodal loci, and do a little more—see [Alu98] for more examples.

It would be fun to carry this out for more loci, or to actually compute $c_{\rm SM}$ of all orbits within a discriminant, for instance for low degree plane curves. I have done it for degree 3, and it is a very pleasant computation.

5.4. **What now? Shadows of blowup algebras. . .** What keeps me glued to the main theorem is the high codimension puzzle: even if other approaches are successful in dealing with more general cases than hypersurfaces, I simply cannot believe that the formula that has come up over and over in these lectures does not admit a straightforward generalization.

As I see it, the factors that prevent a generalization of the formula are its dependence on the ambient space and on the line bundle: these are elements that make sense only if X is a hypersurface. A satisfactory formula should not depend on this information. There must be a way to reformulate the main statement, independent of these factors.

The most convincing thought I have had recently in this direction is the following. First, I need a name for the operation involved in the Lagrangian formulation (back from the third lecture). If $\mathbb{P}E \stackrel{\epsilon}{\to} S$ is a projective bundle of rank e over a scheme S, then there is a structure theorem for its Chow group: every class $C \in A_r\mathbb{P}E$ can be written uniquely as $C = \sum_{j=0}^{e} c_1(\mathcal{O}(1))^j \cap \epsilon^*(C_{r-e+j})$ where $\mathcal{O}(1)$ is the tautological line bundle and $C_{r-e+j} \in A_{r-e+j}S$ ([Ful84], §3.3). So every class in $\mathbb{P}E$ has a counterpart in S, although dimensions 'get spread out', introducing lots of fuzz. The class C can be reconstructed from $C_{r-e}+\cdots+C_r$ if the dimension of C is known, but not otherwise. I call this counterpart in $\mathcal{A}S$ the 'shadow' of C.

Now I claim that we have encountered this operation already. The lagrangian setup expressed $c_*(\varphi)$ in terms of a certain cycle $\mathrm{Ch}(\varphi)$ associated with the constructible function φ, in a projective bundle (Ch was defined in a vector bundle but was conical, so its information is equivalent to the corresponding cycle in the projectivization, and now I'll put it there). Actually one has to introduce a dualization in the process, changing every other sign (something like $^\vee$ in my favorite notations). So I'll call $\check{c}_*(\varphi)$ this 'dualized' class.

Claim. $\check{c}_*(\varphi)$ is the shadow of the characteristic cycle $\mathrm{Ch}(\varphi)$.

This remark has the only beneficial effect of providing me with an element of language that does not immediately seem to depend on the ambient variety. If I manage to realize the characteristic cycle of X inside any bundle, I will know what to do in order to extract the information from it without invoking '$c(TM)\cap\cdots$'.

But by far the most important item is the realization of the characteristic cycle. As we have seen, the formula is equivalent to the statement that the characteristic cycle of X can be obtained by pulling X back via the blow-up of the ambient M along Y, and then taking away one copy of the exceptional divisor. Every word in this sentence requires M to be there.

I have found a way around this, but (to my total amazement) it seems to rely on a strange condition on X. I have called this the $\times$-condition, in a futile attempt at being funny, and because nodal curves are the prototypical example of hypersurfaces that satisfies it. In its gory details, the condition relies on the system of differential equations satisfied by a defining equation for the hypersurface; more geometrically (but probably not equivalently), the $\times$-condition is satisfied if the singularity subscheme is *linearly embedded* in the ambient space. Many examples of this are known (see [Alu02]) Exercise: find a hypersurface that does *not* satisfy the $\times$-condition.

What's good about surfaces satisfying the $\times$-condition?

Claim. Up to signs and duals, $c_{\mathrm{SM}}(X)$ is the shadow of $\mathbb{P}(\mathrm{Sym}_X(Y))$.

The algebra $\mathrm{Sym}_X(Y)$ could not care less about X being a hypersurface. So this goes one step in the right direction. Note: $\mathbb{P}(\mathrm{Rees}_X(Y))$ gives $c_{\mathrm{Ma}}(X)$, a further indication that I may be on the right track.

5.5. **. . . and maybe a new Chern class.** That looks promising, although there are still a couple of problems with it: (1) I don't understand the '$\times$-condition'; (2) the whole operation is still not really independent of the ambient variety.

Comments: (1) if a hypersurface X does not satisfy the $\times$-condition, then I can construct an algebra whose shadow is $c_{\rm SM}(X)$, but this algebra is not as simple-minded as $\mathrm{Sym}_X Y$ then; it 'interpolates' between $\mathrm{Rees}_X(Y)$ and $\mathrm{Sym}_X(Y)$, with which it agrees when Y is linearly embedded in the ambient variety. Details are in [Alu02]. I am hoping commutative algebraists will find some use for this new 'blow-up algebra'.

(2) There is a subtle point about the operation: the shadow does depend on the bundle in whose projectivization $\mathbb{P}(\mathrm{Sym}_X Y)$ is immersed, and there are more than one choice for this. Knowing which one to choose seems to require knowing the line bundle of X, hence, it still relies on X being a hypersurface.

I can be a little more precise. The ideal of Y in X is generated by the components of a section of $\mathcal{P}^1_M\mathcal{L}|_X$; this realizes $\mathrm{Sym}_Y X$ as a quotient of $\mathrm{Sym}(\mathcal{P}^1_M\mathcal{L}|_X)^\vee$, hence places the proj in $\mathbb{P}(\mathcal{P}^1_M\mathcal{L}|_X)$. We have already seen that it can be juggled back into $\mathbb{P}(T^*M)$, and *this* is where I want to cast the shadow. But to go from one to the other I have to tensor by $\mathcal{L}^\vee$: how to do that if I don't have a $\mathcal{L}$ around?

Here is a way out. If X is *any* subscheme of a variety M, with ideal $\mathcal{I}$, there is a surjection $\Omega^1_M|_X \to \Omega^1_X \to 0$; taking $\mathrm{Hom}(-,\mathcal{O}_X)$, we get a morphism

$$\mathrm{Hom}(\Omega^1_X,\mathcal{O}_X) \to \mathrm{Hom}(\Omega^1_M|_X,\mathcal{O}_X) \quad .$$

What could be more natural than giving the following:

Definition. Denote by $\mathcal{N}_X M$ the cokernel of this morphism.

What good does this do for us?

Claim. If X is a hypersurface, then $\mathrm{Proj}(\mathrm{Sym}_X Y) = \mathrm{Proj}(\mathrm{Sym}_X(\mathcal{N}_X M))$.

This very simple observation solves the line bundle puzzle: because $\mathcal{N}_X M$ is defined for *every* X, and it already lives in the right place (no $\mathcal{L}$ involved!). Putting everything together, the main statement becomes

Theorem 5.1. *If X is a hypersurface satisfying the $\times$-condition, then $c_{\rm SM}(X)$ is (essentially) the shadow of* $\mathrm{Proj}(\mathrm{Sym}_X(\mathcal{N}_X M))$.

The point here is of course that the shadow of $\mathrm{Proj}(\mathrm{Sym}_X(\mathcal{N}_X M))$ is defined *even if X is not a hypersurface!* Might this be the statement I have been looking for? Not quite: for example, because of the qualifier '. . . satisfying the $\times$-condition'. I feel that any advance on a statement of this sort has to wait until the $\times$-condition is better understood.

But this construction raises a tantalizing point. The 'shadow' in the last statement can be seen as an operation performed on Ω^1_X: dominate it with the locally free $\Omega^1_M|_X$, then take kernel, then $\mathrm{Hom}(-,\mathcal{O}_X)$, etc. This operation can be performed on *every* coherent sheaf $\mathcal{F}$, and the upshot of the above considerations is that if X is a hypersurface satisfying the $\times$-condition, then doing it to Ω^1_X essentially yields $c_{\rm SM}(X)$.

Exercise: if $\mathcal{F}$ is locally free, this operation gives (essentially) the total 'homology' Chern class of $\mathcal{F}$. So this is a new 'Chern class' for coherent sheaves, on possibly singular schemes, which is closely related to Chern-Schwartz-MacPherson classes in an interesting class of examples.

I will then end with what is now the obvious next question: *what is the exact relation between the new Chern class and the functorial Chern-Schwartz-MacPherson class?*

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