INCLUSION-EXCLUSION AND SEGRE CLASSES

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Dedicated to Steven Kleiman on the occasion of his 60th birthday

Abstract. We propose a variation of the notion of Segre class, by forcing a naive 'inclusion-exclusion' principle to hold. The resulting class is computationally tractable, and is closely related to Chern-Schwartz-MacPherson classes. We deduce several general properties of the new class from this relation, and obtain an expression for the Milnor class of an arbitrary scheme in terms of this class.

1. Introduction

Notwithstanding their fundamental rôle in modern intersection theory (cf. [Ful84], Chapters 4 and 6), *Segre classes* remain a somewhat esoteric concept: with a few notable exceptions, they have been used more for foundational purposes than for actual computations of concrete intersection products. This is due to the effective inaccessibility of Segre classes: essentially no techniques are known to compute the Segre class $s(Z, M)$ of a scheme Z in a scheme M, other than its raw definition; which is perhaps a little too close to the ideal of Z for comfort, in almost every geometrically significant problem. The main virtue of the Segre class—that is, its sensitivity to the fine structure of Z—turns out to be the main problem in handling it in concrete situations.

In this article we propose an a variation ($s^\circ(Z, M)$, Definition 2.2) of the notion of Segre class of a subscheme Z in a *nonsingular variety* M, by imposing on it an *inclusion-exclusion principle,* which makes $s^\circ(Z, M)$ well-behaved with respect to naive set-theoretic operations. The class we define does not work as a Segre class in a definition of an intersection product, but shares with Segre classes several notable properties: our $s^\circ(Z, M)$ agrees with $s(Z, M)$ if Z is nonsingular; and behaves with respect to different embeddings of Z in nonsingular varieties or to smooth maps in precisely the same way the ordinary Segre class would.

We collect these and other properties in Theorem 2.3. We find these observations very remarkable. First, it is very puzzling that the definition of $s^\circ(Z, M)$ makes sense at all. The definition of $s^\circ(Z, M)$ is given in terms of any collection of hypersurfaces

2000 Mathematics Subject Classification: Primary 14C17; Secondary 14J17, 57R20

cutting out Z, and consists of a rather complicated combination of conventional Segre classes. That the end-result should not depend on the choices of the hypersurfaces must amount to massive cancellations occurring among the Segre classes involved in the definition; to our knowledge, there is no direct explanation for these cancellations. Second, the definition turns out to *depend only on the support of Z;* given the sensitivity to scheme structure of ordinary Segre classes, we find this fact rather astonishing, again amounting to remarkable cancellations which must take into account and then eliminate the contributions of nilpotents to the class. We illustrate such 'cancellations' with a couple of simple examples; the reader is warmly invited to work out more complex ones.

Our perspective is that the properties listed in Theorem 2.3 must reflect some unknown and powerful features of ordinary Segre classes. We view Theorem 2.3 as 'experimental evidence' for these features, and the main purpose of this article is to advertise this evidence. With this understood, it is remarkable that Theorem 2.3 can be proved without uncovering or even being able to state precisely these features. In fact, the proof of Theorem 2.3 is essentially immediate once it is realized that $s^\circ(Z, M)$ is closely tied to another important class, defined for all singular varieties. This relationship is exposed in Theorem 3.1; Theorem 2.3 follows easily from this.

Unfortunately, being able to prove something is not the same as understanding it. While we do prove Theorem 2.3, the properties of Segre classes which must be responsible for it remain just as unknown after the fact. We believe that establishing these properties would be exceedingly interesting. If Segre classes were computable objects, then a great many problems in enumerative geometry would become a routine exercise. Given the (unreasonable?) relevance of enumerative geometry in recent developments in algebraic geometry, clarifying the notion of Segre class seems a very worthwhile goal.

This goal seems to us largely unmet—with the exception of the seminal work of Steven Kleiman and his collaborators, in which Segre classes play an important part (cf. for example [Kle94] and [KT96]).

This is the first article in a series planned to explore 'inclusion-exclusion' phenomena in the theory of Segre classes. In [Alu02b] we will propose a different variation on the theme of Segre classes, also satisfying an inclusion-exclusion principle, and yielding a simple computation of $s^\circ(Z, M)$ in certain cases. The inclusion-exclusion principle is used in [Alu02a] to obtain an explicit computational tool for characteristic classes of projective schemes. For example, the algorithm computes the topological Euler characteristic of a subscheme of $\mathbb{P}^n\mathbb{C}$ from the generators of a defining homogeneous ideal.

Acknowledgments. I thank the Max-Planck-Institut für Mathematik in Bonn, Germany, for the hospitality and support. Thanks are also due to the referee, who spotted several inaccuracies in a preliminary version of this paper.

2. SM-Segre classes

Throughout this section, M denotes a nonsingular variety (over an algebraically closed field of characteristic 0, although this does not seem to be essential).

We will consider a proper subscheme Z of M, and a finite family $\{X_i\}_{i=1,\dots,r}$ of hypersurfaces cutting out Z in M:

$$Z = X_1 \cap \cdots \cap X_r \quad ;$$

note that we are putting no restriction on r, and no other restrictions on the hypersurfaces (they may be nonreduced, there may be repetitions in the list, etc.). In fact, the requirement on the hypersurfaces will be further relaxed later on, cf. Remark 2.4.

For a hypersurface X, we define its *SM-Segre class* as follows[1]. Let Y be the singularity subscheme of X, that is, the subscheme locally defined by the partial derivatives of a local generator for the ideal of X.

Definition 2.1. The SM-Segre class of X in M is the class

$$s^\circ(X,M) = s(X,M) + c(\mathcal{O}(X))^{-1} \cap (s(Y,M)^\vee \otimes_M \mathcal{O}(X)) \quad .$$

This definition uses notations—e.g., for the tensor of a rational equivalence class by a line bundle—introduced in [Alu94], Def. 2; in more conventional (but less manageable) terms, the component of dimension m in $s^\circ(X,M)$ is

$$s(X,M)_m + (-1)^{n-m} \sum_{j=0}^{n-m} \binom{n-m}{j} X^j \cdot s(Y,M)_{m+j}$$

where $n = \dim M$; a formula that is reminiscent of classical residual intersection formulas (cf. [Ful84], Prop. 9.2). In [Alu94] and [Alu99] the class $s^\circ(X,M)$ is denoted $s(X \setminus Y, M)$; no analogs for schemes other than hypersurfaces were considered there.

As given in Definition 2.1, the SM-Segre class *of a hypersurface* X lives naturally in the Chow group A_*X of X. In view of the upgrade to arbitrary subschemes Z of M that follows, however, it will be more natural to view it for the moment as an element of the Chow group A_*M of the ambient nonsingular variety. We will omit such evident push-forwards from our notations, as well as evident pull-backs.

Definition 2.2. Let Z be a proper subscheme of M, and let $X_1, \dots, X_r$ be hypersurfaces cutting out Z:

$$Z = X_1 \cap \cdots \cap X_r \quad . \tag{1}$$

Then the *SM-Segre class of Z in M* is obtained by applying inclusion-exclusion to the SM-Segre classes of the hypersurfaces X_i. Explicitly, we set

$$s^\circ(Z,M) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1<\cdots<i_s} s^\circ(X_{i_1} \cup \cdots \cup X_{i_s}, M) \quad \in A_*M \quad .$$

[1] *SM* is supposed both to evoke the connection with Schwartz-MacPherson classes, which is the key to the main properties of the class, and the fact that we view this notion as a '*sm*oothing' of the notion of conventional Segre class

Here $X_{i_1} \cup \cdots \cup X_{i_s}$ is the hypersurface whose ideal is the product of the ideals of $X_{i_1}, \dots, X_{i_s}$ (so $X \cup X \neq X$!); but see Remark 2.4 below. Also, while we are defining the class in A_*M at this stage, Theorem 3.1 will imply that it is the image of class naturally defined on Z itself; also cf. Remark 2.6.

The terminology *inclusion-exclusion* is adapted from the similar-looking formula for the number of elements in the intersection Z of r finite sets $X_1, \dots, X_r$:

$$\#Z = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1 < \cdots < i_s} \#(X_{i_1} \cup \cdots \cup X_{i_s}) \quad ;$$

this is immediately proved by induction on r.

Concerning Definition 2.2, the reader should now expect a proof that $s^\circ(Z, M)$ does not depend on the specific choice of hypersurfaces cutting out Z. We isolate this and other properties of the definition in the following statement.

Theorem 2.3. (1) *The definition of $s^\circ(Z, M)$ is independent of the choices.*

(2) *If Z_{red} is the support of Z, then $s^\circ(Z, M) = s^\circ(Z_{red}, M)$.*

(3) *If Z is nonsingular, then $s^\circ(Z, M) = s(Z, M) = c(N_Z M)^{-1} \cap [Z]$.*

(4) *If $Z \subset M \subset M'$, where M' is a nonsingular variety and $M \subset M'$ is a closed embedding, then*

$$s^\circ(Z, M') = c(N_M M')^{-1} \cap s^\circ(Z, M) \quad .$$

(5) *If $Z \subset U \subset M$, where $U \xrightarrow{j} M$ is an open embedding, then $s^\circ(Z, U) = j^* s^\circ(Z, M)$.*

(6) *More generally: if $p : M' \to M$ is a smooth morphism, $Z \subset M$ a subscheme, and $Z' = p^{-1}(Z)$ its inverse image, then $s^\circ(Z', M') = p^* s^\circ(Z, M)$.*

(7) *The class $s^\circ(Z, M)$ satisfies a full inclusion-exclusion principle, in the sense that if $Z_1, \dots, Z_r$ are subschemes of M such that $Z = Z_1 \cap \cdots \cap Z_r$, then*

$$s^\circ(Z, M) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1 < \cdots < i_s} s^\circ(Z_{i_1} \cup \cdots \cup Z_{i_s}, M) \quad ,$$

where $Z_{i_1} \cup \cdots \cup Z_{i_s}$ is the subscheme of M whose ideal is the product of the ideals of $Z_{i_1}, \dots, Z_{i_s}$.

We are separating these statements from their *Ursprung,* which is Theorem 3.1 below, in the attempt to highlight them independently of our technical bias. Theorem 2.3 will follow from Theorem 3.1, but we strongly feel that these statements are of substantial independent interest, and call for a straightforward intersection-theoretic proof; with the exclusion of parts 5.-7., which are formal exercises (left to the reader), we do not know such a proof.

We delay the (rather anticlimatic) proof of Theorem 2.3 until the next section. The rest of this section is taken by several comments meant to further highlight the content of the theorem.

Remark 2.4. By part 2., the equality (1) in Definition 2.2 need only hold set-theoretically. By the same token, we can in fact replace the unions $X_{i_1} \cup \cdots \cup X_{i_s}$ in that definition, or the union $Z_{i_1}\cup\cdots\cup Z_{i_s}$ in part 7., by any other schemes supported on such unions (for example, we could take the ideals defined by the *intersection* of the ideals, rather than their product, or take radicals throughout). □

Remark 2.5. In view of parts 3. and 4, it is consistent to define $s^\circ(M,M)=[M]$. □

Remark 2.6. The formulas in part 4. and 5., ruling the behavior of the SM-Segre class under different embeddings of Z, hold in precisely the same terms for the ordinary Segre class (this follows from [Ful84], Example 4.2.6(a)). In fact, part 4 suggests that $s^\circ(Z,M)$ should be, up to a correction factor $c(TM)$, a class intrinsic to Z (and living in A_*Z). This is precisely the case, as will follow from Theorem 3.1. □

Remark 2.7. By part 3. and 5., if Z is reduced then the difference between $s^\circ(Z,M)$ and $s(Z,M)$ is supported within the singular locus Z_s of Z. Indeed, letting $U = M\setminus Z_s$, the classes $s^\circ(Z,M)$ and $s(Z,M)$ restrict (by part 5.) to the same (by part 3.) class in A_*U, so the difference comes from A_*Z_s by [Ful84], Proposition 1.8. □

Remark 2.8. The behavior of s° under smooth morphisms, prescribed by part 6., also matches the behavior of conventional Segre classes. In fact, for Segre classes the same behavior extends to the more general case of *flat* maps, by [Ful84], Proposition 4.2(b); this is not so for SM-Segre classes. Another key property of Segre classes, that is, birational invariance (cf. Proposition 4.2(a) in [Ful84]) also does not hold for SM-Segre classes.

On the other hand, by part 2. of Theorem 2.3, the stated equality for SM-Segre classes under smooth morphisms holds as soon as Z' equals $p^{-1}(Z)$ set-theoretically; for the ordinary Segre class, this equality has to hold scheme-theoretically. □

To give a sense of what might go into a 'direct argument' as envisioned above, here is a direct proof of a very particular case of the innocuous-looking part 3.

Proof. We will prove part 3. in the particular case in which Z is the complete intersection of two transversal nonsingular hypersurfaces X_1, X_2. In this case, Definition 2.1 gives

$$s^\circ(X_1,M)=\frac{[X_1]}{1+X_1}\quad,\quad s^\circ(X_2,M)=\frac{[X_2]}{1+X_2}$$

(where we employ the convenient shorthand $\frac{[X]}{1+X}=[X]-[X^2]+[X^3]-\cdots=s(X,M)$). As for $X_1\cup X_2$, it is easily verified that Z is itself the singularity subscheme of this hypersurface; hence

$$\begin{aligned}
s^\circ(X_1\cup X_2,M) &= \frac{[X_1+X_2]}{1+X_1+X_2}+\frac{1}{1+X_1+X_2}\left(s(Z,M)^\vee\otimes\mathcal{O}(X_1+X_2)\right)\\
&= \frac{[X_1+X_2]}{1+X_1+X_2}+\frac{1}{1+X_1+X_2}\left(\frac{[X_1X_2]}{(1-X_1)(1-X_2)}\otimes\mathcal{O}(X_1+X_2)\right)\\
&= \frac{[X_1+X_2]}{1+X_1+X_2}+\frac{1}{1+X_1+X_2}\frac{[X_1X_2]}{(1+X_2)(1+X_1)}
\end{aligned}$$

where we have used Proposition 1 in [Alu94]. Thus, according to Definition 2.2:

$$\begin{aligned} s^\circ(Z,M) &= \frac{[X_1]}{1+X_1} + \frac{[X_2]}{1+X_2} - \left(\frac{[X_1+X_2]}{1+X_1+X_2} + \frac{1}{1+X_1+X_2}\frac{[X_1X_2]}{(1+X_2)(1+X_1)}\right) \\ &= \frac{[X_1X_2]}{(1+X_1)(1+X_2)} \end{aligned}$$

by trivial formal manipulations. Since Z is the complete intersection of X_1 and X_2, the right-hand-side is $s(Z,M)$, as prescribed by part 3 of Theorem 2.3. □

The whole of Theorem 2.3, especially the crucial parts 1. and 2., ought to have a similarly direct explanation, but none is available to us at this time. For Theorem 2.3 to hold, drastic cancellations (of which the simplifications occurring in the proof presented a moment ago must be the simplest instance) must be at work behind the scenes. This can be also be observed in any concrete computation of SM-Segre classes; we give two simple examples for illustration purposes.

Example 2.9. The curve C consisting of two lines meeting at a point in $\mathbb{P}^3$ can be realized as the intersection of a nonsingular quadric Q and a tangent plane H. Applying the definition of SM-Segre classes (noting that $Y = \emptyset$ for a nonsingular hypersurface) gives

$$\begin{aligned} s^\circ(H,\mathbb{P}^3) &= s(H,\mathbb{P}^3) = [\mathbb{P}^2] - [\mathbb{P}^1] + [\mathbb{P}^0] \\ s^\circ(Q,\mathbb{P}^3) &= s(Q,\mathbb{P}^3) = 2[\mathbb{P}^2] - 4[\mathbb{P}^1] + 8[\mathbb{P}^0] \quad . \end{aligned}$$

The singularity subscheme Y of $H \cup Q$ consists of the two lines, with an embedded point at the intersection point. A blow-up computation gives

$$s(Y,\mathbb{P}^3) = 2[\mathbb{P}^1] - 5[\mathbb{P}^0]$$

yielding, according to Definition 2.1,

$$s^\circ(H\cup Q,\mathbb{P}^3) = 3[\mathbb{P}]^2 - 7[\mathbb{P}]^1 + 14[\mathbb{P}^0] \quad .$$

Thus (Definition 2.2):

$$\begin{aligned} s^\circ(C,\mathbb{P}^3) &= ([\mathbb{P}^2] - [\mathbb{P}^1] + [\mathbb{P}^0]) + (2[\mathbb{P}^2] - 4[\mathbb{P}^1] + 8[\mathbb{P}^0]) - (3[\mathbb{P}]^2 - 7[\mathbb{P}]^1 + 14[\mathbb{P}^0]) \\ &= 2[\mathbb{P}^1] - 5[\mathbb{P}^1] \quad . \end{aligned}$$

On the other hand, C is itself a hypersurface, in $M' = \mathbb{P}^2$, with singularity subscheme a point. By Definition 2.1,

$$s^\circ(C,\mathbb{P}^2) = \frac{[C]}{1+C} + [\mathbb{P}^0] = 2[\mathbb{P}^1] - 3[\mathbb{P}^0] \quad .$$

As prescribed by part 4. of Theorem 2.3,

$$s^\circ(C,\mathbb{P}^3) = c(N_{\mathbb{P}^2}\mathbb{P}^3)^{-1} \cap s^\circ(C,\mathbb{P}^2) \quad .$$

Example 2.10. Now let C be the subscheme defined by the homogeneous ideal (xy, x^2) in $\mathbb{P}^2$; that is, a line with an embedded point. Thus, C is the intersection of X_1 and X_2, where X_1 is a union of two lines, and X_2 is the double line with ideal (x^2). By Example 2.9 we have

$$s^\circ(X_1,\mathbb{P}^2) = 2[\mathbb{P}^1] - 3[\mathbb{P}^0] \quad .$$

The singularity subscheme of X_2 consists of a (single) line, so Definition 2.1 gives

$$\begin{aligned} s^\circ(X_2, \mathbb{P}^2) &= \frac{[X_2]}{1+X_2} + \frac{1}{1+X_2}\left((s(\mathbb{P}^1, \mathbb{P}^2)^\vee \otimes_{\mathbb{P}^2} \mathcal{O}(X_2)\right) \\ &= 2[\mathbb{P}^1] - 4[\mathbb{P}^0] - ([\mathbb{P}^1] - 3[\mathbb{P}^0]) = [\mathbb{P}^1] - [\mathbb{P}^0] \quad . \end{aligned}$$

Next, $X_1 \cup X_2$ has ideal (x^3y), hence its singularity subscheme has ideal (x^2y, x^3). Computing (conventional) Segre classes and using again Definition 2.1 gives

$$s^\circ(X_1 \cup X_2, \mathbb{P}^2) = 2[\mathbb{P}^1] - 3[\mathbb{P}^0] \quad .$$

Thus

$$s^\circ(C, \mathbb{P}^2) = (2[\mathbb{P}^1] - 3[\mathbb{P}^0]) + ([\mathbb{P}^1] - [\mathbb{P}^0]) - (2[\mathbb{P}^1] - 3[\mathbb{P}^0]) = [\mathbb{P}^1] - [\mathbb{P}^0] \quad .$$

This is in agreement with parts 2. and 3. of Theorem 2.3: the support of C is $\mathbb{P}^1$, hence

$$s^\circ(C, \mathbb{P}^2) = s^\circ(\mathbb{P}^1, \mathbb{P}^2) = s(\mathbb{P}^1, \mathbb{P}^2) = [\mathbb{P}^1] - [\mathbb{P}^0]$$

as obtained 'by hand'.

3. The proof, and comments about Milnor classes

Theorem 2.3 can be proved rather indirectly, by applying one previous result and a piece of a well-established theory. Once more, our goal in this article is really to suggest that Theorem 2.3 ought to have a direct proof in terms of the theory of Segre classes, and that finding this argument would tell us something very interesting about Segre classes. Our proof does not shed much light in this direction.

We denote by $c_{\rm SM}$ the *Chern-Schwartz-MacPherson* class, cf. [Mac74] (and [Ken90] for a treatment over an arbitrary algebraically closed field of characteristic 0).

Theorem 3.1. *Let Z be a subscheme of a nonsingular variety M. Then*

$$c_{\rm SM}(Z_{red}) = c(TM) \cap s^\circ(Z, M) \quad .$$

Proof. The Chern-Schwartz-MacPherson class satisfies inclusion-exclusion. Indeed, if

$$Z = X_1 \cup \cdots \cup X_n \quad ,$$

then one easily checks that

$$1\!\!1_Z = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1 < \cdots < i_s} 1\!\!1_{X_{i_1} \cup \cdots \cup X_{i_s}} \quad ,$$

where $1\!\!1_X$ denotes the constructible function which is 1 at points of X, and 0 outside of X; and note $1\!\!1_X = 1\!\!1_{X_{\rm red}}$ trivially. Applying MacPherson's natural transformation gives then

$$c_{\rm SM}(Z_{\rm red}) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1 < \cdots < i_s} c_{\rm SM}((X_{i_1} \cup \cdots \cup X_{i_s})_{\rm red})$$

in A_*M. This observation reduces the statement of the theorem to the case in which Z is a hypersurface in a nonsingular variety, which is shown in [Alu99] (Theorem I.1 and Corollary II.2). □

Theorem 2.3 follows immediately as a corollary; the details are left to the reader.

We also note that since $s^\circ(Z,M) = c(TM)^{-1} \cap c_{\rm SM}(Z_{\rm red})$, it follows that $s^\circ(Z,M)$ is naturally the image of a class in A_*Z, although this is far from clear from Definition 2.2 (cf. Remark 2.6).

We end with comments regarding the so-called *Milnor class* of a scheme.

Theorem 3.1 writes the Chern-Schwartz-MacPherson class in a way that the informed reader will recognize immediately as a companion to the definition of an intrinsic class given by William Fulton (cf. [Ful84], 4.2.6 for the definition of this class and of a kindred notion, the *Fulton-Johnson* class): if Z is a subscheme of a nonsingular variety M, Fulton's class is defined by

$$c_F(Z) = c(TM) \cap s(Z,M) \quad .$$

According to several authors, but with slightly different conventions of sign and context, the *Milnor class* measures the difference between the Chern-Schwartz-MacPherson class of a variety and other classes such as Fulton's or Fulton-Johnson's. Determining this discrepancy has been identified as a Verdier-Riemann-Roch type problem, cf. [Yok99]. The hypersurface case is rather well understood, cf. [PP01]; complete intersections are treated in [BLSS99] and in the recent [Sch].

Our viewpoint on the problem is perhaps a little different from the one taken by these authors. For us, the Milnor class of Z should be measured by a Segre-class type of invariant defined on the singularity subscheme of Z; in fact our motivation in pursuing any such formula is precisely to learn something new about Segre classes. In this sense, the problem seems wide open for anything but hypersurfaces.

In the context of the present article, the relevant remark is the following consequence of Theorem 3.1: the Milnor class of a reduced scheme Z embedded in a nonsingular variety M is (up to sign)

$$c(TM) \cap m(Z,M) \quad ,$$

where

$$m(Z,M) := s^\circ(Z,M) - s(Z,M) \quad .$$

Note that the class $m(Z,M)$ is localized on the singularities of Z (see Remark 2.7).

From our perspective, the main problem in the study of Milnor classes is the explicit determination of $m(Z,M)$ in terms of conventional intersection-theoretic operations.

Example 3.2. If X is a hypersurface, with singularity subscheme Y, then

$$m(X,M) = c(\mathcal{O}(X))^{-1} \cap (s(Y,M)^\vee \otimes_M \mathcal{O}(X)) \quad .$$

This formula, now incorporated in the definition of s°, is the main result of [Alu94] and [Alu99]. Similarly explicit formulas for $m(Z,M)$ for more general Z, in terms of Chern/Segre-class type invariants of the singularity subscheme of Z, would be highly desirable. To our knowledge, it is not even known whether $m(Z,M)$—and hence the Milnor class of Z—is determined by the singularity subscheme of Z, *even when Z is a complete intersection of codimension 2.*

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