A MOTIVIC APPROACH TO PHASE TRANSITIONS IN POTTS MODELS

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Abstract. We describe an approach to the study of phase transitions in Potts models based on an estimate of the complexity of the locus of real zeros of the partition function, computed in terms of the classes in the Grothendieck ring of the affine algebraic varieties defined by the vanishing of the multivariate Tutte polynomial. We give completely explicit calculations for the examples of the chains of linked polygons and of the graphs obtained by replacing the polygons with their dual graphs. These are based on a deletion–contraction formula for the Grothendieck classes and on generating functions for splitting and doubling edges.

Contents

1. Introduction

It is well known that the partition function of a Potts model with q spin states on a graph G is given by the value at q of the multivariate Tutte polynomial of the graph, a famous combinatorial invariant of graphs. The problem of phase transitions in Potts models is related to the behavior of the sets of complex zeros and real zeros of these polynomials, see for instance [24].

In this paper we propose a new approach, based on algebraic geometry, and especially on motivic invariants such as classes in the Grothendieck ring of varieties, to study how the set of zeros of the partition function of a Potts model changes for a nested family of finite graphs that grow in size to approximate an infinite graph.

We aim at estimating the "topological complexity" of the set of real zeros by computing its Euler characteristic with compact support, which is also known to provide a lower bound for the algorithmic complexity of the real algebraic set. In order to compute this invariant and estimate its growth over certain explicit families of graphs, we use the fact that the invariant is not just topological but also "motivic", which means that it defines a ring homomorphism from the Grothendieck ring of varieties to the integers.

Thus, we proceed first to compute explicitly the classes in the Grothendieck ring of the varieties defined by the zeros of the Potts models partition functions, following techniques recently developed to treat a similar problem arising in perturbative quantum field theory, for algebraic varieties associated to the parametric form of Feynman integrals. The varieties arising in quantum field theory can be viewed, up to a duality, as a limit case of the ones arising from the Potts model partition functions.

We first prove an algebro-geometric inclusion-exclusion formula that relates the classes for a given graph to those of the graphs obtained by deleting or contracting one edge in the graph, and a more complicated algebro–geometric term, which is the variety defined by the intersection of the varieties of the deletion and the contraction. This formula is similar to an analogous result proved in [1] for the varieties arising from Feynman diagrams in quantum field theory.

We then show that, when iterating simple operations on graphs, such as splitting an edge or doubling an edge, the more complicated term in the algebro–geometric deletion contraction formula can be simplified due to cancellations in the Grothendieck ring and the resulting operation can be described completely in terms of varieties associated to purely combinatorial data on the graph. The corresponding recursive relation leads to explicit and remarkably simple generating functions for the classes of graphs obtained from a given graph by multiple edge splittings or edge doublings.

We use the formulae obtained in this way to write explicitly the classes in the Grothendieck ring for the loci of zeros of the partition function on Potts models over graphs given by chains of linked polygons. This class of Potts models was already studied by different techniques in the literature (see for instance [23] and references therein). Similarly, we compute the Grothendieck classes explicitly for similar chains where the polygons are replaced by their dual graphs, the banana graphs.

For these illustrative examples, we then use the expression for the Grothendieck class to compute explicitly the Euler characteristic with compact support, and we show that it grows exponentially as the graphs grow in size, thus estimating the corresponding growth in complexity of the set of real zeros of the partition function.

2. Potts models, multivariate Tutte polynomial and hypersurfaces

We recall some basic facts and terminology about Potts models that we need in the rest of the paper. For a more detailed introduction to partition functions of Potts models and their relation to graph combinatorics, we refer the reader to [24], see also the survey [5].

2.1. **Multivariate Tutte polynomial.** The multivariate Tutte polynomial of a finite graph G is defined as follows. Let $V = V(G)$ be the set of vertices of G and $E = E(G)$ the set of edges. We do not assume G to be connected. One assigns to each edge $e \in E$ a variable t_e and then considers the polynomial

$$Z_G(q,t) = \sum_{G' \subseteq G} q^{k(G')} \prod_{e \in E(G')} t_e, \tag{2.1}$$

where $k(G') = b_0(G')$ is the number of connected components, and the sum is over all subgraphs $G' \subseteq G$ that have the same number of vertices $V(G') = V(G)$ of G. Each subgraph G' therefore corresponds to a choice of a subset $A \subseteq E(G)$ of edges of G, so that $E(G') = A$. The variables t_e and the additional variable q are commuting variables.

A detailed account of the relation between the multivariate Tutte polynomial and the physics of Potts models is given in the survey [24]. To briefly recall the main point, in the case where q is a positive integer, one considers a q-state model on a graph G, where each vertex carries a "spin" that can take q possible values (the case $q = 2$ recovers the usual ± 1 states of spin). We let $\mathfrak{A}$ be the set of cardinality q of the possible spin states. A state of the system is an assignment of a spin state to each vertex and the energy H of a state is the sum over all edges of the graph of a quantity that is equal to zero if the spins assigned to the two endpoints of the edge are different and equal to an assigned value $-J_e$ if they are the same. With the notation $t_e = e^{\beta J_e} - 1$, where β is the thermodynamic parameter (an inverse temperature up to the Boltzmann constant), one has $t_e \geq 0$ in the ferromagnetic case ($J_e \geq 0$) and $-1 \leq t_e \leq 0$ in the antiferromagnetic case $-\infty \leq J_e \leq 0$. The partition function of the system is then the sum over all the possible states of the corresponding Boltzmann weight $e^{-\beta H}$, with H the energy of that state. This gives

$$Z_G(q,t) = \sum_{\sigma: V(G) \to \mathfrak{A}} \prod_{e \in E(G)} (1 + t_e \delta_{\sigma(v),\sigma(w)}), \tag{2.2}$$

where the sum is over all maps of vertices to spin states, v and w are the endpoints $\partial(e) = \{v, w\}$, and δ is the Kronecker delta.

It was shown by Fortuin–Kasteleyn [14] that (2.2) is the restriction to positive integer values of q of a polynomial function in q and that this polynomial is, in fact, the multivariate Tutte polynomial (2.1).

2.2. **Deletion–contraction relation.** As in the case of the ordinary Tutte polynomial, the multivariate Tutte polynomial (2.1) satisfies a deletion–contraction relation. Namely, given an edge $e \in E(G)$, let $G \smallsetminus e$ be the graph obtained by deleting the edge e and let G/e be the graph obtained by contracting it. One has the following formula.

Remark 2.1. The polynomial (2.1) satisfies

$$Z_G(q,t) = Z_{G \smallsetminus e}(q, \hat{t}) + t_e Z_{G/e}(q, \hat{t}), \tag{2.3}$$

where $\hat{t}$ consists of the edge variables with t_e removed. The deletion–contraction relation (2.3) covers all cases, including those where the edge e is a bridge or a looping edge.

2.3. **The problem of phase transitions.** Zeros of the multivariate Tutte polynomial are of special interest in relation to the problem of phase transitions of the statistical mechanical system described by the Potts model. In fact, the partition function $Z_G(q,t)$ becomes the normalization factor of the probability distribution on the set of all possible states of the system, and zeros of $Z_G(q,t)$ would signal the presence of a phase transition in the system, for a specific choice of parameters J_e and q, and for certain values of the inverse temperature β.

In the ferromagnetic case, the physical case of interest is where the variables $t_e \in \mathbb{R}_+$. For $q \geq 1$, the polynomial $Z_G(q,t)$ does not have zeros in that domain. The antiferromagnetic case with $-1 \leq t_e \leq 0$ is more interesting, and various results on zero-free regions are given in [16].

Even when there are no zeros in the region of direct physical interest, it is well known (see [16], [24]) that studying the *complex* zeros of the polynomials $Z_G(q,t)$ can provide useful information on phase transitions, not for a single graph G itself, but for a family of finite graphs G_n, such that $G_\infty = \cup_n G_n$ determines an infinite graph on which one still considers a statistical mechanical system obtained as a thermodynamic limit of the finite systems. If loci of complex zeros of the $Z_G(q,t)$ can approach the real locus in the limit, this will result in the presence of phase transitions for the system on G_∞.

Thus, the geometric problem we concentrate on is to understand and estimate how the loci of complex and real zeros, respectively, of the Potts model partition function change over certain families of finite graphs $\{G_n\}$ as above.

Our point of view, in this paper, is to approach this problem from an algebro–geometric and motivic point of view, inspired by recent developments on motivic properties of the loci of zeros of the closely related Kirchhoff graph polynomials in the setting of perturbative quantum field theory, see [1], [2], [3], [8], [9], [19].

2.4. **Potts model hypersurfaces.** Studying the zeros of the polynomial $Z_G(q,t)$, means understanding the geometry of the hypersurface defined by the equation $Z_G(q,t) = 0$. Since the polynomial is not homogeneous in its variables, it does not define a projective hypersurface (unlike the case of the graph polynomials in quantum field theory), but it does define an affine hypersurface, which we refer to as *the Potts model hypersurface*. In fact, we consider two types of hypersurfaces associated to the Potts model, one where the parameter q is treated as a variable $q \in \mathbb{A}$ along with the other edge variables t_e, and one where one specializes to a fixed value of q.

Definition 2.2. *Suppose given a finite graph G, with set of vertices $V(G)$ and set of edges $E(G)$. Let $\mathcal{Z}_G$ be the hypersurface in affine space $\mathbb{A}^{\#E(G)+1}$ defined by*

$$\mathcal{Z}_G := \{(q,t) \in \mathbb{A}^{\#E(G)+1} \,|\, Z_G(q,t) = 0\}. \tag{2.4}$$

For a fixed value of $q \in \mathbb{A}$, the hypersurface $\mathcal{Z}_{G,q}$ in $\mathbb{A}^{\#E(G)}$ is given by

$$\mathcal{Z}_{G,q} := \{t \in \mathbb{A}^{\#E(G)} \,|\, Z_G(q,t) = 0\}. \tag{2.5}$$

The hypersurface $\mathcal{Z}_{G,q}$ is therefore a slice of $\mathcal{Z}_G$ with the hyperplane in $\mathbb{A}^{\#E(G)+1}$ of fixed q-coordinate. In the physically significant cases, one wants to study the complex and the real zeros of the hypersurface $\mathcal{Z}_{G,q}$ where $q \in \mathbb{N}$ is a positive integer corresponding to the number of spin states of the Potts model.

Definition 2.3. *For a finite graph G, the* virtual phase transitions *of the Potts model are the real points $\mathcal{Z}_G(\mathbb{R})$ of the algebraic variety $\mathcal{Z}_G$. For a fixed q, the virtual phase transitions are the points of the real locus $\mathcal{Z}_{G,q}(\mathbb{R})$ of the variety $\mathcal{Z}_{G,q}$.*

We distinguish here between virtual phase transitions (all real zeros of the polynomial $Z_G(q,t)$) and the actual physical phase transitions, which would be constrained by the additional requirement that $q \in \mathbb{N}$ and the edge variables are $t_e \geq 0$ in the ferromagnetic case, or $-1 \leq t_e \leq 0$ in the antiferromagnetic case. Thus, for example, in the case of a finite graph, even if there are no physical phase transitions in the ferromagnetic case, there can still be a non-empty set of virtual phase transitions.

2.5. **The analogy with Quantum Field Theory.** In perturbative quantum field theory, the parametric form of Feynman integrals for massless scalar field theories can be expressed as a (possibly divergent) integral of an algebraic differential form on the complement of an algebraic hypersurface defined by the vanishing of a polynomial associated to the graph, the first Kirchhoff polynomial given by

$$\Psi_G(t) = \sum_{T \subseteq G} \prod_{e \notin T} t_e, \tag{2.6}$$

where $t = (t_1, \dots, t_n)$ are variables assigned to the edges of the graph and T runs over maximal spanning forests, that is, subgraphs of G with $V(T) = V(G)$, which are forests with $b_0(T) = b_0(G)$. (Note, the terminology "spanning forest" is used elsewhere for what we refer to here as "maximal spanning forests".)

In the literature on motivic aspects of Feynman integrals [6], [25], [26], it is also common to consider the related polynomial

$$\Phi_G(t) = \sum_{T \subseteq G} \prod_{e \in T} t_e, \tag{2.7}$$

where the sum is, as above, over the maximal spanning forests, but the product is on edges in the forest, instead of edges in the complement.

Definition 2.4. *We denote by $\mathcal{X}_G \subset \mathbb{A}^{\#E(G)}$ the affine hypersurface defined by the polynomial* (2.6) *and by $\bar{\mathcal{X}}_G \subset \mathbb{P}^{\#E(G)-1}$ the corresponding projective hypersurface.*

Similarly, we denote by $\mathcal{Y}_G \subset \mathbb{A}^{\#E(G)}$ the affine hypersurface defined by the polynomial (2.7) *and by $\bar{\mathcal{Y}}_G \subset \mathbb{P}^{\#E(G)-1}$ the corresponding projective hypersurface.*

Remark 2.5. One obtains $\Psi_G(t)$ from $\Phi_G(t)$ by dividing by $\prod_{e \in E(G)} t_e$ and changing variables by the transformation $t_e \mapsto 1/t_e$.

For a planar graph this operation relates the Kirchhoff polynomial of a graph with that of a dual graph, see the discussion on the Cremona transformation in [3].

Remark 2.6. It is also well known (see for instance [18], [24]) that the graph polynomial $\Phi_G(t)$ of (2.7) can be recovered from the multivariate Tutte polynomial by the following operations:

(1) Clear an overall factor of $q^{k(G)}$ with $k(G) = b_0(G)$ the number of connected components, that is, consider the normalized Potts partition function

$$\tilde{Z}_G(q,t) = q^{-k(G)} Z_G(q,t). \tag{2.8}$$

(2) Take the evaluation $\tilde{Z}_G(q,t)|_{q=0}$. This corresponds to a sum on subgraphs G' with $k(G') = k(G)$.
(3) Of this take then the homogeneous piece with the lowest degree in the $t = (t_e)$ variables. This corresponds to the sum over maximal spanning forests, that is, to the polynomial $\Phi_G(t)$.

In the context of quantum field theory, Tutte polynomials can also occur, for example where one considers scalar field theories on noncommutative spacetimes as in [18].

2.6. **The tangent cone.** Consider an affine hypersurface $X \subset \mathbb{A}^N$ given by the vanishing $X = \{t \in \mathbb{A}^N \mid P(t) = 0\}$ of a (possibly non-homogeneous) polynomial $P(t)$ whose leading term $P_k(t)$ (the term of lowest order in the variables $t = (t_i)$) is of some degree $k \geq 1$. Then the *tangent cone* to X at the origin, $\mathcal{TC}(X) = \mathcal{TC}_0(X)$ is also a hypersurface in $\mathbb{A}^N$, given by

$$\mathcal{TC}(X) = \{t \in \mathbb{A}^N \mid P_k(t) = 0\}, \tag{2.9}$$

the zero locus of the homogeneous polynomial P_k. This corresponds to the *normal cone* $\mathcal{NC}_0(X)$ for the subscheme given by the origin $0 \subset X$. The construction known as *deformation to the normal cone* provides a very useful algebro-geometric replacement of the notion of tubular neighborhoods of embedded subvarieties (subschemes), see [15].

In the case of a closed subscheme $Y \subset X$, one blows up the locus $Y \times \{0\}$ inside $X \times \mathbb{P}^1$. One then considers the complement

$$\tilde{X}_Y := Bl_{Y\times\{0\}}(X \times \mathbb{P}^1) \smallsetminus Bl_Y(X).$$

One obtains in this way a fibration $\tilde{X}_Y \to \mathbb{P}^1$, whose general fiber (away from 0) is naturally isomorphic to X, while the special fiber over zero is the tangent cone $\mathcal{TC}_Y(X)$. This has indeed the effect of deforming X to the tangent cone $\mathcal{TC}_Y(X)$.

For a finite graph G, we denote by P_G the homogeneous polynomial

$$P_G(q,t) = \text{ leading term of } Z_G(q,t), \tag{2.10}$$

in the variables $(q,t) \in \mathbb{A}^{\#E(G)+1}$ and by $\mathcal{V}_G$ the affine variety

$$\mathcal{V}_G = \{(q,t) \in \mathbb{A}^{\#E(G)+1} \mid P_G(q,t) = 0\}. \tag{2.11}$$

Since the polynomial P_G is homogeneous, we can also consider the projective hypersurface $\bar{\mathcal{V}}_G \subset \mathbb{P}^{\#E(G)}$. We also consider the affine hypersurfaces

$$\mathcal{V}_{G,q} = \{t \in \mathbb{A}^{\#E(G)} \mid P_G(q,t) = 0\}, \tag{2.12}$$

for fixed q. These are not homogeneous, in general, except in the case $q = 0$.

We then have the following rephrasing of Remark 2.6.

Lemma 2.7. *The variety $\mathcal{V}_G$ is the tangent cone of the variety $\mathcal{Z}_G$ at zero. It has a component given by the hyperplane $H = \{q = 0\}$ with multiplicity equal to $k(G) = b_0(G)$ and another component $\mathcal{W}_G$, which intersects the hyperplane H along the graph hypersurface $\mathcal{Y}_G$.*

Proof. The first statement follows directly from the definition (2.9) of the tangent cone.

The polynomial $P_G(q,t)$ as in (2.10) is the sum of terms of lowest degree in $Z_G(q,t)$. To see what they parameterize, note that if the subgraph G' determined by a set of edges $A \subseteq E$ is *not* a forest, then one or more of the edges may be removed from A without affecting the number of connected components, i.e., the exponent of q; while the degree of the product $\prod t_e$ decreases accordingly.

Further, assume that A is a forest. Then

$$k(A) + |A| = \#V(G).$$

Indeed, this is clear if $A = \emptyset$; and the left-hand side does not change if we add an edge connecting vertices without closing cycles ($|A|$ increases by 1, $k(A)$ decreases by 1 for each such operation). Therefore, all contributions of forests to $Z_G(q,t)$ have degree equal to the number of vertices of G, and this is the lowest possible degree.

Thus, the polynomial $P_G(q,t)$ collects the contribution of those terms of $Z_G(q,t)$ corresponding to subgraphs that are forests with $V(G') = V(G)$ (spanning forests),

$$P_G(q,t) = \sum_{G' \subseteq G,\, b_1(G')=0,\, \#V(G')=N} q^{k(G')} \prod_{e \in E(G')} t_e. \tag{2.13}$$

The hypersurface $\mathcal{W}_G$ is the locus of zeros of the polynomial $Q_G(q,t)$ satisfying $P_G(q,t) = q^{k(G)} Q_G(q,t)$, where q does not divide $Q_G(q,t)$.

The intersection $H \cap \mathcal{W}_G$ is then given by the locus of zeros of the polynomial

$$Q_G(0,t) = \sum_{G' \subseteq G,\ \max \text{ forest}} \prod_{e \in E(G')} t_e, \tag{2.14}$$

which is the polynomial $\Phi_G(t)$ of (2.7). □

2.7. **The Grothendieck ring of varieties and mixed Hodge structures.** The algebraic varieties $\mathcal{X}_G$ and $\mathcal{Y}_G$ associated to Feynman graphs have been studied extensively in recent years in terms of their classes in the Grothendieck ring of varieties, see for instance [6], [8], [25], [26]. We will be applying here analogous techniques to the Potts model hypersurfaces $\mathcal{Z}_G$ and $\mathcal{Z}_{G,q}$ and apply the results to the problem of phase transitions. Thus, we recall here a few things about the Grothedieck ring of varieties, and a result of [1], whose analog for Potts models hypersurfaces we prove in this paper and will be the basis of our motivic approach to phase transitions.

The Grothendieck ring $K_0(\mathcal{V}_{\mathbb{K}})$ of varieties over a field $\mathbb{K}$ is generated by isomorphism classes $[X]$ of smooth (quasi)projective varieties with the inclusion-exclusion relation

$$[X] = [Y] + [X \smallsetminus Y] \tag{2.15}$$

for any closed embedding of a subvariety $Y \subset X$, and with the product structure given by $[X \times Y] = [X][Y]$.

In the following, we will be interested in considering the Potts model hypersurfaces as varieties defined over $\mathbb{C}$, but we will also be focusing on their real zeros, hence thinking of them as varieties over $\mathbb{R}$. Thus, in the following we simply write $K_0(\mathcal{V})$ for the Grothendieck ring, whenever the arguments do not depend on what field we work over, and we will explicitly mention $\mathbb{C}$ or $\mathbb{R}$ when needed.

The class $[X]$ in the Grothendieck ring is a universal Euler characteristic for algebraic varieties (see [7]), in the sense that any invariant of isomorphism classes of algebraic varieties that satisfies the inclusion-exclusion relation and is multiplicative on products factors through the Grothendieck ring. These invariants are sometimes called *motivic*.

In particular, in the case of complex algebraic varieties and of classes in $K_0(\mathcal{V}_{\mathbb{C}})$, among these motivic invariants that factor through the Grothendieck ring we have the topological Euler characteristic, but also the virtual Hodge polynomials. These will be useful to us in the following so we recall briefly the definition.

The virtual Hodge polynomial of an algebraic variety is defined as

$$e(X)(x,y) = \sum_{p,q=0}^{d} e^{p,q}(X) x^p y^q, \quad \text{with} \quad e^{p,q}(X) = \sum_{k=0}^{2d} (-1)^k h^{p,q}(H^k_c(X)), \tag{2.16}$$

where, for each pair of integers (p,q), the term $h^{p,q}(H^k_c(X))$ is the Hodge number of the mixed Hodge structure on the cohomology with compact support of X. If X is smooth projective, then the virtual Hodge polynomial reduces to the Poincaré polynomial, with $e^{p,q}(X) = (-1)^{p+q} h^{p,q}(X)$ being the classical pure Hodge numbers.

The fact that the virtual Hodge polynomial factors through the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ of varieties means that an explicit formula for the class of a variety in the Grothendieck ring can be used to compute the virtual Hodge polynomial and obtain some explicit information on the Hodge numbers and the mixed Hodge structure.

2.8. **Virtual Betti numbers of real algebraic varieties.** As we mentioned above, in the case of complex algebraic varieties, the (ordinary) topological Euler characteristic is a motivic invariant. This is not true for real algebraic varieties, as the additive property over closed embeddings need not be satisfied. However, it is known that there is a unique motivic invariant that agrees with the topological Euler characteristic on compact smooth real algebraic varieties and is homeomorphism invariant (but not homotopy invariant), see [22] and also [13], [27]. It is defined, for any real (semi)algebraic set S, as

$$\chi_c(S) = \sum_k (-1)^k b_k^{BM}(S), \tag{2.17}$$

where $b_k^{BM}(S)$ are the Borel–Moore Betti numbers, namely the ranks of the relative homologies $H_k(\bar{S}, \infty)$, where $\bar{S}$ is the Alexandrov compactification. Equivalently, they are the ranks of the cohomology with compact support $H_c^*(S)$.

We consider here, as in [20], [21], real algebraic varieties in the sense of [10] and their Grothendieck ring $K_0(\mathcal{V}_\mathbb{R})$. The latter is defined in the usual way, as generated by isomorphism classes $[X]$ (as real varieties) with the inclusion-exclusion relation $[X] = [Y] + [X \smallsetminus Y]$ for closed subvarieties $Y \subset X$.

Example 2.8. Let $\mathbb{L} = [\mathbb{A}^1]$ be the Lefschetz motive, the class of the affine line in $K_0(\mathcal{V})$ and let $\mathbb{T} = [\mathbb{G}_m] = \mathbb{L} - 1$ be the class of the multiplicative group $\mathbb{G}_m = \mathbb{A}^1 \smallsetminus \{0\}$. The topological Euler characteristic $\chi : K_0(\mathcal{V}_\mathbb{C}) \to \mathbb{Z}$ satisfies $\chi(\mathbb{L}) = 1$ and $\chi(\mathbb{T}) = 0$, while the Euler characteristic with compact support $\chi_c : K_0(\mathcal{V}_\mathbb{R}) \to \mathbb{Z}$ satisfies $\chi_c(\mathbb{L}) = -1$ and $\chi_c(\mathbb{T}) = -2$.

Moreover, it is shown in [20] that the Betti numbers with $\mathbb{Z}/2\mathbb{Z}$ coefficients $b_k(X) = \dim H_k(X, \mathbb{Z}/2\mathbb{Z})$, defined in the usual way for compact smooth real algebraic varieties, extend in a unique way to $K_0(\mathcal{V}_\mathbb{R})$, so that one obtains a ring homomorphism

$$\beta : K_0(\mathcal{V}_\mathbb{R}) \to \mathbb{Z}[t], \tag{2.18}$$

such that $\beta(X, t) = \sum_k b_k(X) t^k$ for compact smooth varieties. The coefficients β_k of the ring homomorphism β are called the *virtual Betti numbers*. They are *not* topological invariants. However, they compute the Euler characteristic (2.17), namely,

$$\beta(X, -1) = \chi_c(X) \tag{2.19}$$

for all real algebraic varieties X, with both equal to the ordinary Euler characteristic $\chi(X)$ in the compact smooth case. Notice that, while $\chi_c(X)$ is the alternating sum of the ranks of the Borel–Moore homologies, the virtual Betti numbers $\beta_k(X)$ are in general not equal to the Borel–Moore Betti numbers $b_k^{BM}(X)$ (for instance, the $\beta_k(X)$ can be negative), although their alternating sums agree.

In the case of a compact smooth real algebraic variety, which is the real locus $X(\mathbb{R})$ of a smooth projective complex algebraic variety $X(\mathbb{C})$, there are ways to bound the "topological complexity" of $X(\mathbb{R})$ in terms of invariants of $X(\mathbb{C})$, in the form of Petrovskiĭ–Oleĭnik inequalities: for example, for $X(\mathbb{R})$ a smooth projective real algebraic variety of even dimension $n = 2p$, one has [4], [17]

$$|\chi(X(\mathbb{R})) - 1| < h^{p,p}(H^n(X(\mathbb{C}))). \tag{2.20}$$

This type of result was extended to cases with isolated singularities in [12], where one gets

$$|\chi(X(\mathbb{R})) - 1| \leq \begin{cases} \sum_{0\leq q\leq p} h^{q,q}(H_c^n(X(\mathbb{C}))) & n = 2p \\ \sum_{0\leq q\leq p} h^{q,q}(H_c^n(X(\mathbb{C}))) + h^{p+1,p+1}(H_c^{p+1}(X(\mathbb{C}))) & n = 2p+1. \end{cases} \tag{2.21}$$

However, more generally one does not have a Petrovskiĭ–Oleĭnik type inequalities to estimate the virtual Betti numbers and the Euler characteristic $\chi_c(X)$ of arbitrary real algebraic varieties in terms of the virtual Hodge polynomials of the complex variety. For complex varieties the virtual Betti numbers can be computed in terms of the virtual Hodge polynomial. In fact, one can introduce the weight k Euler characteristic given by setting

$$w_j^k(X(\mathbb{C})) = \sum_{p+q=j} h^{p,q}(H_c^k(X(\mathbb{C}))),$$

which equals $b_k(X)$ for $j = k$ and zero otherwise in the smooth projective case, and are otherwise equal to the ranks of the quotients of the weight filtration $w_j^k(X(\mathbb{C})) = \dim_{\mathbb{C}} W_j^k(X)/W_{j-1}^k(X(\mathbb{C}))$ on the cohomology with compact support $H_c^k(X(\mathbb{C}))$. Then for arbitrary complex algebraic varieties the virtual Betti numbers are given by ([20])

$$\beta_j(X(\mathbb{C})) = (-1)^j \sum_k (-1)^k w_j^k(X(\mathbb{C})).$$

In general one does not have a good way to estimate the virtual Betti numbers $\beta_k(X(\mathbb{R}))$ of a real algebraic variety, nor their alternating sum $\chi_c(X(\mathbb{R}))$, in terms of the virtual Betti numbers $\beta_k(X(\mathbb{C}))$.

Although in general one cannot estimate $\chi_c(X(\mathbb{R}))$ in terms of a suitable Petrovskiĭ–Oleĭnik type inequality, we will show that in certain cases one can compute explicitly both $\chi_c(X(\mathbb{R}))$ and the virtual Hodge numbers of $X(\mathbb{C})$ as a consequence of being able to compute explicitly the class $[X]$ in the Grothendieck ring of varieties.

We will discuss later how these considerations relate to the problem of phase transitions in Potts models. In particular, we will see that, by working with classes in the Grothendieck ring, we obtain some estimates on the topological complexity of the set of virtual phase transitions of the Potts model over certain families of finite graphs G_n approximating some infinite graph $G = \cup_n G_n$.

2.9. **Grothendieck classes of Potts model hypersurfaces.** We also introduce the following notation for the classes in the Grothendieck ring of the Potts model hypersurfaces.

Definition 2.9. *Let* $[\mathcal{Z}_G]$ *be the class in the Grothendieck ring* $K_0(\mathcal{V})$ *of the Potts model hypersurface* (2.4)*. Also let* $\{\mathcal{Z}_G\}$ *denote the class of the hypersurface complement,*

$$\{\mathcal{Z}_G\} = [\mathbb{A}^{\#E(G)+1} \smallsetminus \mathcal{Z}_G] = \mathbb{L}^{\#E(G)+1} - [\mathcal{Z}_G], \tag{2.22}$$

where $\mathbb{L} = [\mathbb{A}^1]$ *is the Lefschetz motive (the class of the affine line). The classes* $[\mathcal{Z}_{G,q}]$ *and* $\{\mathcal{Z}_{G,q}\} = [\mathbb{A}^{\#E(G)} \smallsetminus \mathcal{Z}_{G,q}]$ *are similarly defined for the hypersurface* $\mathcal{Z}_{G,q}$ *of* (2.5)*.*

As in the case of the graph hypersurfaces of Feynman graphs (see [1], [2], [6]), we will see that for Potts models it is simpler to write explicit formulae for the class of hypersurface complement $\{\mathcal{Z}_G\}$ than for the class $[\mathcal{Z}_G]$ of the hypersurface itself, though the information is clearly equivalent due to the simple relation (2.22) between them.

2.10. **The Grothendieck class for fixed q and the fibration condition.** We discuss here the relation between the classes of the hypersurface complement $\{\mathcal{Z}_G\}$ and $\{\mathcal{Z}_{G,q}\}$ for the full Potts model hypersurface and for the one with fixed q. We identify a useful condition, according to which the the class of $\{\mathcal{Z}_G\}$ behaves as one would expect in the case of a fibration on the locus $q \neq 0, 1$. We will later identify specific classes of graphs we want to work with and check that they satisfy this condition. We do not address in this paper the question of how general this condition actually is, nor the question of whether the variety $\mathcal{Z}_G$ itself really is a locally trivial fibration over the locus $q \neq 0, 1$, at least for some specific families of graphs.

One can see directly from the polynomial $Z_G(q, t)$ why $q = 0$ and $q = 1$ should certainly be special values, for the following reasons. Recall that the equation we are dealing with is

$$Z_G(q, t) = \sum_{A \subseteq E(G)} q^{k(A)} \prod_{a \in A} t_a.$$

- For $q = 0$ and G nonempty, this is 0: indeed, $k(A) > 0$ for every subset A of edges in that case. This just says that the hypersurface $Z_G(qt) = 0$ has a component along $q = 0$; this component can be removed (dividing $Z_G(q,t)$ by $q^{k(G)}$) and the residual hypersurface may be studied over $q = 0$. This is the hypersurface $Q_G(0,t) = 0$ of (2.14), which is the dual $\mathcal{Y}_G$ of the graph hypersurface $\mathcal{X}_G$ as in Definition 2.4. This falls back on the case investigated in [2], [1].
- For $q = 1$, the polynomial becomes

$$Z_G(1,t) = \sum_{A \subseteq E} \prod_{a \in A} t_a = \prod_{e \in E} (1 + t_e). \tag{2.23}$$

  This is a union of normal crossing divisors, and in fact it consists of essentially n coordinate hyperplanes in $\mathbb{A}^n$. The complement is the set of n-tuples $(t_1, \ldots, t_n)$ ($n = \#E(G)$) with each $t_i + 1 \neq 0$, a copy of the n-torus. Thus the class of its complement is $\mathbb{T}^n$.

The condition that the class $\{\mathcal{Z}_G\}$ behaves as in the case of a fibration over the locus $q \neq 0, 1$ can then be formulated as the condition that

- The class $\{\mathcal{Z}_{G,q}\}$ is independent of q for $q \neq 0, 1$; this class will be denoted $\{\mathcal{Z}_{G,q\neq 0,1}\}$;
- The following holds:

$$\{\mathcal{Z}_G\} = (\mathbb{T} - 1)\{\mathcal{Z}_{G,q\neq 0,1}\} + \mathbb{T}^{\#E(G)}. \tag{2.24}$$

This accounts for the fact that the complement of $Z_G = 0$ is contained in $q \neq 0$, has a torus fiber over $q = 1$, and (heuristically) fibers over $q \neq 0, 1$, a copy of $\mathbb{T} - 1$, with constant fiber class.

In particular, a necessary condition for (2.24) is that $(\mathbb{T} - 1)$ divides $\{\mathcal{Z}_G\} - \mathbb{T}^{\#E(G)}$ in the Grothendieck ring, so any counterexample to this property would give examples where the fibration condition (2.24) does not hold.

3. Deletion–contraction for classes in the Grothendieck ring

In [1] it was shown that, in the case of the graph hypersurface complements $\mathbb{A}^{\#E(G)} \smallsetminus \mathcal{X}_G$, the classes in the Grothendieck ring satisfy an algebro-geometric analog of a deletion–contraction relation. More precisely, it was proved in [1] that, for a graph G with $n = \#E(G)$ and for a given edge $e \in E(G)$, the classes of the varieties $\mathcal{X}_G$, $\mathcal{X}_{G/e}$ and $\mathcal{X}_{G \smallsetminus e}$ are related by

$$[\mathbb{A}^n \smallsetminus \mathcal{X}_G] = \mathbb{L}\,[\mathbb{A}^{n-1} \smallsetminus (\mathcal{X}_{G\smallsetminus e} \cap \mathcal{X}_{G/e})] - [\mathbb{A}^{n-1} \smallsetminus \mathcal{X}_{G\smallsetminus e}], \tag{3.1}$$

when e is neither a bridge nor a looping edge, and

$$[\mathbb{A}^n \smallsetminus \mathcal{X}_G] = \mathbb{L}\,[\mathbb{A}^{n-1} \smallsetminus \mathcal{X}_{G/e}] = \mathbb{L}\,[\mathbb{A}^{n-1} \smallsetminus \mathcal{X}_{G\smallsetminus e}]$$

when e is a bridge and

$$[\mathbb{A}^n \smallsetminus \mathcal{X}_G] = (\mathbb{L} - 1)[\mathbb{A}^{n-1} \smallsetminus \mathcal{X}_{G/e}] = (\mathbb{L} - 1)[\mathbb{A}^{n-1} \smallsetminus \mathcal{X}_{G\smallsetminus e}]$$

when e is a looping edge, where $\mathbb{L} = [\mathbb{A}^1]$ is the Lefschetz motive, as above.

Notice how (3.1) is not a combinatorial deletion–contraction formula: indeed, the term involving the intersection $\mathcal{X}_{G\smallsetminus e} \cap \mathcal{X}_{G/e}$ of the hypersurfaces of the deletion and the contraction is in general difficult to control explicitly, even if one has an explicit formula for the classes of the deletion and the contraction separately. However, it was also shown in [1] that, for certain families of graphs, such as chains of polygons, one obtains recursive relations in the Grothendieck ring, where the "problematic" term in the deletion–contraction formula cancels out and one obtains an explicit generating function for the classes of the varieties associated to the family of graphs. The result provides a way to control how the class in the Grothendieck ring grows in complexity when the graph is enlarged through some simple operations, such as doubling edges or splitting edges. In the setting of quantum field theory the families of graphs obtained through such simple operations, like the chains of polygons, are typically not complex enough to capture interesting behaviors of the associated periods, but we will argue here that analogous operations performed in the setting of Potts models already gives rise to interesting non-trivial cases.

3.1. **Algebro-geometric deletion–contraction for Potts model hypersurfaces.** We prove here an analog of the deletion-contraction formula (3.1) for the classes in the Grothendieck ring of the Potts model hypersurfaces. We first analyze the case of the full $\mathcal{Z}_G$ and then the case of $\mathcal{Z}_{G,q}$ with fixed q and of the tangent cone $\mathcal{V}_G$ at zero.

We now state our main result on the deletion–contraction properties for Potts model hypersurfaces. Notice that, in the following statement, there is no distinction between the case of bridges or looping edges and all the other edges, just as in the combinatorial deletion–contraction relation for the multivariate Tutte polynomials.

Theorem 3.1. *Let G be a finite graph and e an edge of G. Then the class $\{\mathcal{Z}_G\}$ of (2.22) satisfies*

$$\{\mathcal{Z}_G\} = \mathbb{L}\{\mathcal{Z}_{G/e} \cap \mathcal{Z}_{G\smallsetminus e}\} - \{\mathcal{Z}_{G/e}\}. \tag{3.2}$$

Proof. The result follows from the combinatorial deletion–contraction relation for the multivariate Tutte polynomials

$$Z_G(q,t) = Z_{G\smallsetminus e}(q,\hat{t}^{(e)}) + t_e\, Z_{G/e}(q,\hat{t}^{(e)}),$$

where $\hat{t}^{(e)}$ is the set of variables $t = (t_{e'})_{e'\in E(G)}$ with the variable t_e omitted. We then check the various cases.

• If $Z_{G/e}(q,\hat{t}^{(e)}) \neq 0$, then $Z_G(q,t)$ is guaranteed to be $\neq 0$ if t_e does not equal $-Z_{G\smallsetminus e}(q,\hat{t}^{(e)})/Z_{G/e}(q,\hat{t}^{(e)})$. This accounts for a $\mathbb{G}_m$ worth of t_e's for each such $(q,\hat{t}^{(e)})$, contributing a class

$$(\mathbb{L}-1)\{\mathcal{Z}_{G/e}\}.$$

• If $Z_{G/e}(q,\hat{t}^{(e)}) = 0$, then $Z_G(q,t) \neq 0$ if and only if $Z_{G\smallsetminus e}(q,\hat{t}^{(e)}) \neq 0$. This accounts for an $\mathbb{A}^1$ worth of t_e's for each $(q,\hat{t}^{(e)})$ such that $Z_{G/e}(q,\hat{t}^{(e)}) = 0$ and $Z_{G\smallsetminus e}(q,\hat{t}^{(e)}) \neq 0$. This contributes a class

$$\mathbb{L}\cdot[\mathcal{Z}_{G/e} \smallsetminus (\mathcal{Z}_{G/e}\cap \mathcal{Z}_{G\smallsetminus e})].$$

Note that

$$\begin{aligned}[]
[\mathcal{Z}_{G/e}] - [\mathcal{Z}_{G/e}\cap\mathcal{Z}_{G\smallsetminus e}] &= \mathbb{L}^{\#E(G)} - [\mathcal{Z}_{G/e}\cap\mathcal{Z}_{G\smallsetminus e}] - \mathbb{L}^{\#E(G)} + [\mathcal{Z}_{G/e}] \\
&= \{\mathcal{Z}_{G/e}\cap\mathcal{Z}_{G\smallsetminus e}\} - \{\mathcal{Z}_{G/e}\}.
\end{aligned}$$

Thus, the two contributions add up to

$$\{\mathcal{Z}_G\} = (\mathbb{L}-1)\{\mathcal{Z}_{G/e}\} + \mathbb{L}\left(\{\mathcal{Z}_{G/e}\cap\mathcal{Z}_{G\smallsetminus e}\} - \{\mathcal{Z}_{G/e}\}\right) = \mathbb{L}\{\mathcal{Z}_{G/e}\cap\mathcal{Z}_{G\smallsetminus e}\} - \{\mathcal{Z}_{G/e}\},$$

as claimed. ∎

The following properties of the classes of Potts model hypersurfaces are simple consequences of the definitions, or follow easily from Theorem 3.1.

Corollary 3.2. *The classes $\{\mathcal{Z}_G\} \in K_0(\mathcal{V})$ of (2.22) satisfy the following properties:*

1. *If G consists of a single vertex and no edges, then $\{\mathcal{Z}_G\} = \mathbb{L}-1$.*
2. *If G consists of a single edge, with either one or two vertices, then $\{\mathcal{Z}_G\} = (\mathbb{L}-1)^2$.*
3. *If a graph G' is the union $G' = G_1 \cup_v G_2$ of two graphs joined at a vertex v, and G'' denotes the disjoint union of the same two graphs, then $\{\mathcal{Z}_{G'}\} = \{\mathcal{Z}_{G''}\}$.*
4. *If G is obtained by joining G_1 and G_2 with a single edge from a vertex of G_1 to a vertex of G_2, then $\{\mathcal{Z}_G\} = (\mathbb{L}-1)\{\mathcal{Z}_{G'}\}$, with $G' = G_1 \cup_v G_2$ as above.*

(5) *If $\overline{G}$ is obtained from a graph G by appending a single (looping or otherwise) edge to a vertex, then* $\{\mathcal{Z}_{\overline{G}}\} = (\mathbb{L}-1)\{\mathcal{Z}_G\}$.

Proof. (1) If G consists of a single vertex, then $Z_G = q$ defines a point in the affine line $\mathbb{A}^1$ with coordinate q.

(2) If G is a single edge joining two distinct vertices, then $Z_G(q,t) = qt + q^2$ and therefore $\{\mathcal{Z}_G\} = \mathbb{L}^2 - 2\mathbb{L} + 1 = (\mathbb{L}-1)^2$.

If G is a single looping edge, then $\mathcal{Z}_G(q,t) = qt + q$ and again $\{\mathcal{Z}_G\} = (\mathbb{L}-1)^2$.

(3) If G' consists of two graphs G_1 and G_2 joined at a vertex, then $Z_{G'}(q,t) = \frac{1}{q} Z_{G_1} Z_{G_2}$. If G'' consists of the disjoint union of G_1 and G_2, then $Z_{G''}(q,t) = Z_{G_1} Z_{G_2}$. It follows (if none of the graphs is empty) that $\{\mathcal{Z}_{G'}\} = \{\mathcal{Z}_{G''}\}$, and that $\{\mathcal{Z}_{G'} \cap \mathcal{Z}_{G''}\} = \{\mathcal{Z}_{G'}\}$.

(4) Applying Theorem 3.1, this implies that if G is obtained by joining G_1 and G_2 with a single edge from a vertex of G_1 to a vertex of G_2, then $\{\mathcal{Z}_G\} = (\mathbb{L}-1)\{\mathcal{Z}_{G'}\}$.

(5) Let G' be the disjoint union of G and of a single vertex. Then $Z_{G'} = qZ_G$, and in particular $\{Z_{G'}\} = \{Z_G\}$, and $Z_{G'} \cap Z_G = Z_G$. Applying Theorem 3.1, we see that if $\overline{G}$ is obtained by appending a single non-looping edge to a vertex of G, then $\{\mathcal{Z}_{\overline{G}}\} = (\mathbb{L}-1)\{\mathcal{Z}_G\}$. The same conclusion is reached if $\overline{G}$ is obtained from G by adding a looping edge. □

Notice that, unlike what happens with the graph hypersurfaces of Feynman diagrams (see [2]), in the case of a disjoint union of graphs the class of $\{\mathcal{Z}_{G_1\cup G_2}\}$ is not the product of the classes of the hypersurface complements of the two graphs, since here the polynomials Z_{G_1} and Z_{G_2} have the same variable q in common. However, if the graph $G = G_1 \cup G_2$ is given by the disjoint union of two graphs G_1 and G_2, and all the graphs involved satisfy the fibration condition (2.24), then we can obtain an explicit formula for the class $\{\mathcal{Z}_{G_1\cup G_2}\}$.

Corollary 3.3. *Let $G = G_1 \cup G_2$ be the disjoint union of two finite graphs G_1 and G_2, and assume that the classes $\{\mathcal{Z}_G\}$ and $\{\mathcal{Z}_{G_i}\}$ satisfy the fibration condition* (2.24). *Then the class $\{\mathcal{Z}_G\}$ can be expressed explicitly in terms of the classes $\{\mathcal{Z}_{G_1}\}$ and $\{\mathcal{Z}_{G_2}\}$ by*

$$\{\mathcal{Z}_{G_1\cup G_2}\} = \frac{\{\mathcal{Z}_{G_1}\}\cdot\{\mathcal{Z}_{G_2}\} - \mathbb{T}^{\#E(G_1)}\cdot\{\mathcal{Z}_{G_2}\} - \mathbb{T}^{\#E(G_2)}\cdot\{\mathcal{Z}_{G_1}\} + \mathbb{T}^{\#E(G_1)+\#E(G_2)+1}}{\mathbb{T}-1}. \tag{3.3}$$

Proof. For q fixed, the remaining variables are indeed distinct for disjoint G_1, G_2. Thus, the classes satisfy

$$\{\mathcal{Z}_{G_1\cup G_2,q}\} = \{\mathcal{Z}_{G_1,q}\}\cdot\{\mathcal{Z}_{G_2,q}\}. \tag{3.4}$$

(This holds for all q, including the special values $q=0$ and $q=1$.) If the classes $\{\mathcal{Z}_{G_i,q}\}$ are independent of $q \neq 0,1$ and the formula (2.24) holds, then we get

$$\begin{aligned}
\{\mathcal{Z}_{G_1\cup G_2}\} &= (\mathbb{T}-1)\{\mathcal{Z}_{G_1\cup G_2,q\neq 0,1}\} + \mathbb{T}^{\#E(G_1)+\#E(G_2)} \\
&= (\mathbb{T}-1)(\{\mathcal{Z}_{G_1,q\neq 0,1}\}\cdot\{\mathcal{Z}_{G_2,q\neq 0,1}\}) + \mathbb{T}^{\#E(G_1)+\#E(G_2)} \\
&= (\mathbb{T}-1)\left(\frac{\{\mathcal{Z}_{G_1}\} - \mathbb{T}^{\#E(G_1)}}{\mathbb{T}-1}\cdot\frac{\{\mathcal{Z}_{G_2}\} - \mathbb{T}^{\#E(G_2)}}{\mathbb{T}-1}\right) + \mathbb{T}^{\#E(G_1)+\#E(G_2)} \\
&= \frac{\{\mathcal{Z}_{G_1}\}\cdot\{\mathcal{Z}_{G_2}\} - \mathbb{T}^{\#E(G_1)}\cdot\{\mathcal{Z}_{G_2}\} - \mathbb{T}^{\#E(G_2)}\cdot\{\mathcal{Z}_{G_1}\} + \mathbb{T}^{\#E(G_1)+\#E(G_2)+1}}{\mathbb{T}-1}
\end{aligned}$$

□

3.2. **Deletion–contraction for fixed q.** Deletion–contraction works exactly as in the case of the full Potts model hypersurface $\mathcal{Z}_G$.

Proposition 3.4. *For a finite graph G, the class $\{\mathcal{Z}_{G,q}\}$ for fixed q satisfies*

$$\{\mathcal{Z}_{G,q}\} = (\mathbb{T}+1)\{\mathcal{Z}_{G/e,q} \cap \mathcal{Z}_{G\smallsetminus e,q}\} - \{\mathcal{Z}_{G/e,q}\}. \tag{3.5}$$

The argument is identical to the one used in the proof of Theorem 3.1. (For $q = 0$, all classes equal 0.) We then also have the analog of Corollary 3.2.

Corollary 3.5. *For $q \neq 0$, the classes $\{\mathcal{Z}_{G,q}\}$ satisfy the following properties:*

(1) *For G a single vertex, $\{\mathcal{Z}_{G,q}\} = 1$.*
(2) *For G a single edge joining either one or two vertices, $\{\mathcal{Z}_{G,q}\} = \mathbb{T}$.*
(3) *If G' consists of two graphs G_1 and G_2 joined at a vertex, or disjoint, then $\{\mathcal{Z}_{G',q}\} = \{\mathcal{Z}_{G_1,q}\} \cdot \{\mathcal{Z}_{G_2,q}\}$.*
(4) *If G is obtained by joining G_1 and G_2 with a single edge from a vertex of G_1 to a vertex of G_2, then $\{\mathcal{Z}_{G,q}\} = \mathbb{T}\{\mathcal{Z}_{G',q}\}$.*
(5) *Appending a single edge to a vertex of G multiplies the class $\{\mathcal{Z}_{G,q}\}$ by $\mathbb{T}$.*

Proof. (1) For G a single vertex, $Z_G(q,t) = q \neq 0$; thus $\mathcal{Z}_{G,q} = \emptyset \subset \mathbb{A}^0$, and $\{\mathcal{Z}_{G,q}\} = 1$.

(2) For G a single edge joining two distinct vertices, $Z_G(q,t) = qt + q^2$ defines a point in $\mathbb{A}^1$; thus $\{\mathcal{Z}_{G,q}\} = \mathbb{T}$. For G a single looping edge, $Z_G(q,t) = qt + q$ and again $\{\mathcal{Z}_{G,q}\} = \mathbb{T}$.

(3) This case is simpler than the case for $\{\mathcal{Z}_G\}$ with indeterminate q. It follows, as in (3.4), from the fact that the polynomials $Z_{G_1}(q,t)$ and $Z_{G_2}(q,t)$ have none of the variables other than the fixed q in common.

(4) This again follows from the simpler formula for unions as in (3), and from the computation for a single edge. The relation is the same as in the indeterminate-q case.

(5) This is again the same formula as in the free case, which here follows from (3) and the case of a single edge. □

3.3. **Deletion–contraction for the tangent cone.** In the case of the tangent cone $\mathcal{V}_G$ at zero of the variety $\mathcal{Z}_G$, which, as we have seen interpolates between the Potts model and the graph hypersurfaces considered in the quantum field theory context, it is convenient to introduce the following notation for the classes in the Grothendieck ring.

We still denote by $[\mathcal{V}_G]$ the class of $\mathcal{V}_G$ and by $\{\mathcal{V}_G\} = [\mathbb{A}^{\#E(G)+1} \smallsetminus \mathcal{V}_G]$ the class of the complement. We also use the notation $\mathcal{Y}_G$ for the graph hypersurface given by the intersection of the component $\mathcal{W}_G = \{Q_G(q,t) = 0\}$ with the hyperplane $H = \{q = 0\}$. Thus, $\mathcal{Y}_G$ is the locus of zeros of $Q_G(0,t)$, where $P_G(q,t) = q^{k(G)} Q_G(q,t)$. This gives, at the level of the classes

$$[\mathcal{V}_G] = [\mathcal{W}_G] + \mathbb{L}^{\#E(G)} - [\mathcal{Y}_G], \tag{3.6}$$

where $\mathbb{L}^{\#E(G)} = [H]$ and $\mathcal{V}_G = \mathcal{W}_G \cup H$ with $\mathcal{Y}_G = \mathcal{W}_G \cap H$. Thus, we obtain the following.

Lemma 3.6. *The class $\{\mathcal{V}_G\}$ of the complement of the tangent cone $\mathcal{V}_G$ in $\mathbb{A}^{\#E(G)+1}$ is given by the class $\{\mathcal{V}_G\} = \{\mathcal{W}_G\} - \{\mathcal{Y}_G\}$.*

Proof. Using (3.6) and taking complements, we obtain

$$\mathbb{L}^{\#E(G)+1} - [\mathcal{V}_G] = \mathbb{L}^{\#E(G)+1} - [\mathcal{W}_G] - \mathbb{L}^{\#E(G)} + [\mathcal{Y}_G] = \{\mathcal{W}_G\} - \{\mathcal{Y}_G\}.$$

Alternatively, one may simply observe that the complement of $\mathcal{W}_G$ is the disjoint union of $\mathcal{V}_G$ and cY_G. □

We can then see directly that the class $\{\mathcal{V}_G\}$ satisfies the following simple properties.

Lemma 3.7. *The class $\{\mathcal{V}_G\}$ satisfies:*

(1) *If G' is obtained by attaching a looping edge to a vertex of G, then*
$$\{\mathcal{V}_{G'}\} = (\mathbb{T} + 1)\{\mathcal{V}_G\}.$$
(2) *If G' is obtained by appending a non-looping edge to a vertex of G, then*
$$\{\mathcal{V}_{G'}\} = \mathbb{T}\{\mathcal{V}_G\}.$$

(3) *More generally, if G is obtained by connecting two disjoint graphs by a bridge e, then $\{\mathcal{V}_G\} = \mathbb{T}\{\mathcal{V}_{G/e}\}$.*

(4) *If G' is obtained by attaching an edge parallel to one of the edges of G, then*

$$\{\mathcal{V}_{G'}\} = (\mathbb{T}+1)\{\mathcal{V}_G\}.$$

Proof. (1) The polynomial P_G is only counting forests and loops are excluded from forests, so the polynomial $P_{G'}$ equals P_G, but it is viewed in one dimension higher. This simply multiplies everything by $\mathbb{L} = \mathbb{T}+1$.

(2) Attaching an unconnected edge e multiplies P_G by $(q+t_e)$. The condition $(t+q)P_G \neq 0$ implies $P_G \neq 0$ and $t+q \neq 0$. This says that the complement to $P_{G'} = 0$ fibers over the complement to $P_G = 0$, with $\mathbb{T}$ fibers.

(3) The same argument proves this assertion.

(4) Let e be the edge of G that we are doubling, and call f the new parallel edge. Then $P_{G'}$ is obtained from P_G by replacing t_e by $t_e + t_f$. This operation amounts to taking a cylinder, hence multiplying everything by $\mathbb{L} = \mathbb{T}+1$. □

We then look at the deletion contraction formula. In the cases of $\{\mathcal{Z}_G\}$ and $\{\mathcal{Z}_{G,q}\}$ considered above, we did not have to make a special case for bridges and looping edges because the combinatorial deletion–contraction formula for the multivariate Tutte polynomial does not make such a distinction. However, in the case of the tangent cone, as in the case of the graph hypersurfaces of Feynman diagrams of [1], we need to take these two special cases into account separately.

When the edge e is neither a bridge nor a looping edge, the deletion-contraction formula for the multivariate Tutte polynomial specializes to one for P_G:

$$P_G(q,t) = P_{G\smallsetminus e}(q,\hat{t}^{(e)}) + t_e\, P_{G/e}(q,\hat{t}^{(e)}). \tag{3.7}$$

In this case the numbers of connected components of G, $G\smallsetminus e$, G/e are all equal. Therefore, the same formula holds for Q_G. It also holds once q is set to 0 in the latter.

We say that an edge is a *regular edge* if it is neither a bridge nor a looping edge.

Proposition 3.8. *Assume e is a regular edge of G. Then*

$$\{\mathcal{W}_G\} = \mathbb{L}\cdot[\mathbb{A}^{\#E(G)} - (\mathcal{W}_{G\smallsetminus e}\cap \mathcal{W}_{G/e})] - \{\mathcal{W}_{G/e}\} \quad ,$$
$$\{\mathcal{Y}_G\} = \mathbb{L}\cdot[\mathbb{A}^{\#E(G)-1} - (\mathcal{Y}_{G\smallsetminus e}\cap \mathcal{Y}_{G/e})] - \{\mathcal{Y}_{G/e}\}.$$

and the formula for $\{\mathcal{V}_G\}$ is then the difference of these.

Proof. The proof is entirely analogous to the one given for $\{\mathcal{Z}_G\}$, applied to the polynomials Q_G and $Q_G|_{q=0}$. □

The cases of bridges and looping edges are already dealt with in Lemma 3.7; we repeat them here for clarity:

Proposition 3.9. *If e is a looping edge of G, then*

$$\{\mathcal{V}_G\} = (\mathbb{T}+1)\{\mathcal{V}_{G\smallsetminus e}\}.$$

If e is a bridge, then

$$\{\mathcal{V}_G\} = \mathbb{T}\,\{\mathcal{V}_{G/e}\}.$$

4. Edge splitting

We now use the deletion–contraction formula of Theorem 3.1 to describe the effect of splitting an edge in a graph.

Definition 4.1. *Given a finite graph G and an edge $e \in E(G)$, let ${}^0G$ denote the contraction G/e; let ${}^1G = G$; and more generally let ${}^kG$ be the graph obtained by replacing the edge e with a chain of $k \geq 2$ edges.*

4.1. **Splitting an edge.** We have the following formula for the class of the hypersurface complement of the graph ${}^2G$. Here and in the following, we denote by $V(f_1, \ldots, f_k)$ the zero locus $\{f_1 = \cdots = f_k = 0\}$ of the ideal generated by $(f_1, \ldots, f_k)$.

Theorem 4.2. *For G a finite graph and ${}^2G$ the graph obtained by splitting an edge e in G, the class $\{Z_{{}^2G}\}$ satisfies*

$$\{\mathcal{Z}_{{}^2G}\} = (\mathbb{T}-2)\{\mathcal{Z}_{{}^1G}\} + (\mathbb{T}-1)\{\mathcal{Z}_{{}^0G}\} + (\mathbb{T}+1)\left(\{\mathcal{Z}_{G\smallsetminus e}\} + \{A^e_G\}\right), \tag{4.1}$$

where $A^e_G = V(q + t_e, Z_{G\smallsetminus e} - qZ_{G/e})$.

The term $\{A^e_G\}$ appears to be difficult to evaluate geometrically. Equation 4.1 gives a relation between this term and the terms $\{Z_{{}^mG}\}$, which will allow us to obtain recursive formulas for these classes which are independent of $\{A^e_G\}$.

Proof. First observe that the effect of attaching an edge e to a vertex of a graph G is to multiply the polynomial $Z_G(q,t)$ by $(t_e + q)$ in the case of a non-looping edge, and by $(t_e + 1)$ in the case of a looping edge. This has the effect, in both cases, of multiplying the class $\{\mathcal{Z}_G\}$ by the factor $\mathbb{L} - 1$ as seen in Corollary 3.2, (5).

Applying Theorem 3.1 to the graph ${}^2G$ requires handling the contraction, ${}^1G$ in this case, and the deletion, which is obtained from $G \smallsetminus e$ by adding a non-looping edge. In terms of equations, these are given respectively by the vanishing of $Z_G(q,t)$ and of $Z_{G\smallsetminus e}(q, \hat{t}^{(e)}) \cdot (q + t_e)$. The most interesting term is, of course, the class of the intersection of these two hypersurfaces, with ideal

$$(Z_{G\smallsetminus e}(q, \hat{t}^{(e)})(q + t_e), Z_G(q,t)). \tag{4.2}$$

This ideal defines a subscheme of $\mathbb{A}^{\#E(G)+1}$, while $\mathcal{Z}_{{}^2G}$ lives in $\mathbb{A}^{\#E(G)+2}$. Deletion-contraction applied to G gives

$$Z_G = Z_{G\smallsetminus e} + t_e Z_{G/e},$$

therefore the zero locus $V(Z_{G\smallsetminus e}(q+t_e), Z_G)$ equals

$$\begin{aligned} V(Z_{G\smallsetminus e}(q+t_e), Z_{G\smallsetminus e} + t_e Z_{G/e}) &= V(Z_{G\smallsetminus e}, t_e Z_{G/e}) \cup V(q + t_e, Z_{G\smallsetminus e} + t_e Z_{G/e}) \\ &= V(Z_{G\smallsetminus e}, t_e) \cup V(Z_{G\smallsetminus e}, Z_{G/e}) \cup V(q + t_e, Z_{G\smallsetminus e} - qZ_{G/e}). \end{aligned}$$

To apply inclusion-exclusion, we need the double and triple intersections of these components:

$$\begin{aligned} &V(Z_{G\smallsetminus e}, t_e, Z_{G\smallsetminus e}, Z_{G/e}) = V(t_e, Z_{G\smallsetminus e}, Z_{G/e}) \\ &V(Z_{G\smallsetminus e}, t_e, q + t_e, Z_{G\smallsetminus e} - qZ_{G/e}) = V(q, t_e, Z_{G\smallsetminus e}) = V(q, t_e) \\ &V(Z_{G\smallsetminus e}, Z_{G/e}, q + t_e, Z_{G\smallsetminus e} - qZ_{G/e}) = V(q + t_e, Z_{G\smallsetminus e}, Z_{G/e}) \end{aligned}$$

and

$$V(Z_{G\smallsetminus e}, t_e, Z_{G\smallsetminus e}, Z_{G/e}, q + t_e, Z_{G\smallsetminus e} - qZ_{G/e}) = V(q, t_e, Z_{G\smallsetminus e}, Z_{G/e}) = V(q, t_e),$$

where we used the fact that $Z_{G\smallsetminus e}$ and $Z_{G/e}$ are multiples of q. This implies that the triple intersection is in fact a double intersection, causing a useful cancellation at the level of Grothendieck classes:

$$\begin{aligned} [V(Z_{G\smallsetminus e}(q+t_e), Z_G)] = [V(Z_{G\smallsetminus e}, t_e)] &+ [V(Z_{G\smallsetminus e}, Z_{G/e})] + [V(q + t_e, Z_{G\smallsetminus e} - qZ_{G/e})] \\ &- [V(t_e, Z_{G\smallsetminus e}, Z_{G/e})] - [V(q + t_e, Z_{G\smallsetminus e}, Z_{G/e})]. \end{aligned}$$

All but one of the classes on the right-hand side have a clear interpretation:

$$[V(Z_{G\smallsetminus e}, t_e)] = [\mathcal{Z}_{G\smallsetminus e}],$$

where we view $\mathcal{Z}_{G\smallsetminus e}$ as a hypersurface of $\mathbb{A}^{\#E(G)}$;

$$[V(Z_{G\smallsetminus e}, Z_{G/e})] = \mathbb{L} \cdot [\mathcal{Z}_{G\smallsetminus e} \cap \mathcal{Z}_{G/e}],$$

where the intersection is again viewed in $\mathbb{A}^{\#E(G)}$ and the factor of $\mathbb{L}$ is due to the free variable t_e;

$$[V(t_e, Z_{G\smallsetminus e}, Z_{G/e})] = [V(q+t_e, Z_{G\smallsetminus e}, Z_{G/e})] = [\mathcal{Z}_{G\smallsetminus e}\cap \mathcal{Z}_{G/e}],$$

still in $\mathbb{A}^{\#E(G)}$: indeed, t_e and $q+t_e$ may be used to eliminate t_e in both cases, and both ideals define the same locus in $\mathbb{A}^{\#E(G)}$ (with variables $(q, \hat{t}^{(e)})$) after this projection.

The remaining term is

$$[V(q+t_e, Z_{G\smallsetminus e} - qZ_{G/e})] = [V(Z_{G\smallsetminus e} - qZ_{G/e})],$$

where again we use $q+t_e = 0$ to eliminate t_e, and view the right-hand side as the class of a locus in $\mathbb{A}^{\#E(G)}$, with variables $(q, \hat{t}^{(e)})$.

Let A^e_G denote the locus determined by this ideal, as a subset of $\mathbb{A}^{\#E(G)}$. We have obtained that

$$[V(Z_{G\smallsetminus e}(q+t_e), Z_G)] = (\mathbb{L}-2)\cdot[Z_{G\smallsetminus e}\cap Z_{G/e}] + [Z_{G\smallsetminus e}] + [A^e_G].$$

This can be equivalently stated in terms of classes of hypersurface complements as

$$\{V(Z_{G\smallsetminus e}(q+t_e), Z_G)\} = (\mathbb{L}-2)\cdot\{Z_{G\smallsetminus e}\cap Z_{G/e}\} + \{Z_{G\smallsetminus e}\} + \{A^e_G\}. \tag{4.3}$$

Indeed, we have

$$\{V(Z_{G\smallsetminus e}(q+t_e), Z_G)\} = \mathbb{L}^{|E|+1} - [V(Z_{G\smallsetminus e}(q+t_e), Z_G)] \quad ,$$

while the complements of the other loci are taken in $\mathbb{A}^{|E|}$. Thus,

$$\begin{aligned}&\{V(Z_{G\smallsetminus e}(q+t_e), Z_G)\} = \mathbb{L}^{\#E(G)+1} - \big((\mathbb{L}-2)\cdot[\mathcal{Z}_{G\smallsetminus e}\cap\mathcal{Z}_{G/e}] + [\mathcal{Z}_{G\smallsetminus e}] + [A^e_G]\big)\\ &= \mathbb{L}(\mathbb{L}^{\#E(G)} - [\mathcal{Z}_{G\smallsetminus e}\cap\mathcal{Z}_{G/e}]) - 2(\mathbb{L}^{\#E(G)} - [\mathcal{Z}_{G\smallsetminus e}\cap\mathcal{Z}_{G/e}]) + 2\mathbb{L}^{\#E(G)} - ([\mathcal{Z}_{G\smallsetminus e}] + [A^e_G])\end{aligned}$$

with the stated result (4.3).

Thus, we have obtained in this way an explicit calculation of the intersection term needed to apply Theorem 3.1 to the graph ${}^2G$ obtained by splitting an edge of G into two. We obtain

$$\{\mathcal{Z}_{{}^2G}\} = \mathbb{L}\left((\mathbb{L}-2)\{\mathcal{Z}_{G\smallsetminus e}\cap\mathcal{Z}_{G/e}\} + \{\mathcal{Z}_{G\smallsetminus e}\} + \{A^e_G\}\right) - \{\mathcal{Z}_{{}^1G}\} \quad . \tag{4.4}$$

We then apply again Theorem 3.1 to G to provide an alternative expression for the intersection term $\{\mathcal{Z}_{G\smallsetminus e}\cap\mathcal{Z}_{G/e}\}$ and we obtain

$$\mathbb{L}\cdot\{\mathcal{Z}_{G\smallsetminus e}\cap\mathcal{Z}_{G/e}\} = \{\mathcal{Z}_{G/e}\} + \{\mathcal{Z}_G\} = \{\mathcal{Z}_{{}^0G}\} + \{\mathcal{Z}_{{}^1G}\}. \tag{4.5}$$

□

The locus A^e_G determined by the ideal $(q+t_e, Z_{G\smallsetminus e} - qZ_{G/e})$ has an interpretation in terms of the combinatorics of the graph G. Let Z' denote the sum

$$\sum_{A\subseteq E} q^{k(A)} \prod_{a\in A} t_a \tag{4.6}$$

where the sum is restricted to the subgraphs not including e and connecting the endpoints of e; and let Z'' denote the same expression, where the sum is restricted to the subgraphs not including e and *not* connecting the endpoints of e. Notice that there is a bijection between the monomial of $Z_{G/e}$ and the monomials of $Z_{G\smallsetminus e}$; in fact,

$$Z_{G\smallsetminus e} = Z' + Z'' \quad , \quad Z_{G/e} = Z' + \frac{Z''}{q}. \tag{4.7}$$

Indeed, the graphs in Z'' lose one connected component when e is contracted.

Lemma 4.3. *The locus A^e_G may be described as $V(q+t_e, (1+t_e)Z')$, where the polynomial $(1+t_e)Z'$ is the sum over all subgraphs of G (including e or not) which connect the endpoints of e in some way other than through e.*

Proof. By (4.7),

$$Z_{G\smallsetminus e} - qZ_{G/e} = (1-q)Z'.$$

Modulo $q + t_e$ this equals $(1+t_e)Z'$, and this term has the interpretation detailed in the statement. □

4.2. **Multiple splittings.** We can now formulate the result for multiple splittings of an edge e in a graph G, that is, for all the classes $\{\mathcal{Z}_{{}^mG}\}$.

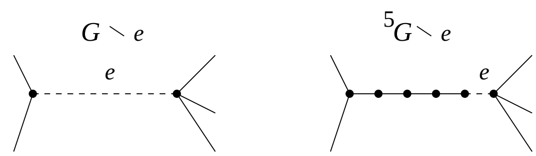
Lemma 4.4. *For all $m \geq 1$, the classes $\{\mathcal{Z}_{{}^{m+1}G}\}$ satisfy*

$$\{\mathcal{Z}_{{}^{m+1}G}\} = (\mathbb{T}-2)\{\mathcal{Z}_{{}^mG}\} + (\mathbb{T}-1)\{\mathcal{Z}_{{}^{m-1}G}\} + (\mathbb{T}+1)\mathbb{T}^{m-1}\left(\{\mathcal{Z}_{G\smallsetminus e}\} + \{A_G^e\}\right). \tag{4.8}$$

Proof. If a polynomial is multiplied by $t+q$, where t is a new indeterminate, then the effect on the class $\{\cdot\}$ of the variety defined by that polynomial is to multiply it by $\mathbb{L}-1 = \mathbb{T} = [\mathbb{G}_m]$ (cf. Corollary 3.2, (5)). Applying this observation to $\mathcal{Z}_{G\smallsetminus e}$ and A_G^e shows that

$$\{\mathcal{Z}_{{}^mG\smallsetminus e}\} + \{A^e_{{}^mG}\} = \mathbb{T}^{m-1}(\{\mathcal{Z}_{G\smallsetminus e}\} + \{A_G^e\}).$$

where e denotes the last edge added in the splitting ${}^mG$. Indeed, $Z_{{}^mG\smallsetminus e}$ is obtained from $Z_{G\smallsetminus e}$ by attaching a chain of $m-1$ edges to $G \smallsetminus e$:



and this has the effect of multiplying $Z_{G\smallsetminus e}$ by a term $\prod_{i=1}^{m-1}(t_i+q)$, where $t_1, \dots, t_{m-1}$ are the variables corresponding to the edges in the chain. Therefore, $\{\mathcal{Z}_{{}^mG\smallsetminus e}\} = \mathbb{T}^{m-1}\{\mathcal{Z}_{G\smallsetminus e}\}$. The effect on A_G^e is precisely the same. Indeed, recall that, by Lemma 4.3, the equation for A_G^e in the hyperplane $t_e = -q$ is $(1-q)Z'$, where Z' is the sum corresponding to subgraphs of $G \smallsetminus e$ which connect the endpoints of e. From this description it is clear that the equation for $A^e_{{}^mG}$ in $\{t_e = -q\}$ is $(1-q)Z'\prod_{i=1}^{m-1}(t_i+q)$, and again it follows that $\{A^e_{{}^mG}\} = \mathbb{T}^{m-1}\{A_G^e\}$. □

Combining Theorem 4.2 and Lemma 4.4 allows us to obtain a 'combinatorial' expression for the class $\{A_G^e\}$. We then have the following recursive formula for the classes $\{\mathcal{Z}_{{}^mG}\}$.

Theorem 4.5. *For all $m \geq 0$:*

$$\{\mathcal{Z}_{{}^{m+3}G}\} = (2\mathbb{T}-2)\{\mathcal{Z}_{{}^{m+2}G}\} - (\mathbb{T}^2 - 3\mathbb{T}+1)\{\mathcal{Z}_{{}^{m+1}G}\} - \mathbb{T}(\mathbb{T}-1)\{\mathcal{Z}_{{}^mG}\}. \tag{4.9}$$

Proof. Starting with (4.8), Theorem 4.2 allows us to express the term $\{\mathcal{Z}_{G\smallsetminus e}\} + \{A_G^e\}$ in terms of splittings:

$$(\mathbb{T}+1)(\{\mathcal{Z}_{G\smallsetminus e}\} + \{A_G^e\}) = \{\mathcal{Z}_{{}^2G}\} - (\mathbb{T}-2)\{\mathcal{Z}_{{}^1G}\} - (\mathbb{T}-1)\{\mathcal{Z}_{{}^0G}\}. \tag{4.10}$$

Using the case $m = 2$ in Lemma 4.4 we then get

$$\{\mathcal{Z}_{{}^3G}\} = (\mathbb{T}-2)\{\mathcal{Z}_{{}^2G}\} + (\mathbb{T}-1)\{\mathcal{Z}_{{}^1G}\} + \mathbb{T}(\{\mathcal{Z}_{{}^2G}\} - (\mathbb{T}-2)\{\mathcal{Z}_{{}^1G}\} - (\mathbb{T}-1)\{\mathcal{Z}_{{}^0G}\}),$$

that is,

$$\{\mathcal{Z}_{{}^3G}\} = (2\mathbb{T}-2)\{\mathcal{Z}_{{}^2G}\} - (\mathbb{T}^2 - 3\mathbb{T}+1)\{\mathcal{Z}_{{}^1G}\} - \mathbb{T}(\mathbb{T}-1)\{\mathcal{Z}_{{}^0G}\}.$$

Then applying this formula to ${}^{m+1}G$ rather than ${}^1G$ one obtains the recursion (4.9). □

One can then write a generating function for the classes $\{\mathcal{Z}_{{}^mG}\}$, in a way similar to the corresponding result given in [1] for the graph hypersurfaces of Feynman graphs.

Theorem 4.6. *The generating function of the classes $\{\mathcal{Z}_{{}^mG}\}$ is given by*

$$\sum_{m\geq 0}\{Z_{{}^mG}\}\frac{s^m}{m!} = \left(e^{(\mathbb{T}-1)s} - (\mathbb{T}-1)\cdot\frac{e^{\mathbb{T}s}-e^{-s}}{\mathbb{T}+1}\right)\{Z_{{}^0G}\} + \left((\mathbb{T}-1)\cdot\frac{e^{(\mathbb{T}-1)s}-e^{-s}}{\mathbb{T}} - (\mathbb{T}-2)\cdot\frac{e^{\mathbb{T}s}-e^{-s}}{\mathbb{T}+1}\right)\{Z_{{}^1G}\} + \left(-\frac{e^{(\mathbb{T}-1)s}-e^{-s}}{\mathbb{T}} + \frac{e^{\mathbb{T}s}-e^{-s}}{\mathbb{T}+1}\right)\{Z_{{}^2G}\}. \tag{4.11}$$

Proof. Putting

$$G_e(s) := \sum_{m\geq 0}\{\mathcal{Z}_{{}^mG}\}\frac{s^m}{m!} \quad , \tag{4.12}$$

the recursion formula (4.9) translates into the differential equation

$$G_e'''(s) = (2\mathbb{T}-2)\,G_e''(s) - (\mathbb{T}^2-3\mathbb{T}+1)\,G_e'(s) - \mathbb{T}(\mathbb{T}-1)\,G_e(s), \tag{4.13}$$

with solution

$$G_e(s) = A\,e^{-s} + B\,e^{\mathbb{T}s} + C\,e^{(\mathbb{T}-1)s}. \tag{4.14}$$

We can impose that this series begins with three undetermined coefficients, and solving for A, B, C gives

$$A = \{\mathcal{Z}_{{}^0G}\} + \frac{\{\mathcal{Z}_{{}^2G}\}+\{\mathcal{Z}_{{}^1G}\}}{\mathbb{T}} - \frac{\{\mathcal{Z}_{{}^2G}\}+3\{\mathcal{Z}_{{}^1G}\}+2\{\mathcal{Z}_{{}^0G}\}}{\mathbb{T}+1}$$
$$B = -\{\mathcal{Z}_{{}^1G}\} - \{\mathcal{Z}_{{}^0G}\} + \frac{\{\mathcal{Z}_{{}^2G}\}+3\{\mathcal{Z}_{{}^1G}\}+2\{\mathcal{Z}_{{}^0G}\}}{\mathbb{T}+1}$$
$$C = \{\mathcal{Z}_{{}^1G}\} + \{\mathcal{Z}_{{}^0G}\} - \frac{\{\mathcal{Z}_{{}^2G}\}+\{\mathcal{Z}_{{}^1G}\}}{\mathbb{T}}$$

This then gives the form (4.11) for the generating function $G_e(s)$. □

One can also write (4.11) in a form that involves explicitly the term A^e_G, with slightly simpler form of the coefficients, as follows.

Corollary 4.7. *The generating function* (4.11) *can be equivalently written as*

$$\sum_{m\geq 0}\{Z_{{}^mG}\}\frac{s^m}{m!} = e^{-s}\left(\left(1+\frac{e^{\mathbb{T}s}-1}{\mathbb{T}}\right)\{Z_{{}^0G}\} + \frac{e^{\mathbb{T}s}-1}{\mathbb{T}}\{Z_{{}^1G}\} + \left(e^{(\mathbb{T}+1)s} - e^{\mathbb{T}s} - \frac{e^{\mathbb{T}s}-1}{\mathbb{T}}\right)\left(\{Z_{G\smallsetminus e}\}+\{A^e_G\}\right)\right). \tag{4.15}$$

Proof. This follows directly from (4.11) and (4.10). □

4.3. **Edge splitting for fixed q.** The discussion is entirely parallel to the one given above for the class $\{\mathcal{Z}_G\}$ with variable q. One has the analog of Theorem 4.2 in the case with fixed q, given by the following.

Theorem 4.8. *Let ${}^2G$ be the graph obtained by splitting an edge e in a graph G. Then the class $\{\mathcal{Z}_{{}^2G,q}\}$ satisfies*

$$\{\mathcal{Z}_{{}^2G,q}\} = (\mathbb{T}-2)\{\mathcal{Z}_{{}^1G,q}\} + (\mathbb{T}-1)\{\mathcal{Z}_{{}^0G,q}\} + (\mathbb{T}+1)\left(\{\mathcal{Z}_{G\smallsetminus e,q}\}+\{A^e_{G,q}\}\right). \tag{4.16}$$

Proof. As in the proof of Theorem 4.2, the main point is the computation of the class of $V(Z_{G\smallsetminus e}(q+t_e), Z_G)$.

Lemma 4.9. *With $q \neq 0$ fixed, the class of the locus $V(Z_{G\smallsetminus e}(q+t_e), Z_G)$ is given by*

$$\{V(Z_{G\smallsetminus e}(q+t_e), Z_G)\} = (\mathbb{T}-1)\cdot\{\mathcal{Z}_{G\smallsetminus e,q}\cap\mathcal{Z}_{G/e,q}\} + \{\mathcal{Z}_{G\smallsetminus e,q}\} + \{A^e_{G,q}\},$$

where $A^e_{G,q} \subseteq \mathbb{A}^{\#E(G)-1}$ is the zero locus of the polynomial $Z_{G\smallsetminus e} - qZ_{G/e} = (1-q)Z'$, with fixed q, with Z' as in (4.6) *(with fixed q).*

Proof. The argument here parallels closely the computation in the proof of Theorem 4.2. In that computation (for variable q) there is a key cancellation of a class $[V(q,t_e)]$ due to inclusion-exclusion; for fixed $q \neq 0$ this locus is empty. All other classes admit the same interpretation for fixed q as for variable q.

In particular, the term $A^e_{G,q}$ is the zero locus of $(1-q)Z'$ with Z' as in Lemma 4.3 (and q is now a fixed nonzero number). Note that this is 0 when $q=1$; in this case the last summand in the formula in Lemma 4.9 is 0. In general, arguing as in Lemma 4.3, one sees that Z' is given by the sum (4.6), over the range specified there. □

Implementing deletion-contraction gives then the same formula as in the case with variable q, so that one obtains the following.

Corollary 4.10. *Let ${}^2G$ be the graph obtained by splitting an edge e in a graph G. Then the class $\{\mathcal{Z}_{{}^2G,q}\}$ satisfies*

$$\{\mathcal{Z}_{{}^2G,q}\} = \mathbb{L}\left((\mathbb{L}-2)\{\mathcal{Z}_{G\smallsetminus e,q}\cap\mathcal{Z}_{G/e,q}\} + \{\mathcal{Z}_{G\smallsetminus e,q}\} + \{A^e_{G,q}\}\right) - \{\mathcal{Z}_{{}^1G,q}\}. \tag{4.17}$$

Note again that the term $\{A^e_{G,q}\}$ would be missing in the case $q=1$.

Next, use Proposition 3.4 to get a different expression for $\{\mathcal{Z}_{G\smallsetminus e}\cap\mathcal{Z}_{G/e}\}$, and this gives a perfect parallel to Theorem 4.2, and completes the proof of Theorem 4.8. □

We can now pass to the case of multiple splittings, which again is analogous to the case with variable q.

Theorem 4.11. *Let ${}^mG$ be the graph obtained by multiple splitting on an edge e in G. Then, for all $m \geq 0$, the classes for fixed q satisfy*

$$\{\mathcal{Z}_{{}^{m+1}G,q}\} = (\mathbb{T}-2)\{\mathcal{Z}_{{}^mG,q}\} + (\mathbb{T}-1)\{\mathcal{Z}_{{}^{m-1}G,q}\} + (\mathbb{T}+1)\mathbb{T}^{m-1}\left(\{\mathcal{Z}_{G\smallsetminus e,q}\} + \{A^e_{G,q}\}\right),$$

which then gives the recursive relation

$$\{\mathcal{Z}_{{}^{m+3}G,q}\} = (2\mathbb{T}-2)\{\mathcal{Z}_{{}^{m+2}G,q}\} - (\mathbb{T}^2-3\mathbb{T}+1)\{\mathcal{Z}_{{}^{m+1}G,q}\} - \mathbb{T}(\mathbb{T}-1)\{\mathcal{Z}_{{}^mG,q}\}, \tag{4.18}$$

Proof. The argument is completely analogous to the case with variable q. The key step is the relation

$$\{A^e_{{}^mG,q}\} = \mathbb{T}^{m-1}\{A^e_{G,q}\},$$

that one can see continues to hold in this case by the same argument used before. Then (4.18) is proved by using Theorem 4.8 to solve for the class $(\{\mathcal{Z}_{G\smallsetminus e,q}\} + \{A^e_{G,q}\})$. □

The bottom line is that *the same recursion* holds for any fixed $q \neq 0$ as for the free q case. What will change are the seeds of this recursion, that is, the values of $\{\mathcal{Z}_{{}^mG,q}\}$ for $m = 0,1,2$; these will naturally be different for fixed q.

Nothing in the proof of the recursion excludes the case $q=1$, and indeed $\{\mathcal{Z}_{{}^mG,q=1}\} = \mathbb{T}^{\#E(G)+m-1}$ is a solution of the recursion

$$(2\mathbb{T}-2)\mathbb{T}^{\#E(G)+m+1} - (\mathbb{T}^2-3\mathbb{T}+1)\mathbb{T}^{\#E(G)+m} - \mathbb{T}(\mathbb{T}-1)\mathbb{T}^{\#E(G)+m-1} = \mathbb{T}^{\#E(G)+m+2}.$$

With respect to the question of when the fibration condition (2.24) relating $\{\mathcal{Z}_G\}$ to the $\{\mathcal{Z}_{G,q}\}$ holds, notice that, although the recursion is the same for all fixed q (all being the same as for the case of variable q), one does not *a priori* know whether the seeds of the recursions are independent of q.

4.4. **Edge splitting and the tangent cone.** Again we consider first the operation of splitting one edge in two and then the case of multiple splittings. In the case of the tangent cone, we need to distinguish regular edges from bridges and looping edges.

Lemma 4.12. *If the edge e of G is either a bridge or a looping edge and ${}^mG$ denotes the graph obtained by iterated splitting of the edge e, then for $m \geq 1$ the class of the tangent cone complement satisfies $\{\mathcal{V}_{^{m+1}G}\} = \mathbb{T}^m\{\mathcal{V}_G\}$.*

Proof. If e is a bridge, then splitting e amounts to inserting a new bridge; by the deletion-contraction formulas for bridges (Proposition 3.9)

$$\{\mathcal{V}_{^2G}\} = \mathbb{T} \cdot \{\mathcal{V}_G\}.$$

Thus, in the case of a bridge one also already sees that splitting the edge multiple times just has the effect of multiplying $\{\mathcal{V}_G\}$ by a power of $\mathbb{T}$.

If e is a looping edge, adding a loop to a graph G' multiplies the class $\{\mathcal{V}_{G'}\}$ by $\mathbb{T}+1$ as seen in Lemma 3.7. Attaching a split loop amounts to multiplying $Z_{G'}$ (and hence $P_{G'}$) by $(q+t_e+t_f)$, where t_e and t_f are the variables corresponding to the two new edges; this is simply because $q^2+t_eq+t_fq$ is the Z-polynomial for a 2-banana. Here G' is the graph obtained from G by removing the loop. From

$$P_{^2G} = P_{G'}\,(q+t_e+t_f)$$

we see that $P_{^2G} \neq 0$ implies $P_{G'} \neq 0$ and $t_e \neq -(q+t_f)$.

Therefore, for any point (q,t) for which $P_{G'} \neq 0$ we have an $\mathbb{A}^1$ worth of choices for t_f and an $\mathbb{A}^1 \smallsetminus \mathbb{A}^0$ worth of choices for t_e. So we can conclude that

$$\{\mathcal{W}_{^2G}\} = \mathbb{L}(\mathbb{L}-1)\cdot\{\mathcal{W}_{G'}\} = \mathbb{T}\cdot\{\mathcal{W}_G\}.$$

The same analysis goes through after setting q to 0, and hence

$$\{\mathcal{Y}_{^2G}\} = \mathbb{L}(\mathbb{L}-1)\cdot\{\mathcal{Y}_{G'}\} = \mathbb{T}\cdot\{\mathcal{Y}_G\}.$$

The conclusion is that

$$\{\mathcal{V}_{^2G}\} = \mathbb{T}\cdot\{\mathcal{V}_G\}.$$

Thus, splitting a loop once has again the effect of multiplying the corresponding $\{\mathcal{V}_G\}$ by $\mathbb{T}$, as for bridges. □

We then check the more interesting case of regular edges.

Theorem 4.13. *Let e be a regular edge of G, and let ${}^2G$ be the graph obtained by introducing a 2-valent vertex in e, thereby splitting it. With other notation as above,*

$$\{\mathcal{V}_{^2G}\} = (\mathbb{T}-2)\{\mathcal{V}_G\} + (\mathbb{T}-1)\{\mathcal{V}_{G/e}\} + (\mathbb{T}+1)(\{\mathcal{V}_{G\smallsetminus e}\} + \{V(Q_{G\smallsetminus e} - q\,Q_{G/e})\}).$$

Proof. Let e be a regular edge, and call e (again) and f the two edges created in the process. The polynomial for $\mathcal{W}_{^2G}$ is found easily by applying deletion-contraction:

$$Q_{^2G} = Q_{G\setminus e}\cdot(q+t_e) + t_f\cdot Q_G.$$

Indeed, deleting f leaves the graph $G\setminus e$ with a dangling edge e attached, and attaching an edge e to a vertex has the effect of multiplying the corresponding polynomial by $(q+t_e)$; contracting f gives G back. By Proposition 3.8, therefore we have

$$\{\mathcal{W}_{^2G}\} = \mathbb{L}\cdot[\mathbb{A}^{\#E(G)+1} - \mathcal{W}_\cap] - \{\mathcal{W}_G\}, \tag{4.19}$$

where $\mathcal{W}_\cap$ is the intersection in $\mathbb{A}^{\#E(G)+1}$ of the hypersurface $\mathcal{W}_G$, with equation $Q_G = 0$, and the hypersurface with equation $Q_{G\smallsetminus e}(q+t_e) = 0$.

Set theoretically, the locus $\mathcal{W}_\cap$ is given by

$$\begin{aligned}\mathcal{W}_\cap &= V(Q_{G\smallsetminus e}(q+t_e), Q_G) = V(Q_{G\smallsetminus e}(q+t_e), Q_{G\smallsetminus e}+t_eQ_{G/e})\\ &= V(Q_{G\smallsetminus e}, t_eQ_{G/e}) \cup V(q+t_e, Q_{G\smallsetminus e}+t_eQ_{G/e})\\ &= V(Q_{G\smallsetminus e}, Q_{G/e}) \cup V(t_e, Q_{G\smallsetminus e}) \cup V(q+t_e, Q_{G\smallsetminus e}-qQ_{G/e}),\end{aligned}$$

where all ideals are viewed in $\mathbb{A}^{\#E(G)+1}$, with coordinates q and t_a, $a\in E(G)$. Inclusion–exclusion then gives $[\mathcal{W}_\cap]$ as the sum of the three loci indicated here, minus the sum of the three pairwise intersections, plus the triple intersection. The three pairwise intersections are

$$\begin{gathered}V(t_e, Q_{G\smallsetminus e}, Q_{G/e})\\ V(q+t_e, Q_{G\smallsetminus e}, Q_{G/e})\\ V(q, t_e, Q_{G\smallsetminus e})\end{gathered}$$

and the triple intersection is

$$V(q, t_e, Q_{G\smallsetminus e}, Q_{G/e}) \quad .$$

This is very similar to the situation we have seen for the case of $\mathcal{Z}_G$. We find

$$[V(Q_{G\smallsetminus e}, Q_{G/e})] = \mathbb{L}\cdot[\mathcal{W}_{G\smallsetminus e}\cap\mathcal{W}_{G/e}]$$

as $G\smallsetminus e$ and G/e have $E(G)\smallsetminus\{e\}$ as index set, while

$$[V(t_e, Q_{G\smallsetminus e}, Q_{G/e})] = [V(q+t_e, Q_{G\smallsetminus e}, Q_{G/e})] = [\mathcal{W}_{G\smallsetminus e}\cap\mathcal{W}_{G/e}].$$

In both cases, the first equation eliminates t_e, doing nothing to the rest since the rest does not depend on t_e. We also have

$$[V(t_e, Q_{G\smallsetminus e})] = [\mathcal{W}_{G\smallsetminus e}] \quad , \quad [V(q, t_e, Q_{G\smallsetminus e})] = [\mathcal{Y}_{G\smallsetminus e}],$$

and

$$[V(q, t_e, Q_{G\smallsetminus e}, Q_{G/e})] = [\mathcal{Y}_{G\smallsetminus e}\cap\mathcal{Y}_{G/e}].$$

That leaves us with

$$V(q+t_e, Q_{G\smallsetminus e}-qQ_{G/e}) \subseteq \mathbb{A}^{\#E(G)+1},$$

or simply

$$V(Q_{G\smallsetminus e}-qQ_{G/e}) \subseteq \mathbb{A}^{\#E(G)},$$

since the only effect of the first equation is to eliminate t_e.

Applying then inclusion–exclusion we obtain

$$[\mathcal{W}_\cap] = (\mathbb{L}-2)[\mathcal{W}_{G\smallsetminus e}\cap\mathcal{W}_{G/e}] + [\mathcal{W}_{G\smallsetminus e}] + [V(Q_{G\smallsetminus e}-q\,Q_{G/e})] - [\mathcal{Y}_{G\smallsetminus e}] + [\mathcal{Y}_{G\smallsetminus e}\cap\mathcal{Y}_{G/e}].$$

Passing to the classes of the complements and implementing the resulting expression for ${}^2G$ then gives

$$\begin{aligned}\{\mathcal{W}_{{}^2G}\} = (\mathbb{L}-2)\mathbb{L}(\mathbb{L}^{\#E(G)} - [\mathcal{W}_{G\smallsetminus e}\cap\mathcal{W}_{G/e}]) + \mathbb{L}\{\mathcal{W}_{G\smallsetminus e}\}\\ + \mathbb{L}\{V(Q_{G\smallsetminus e}-q\,Q_{G/e})\} - \mathbb{L}\{\mathcal{Y}_{G\smallsetminus e}\} + \mathbb{L}(\mathbb{L}^{\#E(G)-1} - [\mathcal{Y}_{G\smallsetminus e}\cap\mathcal{Y}_{G/e}]) - \{\mathcal{W}_G\}.\end{aligned}$$

Using Proposition 3.8 and computing the difference $\{\mathcal{V}_G\} = \{\mathcal{W}_G\} - \{\mathcal{Y}_G\}$ one gets

$$\begin{aligned}\{\mathcal{V}_{{}^2G}\} &= \{\mathcal{W}_{{}^2G}\} - \{\mathcal{Y}_{{}^2G}\}\\ &= (\mathbb{L}-3)\{\mathcal{W}_G\} + (\mathbb{L}-2)\{\mathcal{W}_{G/e}\} + \mathbb{L}(\{\mathcal{W}_{G\smallsetminus e}\} - \{\mathcal{Y}_{G\smallsetminus e}\}) + \{\mathcal{Y}_G\}\\ &\quad + \{\mathcal{Y}_{G/e}\} + \mathbb{L}\{V(Q_{G\smallsetminus e}-q\,Q_{G/e})\} - (\mathbb{L}-2)\{\mathcal{Y}_G\} - (\mathbb{L}-1)\{\mathcal{Y}_{G/e}\} - \mathbb{L}\{\mathcal{Y}_{G\smallsetminus e}\}\\ &= (\mathbb{L}-3)(\{\mathcal{W}_G\} - \{\mathcal{Y}_G\}) + (\mathbb{L}-2)(\{\mathcal{W}_{G\smallsetminus e}\} - \{\mathcal{Y}_{G\smallsetminus e}\})\\ &\quad + \mathbb{L}(\{\mathcal{W}_{G\smallsetminus e}\} - \{\mathcal{Y}_{G\smallsetminus e}\}) + \mathbb{L}(\{V(Q_{G\smallsetminus e}-q\,Q_{G/e})\} - \{\mathcal{Y}_{G\smallsetminus e}\})\\ &= (\mathbb{L}-3)\{\mathcal{V}_G\} + (\mathbb{L}-2)\{\mathcal{V}_{G/e}\} + \mathbb{L}\{\mathcal{V}_{G\smallsetminus e}\} + \mathbb{L}\{V(Q_{G\smallsetminus e}-q\,Q_{G/e})\}.\end{aligned}$$

∎

As in the case of $\mathcal{Z}_G$ and $\mathcal{Z}_{G,q}$, the polynomial $Q_{G \smallsetminus e} - q\, Q_{G/e}$ can be given a combinatorial interpretation as

$$Q_{G \smallsetminus e} - q\, Q_{G/e} = \sum_{A \subseteq E \smallsetminus \{e\} \text{ forest connecting endpoints of } e} q^{k(A)-k(C)} \prod_{a \in A} t_a .$$

We then can proceed as in the case of $\mathcal{Z}_G$ and $\mathcal{Z}_{G,q}$ and obtain the following recursions.

Proposition 4.14. *With notation as above, and assuming e is not a looping edge,*

$$\{\mathcal{Y}_{m+3}{}_G\} = (2\mathbb{T}-1)\{\mathcal{Y}_{m+2}{}_G\} - \mathbb{T}(\mathbb{T}-2)\{\mathcal{Y}_{m+1}{}_G\} - \mathbb{T}^2\{\mathcal{Y}_m{}_G\}$$

for all $m \geq 0$. The generating function

$$G_e^{\mathcal{Y}}(s) := \sum_{m \geq 0} \{\mathcal{Y}_m{}_G\} \frac{s^m}{m!}$$

for the classes $\{\mathcal{Y}_m{}_G\}$ is then of the form

$$G_e^{\mathcal{Y}}(s) = A\, e^{-s} + B\, s\, e^{\mathbb{T}s} + C\, e^{\mathbb{T}s},$$

with the constants A, B, and C satisfying

$$\begin{aligned}
A &= \{\mathcal{Y}_0{}_G\} - 2\frac{\{\mathcal{Y}_1{}_G\} + \{\mathcal{Y}_0{}_G\}}{\mathbb{T}+1} + \frac{\{\mathcal{Y}_2{}_G\} + 2\{\mathcal{Y}_1{}_G\} + \{\mathcal{Y}_0{}_G\}}{(\mathbb{T}+1)^2} \\
B &= -\{\mathcal{Y}_1{}_G\} - \{\mathcal{Y}_0{}_G\} + \frac{\{\mathcal{Y}_2{}_G\} + 2\{\mathcal{Y}_1{}_G\} + \{\mathcal{Y}_0{}_G\}}{\mathbb{T}+1} \\
C &= 2\frac{\{\mathcal{Y}_1{}_G\} + \{\mathcal{Y}_0{}_G\}}{\mathbb{T}+1} - \frac{\{\mathcal{Y}_2{}_G\} + 2\{\mathcal{Y}_1{}_G\} + \{\mathcal{Y}_0{}_G\}}{(\mathbb{T}+1)^2}
\end{aligned}$$

Similarly,

$$\{\mathcal{V}_{m+3}{}_G\} = (2\mathbb{T}-2)\{\mathcal{V}_{m+2}{}_G\} - (\mathbb{T}^2 - 3\mathbb{T}+1)\{\mathcal{V}_{m+1}{}_G\} - \mathbb{T}(\mathbb{T}-1)\{\mathcal{V}_m{}_G\},$$

for all $m \geq 0$, with generating function

$$G_e^{\mathcal{V}}(s) := \sum_{m \geq 0} \{\mathcal{V}_m{}_G\} \frac{s^m}{m!}$$

given by

$$G_e^{\mathcal{V}}(s) = A\, e^{-s} + B\, e^{\mathbb{T}s} + C\, e^{(\mathbb{T}-1)s}$$

with the terms A, B and C satisfying

$$\begin{aligned}
A &= \{\mathcal{V}_0{}_G\} + \frac{\{\mathcal{V}_2{}_G\} + \{\mathcal{V}_1{}_G\}}{\mathbb{T}} - \frac{\{\mathcal{V}_2{}_G\} + 3\{\mathcal{V}_1{}_G\} + 2\{\mathcal{V}_0{}_G\}}{\mathbb{T}+1} \\
B &= -\{\mathcal{V}_1{}_G\} - \{\mathcal{V}_0{}_G\} + \frac{\{\mathcal{V}_2{}_G\} + 3\{\mathcal{V}_1{}_G\} + 2\{\mathcal{V}_0{}_G\}}{\mathbb{T}+1} \\
C &= \{\mathcal{V}_1{}_G\} + \{\mathcal{V}_0{}_G\} - \frac{\{\mathcal{V}_2{}_G\} + \{\mathcal{V}_1{}_G\}}{\mathbb{T}}.
\end{aligned}$$

The argument is essentially analogous to the cases of $\mathcal{Z}_G$ and $\mathcal{Z}_{G,q}$ analyzed before and we do not reproduce it explicitly here.

The generating function for $\mathcal{Y}_m{}_G$ is 'dual' to the one for $\mathcal{X}_m{}_G$ given in [1]: the effect of splitting edges on the class $\{\mathcal{Y}_m{}_G\}$ is analogous to the effect of multiplying edges on the class $\{\mathcal{X}_m{}_G\}$.

5. Polygons and linked polygons

We now focus on a particularly simple class of graphs for which we can compute everything explicitly. These will be polygons and graphs constructed out of chains of linked polygons. We will later focus especially on this class of graphs to provide an explicit example of how to apply these motivic techniques to analyze (virtual) phase transitions in the corresponding Potts models.

We start by using the formulae for edge splittings obtained in the previous section to compute explicitly the classes $\{\mathcal{Z}_G\}$ for polygon graphs.

Proposition 5.1. *Let ${}^mG$ be an $(m+1)$-sided polygon. Then the classes $\{\mathcal{Z}_{{}^mG}\}$ are given explicitly by the formula*

$$\{\mathcal{Z}_{{}^mG}\} = \mathbb{T}^{m+2} + \mathbb{T}(\mathbb{T}-1)(\mathbb{T}^m - (\mathbb{T}-1)^m) + (\mathbb{T}-1)\frac{(\mathbb{T}-1)^m - (-1)^m}{\mathbb{T}} \quad . \tag{5.1}$$

Proof. Let us check directly the initial cases that are needed to use the recursive formula for edge splittings. The first graphs are ${}^0G$ a single loop, ${}^1G$ a 2-banana (two vertices with two parallel edges between them), ${}^2G$ a triangle. The equations $Z_G(q,t) = 0$ are of the form

$$\begin{aligned} {}^0G \quad &: \quad q + qt = 0 \quad (\text{in } \mathbb{A}^{1+1}) \\ {}^1G \quad &: \quad q^2 + (t_1+t_2+t_1t_2)q = 0 \quad (\text{in } \mathbb{A}^{2+1}) \\ {}^2G \quad &: \quad q^3 + (t_1+t_2+t_3)q^2 + (t_1t_2+t_1t_3+t_2t_3+t_1t_2t_3)q = 0 \quad (\text{in } \mathbb{A}^{3+1}) \end{aligned}$$

The corresponding classes $\{\mathcal{Z}_G\}$ can then be computed directly in these cases by applying the basic facts listed in Corollary 3.2 and Theorem 4.2. One obtains

$$\begin{aligned} \{Z_{{}^0G}\} &= \mathbb{T}^2 \\ \{Z_{{}^1G}\} &= \mathbb{T}^3 + \mathbb{T}^2 - 1 \\ \{Z_{{}^2G}\} &= \mathbb{T}^4 + 2\mathbb{T}^3 - 2\mathbb{T}^2 - 2\mathbb{T} + 2. \end{aligned}$$

The expression (5.1) is then $m!$ times the coefficient of s^m in the expansion of the right-hand side of the generating function (4.11), with these initial conditions. ☐

5.1. **Polygons at fixed q.** We will also need in the following the classes of the polygon graphs for a fixed value of q. These are obtained as follows.

Proposition 5.2. *Let ${}^mG$ be an $(m+1)$-sided polygon. Then the classes $\{\mathcal{Z}_{{}^mG,q}\}$ for fixed $q \neq 0, 1$ are given explicitly by the formula*

$$\{\mathcal{Z}_{{}^mG,q}\} = \mathbb{T}^{m+1} + \mathbb{T}(\mathbb{T}^m - (\mathbb{T}-1)^m) + \frac{(\mathbb{T}-1)^m - (-1)^m}{\mathbb{T}}. \tag{5.2}$$

The polygon graphs ${}^mG$ satisfy the fibration condition (2.24).

Proof. As above, the seeds of the recursion are a single loop, a 2-banana, and a triangle.

The single loop has class $\mathbb{T}$ from Corollary 3.5.

The 2-banana hypersurface has equation $qt_1t_2 + qt_1 + qt_2 + q^2 = 0$. Factoring out a q (assumed to be nonzero to begin with) this is equivalent to

$$(t_1+1)(t_2+1) = 1 - q.$$

We are assuming that $q \neq 1$, so up to a variable change this equation is

$$u_1u_2 = r \neq 0.$$

This forces $u_1 \neq 0$, and determines u_2 once u_1 is fixed; thus the class of this locus in the Grothendieck group is $\mathbb{L} - 1$; its complement in $\mathbb{A}^2$ has class

$$\mathbb{L}^2 - \mathbb{L} + 1 = \mathbb{T}^2 + \mathbb{T} + 1.$$

This is independent of $q \neq 0, 1$.

The triangle hypersurface has equation

$$q(t_1t_2t_3 + t_1t_2 + t_1t_3 + t_2t_3 + q(t_1 + t_2 + t_3) + q^2) = 0.$$

Changing variables: $u_i = t_i + 1$, $r = q - 1$, and keeping in mind $q \neq 0$, this is equivalent to

$$u_1u_2u_3 + (u_1 + u_2 + u_3)r = r(1 - r). \tag{5.3}$$

Solving for u_3 gives

$$u_3 = \frac{r(1 - r - u_1 - u_2)}{u_1u_2 + r}.$$

Thus, the variety contains an open subvariety isomorphic to

$$\mathbb{A}^2 \smallsetminus V(u_1u_2 + r) \quad ;$$

this locus has class $\mathbb{L}^2 - \mathbb{L} + 1$, as we just computed above, since $r \neq 0$ (as $q \neq 1$). If $u_1u_2 = -r$, then (5.3) is equivalent to

$$u_1 + u_2 = 1 - r,$$

implying easily that (u_1, u_2) equals $(1, r)$ or $(r, 1)$, while u_3 is free in this case. That is, the complement in (5.3) of the open subvariety determined above consists of the loci

$$\{(1, -r, u_3)\} \cong \mathbb{A}^1 \quad , \quad \{(-r, 1, u_3)\} \cong \mathbb{A}^1.$$

These lines are distinct, since $r \neq -1$ (as $q \neq 0$). The conclusion is that the class of (5.3) equals

$$\mathbb{L}^2 - \mathbb{L} + 1 + 2\mathbb{L} = \mathbb{L}^2 + \mathbb{L} + 1 \quad ,$$

and hence its complement in $\mathbb{A}^3$ has class

$$\mathbb{L}^3 - \mathbb{L}^2 - \mathbb{L} - 1 = \mathbb{T}^3 + 2\mathbb{T}^2 - 2 \quad .$$

Again this is independent of q.

These cases are compatible with the fibration condition (2.24). For instance, in the triangle case, we get

$$(\mathbb{T} - 1)(\mathbb{T}^3 + 2\mathbb{T}^2 - 2) + \mathbb{T}^3 = \mathbb{T}^4 + 2\mathbb{T}^3 - 2\mathbb{T}^2 - 2\mathbb{T} + 2$$

in agreement with the class for a triangle in the free q case, used in Proposition 5.1.

The recursion then gives the classes (5.2),

$$\begin{aligned}\{\mathcal{Z}_{^mG,q}\} &= \frac{\mathbb{T}^{m+2} + \mathbb{T}(\mathbb{T} - 1)(\mathbb{T}^m - (\mathbb{T} - 1)^m) + (\mathbb{T} - 1)\frac{(\mathbb{T}-1)^m - (-1)^m}{\mathbb{T}} - \mathbb{T}^{m+1}}{\mathbb{T} - 1} \\ &= \mathbb{T}^{m+1} + \mathbb{T}(\mathbb{T}^m - (\mathbb{T} - 1)^m) + \frac{(\mathbb{T} - 1)^m - (-1)^m}{\mathbb{T}} \quad .\end{aligned}$$

One can verify explicitly the compatibility of (5.2) and (5.1) with the fibration condition (2.24): the polynomial

$$\{Z_{^mG}\} = \mathbb{T}^{m+2} + \mathbb{T}(\mathbb{T} - 1)(\mathbb{T}^m - (\mathbb{T} - 1)^m) + (\mathbb{T} - 1)\frac{(\mathbb{T} - 1)^m - (-1)^m}{\mathbb{T}}$$

factors exactly as predicted by (2.24) in terms of the classes (5.2). □

5.2. **Polygons and the class of the tangent cone.** The explicit formula for the classes $\{\mathcal{V}_G\}$ of the complement of the tangent cone of the Potts model hypersurface for polygons is obtained as follows.

Proposition 5.3. *Let ${}^mG$ be the polygon with $m+1$ edges. Then*

$$\{\mathcal{V}_{{}^mG}\} = (\mathbb{T}-1)(-1)^m + 2\mathbb{T}^{m+2} - (\mathbb{T}+1)(\mathbb{T}-1)^{m+1}.$$

Proof. For the seed of the recursion we have in this case

$$\begin{array}{lcl} {}^0G & : & q = 0 \quad (\text{in } \mathbb{A}^{1+1}) \\ {}^1G & : & q^2 + (t_1+t_2)q = 0 \quad (\text{in } \mathbb{A}^{2+1}) \\ {}^2G & : & q^3 + (t_1+t_2+t_3)q^2 + (t_1t_2+t_1t_3+t_2t_3)q = 0 \quad (\text{in } \mathbb{A}^{3+1}) \end{array}$$

for which we then get

$$\begin{aligned} \{\mathcal{V}_{{}^0G}\} &= \mathbb{L}^2 - \mathbb{L}^1 = \mathbb{T}(\mathbb{T}+1) \\ \{\mathcal{V}_{{}^1G}\} &= \mathbb{L}^3 - 2\mathbb{L}^2 + \mathbb{L}^1 = \mathbb{T}^2(\mathbb{T}+1) \\ \{\mathcal{V}_{{}^2G}\} &= \mathbb{T}^4 + 2\mathbb{T}^3 - \mathbb{T} = \mathbb{T}(\mathbb{T}+1)(\mathbb{T}^2+\mathbb{T}-1). \end{aligned}$$

The first two expressions are immediate; for the third, note that $\mathcal{V}_{{}^2G}$ is a cone over a nonsingular quadric in $\mathbb{P}^3$; the computation is then straightforward. We then have

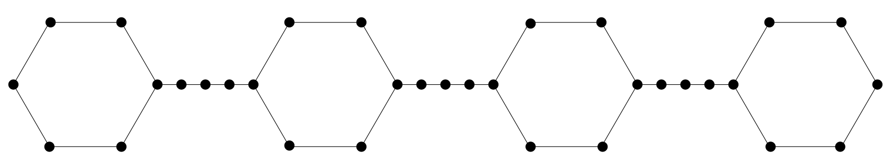
$$\begin{aligned} G_e^{\mathcal{V}}(s) = e^{-s}\Bigg(\left(1+\frac{e^{\mathbb{T}s}-1}{\mathbb{T}}\right)\mathbb{T}(\mathbb{T}+1) + \frac{e^{\mathbb{T}s}-1}{\mathbb{T}}\mathbb{T}^2(\mathbb{T}+1) \\ + \left(e^{(\mathbb{T}+1)s} - e^{\mathbb{T}s} - \frac{e^{\mathbb{T}s}-1}{\mathbb{T}}\right)\cdot 2\mathbb{T}^2\Bigg) \quad . \end{aligned}$$

that is

$$G_e^{\mathcal{V}}(s) = (\mathbb{T}-1)e^{-s} + 2\mathbb{T}^2 e^{\mathbb{T}s} - (\mathbb{T}^2-1)e^{(\mathbb{T}-1)s} \quad .$$

Reading off the coefficient of $s^m/m!$ then gives the result. □

5.3. **Chains of linked polygons.** A class of graphs that have been extensively studied from the point of view of the statistical mechanics of Potts models is the case of chains of linked polygons. These are graphs consisting of N equal polygons ${}^mG$, each with $m+1$ edges, attached to one another by chains of $k \geq 0$ edges. The case $k=0$ corresponds to polygons joined at vertices. The corresponding Potts models were studied, from the point of view of the properties of ground state entropy, in [23]: see also the references therein for several other results on this class of Potts models.



Definition 5.4. *Let ${}^{(m,k)}G^N$ denote the graph obtained by joining N polygons, each with $m+1$ sides, with every two nearby polygons connected by a chain of $k \geq 0$ edges.*

This family of graphs has three parameters m, k, and N, each of which can be independently sent to ∞ to create an infinite graph. Thus, we will use this family as our main example on which to analyze the topological complexity of the corresponding set of virtual phase transitions.

The classes $\{\mathcal{Z}_{G,q}\}$ can be computed explicitly for this family from the result for polygons in Proposition 5.2 and the other basic properties of Corollary 3.5.

Proposition 5.5. *For ${}^{(m,k)}G^N$ as in Definition 5.4, the classes $\{\mathcal{Z}_{G,q}\}$ with fixed $q \neq 0, 1$ are given by*

$$\left(\mathbb{T}^{m+1} + \mathbb{T}(\mathbb{T}^m - (\mathbb{T}-1)^m) + \frac{(\mathbb{T}-1)^m - (-1)^m}{\mathbb{T}}\right)^N \mathbb{T}^{k(N-1)}. \tag{5.4}$$

Proof. By (3) and (4) of Corollary 3.5, we have

$$\{\mathcal{Z}_{{}^{(m,k)}G^N,q}\} = \{\mathcal{Z}_{{}^mG,q}\}^N \ \mathbb{T}^{k(N-1)}.$$

The result then follows from (5.2). □

In this case, since the graphs involved satisfy the fibration condition (2.24), one can also obtain an explicit formula for the classes $\{\mathcal{Z}_{{}^{(m,k)}G^N}\}$ with variable q, from (3.3) of Corollary 3.3.

6. Multiple edge formula

The operation of doubling edges is dual to splitting edges. However, while in the case of the graph hypersurfaces of Feynman graphs analyzed in [1] the corresponding operation on the hypersurface complement classes in the Grothendieck ring is very simple, this is not the case when one considers the Potts model hypersurface.

In fact, the combinatorial deletion–contraction formula for the multivariate Tutte polynomial shows that the polynomial for the graph obtained from G by doubling an edge e is of the form

$$Z_{G\smallsetminus e} + (t_e + t_f + t_e t_f) Z_{G/e}, \tag{6.1}$$

and it is the presence of the extra term $t_e t_f$ here that complicates the matter.

6.1. Edge doubling.

We derive here a formula for the class $\{\mathcal{Z}_G\}$ under the operation of doubling an edge, and then we obtain a recursive formula for the iteration of this operation.

Theorem 6.1. *Let G' be the graph obtained by doubling the edge e in a graph G. Then*

$$\{\mathcal{Z}_{G'}\} = \mathbb{T} \cdot \{\mathcal{Z}_G\} + (\mathbb{T}+1) \cdot \{B^e_G\}, \tag{6.2}$$

where B^e_G is the locus of zeros of $Z_{G\smallsetminus e} - Z_{G/e}$.

Proof. It is convenient to change variables, letting $u_e = 1 + t_e$, $u_f = 1 + t_f$. Then the class (6.1) for the double edged graph is given by

$$Z_{G\smallsetminus e} + (u_e u_f - 1) Z_{G/e} \neq 0. \tag{6.3}$$

If $Z_{G/e} = 0$, then necessarily $Z_{G\smallsetminus e} \neq 0$; u_e and u_f are free, so this accounts for a class

$$(\mathbb{T}+1)^2 \cdot [\mathcal{Z}_{G/e} \smallsetminus (\mathcal{Z}_{G\smallsetminus e} \cap \mathcal{Z}_{G/e})].$$

If $Z_{G/e} \neq 0$, then the condition amounts to

$$u_e u_f \neq 1 - \frac{Z_{G\smallsetminus e}}{Z_{G/e}}.$$

This in turn leads to to two possibilities:

- Either $\frac{Z_{G\smallsetminus e}}{Z_{G/e}} = 1$, and then $u_e u_f \neq 0$; this accounts for $\mathbb{L}^2 - 2\mathbb{L} + 1 = \mathbb{T}^2$;
- Or $\frac{Z_{G\smallsetminus e}}{Z_{G/e}} \neq 1$, and then $u_e u_f \neq c$ for some $c \neq 0$. For $c \neq 0$, $u_e u_f = c$ necessarily gives $u_f \neq 0$, $u_e = c/u_f$; this accounts for $\mathbb{L} - 1$. Thus the class of $u_e u_f \neq c$ is $\mathbb{L}^2 - \mathbb{L} + 1 = \mathbb{T}^2 + \mathbb{T} + 1$.

In total, the class of the complement for the double-edged graph is

$$(\mathbb{T}+1)^2 \cdot [\mathcal{Z}_{G/e} \smallsetminus (\mathcal{Z}_{G\smallsetminus e} \cap \mathcal{Z}_{G/e})] + \mathbb{T}^2[(\mathbb{A}^{|E|} \smallsetminus \mathcal{Z}_{G/e}) \cap V(Z_{G\smallsetminus e} - Z_{G/e})]$$
$$+ (\mathbb{T}^2+\mathbb{T}+1)[(\mathbb{A}^{|E|} \smallsetminus \mathcal{Z}_{G/e}) \smallsetminus V(Z_{G\smallsetminus e} - Z_{G/e})].$$

This expression can be further simplified in the following way:

$$(\mathbb{T}+1)^2 \cdot [\mathcal{Z}_{G/e} \smallsetminus (\mathcal{Z}_{G\smallsetminus e} \cap \mathcal{Z}_{G/e})] + (\mathbb{T}^2+\mathbb{T}+1)[\mathbb{A}^{\#E(G)} \smallsetminus \mathcal{Z}_{G/e}]$$
$$- (\mathbb{T}+1)[(\mathbb{A}^{\#E(G)} \smallsetminus \mathcal{Z}_{G/e}) \cap V(Z_{G\smallsetminus e} - Z_{G/e})]$$

$$= (\mathbb{T}^2+2\mathbb{T}+1) \cdot [\mathcal{Z}_{G/e}] - (\mathbb{T}+1)^2[\mathcal{Z}_{G\smallsetminus e} \cap \mathcal{Z}_{G/e}] + (\mathbb{T}^2+\mathbb{T}+1)[\mathbb{A}^{\#E(G)} \smallsetminus \mathcal{Z}_{G/e}]$$
$$- (\mathbb{T}+1)[(\mathbb{A}^{\#E(G)} \smallsetminus \mathcal{Z}_{G/e}) \cap V(Z_{G\smallsetminus e} - Z_{G/e})]$$

$$= \mathbb{T} \cdot [\mathcal{Z}_{G/e}] - (\mathbb{T}+1)^2[\mathcal{Z}_{G\smallsetminus e} \cap \mathcal{Z}_{G/e}] + (\mathbb{T}^2+\mathbb{T}+1)\mathbb{L}^{\#E(G)}$$
$$- (\mathbb{T}+1)[(\mathbb{A}^{\#E(G)} \smallsetminus \mathcal{Z}_{G/e}) \cap V(Z_{G\smallsetminus e} - Z_{G/e})]$$

$$= (\mathbb{T}+1)^2(\mathbb{L}^{\#E(G)} - [\mathcal{Z}_{G\smallsetminus e} \cap \mathcal{Z}_{G/e}]) - \mathbb{T}(\mathbb{L}^{\#E(G)} - [\mathcal{Z}_{G/e}])$$
$$- (\mathbb{T}+1)[(\mathbb{A}^{\#E(G)} \smallsetminus \mathcal{Z}_{G/e}) \cap V(Z_{G\smallsetminus e} - Z_{G/e})].$$

We then use Theorem 3.1 to express $\{\mathcal{Z}_{G\smallsetminus e} \cap \mathcal{Z}_{G/e}\}$, obtaining

$$(\mathbb{T}+1)(\{\mathcal{Z}_G\} + \{\mathcal{Z}_{G/e}\}) - \mathbb{T}(\{\mathcal{Z}_{G/e}\}) - (\mathbb{T}+1)[(\mathbb{A}^{\#E(G)} \smallsetminus \mathcal{Z}_{G/e}) \cap V(Z_{G\smallsetminus e} - Z_{G/e})]$$

and hence simply

$$(\mathbb{T}+1)\{\mathcal{Z}_G\} + \{\mathcal{Z}_{G/e}\} - (\mathbb{T}+1)[(\mathbb{A}^{\#E(G)} \smallsetminus \mathcal{Z}_{G/e}) \cap V(Z_{G\smallsetminus e} - Z_{G/e})].$$

Now

$$(\mathbb{A}^{\#E(G)} \smallsetminus \mathcal{Z}_{G/e}) \cap V(Z_{G\smallsetminus e} - Z_{G/e}) = V(Z_{G\smallsetminus e} - Z_{G/e}) \smallsetminus (\mathcal{Z}_{G\smallsetminus e} \cap \mathcal{Z}_{G/e}),$$

and again using Theorem 3.1 we get that the class of the complement for the double-edged graph is

$$\mathbb{T} \cdot \{\mathcal{Z}_G\} - (\mathbb{T}+1) \cdot \{V(Z_{G\smallsetminus e} - Z_{G/e})\}.$$

□

As in the case of the term A^e_G in the formula for edge splitting, the term B^e_G here also has an interpretation in terms of the combinatorics of the graph.

Lemma 6.2. *The locus B^e_G of zeros of $Z_{G\smallsetminus e} - Z_{G/e}$ is equivalently the locus of zeros of a polynomial $(q-1)\overline{Z}''$, where Z'' is the sum of the monomials $\prod_{e\in A} t_e$ corresponding to subgraphs A of G/e that acquire an additional connected component when they are viewed in $G \smallsetminus e$, and $\overline{Z}'' = Z''/q$.*

Proof. Recall from (4.7) that

$$Z_{G\smallsetminus e} = Z' + Z'' \quad , \quad Z_{G/e} = Z' + \frac{Z''}{q} \quad .$$

and hence

$$Z_{G\smallsetminus e} - Z_{G/e} = (q-1)\overline{Z}'' \quad ,$$

where $\overline{Z}'' = Z''/q$ is indeed the sum of the standard monomials over the subgraphs of G/e which acquire an additional connected component when they are viewed in $G \smallsetminus e$. □

Thus, the description of B^e_G is somewhat complementary to that of A^e_G: both are $(q-1)$ times a sum of terms having to do with subgraphs of G/e. For A, one looks at graphs for which the number of connected components is the same when the subgraph is viewed in $G \smallsetminus e$; for B, one looks at the graphs for which the number of connected components increases.

Example 6.3. When e is a looping edge, then $B^e_G = 0$: indeed, $G \smallsetminus e = G/e$ in this case. Thus $\{B^e_G\} = 0$, and the formulas simplifies to $\{Z_{G'}\} = \mathbb{T} \cdot \{Z_G\}$, which recovers the case (5) of Corollary 3.2, namely attaching a new looping edge to G.

6.2. **Multiple edge formulas.** We now consider the case where the operation of doubling an edge is iterated, that is, where an edge e in a graph is replaced by m parallel edges, between the same endpoints. This can be seen also as replacing an edge e with the m-th banana graph (see [3]).

Theorem 6.4. *Let $G^{(m)}$ denote the graph obtained by adding m edges parallel to e in G. (So $G = G^{(0)}$, $G' = G^{(1)}$.) Then, for $m \geq 0$, the classes $\{\mathcal{Z}_{G^{(m)}}\}$ satisfy*

$$\{\mathcal{Z}_{G^{(m+2)}}\} = (2\mathbb{T}+1)\{\mathcal{Z}_{G^{(m+1)}}\} - \mathbb{T}(\mathbb{T}+1)\{\mathcal{Z}_{G^{(m)}}\}. \tag{6.4}$$

Proof. The key case to consider is that in which we *triple* a given edge of G: let G' denote (as in Theorem 6.1) the graph obtained from G by doubling e, and let G'' be the graph obtained from G' by doubling e again. Applying Theorem 6.1 yields

$$\{\mathcal{Z}_{G''}\} = \mathbb{T} \cdot \{\mathcal{Z}_{G'}\} + (\mathbb{T}+1) \cdot \{B^e_{G'}\}.$$

Thus, we have to understand $\{B^e_{G'}\}$. According to Lemma 6.2, this hypersurface has equation $(q-1)\overline{Z}''$, where $\overline{Z}''$ collects monomial according to the subgraphs of G'/e which acquire a component when viewed in $G' \smallsetminus e$. The new edge in G' parallel to e cannot be part of any such subgraphs, since it does join the endpoints of e, so it prevents a new component from forming as we remove e. Therefore, the $\overline{Z}''$ for G' actually equals on the nose the $\overline{Z}''$ for G; the only difference between B^e_G and $B^e_{G'}$ is that the latter is contained in a space of dimension one higher, and it may be described as a cylinder on B^e_G.

Thus, we have

$$\{\mathcal{Z}_{G''}\} = \mathbb{T} \cdot \{\mathcal{Z}_{G'}\} + (\mathbb{T}+1) \cdot \{B^e_{G'}\}$$

and $\{W^e_{G'}\} = (\mathbb{T}+1)\{W^e_G\}$. By Theorem 6.1,

$$(\mathbb{T}+1)\{W^e_G\} = \{\mathcal{Z}_{G'}\} - \mathbb{T}\{\mathcal{Z}_G\}.$$

Thus, we obtain

$$\begin{aligned}\{\mathcal{Z}_{G''}\} &= \mathbb{T}\{\mathcal{Z}_{G'}\} + (\mathbb{T}+1)(\{\mathcal{Z}_{G'}\} - \mathbb{T}\{\mathcal{Z}_G\}) \\ &= (2\mathbb{T}+1)\{\mathcal{Z}_{G'}\} - \mathbb{T}(\mathbb{T}+1)\{\mathcal{Z}_G\}.\end{aligned}$$

The stated formula follows by applying this formula to $G^{(m)}$ instead of G. ∎

We can form a generating function for the classes of graphs with multiple edges.

Theorem 6.5. *The generating function of the classes $\{Z_{G^{(m)}}\}$ is given by*

$$\sum_{m\geq 0} \{\mathcal{Z}_{G^{(m)}}\} \frac{s^m}{m!} = ((\mathbb{T}+1)\{\mathcal{Z}_G\} - \{\mathcal{Z}_{G'}\})\, e^{\mathbb{T}s} + (\{\mathcal{Z}_{G'}\} - \mathbb{T}\{\mathcal{Z}_G\})\, e^{(\mathbb{T}+1)s}. \tag{6.5}$$

Proof. The recurrence relation (6.4) of Theorem 6.4 translates into the differential equation

$$g''(s) = (2\mathbb{T}+1)g'(s) - \mathbb{T}(\mathbb{T}+1)g(s) \tag{6.6}$$

for the generating function

$$g(s) = \sum \{\mathcal{Z}_{G^{(m)}}\} \frac{s^m}{m!}, \tag{6.7}$$

solving which reveals that

$$\sum_{m\geq 0}\{\mathcal{Z}_{G^{(m)}}\}\frac{s^m}{m!}=Ae^{\mathbb{T}s}+Be^{(\mathbb{T}+1)s}.$$

Solving for the constants A, B in terms of $\{\mathcal{Z}_G\}$ and $\{\mathcal{Z}_{G'}\}$ gives (6.5). □

Example 6.6. The m-th banana graph is a graph with two vertices and m parallel edges between them. To compute the class for bananas, we can start with G a single non-looping edge, for which $\{\mathcal{Z}_G\}=\mathbb{T}^2$, and G' a 2-banana, for which $\{\mathcal{Z}_{G'}\}=\mathbb{T}^3+\mathbb{T}^2-1$. Then from (6.5) we have

$$\sum_{m\geq 0}\{\mathcal{Z}_{G^{(m)}}\}\frac{s^m}{m!}=e^{\mathbb{T}s}+(\mathbb{T}^2-1)e^{(\mathbb{T}+1)s}\quad ,$$

from which we obtain

$$\{\mathcal{Z}_{G^{(m)}}\}=\mathbb{T}^m+(\mathbb{T}-1)(\mathbb{T}+1)^{m+1} \tag{6.8}$$

for the class of the m-th banana graph. A generating function for the analogous class for graph hypersurfaces $\mathcal{X}_G$ is given in [1].

6.3. **Multiple edges for fixed q.** Again the argument for variable q carries over almost identically to cover the case with fixed $q\neq 0,1$. We obtain the following results.

Proposition 6.7. *Let G' be the graph obtained by doubling the edge e in a graph G. Then*

$$\{\mathcal{Z}_{G',q}\}=\mathbb{T}\cdot\{\mathcal{Z}_{G,q}\}+(\mathbb{T}+1)\cdot\{W^e_{G,q}\},$$

where the locus $W^e_{G,q}\subset\mathbb{A}^{\#E(G)-1}$, for $q\neq 0,1$, is given by the vanishing of the polynomial Z'' adding monomials over the subgraphs of G/e which acquire an additional connected component when they are viewed in $G\smallsetminus e$.

Corollary 6.8. *Let $G^{(m)}$ denote the graph obtained by adding m edges parallel to e in G. (So $G=G^{(0)}$, $G'=G^{(1)}$.) Then for $m\geq 0$*

$$\{\mathcal{Z}_{G^{(m+2)},q}\}=(2\mathbb{T}+1)\{\mathcal{Z}_{G^{(m+1)},q}\}-\mathbb{T}(\mathbb{T}+1)\{\mathcal{Z}_{G^{(m)},q}\}.$$

The general solution of this recursion also matches the one for free-q:

$$\sum_{m\geq 0}\{\mathcal{Z}_{G^{(m)},q}\}\frac{s^m}{m!}=\left((\mathbb{T}+1)\{\mathcal{Z}_{G,q}\}-\{\mathcal{Z}_{G',q}\}\right)e^{\mathbb{T}s}+\left(\{\mathcal{Z}_{G',q}\}-\mathbb{T}\{Z_{G,q}\}\right)e^{(\mathbb{T}+1)s}.$$

Example 6.9. Consider the case of the *banana graphs*. The seeds for bananas are a single non-looping edge, with class $\mathbb{T}$ and the 2-banana, with class $\mathbb{T}^2+\mathbb{T}+1$, as computed above. Plugging into the last formula, we get the generating function for the classes of Potts model complements of banana graphs for fixed q:

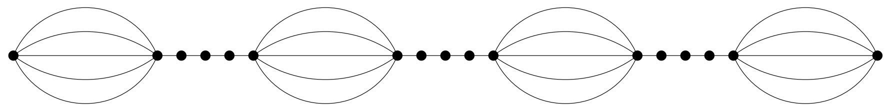
$$\sum_{m\geq 0}\{\mathcal{Z}_{G^{(m)},q}\}\frac{s^m}{m!}=(\mathbb{T}+1)e^{(\mathbb{T}+1)s}-e^{\mathbb{T}s}\quad ;$$

extracting the term of degree m gives the very simple class for the $(m+1)$-banana:

$$(\mathbb{T}+1)^{m+1}-\mathbb{T}^m \tag{6.9}$$

in agreement with the fibration condition (2.24) for (6.8), and in particular independent of $q\neq 0,1$.

6.4. **Chains of linked banana graphs.** By analogy to the example of the chains of linked polygon graphs considered in §5.3, we consider here a similar family but with the polygons replaced by banana graphs.



Definition 6.10. *The graphs* ${}^kG^{(m),N}$ *are obtained by connecting* N *banana graphs* $G^{(m)}$*, each with* m *parallel edges, each connected to the next by a chain of* $k \geq 0$ *edges (connected by joining vertices in the case* $k = 0$*).*

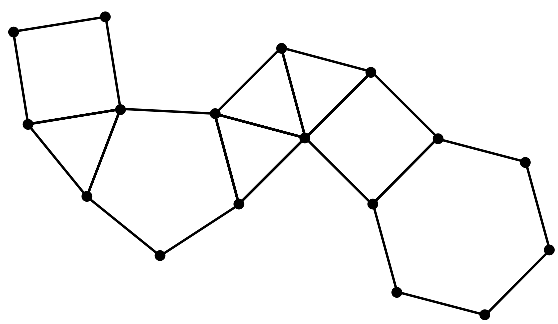
We can compute the classes $\{\mathcal{Z}_{G,q}\}$ for this family of graphs using the explicit formula (6.9) for the banana graphs.

Proposition 6.11. *Let* ${}^kG^{(m),N}$ *be the graphs of Definition 6.10. Then the corresponding classes are given by*

$$\{\mathcal{Z}_{{}^kG^{(m),N},q}\} = ((\mathbb{T}+1)^{m+1} - \mathbb{T}^m)^N \mathbb{T}^{k(N-1)}. \tag{6.10}$$

Proof. The result follows immediately by applying (3) and (4) of Corollary 3.5 to the explicit formula (6.9) in Example 6.9. □

6.5. **Polygon chains.** A class of graphs that can be obtained by alternating edge splitting and edge doubling operations in different orders are the chains of polygons of various sizes joined along edges.



In the case of the graph hypersurfaces of Feynman graphs, explicit formulae for the Grothendieck classes of the hypersurface complements for this type of graphs were obtained in [1]. However, in the case of the Potts model hypersurfaces, the recursion relation becomes more complicated: if one combines the formula for doubling an edge with the formula for splitting it, the resulting class becomes of the form

$$(\mathbb{T}^2 - \mathbb{T} + 1)\{\mathcal{Z}_G\} + (\mathbb{T}-1)\mathbb{T}\{\mathcal{Z}_{G/e}\} + (\mathbb{T}+1)\{C_G^e\} + (\mathbb{T}-2)(\mathbb{T}+1)\{D_G^e\},$$

where the term C_G^e is obtained combinatorially from the subgraphs of G that connect the endpoints of e, and D_G^e is obtained from paths in $G \smallsetminus e$ which do not connect the endpoints of e. There is then no obvious recursive procedure of the type used for either the splitting or the doubling alone, which takes care of eliminating both of these additional terms. This remains an interesting case of graphs to investigate.

7. Estimating topological complexity of virtual phase transitions

We show here, in the concrete example of the chains of linked polygon graphs ${}^{(m,k)}G^N$ and in the simpler example of the banana graphs, how one can use our calculations of classes in the Grothendieck ring to estimate how the topological complexity of the set of virtual phase transition grows as the graphs grow in size within the given family.

Good indicators of topological complexity are homologies and cohomologies and the associated Euler characteristics. In the case of the real algebraic varieties $\mathcal{Z}_{G,q}(\mathbb{R})$, which can in general be singular and non-compact, therefore, we can take as a good indicator the Euler characteristic with compact support $\chi_c(\mathcal{Z}_{G,q}(\mathbb{R}))$, which is known to give a lower bound for the complexity. As we discussed in §2.8, this is the unique invariant of real algebraic varieties that is both a topological invariant and a motivic invariant. Using the fact that it factors through the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{R}})$, we obtain the following results.

Proposition 7.1. *Let ${}^{(m,k)}G^N$ be the chain of linked polygons graphs of Definition 5.4. Then the Euler characteristic with compact support $\chi_c(\mathcal{Z}_{{}^{(m,k)}G^N,q}(\mathbb{R}))$ of the set of virtual phase transitions $\mathcal{Z}_{{}^{(m,k)}G^N,q}(\mathbb{R})$ of the Potts model is given by*

$$(-1)^{mN+kN-k}\left((-1)^N - 2^{kN-k-N}\left(3^{m+1}+1-2^{m+3}\right)^N\right). \tag{7.1}$$

Proof. As we have seen in Example 2.8, for real varieties $\chi_c(\mathbb{T}) = \chi_c(\mathbb{G}_m(\mathbb{R})) = -2$. Using the fact that χ_c is a ring homomorphism $\chi_c : K_0(\mathcal{V}_{\mathbb{R}}) \to \mathbb{Z}$ we then obtain from (5.4) of Proposition 5.5,

$$(-1)^{mN+kN-k}\, 2^{kN-k-N}\left(3^{m+1}+1-2^{m+3}\right)^N.$$

We then use additivity again to obtain

$$\chi_c(\mathcal{Z}_{{}^{(m,k)}G^N,q}(\mathbb{R})) = \chi_c(\mathbb{A}^{\#E({}^{(m,k)}G^N)}) - \chi_c(\{\mathcal{Z}_{{}^{(m,k)}G^N,q}\}),$$

where $\#E({}^{(m,k)}G^N) = N(m+1)+k(N-1)$ and $\chi_c(\mathbb{A}^1) = \chi_c(\mathbb{R}) = -1$, so that we obtain (7.1). $\square$

We obtain a similar result for the class of graphs ${}^kG^{(m),N}$ of Definition 6.10, obtained by chains of linked banana graphs.

Proposition 7.2. *Let ${}^kG^{(m),N}$ be the graphs of Definition 6.10. Then the Euler characteristic with compact support $\chi_c(\mathcal{Z}_{{}^kG^{(m),N}}(\mathbb{R}))$ of the set of virtual phase transitions in the model is given by*

$$(-1)^{mN+kN+N-k}\left(1-2^{k(N-1)}\left(2^m+1\right)^N\right). \tag{7.2}$$

Proof. The argument is exactly as in the previous case, using the expression (6.10) for the classes in the Grothendieck ring. $\square$

Even in the absence of Petrovskiĭ-Oleĭnik inequalities comparing the Euler characteristic of the locus of real zeros and the Hodge numbers of the locus of complex zeros, using the explicit form of the class in the Grothendieck ring and the motivic nature of the invariants, one can also compute for these same examples the virtual Hodge polynomials of the complex variety $\mathcal{Z}_{G,q}(\mathbb{C})$. In the two cases analyzed above one obtains the following.

Proposition 7.3. *Let ${}^{(m,k)}G^N$ be the chain of linked polygons graphs of Definition 5.4. Then the virtual Hodge polynomial $e(\mathcal{Z}_{{}^{(m,k)}G^N,q})(\mathbb{C})$ is given by*

$$e(\mathcal{Z}_{{}^{(m,k)}G^N,q})(\mathbb{C})(x,y) = (xy-1)^{k(N-1)}\left(2(xy-1)^{m+1} - \frac{(-1)^m+(xy-2)^{m+1}xy}{xy-1}\right)^N. \tag{7.3}$$

Let ${}^kG^{(m),N}$ be the graphs of Definition 6.10. Then the virtual Hodge polynomial is given by

$$e(\mathcal{Z}_{{}^kG^{(m),N}}(\mathbb{C}))(x,y) = (xy-1)^{k(N-1)}(xy^{m+1} - (xy-1)^m)^N. \tag{7.4}$$

Proof. Using the explicit form of the classes (5.4) and (6.10) in the Grothendieck, the fact that the virtual Hodge polynomial is a ring homomorphism $e : K_0(\mathcal{V}_\mathbb{C}) \to \mathbb{Z}[x,y]$ and that $e(\mathbb{L}) = xy$, we obtain the result. □

7.1. **Decision complexity and topological complexity.** There is, in fact, a more technical sense, from the point of view of complexity theory, according to which the Euler characteristic $\chi_c(\mathcal{Z}_{G,q}(\mathbb{R}))$ really gives a lower bound for the complexity of the real algebraic variety $\mathcal{Z}_{G,q}(\mathbb{R})$ of virtual phase transitions of the Potts model over the graph G. We describe it briefly following the survey [11].

An algebraic circuit over $\mathbb{R}$ is an acyclic directed graph where each node has in-degree either 0 or 1, or 2. The nodes of in-degree 0 are the input nodes and they are labelled by real variables; the nodes of in-degree 1 are either output nodes (out-degree equal to zero) or sign nodes: these are nodes that, to an input x assign output 1 if $x \geq 0$ and zero otherwise; and the nodes of in-degree 2 are labelled by an operation $+$, $-$, $\times$, or $/$ and are called arithmetic nodes. The size $\sigma(\mathcal{C})$ of an algebraic circuit $\mathcal{C}$ is the number of nodes. To an algebraic circuit $\mathcal{C}$ with n input nodes and m output nodes one can associate a function $\varphi_\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$, the function computed by the circuit. A decision circuit is an algebraic circuit with only one output node returning values of 0 or 1. To each decision circuit one can associate a real (semi)algebraic set $S_\mathcal{C} = \{x \in \mathbb{R}^n \,|\, \varphi_\mathcal{C}(x) = 1\}$ and, conversely, it is known that each (semi)algebraic set in $\mathbb{R}^n$ is realized by some decision circuit. The *decision complexity* of a real (semi)algebraic set S is defined as

$$C(S) = \min\{\sigma(\mathcal{C}) \,|\, S_\mathcal{C} = S\}. \tag{7.5}$$

It is known by [28] that there is a lower bound on the decision complexity of a (semi)algebraic set S of the form

$$C(S) \geq \frac{1}{3}(\log_3 \chi_c(S) - n - 4), \tag{7.6}$$

in terms of the Euler characteristic with compact support. A similar, more refined estimate exists in terms of the sum of the Borel–Moore Betti numbers.

Instead of considering the real algebraic varieties of virtual phase transitions $\mathcal{Z}_{G,q}(\mathbb{R})$, one can look at the actual physical phase transitions. For a finite graph G, this means considering, in the antiferromagnetic case, the semialgebraic set given by the intersection of $\mathcal{Z}_{G,q}(\mathbb{R})$ with the semialgebraic set $S = \{t \in \mathbb{A}^{\#E(G)} \,|\, -1 \leq t_e \leq 0\}$.

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