SHADOWS OF CHARACTERISTIC CYCLES, VERMA MODULES, AND POSITIVITY OF CHERN-SCHWARTZ-MACPHERSON CLASSES OF SCHUBERT CELLS

PAOLO ALUFFI, LEONARDO C. MIHALCEA, JÖRG SCHÜRMANN, AND CHANGJIAN SU

Abstract. Chern-Schwartz-MacPherson (CSM) classes generalize to singular varieties the classical total homology Chern class of the tangent bundle of a smooth compact complex algebraic variety. The theory of CSM classes has been extended to the equivariant setting by Ohmoto. We prove that for an arbitrary complex algebraic variety X, the homogenized, torus equivariant CSM class of a constructible function φ is the restriction of the characteristic cycle of φ via the zero section of the cotangent bundle of X. In the process we relate the CSM class in question to a Segre operator applied to the characteristic cycle. This extends to the equivariant setting results of Ginzburg and of Sabbah. We specialize X to be a (generalized) flag manifold G/B. In this case CSM classes are determined by a Demazure-Lusztig (DL) operator. We prove a 'Hecke orthogonality' of CSM classes, determined by the DL operator and its adjoint, and a 'geometric orthogonality' between CSM and Segre-MacPherson classes. This implies a remarkable formula for the CSM class of a Schubert cell in terms of the Segre class of the characteristic cycle of a holonomic Verma $\mathcal{D}_X$-module. We deduce a positivity property for CSM classes previously conjectured by Aluffi and Mihalcea, and extending positivity results by J. Huh in the Grassmann manifold case. As an application, we prove positivity for certain Kazhdan-Lusztig classes, and for some instances of Mather classes, of Schubert varieties. We also establish an equivalence between CSM classes and stable envelopes; this reproves results of Rimányi and Varchenko. Finally, we generalize all of this to partial flag manifolds G/P.

2010 *Mathematics Subject Classification.* Primary 14C17, 14M15; Secondary 32C38, 14F10, 17B10, 14N15.

Key words and phrases. Chern-Schwartz-MacPherson class, characteristic cycle, shadow, stable envelope, Verma D-module, flag manifold, Schubert cells, Demazure-Lusztig operators.

P. Aluffi was supported in part by NSA Award H98230-16-1-0016 and Simons Collaboration Grants #245301 and #625561; L. C. Mihalcea was supported in part by NSA Young Investigator Award H98320-16-1-0013, the Simons Collaboration Grant #581675, and the NSF award DMS-2152294; J. Schürmann was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Project-ID 427320536 – SFB 1442, as well as under Germany's Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics – Geometry – Structure.

Contents

1. Introduction

According to a conjecture attributed to Deligne and Grothendieck, there is a unique natural transformation $c_* : \mathcal{F} \to H_*$ from the functor of constructible functions to homology, over the category of compact complex algebraic varieties and proper morphisms, such that if X is smooth then $c_*(1\!\!1_X) = c(TX) \cap [X]$. This conjecture was proved by MacPherson [Mac74]; the class $c_*(1\!\!1_X)$ for possibly singular X was later shown to coincide with a class defined earlier by M.-H. Schwartz [Sch65a, Sch65b, BS81]. For any constructible subset $W \subseteq X$, we call the class $c_{SM}(W) := c_*(1\!\!1_W) \in H_*(X)$ the *Chern-Schwartz-MacPherson* (CSM) class of W in X. The theory of CSM classes was later extended to the equivariant setting by Ohmoto [Ohm06]. We denote by $c^T_{\mathrm{SM}}(W) := c^T_*(1\!\!1_W)$ the torus equivariant CSM class.

The main objects of study in this paper are the (torus) equivariant CSM classes of Schubert cells in flag manifolds. These classes were computed in various generality: for Grassmannians, in the non-equivariant specialization, by Aluffi and Mihalcea [AM09, Mih15], and Jones [Jon10]; for type A partial flag manifolds by Rimányi and Varchenko [RV18], using the fact that they coincide with certain weight functions studied in [RTV15b, RTV15a, RTV14], and using interpolation properties obtained by Weber [Web12]; and for flag manifolds in all Lie types by Aluffi and Mihalcea [AM16] using Bott-Samelson desingularizations of Schubert varieties.

One of our main goals is to show that the CSM classes of Schubert cells are effective, thus proving the non-equivariant version of a positivity conjecture from [AM16]. This generalizes a similar positivity result proved by J. Huh [Huh16] for the Grassmannian case (in turn proving an earlier conjecture posed in [AM09]). This also implies positivity of the Kazhdan-Lusztig classes (associated to the stalk Euler characteristic of intersection cohomology complexes of Schubert varieties), and in some instances positivity of Mather classes.

On the route of proving the positivity conjecture we revisit, and extend to the equivariant setting, the 'Lagrangian model' for constructing MacPherson's transformation for arbitrary smooth projective varieties X, as developed by Sabbah [Sab85] and by Ginzburg [Gin86]. Our methods and results, using 'shadows' of characteristic cycles in the cotangent bundle of X, may be of independent interest.

For flag manifolds, we apply our methods to relate CSM classes of Schubert cells, characteristic cycles of Verma $\mathcal{D}$-modules, and Maulik and Okounkov stable envelopes for the cotangent bundle of flag manifolds [MO19]. This adds a 'Lagrangian perspective' to previous connections between CSM classes and stable envelopes found by Rimányi and Varchenko [RV18] (see also Su's thesis [Su17b]). We give a more precise account of our results next.

1.1. **Statement of results.**

The first part of the paper (§§2-4) consists of a general discussion of Ohmoto's torus equivariant CSM classes from the point of view of Lagrangian cycles in the cotangent bundle of a smooth complex algebraic variety X. We extend the formalism of *shadows* of characteristic cycles developed in [Alu04] to build a dictionary between $\mathbb{C}^*$-equivariant classes in a vector bundle E endowed with a $\mathbb{C}^*$-action by fiberwise dilation and *homogenizations* of shadows of corresponding classes in the projective completion of E (Proposition 2.7). The homogenization of a class $\alpha = \sum_{i=0}^n \alpha_i$ with $\alpha_i \in H_{2i}(X)$, with respect to the character χ of $\mathbb{C}^*$, is the class

$$\alpha^\chi := \alpha_0 + \chi\alpha_1 + \cdots + \chi^n\alpha_n \in H_0^{\mathbb{C}^*}(X) \quad .$$

Here the character χ is identified with $c_1^{\mathbb{C}^*}(\mathbb{C}_\chi) \in H^2_{\mathbb{C}^*}(\mathrm{pt})$, the $\mathbb{C}^*$-equivariant first Chern class of the $\mathbb{C}^*$-module $\mathbb{C}_\chi$ given by the dilation on $\mathbb{C}$ with character χ; see §2.3 below. Let $\hbar^{\pm 1}$ be the characters defined by $z \mapsto z^{\pm 1}$. Note that $H_*^{\mathbb{C}^*}(X) \cong H_*(X)[\hbar]$ since $\mathbb{C}^*$ acts trivially on X.

We extend the formalism further to smooth varieties X endowed with the action of a torus T. This notion allows us to define a morphism from the group of T-equivariant conical Lagrangian cycles in the cotangent bundle of X to $H_*^T(X)$. One of the main results is the following (Theorem 4.2):

Theorem 1.1. *Let X be a smooth complex algebraic variety, with a T-action. Consider the $\mathbb{C}^*$-action dilating the cotangent fibers with character $\hbar^{-1}$ on $T^*(X)$. Let $\iota : X \to T^*(X)$ be the zero section. Then*

$$\iota^*[\mathrm{CC}(\varphi)]_{T\times\mathbb{C}^*} = c_*^T(\varphi)^\hbar \in H_0^{T\times\mathbb{C}^*}(X).$$

Here $c_*^T(-)^\hbar$ is the homogenization of Ohmoto's equivariant MacPherson natural transformation, and $\mathrm{CC}(\varphi)$ is the characteristic cycle of a constructible function φ; see §3.2. Theorem 1.1 generalizes to the equivariant case results of Ginzburg [Gin86, Appendix] and Sabbah [Sab85]. In fact, even in the non-equivariant case, the theorem gives a transparent interpretation, as the pull-back via the zero section, of Ginzburg's analogous map from *loc. cit.* (as explained in §4.2). Our proof, based on the formalism of shadows, is rather elementary and it avoids the use of equivariant K-theory and the equivariant Riemann-Roch transformation in [Gin86]; at the same time it has a natural equivariant extension. A reader who is not familiar with the work of Ohmoto [Ohm06] (or Ginzburg [Gin86, Appendix] in the non-equivariant case) can take Theorem 1.1 as the starting point for a definition of the (non-)equivariant Chern class transformation c_*^T (resp., c_*).

In intersection theory, the pull-back via the zero section is closely related to a Segre operator; cf. [Ful84, §3.3]. We extend this relation equivariantly, using (equivariant) shadows. The resulting identity is not only used to prove Theorem 1.1, but it also informs the rest of this paper. In order to state it, we first define the Segre operator. Consider the following diagram:

$$\begin{tikzcd}
T^*(X) \arrow[r, hook] \arrow[dr, bend left, "\pi"] & \mathbb{P}(T^*(X) \oplus 1\!\!1) \arrow[d, "\overline{q}"] \\
& X \arrow[ul, bend left, "\iota"]
\end{tikzcd}$$

Here $\mathbb{P}(T^*(X)\oplus 1\!\!1)$ is the completion of the cotangent bundle, $\pi, \overline{q}$, the natural projections, and ι is the inclusion of the zero section. As before, consider a $\mathbb{C}^*$-action on $T^*(X)$ acting on the cotangent fibers with character $\hbar^{-1}$. Let $C \subseteq T^*(X)$ be an equivariant cycle in $T^*(X)$ and let $\overline{C} \subseteq \mathbb{P}(T^*(X)\oplus 1\!\!1)$ be its Zariski closure. Associated with this data, we consider the Segre operator given by

$$\mathrm{s}^T(C) := \overline{q}_*\Big(\frac{[\overline{C}]}{c^T(\mathscr{O}(-1))}\Big) = \overline{q}_*(\sum_{j\geq 0} c_1^T(\mathscr{O}(1))^j \cap [\overline{C}])$$

(as well as its non-equivariant counterpart $\mathrm{s}(C)$). Here c^T denotes the total torus equivariant Chern class of a T-equivariant vector bundle. In non-equivariant homology, the Chern classes c_i, $i \geq 1$ are nilpotent, therefore this Segre operator has values in homology. In the equivariant context the operator has values in the completion $\hat{H}_*^T(X)$. With this notation, we prove in Theorem 3.3 and Corollary 3.5 the following identity.

Theorem 1.2. *Let φ be a T-invariant constructible function and* $\mathrm{CC}(\varphi)$ *its characteristic cycle. Then*

$$c^T(T^*(X)) \cap \mathrm{s}^T(\mathrm{CC}(\varphi)) = \mathrm{Shadow}^T(\mathrm{CC}(\varphi)) = \breve{c}_*^T(\varphi) \in H_*^T(X),$$

where $\breve{c}_*^T(\varphi)$ denotes a 'signed' version of Ohmoto's natural transformation; see (13).

The second part of the paper (§§5-9) applies this formalism to the study of CSM classes of Schubert cells in (generalized) flag manifolds $X = G/B$, where G is a complex, semisimple Lie group and B is a Borel subgroup. Let $T \subseteq B$ be the maximal torus, and W the associated Weyl group. Let R^+ denote the set of positive roots associated to (G, B). Denote by $X(w)^\circ := BwB/B$ the Schubert cell for the Weyl group element $w \in W$. Further, let $\mathcal{M}_w$ be the Verma $\mathcal{D}_X$-module determined by the Verma module of highest weight $-w(\rho)-\rho$, where ρ is half the sum of positive roots. Denote by $\mathrm{Char}(\mathcal{M}_w)$ the characteristic cycle of the holonomic $\mathcal{D}_X$-module $\mathcal{M}_w$; see §8.1. This is a conic $T\times\mathbb{C}^*$-stable Lagrangian cycle in $T^*(X)$.

The relevance of the Verma module comes from the proof of the Kazhdan-Lusztig conjectures by Brylinski-Kashiwara [BK81] and Beilinson-Bernstein [BB81]. There it was shown that $\mathrm{Char}(\mathcal{M}_w)$ is (up to a sign) equal to the the characteristic cycle $\mathrm{CC}(1\!\!1_{X(w)^\circ})$. The main result of the paper involves a formula for the CSM classes in terms of the Segre operator applied to characteristic cycle of the Verma module (Theorem 8.3):

Theorem 1.3. *Let $w \in W$ be an element in the Weyl group. Then the following equality holds:*

$$c_{\mathrm{SM}}^T(X(w)^\circ) = \Big(\prod_{\alpha\in R^+}(1+\alpha)\Big)\,\mathrm{s}^T(\mathrm{Char}(\mathcal{M}_w)) \in \hat{H}_*^T(G/B).$$

In particular, in non-equivariant homology,

$$c_{\mathrm{SM}}(X(w)^\circ) = \mathrm{s}(\mathrm{Char}(\mathcal{M}_w)) \in H_*(G/B).$$

In the non-equivariant case the Segre class is manifestly effective, and this implies the positivity of CSM classes. The proof of this consequence (but *not* of Theorem 1.3!) is independent of other facts in the paper, and will be given next. Let $[X(w)] \in H_*(X)$ be the fundamental class of the Schubert variety $X(w) := \overline{X(w)^\circ}$. The set of fundamental classes $\{[X(w)]\}_{w\in W}$ forms a basis of the $\mathbb{Z}$-module $H_*(X)$.

Corollary 1.4 (Positivity of CSM classes)**.** *Let $w \in W$. Then the non-equivariant CSM class $c_{\mathrm{SM}}(X(w)^\circ)$ is effective, i.e., in the Schubert expansion*

$$c_{\mathrm{SM}}(X(w)^\circ) = \sum_{v\leq w} c(v;w)[X(v)] \in H_*(X),$$

the coefficients $c(v;w)$ are non-negative. Further, let $P \subseteq G$ be any parabolic subgroup. Then the CSM classes of Schubert cells in $H_*(G/P)$ are effective, i.e., its coefficients in the corresponding Schubert expansion are nonnegative.

Proof. Consider first $X = G/B$. The characteristic cycle of the Verma $\mathcal{D}_X$-module $\mathrm{Char}(\mathcal{M}_w)$ in $T^*(X)$ is effective [HTT08, p. 60], and so is its closure in $\mathbb{P}(T^*(X) \oplus \mathbb{1})$. (The Verma module is also holonomic, and in this case effectivity is explicit from [HTT08, p. 119], or from [Sch03, Remark 6.0.4 on p. 389] in terms of perverse sheaves.) The tautological line bundle $\mathscr{O}_{T^*(X)\oplus\mathbf{1}}(1)$ is globally generated, i.e., it is a quotient of a trivial bundle. Indeed, $\mathscr{O}_{T^*(X)\oplus\mathbf{1}}(1)$, is a quotient of $T(X)\oplus\mathbb{1}$, and by homogeneity of X, $T(X)$ is globally generated. Then $c_1(\mathscr{O}_{T^*(G/B)\oplus\mathbf{1}}(1))\cap[\overline{C}]$ is effective, for any effective cycle $\overline{C}$ [Ful84, Ex. 12.1.7 (a)]. The result follows from Theorem 1.3. The effectivity of CSM classes in G/B implies the effectivity in G/P by [AM16, Proposition 3.5]. □

This generalizes the positivity of CSM classes of Schubert cells for Grassmann manifolds, which was proved in several cases by Aluffi and Mihalcea [AM09, Mih15], Jones [Jon10], and Stryker [Str11], and for any Schubert cell in any Grassmann manifold by J. Huh [Huh16]. In fact, Huh proved a stronger version of the positivity conjecture, asserting that each homogeneous component of the CSM class of a Schubert cell is equal to the fundamental class of a non-empty subvariety inside the corresponding Schubert variety. Huh's proof used that Schubert varieties in Grassmann manifolds can be desingularized by varieties with finitely many Borel orbits. Unfortunately, the known desingularizations of Schubert varieties in arbitrary flag manifolds do not satisfy this property in general. Seung Jin Lee [Lee18, Theorem 1.1] proved that the positivity of CSM classes for type A flag manifolds is *implied* by a positivity conjecture in a certain subalgebra of Fomin-Kirillov algebra [FK99] generated by Dunkl elements. Thus CSM positivity can be also regarded as evidence for the Fomin-Kirillov conjecture.

Further consequences of the positivity of CSM classes are the positivity of certain 'Kazhdan-Lusztig' classes, the positivity of Chern-Mather classes for the Schubert varieties in the minuscule Grassmannians of classical Lie type (both discussed in §8.3), and the fact that the *Segre-MacPherson classes* of Schubert cells in G/B are Schubert alternating; see Theorem 7.5.

The proof of Theorem 1.3 is based on Theorem 1.2 and two orthogonality properties. Let $Y(w)^\circ = B^- wB/B \subseteq G/B$ be the Schubert cell determined by the opposite Borel subgroup B^-. Then G/B has two transversal Whitney stratifications: $G/B = \bigsqcup_w X(w)^\circ = \bigsqcup_w Y(w)^\circ$. By Theorem 7.1,

$$\left\langle c^T_{\mathrm{SM}}(X(u)^\circ), \frac{c^T_{\mathrm{SM}}(Y(v)^\circ)}{c^T(T(G/B))} \right\rangle = \delta_{u,v}. \tag{1}$$

Non-equivariantly, this was proved by Schürmann [Sch17] for complex varieties with transversal Whitney stratifications; see Theorem 3.6. In an Appendix to this paper (§10) we include an outline of that proof, including the details required to extend it equivariantly. A similar orthogonality holds for Maulik and Okounkov stable envelopes [MO19], and it was used to prove (1) in an earlier arχiv version of this paper. In the language of *weight functions,* equivalent statements were obtained in [RTV14, RTV15a, RTV15b, GRTV13] for the type A flag manifolds.

The second orthogonality is derived from the fact, proved in [AM16], that the CSM classes may be calculated recursively via Demazure-Lusztig (DL) operators. These generate the degenerate Hecke algebra of W [LLT96, LLT97, Gin98]; see §6. The key fact is that the classes of the adjoint DL operators generate Poincaré dual classes, giving the following (cf. Theorem 7.2):

$$\langle c^T_{\mathrm{SM}}(X(u)^\circ), c^{T,\vee}_{\mathrm{SM}}(Y(v)^\circ) \rangle = \delta_{u,v} \prod_{\alpha \in R^+} (1+\alpha). \tag{2}$$

Here $c^{T,\vee}_{\mathrm{SM}}(Y(v)^\circ)$ is a 'signed' version of the CSM class, where the signs of the homogeneous components are changed according to complex codimension. We call this the 'Hecke orthogonality'. Combining identities (1) and (2) yields the remarkable identity

$$c^{T,\vee}_{\mathrm{SM}}(Y(v)^\circ) = \prod_{\alpha \in R^+} (1+\alpha) \frac{c^T_{\mathrm{SM}}(Y(v)^\circ)}{c^T(T(G/B))}.$$

By Theorem 1.2, the right-hand side of this identity is essentially equal to the Segre operator $\mathrm{s}^T(\mathrm{Char}(\mathcal{M}_w))$, except that *we need to change signs of homogeneous components*. Then the formula from Theorem 1.3 will follow; see Theorem 8.3.

As further applications, observe that Theorem 1.1, applied to the indicator function of the Schubert cell $\varphi = 1\!\!1_{X(w)^\circ}$, implies that

$$c^{T,\hbar}_{\mathrm{SM}}(X(w)^\circ) = (-1)^{\ell(w)} \iota^* [\mathrm{Char}(\mathcal{M}_w)].$$

The Verma characteristic cycle in the right-hand side equals Maulik and Okounkov's *stable envelope* $\mathrm{stab}_+(w)$. This is stated without proof in [MO19, Remark 3.5.3, p. 69], and for completeness we sketch an argument in Lemma 9.4. Combining the two facts one immediately obtains the following corollary (cf. Corollary 9.5):

Corollary 1.5. *Let $w \in W$. Then $\iota^*(\mathrm{stab}_+(w)) = (-1)^{\dim X} c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ)$ as elements in $H_0^{T\times\mathbb{C}^*}(X)$.*

This equality was proved earlier by Rimányi and Varchenko [RV18, Theorem 8.1 and Remark 8.2] (see also [Su17b]), using interpolation properties of CSM classes stemming from Weber's work [Web12] and the defining localization properties of the stable envelopes; cf. §9. For a K theoretic version of these results, see e.g., [AMSS19]. Once this is established, we use the dictionary between CSM classes and stable envelopes to prove various results about the former, notably a localization formula (Corollary 9.8) and a Chevalley formula (Theorem 9.10), which might be of independent interest.

1.2. **Conventions and notation.** We work over $\mathbb{C}$. *Varieties* are reduced and irreducible. *Subvarieties* are assumed to be closed. An *irreducible cycle* is the cycle of a subvariety. We will frequently use the following notation; we indicate here where the notation is defined.

α^χ, homogenization of a homology class α by a character χ .... §2.3 (5)

Shadow^T, equivariant shadow .... §2.4 (9)

c_{SM}^T, equivariant Chern-Schwartz-MacPherson (CSM) class Definition 3.2

c_{Ma}^T, equivariant Chern-Mather class .... Definition 3.2

$\breve{c}_*$, $\breve{c}_*^T$, signed (Ohmoto-)MacPherson natural transformations §3.1, (13)

$c_{\mathrm{SM}}^{T,\vee}$, $c_{\mathrm{Ma}}^{T,\vee}$, dual equivariant CSM and Mather classes .... §3.1, (14)

$c_{\mathrm{SM}}^{T,\hbar}$, $c_{\mathrm{SM}}^{T,\hbar,\vee}$, homogenized equivariant CSM classes/dual CSM classes §8.1, §9.2

$\breve{\mathrm{Eu}}$, signed Euler obstruction .... §3.2

$\hbar$, identity character of $\mathbb{C}^*$ .... §2.1

s^T, equivariant Segre operator .... §2.4, (11)

s_{SM}, s_{SM}^T, (equivariant) Segre-MacPherson class .... §3.1, (23)

$X(w)^\circ$, Schubert cell .... §5

$Y(w)^\circ$, opposite Schubert cell .... §5

$\mathcal{T}_i, \mathcal{T}_i^\vee$, Demazure–Lusztig operators .... §6.1, (30)

$\mathcal{M}_w$, Verma module 8.1

$\mathrm{KL}(w)$, Kazhdan-Lusztig classes 8.6

$\mathrm{stab}_+(w)$, stable envelopes .... §9, Theorem 9.2

Acknowledgments. P. Aluffi is grateful to Caltech for hospitality while part of this work was carried out. L. C. Mihalcea wishes to thank D. Anderson, A. Knutson, S. J. Lee, M. Shimozono, D. Orr and R. Rimányi for stimulating discussions; and further A. Knutson for being a catalyst to this collaboration. C. Su wishes to thank his advisor A. Okounkov for encouragements and useful discussions. Part of this work was carried while at the 2015 PCMI Summer session in Park City; L. C. Mihalcea and C. Su are both grateful for support and for providing an environment conducive to research. Finally, we are indebted to two anonymous referees, whose careful reading and many valuable suggestions helped us improve substantially the structure of this paper.

2. Shadows and equivariant (co)homology

2.1. **Equivariant (co)homology.** In this paper we work in the complex algebraic context and utilize $H_*(X)$, the Borel-Moore homology group of X, and $H^*(X)$, the cohomology ring. The reader so inclined may use Chow (co)homology instead: there is a homology degree doubling cycle map between the Chow and Borel-Moore groups, and our constructions are compatible with this map. This map is an isomorphism in some important situations, such as the complex flag manifolds, studied later in this paper. We refer to [Ful84, §19.1] and [Gin98, §2.6] for more details about Borel-Moore homology and its relation to Chow groups. In case we speak of (co)dimension we always assume that our spaces are pure dimensional; in addition, by (co)dimension we will mean the *complex* (co)dimension. Thus any subvariety of (complex) dimension k has a fundamental class $[Y] \in H_{2k}(X)$. Whenever X is smooth, we can and will identify the Borel-Moore homology and cohomology via Poincaré duality.

Let T be a torus and let X be a variety with a T-action. Then the equivariant cohomology $H_T^*(X)$ is the ordinary cohomology of the Borel mixing space $X_T := (ET \times X)/T$, where ET is the universal T-bundle and T acts by $t \cdot (e,x) = (et^{-1}, tx)$. It is an algebra over $H_T^*(\mathrm{pt})$, the polynomial ring $\mathrm{Sym}_{\mathbb{Z}}\, \mathfrak{X}(T)$ in the character group $\mathfrak{X}(T)$; see e.g., [And12, AF23] or [Kum02, §11.3.5]. In this paper we utilize the T-equivariant version of Borel Moore homology theory, and one may alternatively work with the T-equivariant Chow groups. The two theories are related by an equivariant cycle map. The space $ET \simeq (\mathbb{C}^\infty \setminus 0)^{\mathrm{rank}\ T}$ is infinite dimensional, and the latter theories utilize finite dimensional approximations $U \simeq (\mathbb{C}^N \setminus 0)^{\mathrm{rank}\ T} \subset ET$, giving approximations $U \times_T X$ of the mixing space X_T; see [EG98].[1] Every k-dimensional subvariety $Y \subseteq X$ that is stable under the T action determines an equivariant fundamental class $[Y]_T$ in $H_{2k}^T(X)$. As in the non-equivariant case, whenever X is smooth, we will identify $H_*^T(X)$ and

[1] In general, the algebraic approximation $(U \times X)/T$ of the Borel mixing space X_T is only a separated algebraic space, but if X is a quasi-projective scheme, then $(U \times X)/T$ is again quasi projective; see [Ohm06, §2.2] or [Ohm12].

$H^*_T(X)$. In particular, when $X = \mathrm{pt}$, the identification sends $a \in H^*_T(\mathrm{pt})$ to $a \cap [\mathrm{pt}]_T$. We address the reader to [And12, AF23, Knu], or [Ohm06] for basic facts on equivariant cohomology and homology. Equivariant vector bundles have equivariant Chern classes $c_j^T(-) \in H^{2j}_T(X)$, such that $c_j^T(E) \cap -$ is an operator $H_i^T(X) \to H_{i-2j}^T(X)$; see [And12, §1.3], [EG98, §2.4]. In terms of the Borel construction, $c_j^T(E)$ corresponds to $c_j(E_T) \in H^{2j}(X_T)$ for the vector bundle $E_T \to X_T$. Then the above identification $H^*_T(\mathrm{pt}) \cong \mathrm{Sym}_{\mathbb{Z}}\,\mathfrak{X}(T)$ is induced by the isomorphism $\mathfrak{X}(T) \cong H^2_T(\mathrm{pt})$, which associates to a character $\chi \in \mathfrak{X}(T)$ the first equivariant Chern class $c_1^T(\mathbb{C}_\chi) \in H^2_T(\mathrm{pt})$ of the T-module $\mathbb{C}_\chi$ given by the dilation on $\mathbb{C}$ with character χ (regarded as an equivariant line bundle over a point).

Let $\pi : E \to X$ be a vector bundle of rank $e+1$ on X. We consider the action of $\mathbb{C}^*$ on E by fiberwise dilation with character χ, and denote by E^χ the vector bundle E endowed with this $\mathbb{C}^*$-action. Equivalently, $E^\chi \cong E \otimes \mathbb{C}_\chi$, where $\mathbb{C}^*$ acts trivially on E and X and via χ on $\mathbb{C}_\chi$. The natural projection $\pi : E^\chi \to X$ is equivariant, where $\mathbb{C}^*$ acts trivially on X. We may take $E\mathbb{C}^* = \mathbb{C}^\infty \setminus 0$; since the action of $\mathbb{C}^*$ on X is trivial, the Borel mixing space $X_{\mathbb{C}^*}$ is isomorphic to $B\mathbb{C}^* \times X = \mathbb{P}^\infty \times X$. Here and in the following, we will denote by $\mathbb{P}^\infty$ any approximation $\mathbb{P}^N$ with $N \gg 0$ sufficiently large; see e.g., [And12, §1.2]. We will give the results in the ordinary and equivariant Borel-Moore homology groups. Since $X_{\mathbb{C}^*} \cong \mathbb{P}^\infty \times X$,

$$H_*^{\mathbb{C}^*}(X_{\mathbb{C}^*}) \cong H_*(X)[\hbar] \quad ,$$

where $\hbar := c_1(\mathscr{O}_{\mathbb{P}^\infty}(-1)) \in H^2(\mathbb{P}^\infty) = H^2_{\mathbb{C}^*}(\mathrm{pt})$ corresponds to the equivariant first Chern class $\hbar := c_1^{\mathbb{C}^*}(\mathbb{C}_{\chi_1}) \in H^2_{\mathbb{C}^*}(\mathrm{pt})$ of the $\mathbb{C}^*$-module $\mathbb{C}_{\chi_1}$ with χ_1 the character $z \mapsto z^1$. Next, denote by

$$\rho : X_{\mathbb{C}^*} = \mathbb{P}^\infty \times X \longrightarrow X$$

the projection. If χ is the character χ_a given by $z \mapsto z^a$, a standard computation shows that the mixing space $E^\chi_{\mathbb{C}^*}$, along with the natural projection to $X_{\mathbb{C}^*}$, is isomorphic to the vector bundle $\pi^\chi : \rho^*E \otimes \mathscr{O}_{\mathbb{P}^\infty}(-a) \to X_{\mathbb{C}^*}$. We have the diagram

$$\begin{tikzcd} E^\chi_{\mathbb{C}^*} = \rho^*E \otimes \mathscr{O}_{\mathbb{P}^\infty}(-a) \arrow[d, "\pi^\chi"'] & E^\chi \arrow[d, "\pi"] \\ X_{\mathbb{C}^*} = \mathbb{P}^\infty \times X \arrow[r, "\rho"] & X \end{tikzcd} \tag{3}$$

Lemma 2.1. *The projection π induces by flat pull-back a codimension-preserving isomorphism $\pi^* : H_*^{\mathbb{C}^*}(X) \xrightarrow{\sim} H_*^{\mathbb{C}^*}(E^\chi)$.*

The embedding $\iota : X \to E$ of the zero section induces a codimension-preserving isomorphism

$$\iota^* : H_*^{\mathbb{C}^*}(E^\chi) \xrightarrow{\sim} H_*^{\mathbb{C}^*}(X) \cong H_*(X)[\hbar] \quad ,$$

satisfying $\iota^* = (\pi^*)^{-1}$.

Proof. This follows from [Ful84, Theorem 3.3(a)] applied to the projection $\pi^\chi : E^\chi_{\mathbb{C}^*} \to X_{\mathbb{C}^*}$. □

2.2. **Shadows: definition and basic properties.** Let $\pi : E \to X$ be a vector bundle of rank $e+1$ on a variety X, and consider the projective bundle of lines $q : \mathbb{P}(E) \to X$. Let $\zeta = c_1(\mathscr{O}_E(1))$. Then $H_*(\mathbb{P}(E))$ is a direct sum of $e+1$ copies of $H_*(X)$: every class α of codimension k in $H_*(\mathbb{P}(E))$ may be written as

$$\alpha = \sum_{j=0}^{e} \zeta^j q^*(\underline{\alpha}^{k-j}) \tag{4}$$

for uniquely defined classes $\underline{\alpha}^{k-j}$ of codimension $k-j$ in $H_*(X)$ (cf. [Ful84, Theorem 3.3(b)]). In fact, we have the relation

$$\zeta^{e+1} + \zeta^e q^* c_1(E) + \cdots + \zeta q^* c_e(E) + q^* c_{e+1}(E) = 0$$

in $H_*(\mathbb{P}(E))$ ([Ful84, Remark 3.2.4]).

Following [Alu04], we call the (non-homogeneous) class

$$\mathrm{Shadow}_E(\alpha) := \underline{\alpha}^{k-e} + \underline{\alpha}^{k-e+1} + \cdots + \underline{\alpha}^{k} \in H_*(X)$$

the *shadow* of α in X. By (4), a homogeneous class $\alpha \in H_*(\mathbb{P}(E))$ may be reconstructed from its shadow and its codimension k. We will omit the subscript E from the notation if the ambient projective bundle is understood from the context. The following lemma is useful in relating shadows of classes in different projective bundles, with $c(-)\cap$ the total Chern class operator.

Lemma 2.2. *For a class α in $H_*(\mathbb{P}(E))$ the following hold in $H_*(X)$.*

(i) *The shadow* $\mathrm{Shadow}_E(\alpha)$ *equals*

$$c(E) \cap q_*(c(\mathscr{O}_E(-1))^{-1} \cap \alpha) = c(E) \cap q_* \sum_{j\geq 0} c_1(\mathscr{O}_E(1))^j \cap \alpha \quad ;$$

(ii) *If F is a subbundle of E, and $\alpha \in H_*(\mathbb{P}(F)) \hookrightarrow H_*(\mathbb{P}(E))$, then*

$$\mathrm{Shadow}_E(\alpha) = c(E/F) \cap \mathrm{Shadow}_F(\alpha) \quad ;$$

(iii) *If* $\mathrm{Shadow}_E(\alpha) = \sum_{j=0}^{e} \underline{\alpha}^{k-j}$*, and L is a line bundle on X, then*

$$\mathrm{Shadow}_{E\otimes L}(\alpha) = \sum_{j=0}^{e} c(L)^j \cap \underline{\alpha}^{k-j} \quad ;$$

(iv) *Further, let $E \to F$ be a surjection of bundles, with kernel K, and let C_E be a cycle in $\mathbb{P}(E)$ disjoint from $\mathbb{P}(K)$. Let C_F be the cycle in $\mathbb{P}(F)$ obtained by pushing forward C_E. Then*

$$\mathrm{Shadow}_E([C_E]) = c(K) \cap \mathrm{Shadow}_F([C_F]) \quad .$$

Proof. Part (i) is [Alu04, Lemma 4.2]. Part (ii) follows immediately from (i). Part (iii) is a straightforward computation, which we leave to the reader. For part (iv), let $\underline{q} : \mathbb{P}(F) \to X$ be the projection. The given surjection $E \to F$ induces a rational map $\mathbb{P}(E) \dashrightarrow \mathbb{P}(F)$, which is resolved by blowing up along $\mathbb{P}(K)$; let $\nu : \widetilde{\mathbb{P}} \to \mathbb{P}(E)$ be this blow-up, and let $\underline{\nu} : \widetilde{\mathbb{P}} \to \mathbb{P}(F)$ be the induced morphism:

$$\begin{tikzcd} & \widetilde{\mathbb{P}} \arrow[dl, "\nu"'] \arrow[dr, "\underline{\nu}"] & \\ \mathbb{P}(E) \arrow[rr, dashed] \arrow[dr, "q"'] & & \mathbb{P}(F) \arrow[dl, "\underline{q}"] \\ & X & \end{tikzcd}$$

Since C_E is disjoint from $\mathbb{P}(K)$ and ν is an isomorphism over $\mathbb{P}(E) \smallsetminus \mathbb{P}(K)$, the cycle C_E determines a cycle $\widetilde{C}_E$ in $\widetilde{\mathbb{P}}$, disjoint from the exceptional divisor, such that $[C_E] = \nu_*[\widetilde{C}_E]$ and $[C_F] = \underline{\nu}_*[\widetilde{C}_E]$. By part (i) and the projection formula,

$$\mathrm{Shadow}_E([C_E]) = c(E) \cap q_*\nu_*(c(\nu^*\mathscr{O}_E(-1))^{-1} \cap [\widetilde{C}_E]) \quad .$$

Now note that $\nu^*\mathscr{O}_E(-1)$ and $\underline{\nu}^*\mathscr{O}_F(-1)$ differ by a term supported on the exceptional divisor in $\widetilde{\mathbb{P}}$, hence they agree on $\widetilde{C}_E$. Therefore

$$\begin{aligned} \mathrm{Shadow}_E([C_E]) &= c(E) \cap \underline{q}_*\underline{\nu}_*(c(\underline{\nu}^*\mathscr{O}_F(-1))^{-1} \cap [\widetilde{C}_E]) \\ &= c(E) \cap \underline{q}_*(c(\mathscr{O}_F(-1))^{-1} \cap [C_F]) \\ &= c(K) \cap \mathrm{Shadow}_F([C_F]) \end{aligned}$$

again by the projection formula and part (i). □

The formula in part (i) may be expressed concisely in terms of a 'Segre class operator', which we will introduce (in the equivariant setting) in §2.4.

Shadows are compatible with the operation of taking a cone. More precisely, let $1\!\!1$ denote the trivial rank-1 line bundle on X, and consider the projective completion $\mathbb{P}(E \oplus 1\!\!1)$; E may be identified with the complement of $\mathbb{P}(E \oplus 0)$ in $\mathbb{P}(E \oplus 1\!\!1)$. Consider a $\mathbb{C}^*$-action on E by fiberwise dilation, and the trivial $\mathbb{C}^*$-action on $1\!\!1$. This induces a $\mathbb{C}^*$-action on $\mathbb{P}(E \oplus 1\!\!1)$ such that the inclusion $E \subseteq \mathbb{P}(E \oplus 1\!\!1)$ is $\mathbb{C}^*$-equivariant, and the trivial action on $\mathbb{P}(E) = \mathbb{P}(E \oplus 0)$. A class α in $H_*(\mathbb{P}(E))$ determines a $\mathbb{C}^*$-stable class $C(\alpha)$ in $H_*(\mathbb{P}(E \oplus 1\!\!1))$, obtained by taking the cone with vertex along the zero-section $X = \mathbb{P}(0 \oplus 1\!\!1)$.

Lemma 2.3. $\mathrm{Shadow}_E(\alpha) = \mathrm{Shadow}_{E\oplus\mathbf{1}}(C(\alpha))$.

Proof. This follows from Lemma 2.2 (i). Indeed $c(E \oplus 1\!\!1) = c(E)$, and $\mathbb{P}(E) \cong \mathbb{P}(E \oplus 0)$ represents $c_1(\mathscr{O}_{E\oplus\mathbf{1}}(1))$, so that

$$\sum_{j\geq 1} c_1(\mathscr{O}_{E\oplus\mathbf{1}}(1))^j \cap C(\alpha) = \sum_{j\geq 0} c_1(\mathscr{O}_E(1))^j \cap \alpha;$$

(the remaining term vanishes in the push-forward for dimensional reasons). $\square$

Remark 2.4. Not all $\mathbb{C}^*$-stable classes in $H_*(\mathbb{P}(E \oplus 1\!\!1))$ are obtained from classes in $H_*(\mathbb{P}(E))$ as in Lemma 2.3. For instance, the class of the zero section $X = \mathbb{P}(0 \oplus 1\!\!1)$ is $\mathbb{C}^*$-fixed and not of this form. For any subvariety $V \subseteq X = \mathbb{P}(0 \oplus 1\!\!1) \subseteq \mathbb{P}(E \oplus 1\!\!1)$,

$$\mathrm{Shadow}_{E\oplus\mathbf{1}}([V]) = c(E) \cap [V]$$

by Lemma 2.2 (i), since $\mathscr{O}_{E\oplus\mathbf{1}}(-1)$ is trivial along the zero-section. $\lrcorner$

Denote by $i : E \to \mathbb{P}(E \oplus 1\!\!1)$ the embedding of E as the complement of $\mathbb{P}(E \oplus 0)$, and by $\overline{q} : \mathbb{P}(E \oplus 1\!\!1) \to X$ the projection.

Lemma 2.5. *If $\alpha \in H_*(\mathbb{P}(E \oplus 1\!\!1))$ has codimension k, then $i^*(\alpha) = \pi^*(\underline{\alpha}^k)$ where $\underline{\alpha}^k$ is the component of* $\mathrm{Shadow}_{E\oplus\mathbf{1}}(\alpha)$ *of codimension k.*

Proof. Indeed, $\alpha = \sum_{j=0}^{e+1} \eta^j \overline{q}^*(\underline{\alpha}^{k-j})$, where $\mathrm{Shadow}(\alpha) = \sum_{j=0}^{e+1} \underline{\alpha}^j$ and $\eta = c_1(\mathscr{O}_{E\oplus\mathbf{1}}(1))$. Since $\mathbb{P}(E \oplus 0)$ represents η and is disjoint from E, $i^*(\eta) = 0$. Therefore $i^*(\alpha) = i^*\overline{q}^*(\underline{\alpha}^k) = \pi^*(\underline{\alpha}^k)$ as stated. $\square$

2.3. **Homogenization of shadows and $\mathbb{C}^*$-equivariant homology.** Consider a class $\alpha = \sum_{j=0}^{\ell} \alpha^j \in H_*(X)$, where α^j denotes the homogeneous component of α of codimension j in X. The choice of a codimension $k \geq \ell$ and of a character χ of $\mathbb{C}^*$ determine the homogeneous class

$$\alpha^\chi := \sum_{j=0}^{\ell} \chi^{k-j} \alpha^j \in H^{\mathbb{C}^*}_{2(\dim X - k)}(X); \tag{5}$$

as usual we identify the character $\chi = \chi_a$ given by $z \mapsto z^a$ with its equivariant first Chern class $c_1^{\mathbb{C}^*}(\mathbb{C}_\chi) = a\hbar \in H^2_{\mathbb{C}^*}(\mathrm{pt})$. We will call α^χ the *'(χ-)homogenization'* of degree k of α; the fixed codimension k will often be clear from the context. We will use the additive notation for characters in the context of homogenizations, and the multiplicative notation in the context of group actions. For instance, $\alpha^{-\hbar} = \sum_{j=0}^{\ell} (-\hbar)^{k-j} \alpha^j$ is determined by the character $\hbar^{-1}$ given by $z \mapsto z^{-1}$.

Example 2.6. A key example is given by the homogenization of the total Chern class of the bundle E^χ. If $x_1, \dots, x_{e+1}$ are the (non-equivariant) Chern roots of E then the ($\mathbb{C}^*$-equivariant) Chern roots of $E^\chi \cong E \otimes \mathbb{C}_\chi$ are $x_1 + \chi, \dots, x_{e+1} + \chi$. It follows that for every subvariety $V \subseteq X$,

$$c^{\mathbb{C}^*}_{e+1}(E^\chi) \cap [V]_{\mathbb{C}^*} = (c(E) \cap [V])^\chi \in H^{\mathbb{C}^*}_{2(\dim V - (e+1))}(X) \tag{6}$$

(note that $[V]$ may be identified with $[V]_{\mathbb{C}^*}$ since the $\mathbb{C}^*$-action on X is trivial). I.e., the homogenization of the total Chern class of E is naturally a top equivariant Chern class. $\lrcorner$

Now let C be a $\mathbb{C}^*$-stable cycle of codimension k in $E = E^\chi$. Viewing E as an open subset of $\mathbb{P}(E \oplus 1\!\!1)$ as above, the closure of C is a codimension-k cycle $\overline{C}$ in $\mathbb{P}(E \oplus 1\!\!1)$. The next result compares the class $[C]_{\mathbb{C}^*}$ of C in the equivariant Borel-Moore homology group $H_*^{\mathbb{C}^*}(E^\chi)$ and the class $[\overline{C}]$ in the ordinary Borel-Moore homology group $H_*(\mathbb{P}(E \oplus 1\!\!1))$.

Proposition 2.7. *Let C be a $\mathbb{C}^*$-stable cycle of codimension k in E^χ, as above. Then $[C]_{\mathbb{C}^*} \in H_*^{\mathbb{C}^*}(E^\chi) \overset{\iota^*}{\cong} H_*(X)[\hbar]$ is the χ-homogenization of degree k of the shadow of $[\overline{C}]$:*

$$\iota^*([C]_{\mathbb{C}^*}) = (\mathrm{Shadow}([\overline{C}]))^\chi \quad .$$

Remark 2.8. In particular, this shows that equivariant fundamental classes of subvarieties of a vector bundle E^χ of rank $e+1$ are of the form

$$\alpha^k + \chi\pi^*(\alpha^{k-1}) + \cdots + \chi^{e+1}\pi^*(\alpha^{k-e-1})$$

i.e., combinations of powers $\chi^j = \left(c_1^{\mathbb{C}^*}(\mathbb{C}_\chi)\right)^j$ with $0 \le j \le \mathrm{rk}\, E$. In other words, among all equivariant classes, Proposition 2.7 distinguishes the fundamental classes of fixed codimension equivariant subvarieties in E as those classes determined by no more than $\mathrm{rk}\, E+1$ homogeneous classes in $H_*(X)$, a quantity independent of $\dim X$. It will in fact follow from the proof that if such a subvariety is not supported within the zero section of E, then $\alpha^{k-e-1} = 0$ (cf. (8)). ⌟

Proof of Proposition 2.7. By linearity we may assume that C is the cycle of a $\mathbb{C}^*$-stable subvariety V of E. First assume that the subvariety is contained in the zero-section, so that $C = \iota_*([V])$ for a subvariety V of X. By the (equivariant) self-intersection formula,

$$\iota^*([C]_{\mathbb{C}^*}) = \iota^*(\iota_*[V]_{\mathbb{C}^*}) = c_{e+1}^{\mathbb{C}^*}(E^\chi) \cap [V]_{\mathbb{C}^*} = (c(E) \cap [V])^\chi,$$

where the last equality follows from (6). ($[V]_{\mathbb{C}^*}$ may be identified with $[V]$ since the $\mathbb{C}^*$-action on X is trivial.) On the other hand, $\overline{C} = C$ is also the push-forward of V to the zero-section $X = \mathbb{P}(0 \oplus 1\!\!1) \subseteq \mathbb{P}(E \oplus 1\!\!1)$. As in Remark 2.4, we deduce that

$$\mathrm{Shadow}([\overline{C}]) = c(E \oplus 1\!\!1) \cap [V] = c(E) \cap [V]$$

from which the claim follows.

Next, assume V is *not* supported on the zero-section of E and has codimension k. Since V is $\mathbb{C}^*$-stable, V determines and is determined by a subvariety $\mathbb{P}(V)$ of $\mathbb{P}(E)$; in this case, $\overline{V} \subseteq \mathbb{P}(E \oplus 1\!\!1)$ is the cone over $\mathbb{P}(V)$ with vertex along $\mathbb{P}(0 \oplus 1\!\!1)$. Let $\mathrm{Shadow}_E(\mathbb{P}(V)) = \sum_{j=0}^{e} \underline{\alpha}^{k-j}$, with $\underline{\alpha}^{k-j}$ of codimension $(k-j)$. By Lemma 2.3 this is also the shadow of $\overline{V}$, so that

$$(\mathrm{Shadow}_{E\oplus\mathbf{1}}([\overline{V}]))^\chi = \sum_{j=0}^{e} \chi^j \underline{\alpha}^{k-j} \tag{7}$$

in $H^{\mathbb{C}^*}_{2(\dim X-k)}(X)$. Denote by $i^\chi : E^\chi \to \mathbb{P}(E^\chi \oplus 1\!\!1)$ the open embedding with complement $\mathbb{P}(E^\chi \oplus 0)$. This is a flat $\mathbb{C}^*$-equivariant map, therefore $[V]_{\mathbb{C}^*} = (i^\chi)^*[\overline{V}]_{\mathbb{C}^*}$. We calculate $(i^\chi)^*[\overline{V}]_{\mathbb{C}^*}$ using mixing spaces.

We will let χ be the character $z \mapsto z^a$ and use notation as in diagram (3):

$$E^\chi_{\mathbb{C}^*} = \rho^* E \otimes \mathscr{O}_{\mathbb{P}^\infty}(-a) \xrightarrow{\pi^\chi} \mathbb{P}^\infty \times X \xrightarrow{\rho} X\,.$$

We denote mixing spaces by the subscript $\mathbb{C}^*$. By definition, $[\overline{V}]_{\mathbb{C}^*} = [\overline{V}_{\mathbb{C}^*}]$ under the identification $H^{\mathbb{C}^*}_*(\mathbb{P}(E^\chi \oplus 1\!\!1)) = H_*(\mathbb{P}(E^\chi_{\mathbb{C}^*} \oplus 1\!\!1)$.

The mixing space $\overline{V}_{\mathbb{C}^*}$ is a cone over $\mathbb{P}(V_{\mathbb{C}^*})$. By Lemma 2.5, applied to the open embedding $E^\chi_{\mathbb{C}^*} \to \mathbb{P}(E^\chi_{\mathbb{C}^*} \oplus 1\!\!1)$, and Lemma 2.3,

$$\begin{aligned}(i^\chi)^*[\overline{V}_{\mathbb{C}^*}] &= \text{codimension-}k \text{ component of } (\pi^\chi)^*(\mathrm{Shadow}_{E^\chi_{\mathbb{C}^*}\oplus\mathbf{1}}([\overline{V}_{\mathbb{C}^*})]) \\ &= \text{codimension-}k \text{ component of } (\pi^\chi)^*(\mathrm{Shadow}_{E^\chi_{\mathbb{C}^*}}([\mathbb{P}(V_{\mathbb{C}^*})])\,.\end{aligned}$$

There is a canonical isomorphism

$$\mathbb{P}(E^\chi_{\mathbb{C}^*}) = \mathbb{P}(\rho^*(E) \otimes \mathscr{O}_{\mathbb{P}^\infty}(-a)) \cong \mathbb{P}(\rho^*(E)) = \mathbb{P}^\infty \times \mathbb{P}(E),$$

under which $\mathbb{P}(V_{\mathbb{C}^*}) \subseteq \mathbb{P}(E^\chi_{\mathbb{C}^*})$ is identified with $\mathbb{P}^\infty \times \mathbb{P}(V) \subseteq \mathbb{P}(\rho^*(E))$. We have

$$\mathrm{Shadow}_{\rho^*(E)}([\mathbb{P}^\infty \times \mathbb{P}(V)]) = \sum_{j=0}^{e} \rho^*(\underline{\alpha}^{k-j})\,,$$

therefore, as $E^\chi_{\mathbb{C}^*} = \rho^* E \otimes \mathscr{O}_{\mathbb{P}^\infty}(-a)$,

$$\mathrm{Shadow}_{E^\chi_{\mathbb{C}^*}}([\mathbb{P}(V_{\mathbb{C}^*})]) = \sum_{j=0}^{e} c(\mathscr{O}(-a))^j \cap \rho^*(\underline{\alpha}^{k-j})$$

by Lemma 2.2 (iii). Extracting the codimension k component of this class, we obtain

$$(i^\chi)^*[\overline{V}_{\mathbb{C}^*}] = (\pi^\chi)^* \left(\sum_{j=0}^{e} c_1(\mathscr{O}(-a))^j \cap \rho^*(\underline{\alpha}^{k-j}) \right) \in H_*(E^\chi_{\mathbb{C}^*})\,.$$

Pulling back via the zero-section ι, and viewing the result in the equivariant homology group, we deduce

$$\iota^*([V]_{\mathbb{C}^*}) = \sum_{j=0}^{e} \chi^j \underline{\alpha}^{k-j} \in H^{\mathbb{C}^*}_*(X)\,, \tag{8}$$

hence by (7)

$$\iota^*([V]_{\mathbb{C}^*}) = (\mathrm{Shadow}_{E\oplus\mathbf{1}}([\overline{V}]))^\chi$$

as needed. $\square$

2.4. **Equivariant shadows.** If X is endowed with the action of a torus T, and $\pi : E \to X$ is a T-equivariant vector bundle on X, then shadows of equivariant classes in $H_*^T(\mathbb{P}(E))$ may be defined in $H_*^T(X)$. Explicitly, a T-stable cycle $C \subseteq E$ determines a T-stable cycle $\overline{C}$ in the T-variety $\mathbb{P}(E \oplus 1\!\!1)$, where T acts trivially on $1\!\!1$. Using the induced cycle $\overline{C}_T$ in $\mathbb{P}(E_T \oplus 1\!\!1)$, we let

$$\mathrm{Shadow}^T(C) := \mathrm{Shadow}_{(E_T \oplus \mathbf{1})}([\overline{C}_T]); \tag{9}$$

this class lives in the (ordinary) homology of the mixing space X_T, and is therefore naturally an element of $H_*^T(X)$. Then Lemma 2.2, Lemma 2.3, Remark 2.4 and Lemma 2.5 extend T-equivariantly, using equivariant classes throughout, as well as the key fact that

$$c_1^T(\mathscr{O}_{E\oplus \mathbf{1}}(1)) \cap [\mathbb{P}(E \oplus 1\!\!1)]_T = [\mathbb{P}(E)]_T \quad \in H_*^T(\mathbb{P}(E)).$$

This follows because $(E \oplus 1\!\!1)_T = E_T \oplus 1\!\!1$, since T acts trivially on the line $1\!\!1$.

For further use, we record the equivariant version of Lemma 2.2 (i), in the way we use it below. For a T-stable cycle C in E,

$$\mathrm{Shadow}^T(C) = c^T(E) \cap \mathrm{s}^T(C) \quad . \tag{10}$$

Here, $\mathrm{s}^T(-)$ is an 'equivariant Segre class' operator, defined as follows: as above, the cycle C determines a T-stable cycle $\overline{C}$ in $\mathbb{P}(E \oplus 1\!\!1)$, and

$$\mathrm{s}^T(C) := \overline{q}_*(c^T(\mathscr{O}_{E\oplus\mathbf{1}}(-1))^{-1} \cap [\overline{C}]) = \sum_{i \geq 0} \overline{q}_*(c_1^T(\mathscr{O}_{E\oplus\mathbf{1}}(1))^i \cap [\overline{C}]) \quad , \tag{11}$$

where $\overline{q} : \mathbb{P}(E \oplus 1\!\!1) \to X$ is the projection. In the non-equivariant case, the Chern classes are nilpotent, therefore the operator is well defined. Equivariantly, this operator has values in the completion $\hat{H}_*^T(X) := \prod_{i \leq \dim X} H_{2i}^T(X)$ of $H_*^T(X)$. The Segre operator will be key in §8.2, where $X = G/B$ is a flag manifold. We refer to [Ful84, Chapter 4] and [KT96] for detailed information on Segre classes and operators in ordinary Chow groups.

If in addition X is also endowed with a trivial $\mathbb{C}^*$ action, the definition given in (5) generalizes to give the homogenization of an *equivariant* (non-homogenous) class: for $\alpha = \sum_{j=0}^{\ell} \alpha^j \in H_*^T(X)$, with α^j of codimension j, and the choice of (a codimension) $k \geq \ell$, the homogenization $\alpha^\chi = \sum_{j=0}^{\ell} \chi^{k-j} \alpha^j$ is a class in $H_{2(\dim X - k)}^{T \times \mathbb{C}^*}(X)$. If as above $E = E^\chi$ is given a $\mathbb{C}^*$-action by fiberwise dilation with character χ, then the natural projection $\pi : E \to X$, and the zero section $\iota : X \to E$ are both $T \times \mathbb{C}^*$-equivariant, every T-stable cone $C \subseteq E$ is also $T \times \mathbb{C}^*$ stable, and since $\mathbb{C}^*$ acts trivially on X, $H_*^{T \times \mathbb{C}^*}(X) \cong H_*^T(X)[\hbar]$. The analogue of Proposition 2.7 is:

Proposition 2.9. *Let $\mathbb{C}^*$ act on fiber of E by dilation with character χ. Then for any T-stable cone $C \subseteq E$ of codimension k,*

$$\mathrm{Shadow}^T(C)^\chi = \iota^*([C]_{T \times \mathbb{C}^*}) \quad ,$$

where the left-hand side is the χ homogenization of degree k.

Proof. Apply Proposition 2.7 to the mixing space X_T. □

3. Equivariant Chern-Schwartz-MacPherson classes

3.1. **Preliminaries.** Let X be a scheme with a torus T action. The group of constructible functions $\mathcal{F}(X)$ consists of functions $\varphi = \sum_W c_W 1\!\!1_W$, where the sum is over a finite set of constructible subsets $W \subseteq X$ and $c_W \in \mathbb{Z}$ are integers. A group $\mathcal{F}^T(X)$ of *equivariant* constructible functions (for tori and for more general groups) is defined by Ohmoto in [Ohm06, §2]. We recall the main properties that we need:

(1) If $W \subseteq X$ is a constructible set which is stable under the T-action, its characteristic function $1\!\!1_W$ is an element of $\mathcal{F}^T(X)$. We will denote by $\mathcal{F}^T_{\text{inv}}(X)$ the subgroup of $\mathcal{F}^T(X)$ consisting of T-invariant constructible functions on X. (The group $\mathcal{F}^T(X)$ also contains other elements, but this will be immaterial for us.)

(2) Every proper T-equivariant morphism $f : Y \to X$ of algebraic varieties induces a homomorphism $f^T_* : \mathcal{F}^T(Y) \to \mathcal{F}^T(X)$ defined by

$$f^T_*(1\!\!1_W)(x) = \chi(f^{-1}(x) \cap W) \tag{12}$$

for $x \in X$ and $W \subseteq Y$ a T-stable subvariety; here χ denotes the topological Euler characteristic. The restriction of f^T_* to $\mathcal{F}^T_{\text{inv}}(X)$ coincides with the ordinary push-forward f_* of constructible functions; cf. [Ohm06, §2.6].

Remark 3.1. We recall that for complex algebraic varieties, the topological Euler characteristic agrees with the Euler characteristic with compact support (see e.g., [Ful93, p. 95, p. 141]) and it therefore satisfies additivity on disjoint unions of locally closed varieties. Every constructible set Z can be partitioned into a finite disjoint union of locally closed subvarieties V_i and one can define $\chi(Z) := \sum_i \chi(V_i)$; this is well-defined since any two partitions have a common refinement. If Z_1 and Z_2 are constructible sets, then there is a partition of $Z_1 \cup Z_2$ into locally closed subvarieties extending partitions of Z_1, Z_2, and $Z_1 \cap Z_2$, and it follows that

$$\chi(Z_1 \cup Z_2) = \chi(Z_1) + \chi(Z_2) - \chi(Z_1 \cap Z_2)\,,$$

that is, the Euler characteristic of constructible functions satisfies inclusion-exclusion.

In other words, the Euler characteristic defines a group homomorphism $\chi(X; -)$ from the group of constructible functions of a variety X to $\mathbb{Z}$. For instance, (12) holds for any T-stable constructible subset W of Y.

This group homomorphism $\chi(X; -) : \mathcal{F}(X) \to \mathbb{Z} = \mathcal{F}(\mathrm{pt})$ can also be seen as the special case for $Y = \mathrm{pt}$ of the push-forward $f_! = f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ for a morphism $f : X \to Y$ of complex algebraic varieties, induced from the corresponding transformations $f_!, f_*$ of the Grothendieck groups $K_0(D^b_c(-))$ of (algebraically) constructible sheaf complexes (of vector spaces) coming from the derived functors $Rf_!, Rf_*$. Here one uses the group epimorphism

$$\chi_{stalk} : K_0(D^b_c(-)) \to \mathcal{F}(-)$$

given by the stalkwise Euler characteristic as in [Sch03, Section 2.3, Section 6.0.6]. Then the equality $f_! = f_* : K_0(D^b_c(X)) \to K_0(D^b_c(Y))$ from [Sch03, Equation (6.41), p. 413] for a morphism $f : X \to Y$ implies the corresponding equality $f_! = f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ for constructible functions (see [Sch03, Equation (6.42), p. 413]). ⌟

Ohmoto proves [Ohm06, Theorem 1.1] that there is an equivariant version of MacPherson's transformation $c_*^T : \mathcal{F}^T(X) \to H_{2*}^T(X)$ (the image is in even homology degrees) that satisfies $c_*^T(1\!\!1_X) = c^T(T_X) \cap [X]_T$ if X is a non-singular variety, and that is functorial with respect to proper push-forwards. The last statement means that for all proper T-equivariant morphisms $f : Y \to X$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}^T(Y) \arrow[r, "c_*^T"] \arrow[d, "f_*^T"'] & H_*^T(Y) \arrow[d, "f_*^T"] \\ \mathcal{F}^T(X) \arrow[r, "c_*^T"] & H_*^T(X) \end{tikzcd}$$

While in our main application X will be a projective (flag) variety, we note that c_*^T may be defined for quasi-projective schemes, or, more generally, for separated algebraic spaces [Ohm06, Ohm12].

Definition 3.2. Let Z be a T-stable constructible subset of X. We denote by $c_{\mathrm{SM}}^T(Z) := c_*^T(1\!\!1_Z) \in H_*^T(X)$ the *equivariant Chern-Schwartz-MacPherson (CSM) class,* and for Z a T-stable algebraic subvariety of X by $c_{\mathrm{Ma}}^T(Z) := c_*^T(\mathrm{Eu}_Z) \in H_*^T(X)$ the *equivariant Chern-Mather* class of Z. ⌟

Here, Eu_Z is MacPherson's local Euler obstruction. This is a constructible function, equal to 1 at non-singular points of Z. It may be defined using transcendental methods in analytic topology ([Mac74, §3]) as well as algebraically ([Ful84, Example 4.2.9]). Both CSM and Chern-Mather classes depend on the chosen ambient space X. However, if $\overline{Z}$ is the closure of Z in X, then the inclusion $\overline{Z} \subseteq X$ is proper, and one may view these classes as (non-homogenous) elements of $H_*^T(\overline{Z})$; the corresponding classes in any T-stable subvariety W of X containing Z may be obtained by pushing-forward these classes (by the functoriality of c_*^T). We will often omit the dependence of the ambient space, when this space is clear from the context. Both $c_{\mathrm{SM}}^T(Z)$ and $c_{\mathrm{Ma}}^T(\overline{Z})$ equal $[\overline{Z}]_T+$ lower dimensional terms, and both classes agree with $c^T(TZ) \cap [Z]_T$ if $Z \subseteq X$ is a nonsingular subvariety (which is closed by our conventions). In [Ohm06, §4.3] Ohmoto gives an explicit geometric construction of the equivariant Chern-Mather class.

It will be useful to consider 'signed' versions $\breve{c}_*$, $\breve{c}_*^T$ of MacPherson's and Ohmoto's natural transformations. These are defined by changing the sign of components of odd (complex) dimension. Thus for every constructible function φ on X we set

$$\breve{c}_*(\varphi) = \sum_{k\geq 0} \breve{c}_*(\varphi)_k := \sum_{k\geq 0} (-1)^k c_*(\varphi)_k,$$

where $c_*(\varphi)_k \in H_{2k}(X)$ is the component of complex dimension k, and similarly for all invariant φ we define

$$\breve{c}^T_*(\varphi) = \sum_{k\geq 0} \breve{c}^T_*(\varphi)_k := \sum_{k\geq 0} (-1)^k c^T_*(\varphi)_k. \tag{13}$$

Note that $c^T_*(\varphi)_k = 0$ for $k < 0$ by [Ohm06, §4.1]. Also,

$$\int_X c^T(\varphi)_0 = \chi(X;\varphi) \quad \text{for } X \text{ compact and } \varphi \in \mathcal{F}^T_{\mathrm{inv}}(X)$$

by functoriality of c^T_* and $\mathbb{Z}$-linearity [Ohm06, p. 127]. Here $\chi(X;\varphi)$ is the Euler characteristic weighted by φ, induced from the Euler characteristic of constructible sets, cf. Remark 3.1.

In the non-equivariant setting, this signed Chern class transformation appears implicitly in e.g., work of Sabbah [Sab85] and Schürmann [Sch05] (see also [Ken90, PP01, Sch05, Sch17]) where MacPherson's transformation is constructed via Lagrangian cycles in the cotangent bundle of X. The equivariant version of this construction is discussed below in §3.2. 'Dual' CSM and Chern-Mather classes are defined by setting, for a subvariety $Z \subseteq X$,

$$c^{T,\vee}_{\mathrm{SM}}(Z) := (-1)^{\dim Z} \breve{c}^T_*(1\!\!1_Z) \quad , \quad c^{T,\vee}_{\mathrm{Ma}}(Z) := (-1)^{\dim Z} \breve{c}^T_*(\mathrm{Eu}_Z) \quad . \tag{14}$$

The sign is introduced so that if Z is T-stable, irreducible, and nonsingular, then both classes agree with the equivariant Chern class of the cotangent bundle of Z, $c^T(T^*(Z)) \cap [Z]_T$. For complete flag manifolds, a geometric interpretation of dual CSM classes in terms of Poincaré duality will be given in §7.

3.2. **CSM classes, shadows of characteristic cycles, and an intersection formula.** In this section we recall a construction of MacPherson's natural transformation by means of characteristic cycles, and extend this construction to the equivariant setting. In the non-equivariant case this construction appears in (among others) [Sab85, Ken90, PP01, Alu04, Sch17]. Our main result is Theorem 3.3, which realizes this construction in terms of equivariant shadows. This also gives a *Lagrangian approach* to Ohmoto's construction of the equivariant MacPherson's transformation [Ohm06].

In addition, we record in Theorem 3.6 certain equivariant intersection formulae for MacPherson's transformation, generalizing the non-equivariant statements proved in [Sch17]. These will be needed in the proof of 'geometric orthogonality' for flag manifolds in §7.1. The details required to extend the proofs from [Sch17] to the equivariant setting are given in the Appendix.

In this section X will denote a smooth (complex) algebraic variety with an action of a torus T. As before, we state our results in equivariant Borel-Moore homology; *mutatis mutandis,* they hold in the Chow group. The construction is illustrated in the following diagram (cf. [Sch17, §3]); the notation is explained next.

$$
\begin{tikzcd}
\mathcal{F}^T_{\mathrm{inv}}(X) \arrow[d, equal] & Z^T_*(X) \arrow[l, "\breve{\mathrm{Eu}}"', "\sim"] \arrow[r, "c^{T,\vee}_{\mathrm{Ma}}"] \arrow[d, "\mathrm{cn}"', "\wr"] & H^T_{2*}(X) \arrow[d, equal] \\
\mathcal{F}^T_{\mathrm{inv}}(X) \arrow[r, "\mathrm{CC}", "\sim"'] & L_T(X) \arrow[r, "\mathrm{Shadow}^T"] & H^T_{2*}(X)
\end{tikzcd} \tag{15}
$$

Here $Z^T_*(X)$ denotes the group of T-stable cycles in X, while $L_T(X)$ denotes the additive group of T-stable conic Lagrangian cycles in the cotangent bundle $T^*(X)$ of X. (This is a T-equivariant bundle, where the T-action is induced from the T-action on X.) The elements of $L_T(X)$ are $\mathbb{Z}$-linear combinations of *conormal cycles* $T^*_Z X := \overline{T^*_{Z^{\mathrm{reg}}} X} \subseteq T^*(X)$, where $Z \subseteq X$ is a T-stable subvariety and Z^{reg} is the smooth part of Z.

The top maps are the 'signed' local Euler obstruction, defined on subvarieties Z by $\breve{\mathrm{Eu}}_Z := (-1)^{\dim Z}\,\mathrm{Eu}_Z$ and extended to cycles by linearity, and the dual equivariant Chern-Mather class $c^{T,\vee}_{\mathrm{Ma}}$, defined as in (14) on T-stable varieties and extended by linearity to T-stable cycles. The homomorphism $\breve{\mathrm{Eu}}$ is an isomorphism, and the composition

$$c^{T,\vee}_{\mathrm{Ma}} \circ \breve{\mathrm{Eu}}^{-1} = \breve{c}^T_*$$

is the signed equivariant MacPherson transformation. (Cf. [Ohm06, Proposition 4.3]. Ohmoto works with non-signed classes, but the signed versions are convenient for us as they come up naturally in the context of characteristic cycles in the cotangent bundle $T^*(X)$.) The map $\mathrm{cn} : Z^T_*(X) \to L_T(X)$ takes an irreducible cycle Z to its conormal cycle $T^*_Z X$. This map is a group isomorphism; see e.g., [Ken90, Lemma 3] or [HTT08, Theorem E.3.6]. By composition we obtain an induced 'characteristic cycle' map $\mathrm{CC} : \mathcal{F}^T_{\mathrm{inv}}(X) \to L_T(X)$ determined on irreducible T-stable cycles Z by

$$\mathrm{CC}(\breve{\mathrm{Eu}}_Z) = T^*_Z X \tag{16}$$

(see [PP01, (11), page 67]). Since both maps cn, $\breve{\mathrm{Eu}}$ are isomorphisms, the characteristic cycle map CC is a group isomorphism as well. For a constructible function φ, the image $\mathrm{CC}(\varphi)$ is a conic Lagrangian cycle in $T^*(X)$ called the *characteristic cycle* of φ; this cycle is clearly T-stable if $\varphi \in \mathcal{F}^T_{\mathrm{inv}}(X)$.

The map $\mathrm{Shadow}^T : L_T(X) \to H^T_{2*}(X)$ in the diagram is the 'equivariant shadow' operation defined in §2.4, see (9).

Theorem 3.3. *Diagram* (15) *commutes, i.e.,*

$$\mathrm{Shadow}^T(\mathrm{CC}(\varphi)) = \breve{c}_*^T(\varphi)$$

for every invariant constructible function φ.

By linearity and the construction of CC, this statement is equivalent to the assertion that

$$c_{\mathrm{Ma}}^{T,\vee}(Z) = \mathrm{Shadow}^T(T_Z^*X)$$

for all T-stable subvarieties Z of X. In particular, for $Z = X$ we recover by (the equivariant version of) Remark 2.4 the normalization

$$c_{\mathrm{Ma}}^{T,\vee}(X) = \mathrm{Shadow}^T(T_X^*X) = c^T(T^*(X)) \cap [X]_T \ .$$

Theorem 3.3 also shows that the map Shadow^T coincides with a map defined by Ginzburg in [Gin86, §A.3], using $\mathbb{C}^*$-equivariant K-theory on $T^*(X)$. In this respect, Theorem 3.3 can be regarded as an alternative to Ginzburg's construction; see also Proposition 2.9.

The non-equivariant version of Theorem 3.3 is [Alu04, Lemma 4.3]; this is essentially a reformulation of [PP01, (12), page 67], which in turn is based on a calculation of Sabbah [Sab85]. For another approach to this non-equivariant version of Theorem 3.3 see also [Sch05]. In the present context, the connection between shadows and these formulae is given by Equation (10) in §2.4, applied to the case when $C \in L_T(X)$.

Remark 3.4. As mentioned in the introduction, a reader who is not familiar with Ohmoto's paper [Ohm06] (or with Ginzburg's [Gin86, Appendix] in the non-equivariant case) can take Theorem 3.3, or its reformulations Corollary 3.5 and Theorem 4.2, as the starting point for a definition of the equivariant Chern class transformation c_*^T. In future work we will explore further functoriality properties of characteristic cycles in the equivariant context, aiming to prove directly the functoriality properties for the formulae in Corollary 3.5 and Theorem 4.2 (as already done in the Appendix §10 for a non-characteristic pullback result).

Proof of Theorem 3.3. Let U denote an approximation space to ET (see [EG98], [Ohm06]), and denote by u the projection $U \times_T X \to U/T$; this is a locally trivial fibration, with fibers isomorphic to X. We have the following cartesian diagram of T-equivariant maps, with pr the projection and q, q' the quotient maps:

$$\begin{tikzcd} U \times_T X \arrow[r, "u"] & U/T \\ U \times X \arrow[u, "q'"] \arrow[r, "pr"'] & U \arrow[u, "q"'] \end{tikzcd}$$

The relative cotangent bundle $T^*u = U \times_T T^*(X)$ is the U-approximation of the bundle $T^*(X)_T$. Every invariant constructible function φ on X determines a constructible function on $U \times_T X$, agreeing with φ on the fibers of u; we denote this function by φ_U. It is uniquely characterized by the requirement that

$$q'^*(\varphi_U) = pr'^*(\varphi) \in \mathcal{F}^T_{\mathrm{inv}}(U \times X)\,,$$

with $pr' : U \times X \to X$ the other projection. Ohmoto defines $c^T_*(\varphi)$ as follows:

$$c^T_*(\varphi) := \varinjlim_U c(T(U) \times_T X)^{-1} \cap c_*(\varphi_U)\,, \tag{17}$$

([Ohm06, p. 122 and Definition 3.2]). The (dual of the) exact sequence of differentials for q pulls back to an exact sequence of equivariant vector bundles on $U \times X$:

$$0 \longrightarrow (Tq) \times X \longrightarrow T(U) \times X \longrightarrow q^*T(U/T) \times X \longrightarrow 0$$

inducing the exact sequence

$$0 \longrightarrow (Tq) \times_T X \longrightarrow T(U) \times_T X \longrightarrow u^*(T(U/T)) \longrightarrow 0$$

on $U \times_T X$. Now, $Tq \cong U \times \mathfrak{t}$ with the adjoint action of T on its Lie algebra $\mathfrak{t}$ (see [EG05, Lemma A.1, p. 580] or [MS15, pp. 2218-2219]). In our case the adjoint action of the torus T on $\mathfrak{t}$ is trivial, since T is abelian, so that the bundle $(Tq) \times_T X$ is trivial. Therefore, the bundle $T(U) \times_T X$ is an extension of $u^*(T(U/T))$ by a trivial bundle, and we can rewrite Ohmoto's definition (17) as

$$c^T_*(\varphi) = \varinjlim_U u^*c(T(U/T))^{-1} \cap c_*(\varphi_U)\,. \tag{18}$$

Elements of $H^T_{2k}(X)$ are limits of classes in complex codimension $\dim X - k$ in $U \times_T X$, i.e., of dimension $k + \dim(U/T)$, therefore (18) gives

$$\check{c}^T_*(\varphi) := \varinjlim_U u^*c(T^*(U/T))^{-1} \cap (-1)^{\dim U/T}\check{c}_*(\varphi_U)\,.$$

By [Alu04, Lemma 4.3],

$$\check{c}_*(\varphi_U) = \mathrm{Shadow}_{T^*(U\times_T X)\oplus\mathbf{1}}(\overline{\mathrm{CC}(\varphi_U)})$$

in $H_*(U \times_T X)$. In order to complete the proof we have to show that

$$(-1)^{\dim U/T} \varinjlim_U u^*c(T^*(U/T))^{-1}\, \mathrm{Shadow}_{T^*(U\times_T X)\oplus\mathbf{1}}(\overline{\mathrm{CC}(\varphi_U)})$$
$$= \mathrm{Shadow}^T(\mathrm{CC}(\varphi))$$

for all T-invariant constructible functions φ. By linearity, we may assume $\varphi = \check{\mathrm{E}}\mathrm{u}_Z$ for a T-invariant subvariety Z of X, so that $(-1)^{\dim U/T}\varphi_U = \check{\mathrm{E}}\mathrm{u}_{U\times_T Z}$. Then $\mathrm{CC}(\varphi) = T^*_Z X$ and $(-1)^{\dim U/T}\ \mathrm{CC}(\varphi_U) = T^*_{U\times_T Z}(U \times_T X)$. By definition (see (9)),

$$\mathrm{Shadow}^T(T^*_Z X) = \mathrm{Shadow}_{(T^*X)_T\oplus\mathbf{1}}(\overline{(T^*_Z X)_T}), \tag{19}$$

where $\overline{(T^*_Z X)_T}$ is the closure in $\mathbb{P}(T^*(X)_T \oplus 1\!\!1)$ of the cycle $(T^*_Z X)_T$ determined by $T^*_Z X$ in the bundle $T^*(X)_T$ over the mixing space. As recalled above, the U-approximation of this bundle is the relative cotangent bundle $T^*u = U \times_T T^*(X)$, and the U-approximation of $(T^*_Z X)_T$ is $U \times_T T^*_Z X$. We rewrite (19) as

$$\mathrm{Shadow}^T(T^*_Z X) = \varinjlim_U \mathrm{Shadow}_{T^*u \oplus \mathbf{1}}(\overline{U \times_T T^*_Z X}),$$

and what is left to prove is the following identity of classes in $H_*(U \times_T X)$:

$$\begin{aligned}(20)\quad \mathrm{Shadow}_{T^*(U\times_T X)\oplus\mathbf{1}}&(\overline{T^*_{U\times_T Z}(U \times_T X)}) \\ &= u^* c(T^*(U/T)) \cap \mathrm{Shadow}_{T^*u\oplus\mathbf{1}}(\overline{U \times_T T^*_Z X}).\end{aligned}$$

If $Z = X$, then the conormal cycle $T^*_X X$ is the zero-section of $T^*(X)$, and the cycles $T^*_{U\times_T X}(U \times_T X)$, $U \times_T T^*_X X$ are the zero-sections of $T^*(U \times_T X)$, T^*u, respectively. In this case, (20) amounts to

$$c(T^*(U \times_T X)) \cap [U \times_T X] = u^* c(T^*(U/T))\, c(T^*u) \cap [U \times_T X]$$

(Remark 2.4), which follows from the Whitney formula and the exact sequence of differentials for u:

$$(21)\qquad 0 \longrightarrow u^*T^*(U/T) \longrightarrow T^*(U \times_T X) \longrightarrow T^*u \longrightarrow 0\,.$$

Therefore, we may assume that $Z \neq X$. We claim that $T^*_{U\times_T Z}(U \times_T X)$ only meets $u^*T^*(U/T)$ in the zero-section.

Indeed, the smooth map $q' : U \times X \to U \times_T X$ induces a closed embedding

$$t\colon\ q'^*T^*(U \times_T X) \hookrightarrow T^*(U \times X) = T^*(U) \oplus T^*(X)$$

of bundles over $U \times X$. Via this embedding, $q'^*u^*T^*(U/T)$ is mapped to a subbundle of $T^*(U) \oplus 0$, while

$$t(q'^*(T^*_{U\times_T Z}(U \times_T X)) = 0 \oplus T^*_Z X,$$

as may be verified by chasing the pull-back via q' of the exact sequence (21). (This equality is also a special case of Theorem 10.1, utilizing that q' is non-characteristic with respect to any closed cone in $T^*(U \times_T X)$, because q' is smooth.) The claim follows. From (21) it also follows that $U \times_T T^*_Z X$ is the image of $T^*_{U\times_T Z}(U \times_T X)$ in T^*u.

Since the intersection of $T^*_{U\times_T Z}(U \times_T X)$ and $u^*T^*(U/T)$ is contained in the zero-section, the projectivization $\mathbb{P}(T^*_{U\times_T Z}(U \times_T X))$ is disjoint from $\mathbb{P}(u^*T^*(U/T))$. By Lemma 2.2 (iv), we have

$$\begin{aligned}\mathrm{Shadow}_{T^*(U\times_T X)}&\left(\mathbb{P}(T^*_{U\times_T Z}(U \times_T X))\right) \\ &= u^* c(T^*(U/T)) \cap \mathrm{Shadow}_{T^*u}\left(\mathbb{P}(U \times_T T^*_Z X)\right).\end{aligned}$$

Applying Lemma 2.3 to both sides of this identity gives (20) as needed. □

Theorem 3.3 and identity (10) imply the following key result, extending analogous results from [Sab85, PP01] to the equivariant setting.

Corollary 3.5. *Let $\varphi \in \mathcal{F}^T_{\text{inv}}(X)$ be an invariant constructible function. Then the following identity holds in $H^T_*(X)$:*

$$\breve{c}^T_*(\varphi) = c^T(T^*(X)) \cap \mathrm{s}^T(\mathrm{CC}(\varphi)).$$

The *signed Segre-MacPherson class* of a T-invariant constructible function $\varphi \in \mathcal{F}^T_{\text{inv}}(X)$ is the class

$$\breve{s}^T_{\text{SM}}(\varphi) := \frac{\breve{c}^T_*(\varphi)}{c^T(T^*(X))} = \mathrm{s}^T(\mathrm{CC}(\varphi)) \in \hat{H}^T_*(X). \tag{22}$$

The (unsigned) classes

$$s^T_{\text{SM}}(\varphi) := \frac{c^T_*(\varphi)}{c^T(TX)} \in \hat{H}^T_*(X)\ , \tag{23}$$

called *Segre-MacPherson* (SM) classes (see [Ohm06, §5.3]), are related to the study of Thom polynomials. In the non-equivariant case they were studied in [Alu03] and are denoted here by $\breve{s}_{\text{SM}}(\varphi)$ resp., $s_{\text{SM}}(\varphi)$. For a (T-stable) constructible subset Z of X, we denote by $s_{\text{SM}}(Z) := s_{\text{SM}}(1\!\!1_Z)$ (resp., $s^T_{\text{SM}}(Z) := s^T_{\text{SM}}(1\!\!1_Z)$) the *(equivariant) Segre-MacPherson class of Z.*

Next we will state an intersection formula for the equivariant MacPherson's transformation, which we will use in §7 to compute Poincaré duals of equivariant CSM classes of Schubert cells (see Theorem 7.1). In the non-equivariant case, this formula was proved in [Sch17]. We indicate how to extend the arguments to the equivariant setting in the Appendix. In fact, there we will use Corollary 3.5 to extend to the equivariant case the more general 'non-characteristic pullback' results for (signed) Segre-MacPherson classes.

We remind the reader that in this section X is assumed to be a smooth complex algebraic variety endowed with an action of a torus T. In particular, an intersection product is defined in $H^T_*(X)$. Also, we say that two locally closed nonsingular subvarieties S, S' of X are *transversal* if $T_x(S)+T_x(S') = T_x(X)$ for all $x \in S \cap S'$.

Theorem 3.6. *Let $\alpha \in \mathcal{F}^T_{\text{inv}}(X)$, resp., $\beta \in \mathcal{F}^T_{\text{inv}}(X)$ be constructible with respect to algebraic Whitney stratifications $\mathcal{S} := \{S \subseteq X\}$, resp., $\mathcal{S}' := \{S' \subseteq X\}$ of X, i.e., $\alpha|S$ and $\beta|S'$ are constant for all strata $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$. Assume that each stratum $S \in \mathcal{S}$ is transversal to each stratum $S' \in \mathcal{S}'$. Then*

$$c^T_*(\alpha \cdot \beta) = c^T_*(\alpha) \cdot s^T_{\text{SM}}(\beta) \in H^T_*(X) \subseteq \hat{H}^T_*(X)\,.$$

In particular, if X is compact, then

$$\langle c^T_*(\alpha), s^T_{\text{SM}}(\beta)\rangle := \int_X c^T_*(\alpha) \cdot s^T_{\text{SM}}(\beta) = \chi(X; \alpha \cdot \beta).$$

Note that if X is compact with a finite T-fixed point set X^T, then

$$\chi(X; \alpha \cdot \beta) = \chi(X^T; \alpha \cdot \beta) = \sum_{x \in X^T} \alpha(x) \cdot \beta(x);$$

see [Sch03, Corollary 3.2.2, p. 174]. This is also equivalent to the fact that $\chi(Z)$ is given by the number $|Z \cap X^T|$ of T-fixed points in Z for any locally closed T-stable subvariety $Z \subseteq X$.

Remark 3.7. We will apply Theorem 3.6 in the situation when $X = G/P$ is a partial flag manifold and the T-stable stratification $\mathcal{S}$ (resp., $\mathcal{S}'$) is given by the (opposite) Schubert cells (see [Ric92, Theorem 1.4]). Since these finitely many Schubert cells are also orbits for a corresponding Borel subgroup in G, the stratifications are automatically Whitney regular; indeed, Whitney regularity holds generically, and then by equivariance also everywhere along a Borel orbit. A constructible function is invariant with respect to such a Borel subgroup if and only if it constructible with respect to the corresponding Whitney stratification by these Borel orbits. ⌟

4. Pulling back characteristic cycles by the zero section

4.1. **Homogenized CSM classes are pull backs of characteristic cycles.** Let X be a nonsingular variety endowed with a T-action. Proposition 2.9, applied to the cotangent bundle $T^*(X)$, gives another construction of the map $L_T(X) \to H_*^T(X)$ in diagram (15), using the equivariant pullback via the zero section $\iota : X \to T^*(X)$ of the cotangent bundle. Recall that we view $T^*(X)$ as a $T \times \mathbb{C}^*$-equivariant bundle, where the T-action is induced from the T-action on X as in §3.2 and the $\mathbb{C}^*$ factor acts on the fibers of $T^*(X)$ by dilation with character χ. In this section we focus on the case when χ is the character $\hbar^{-1}$ given by $z \mapsto z^{-1}$. Proposition 2.9 implies the following statement.

Corollary 4.1. *Let $\mathbb{C}^*$ act on fiber of $T^*(X)$ by dilation with character $\hbar^{-1}$. Then for all $C \in L_T(X)$,*

$$\mathrm{Shadow}^T(C) = \iota^*([C]_{T\times\mathbb{C}^*})|_{\hbar\mapsto -1} \quad .$$

By the commutativity of diagram (15), the same result implies a direct realization of the homogenized CSM class of a constructible function in terms of the equivariant pull-back:

Theorem 4.2. *Let $\iota : X \to T^*(X)$ be the zero section, and let $\mathbb{C}^*$ act on the fibers of $T^*(X)$ by the character $\hbar^{-1}$. Then the following holds for any $\varphi \in \mathcal{F}^T_{\mathrm{inv}}(X)$:*

$$\iota^*[\mathrm{CC}(\varphi)]_{T\times\mathbb{C}^*} = c_*^T(\varphi)^\hbar \in H_0^{T\times\mathbb{C}^*}(X),$$

where $c_*^T(\varphi)^\hbar$ denotes the homogenization of degree $\dim X$ *(cf.* (5)*).*

Proof. By Proposition 2.9 and Theorem 3.3,

$$\iota^*[\mathrm{CC}(\varphi)]_{T\times\mathbb{C}^*} = \mathrm{Shadow}^T(\mathrm{CC}(\varphi))^{-\hbar} = \check{c}_*^T(\varphi)^{-\hbar} \quad .$$

By definition of homogenization and of the signed Chern class,

$$\check{c}_*^T(\varphi)^{-\hbar} = \sum_{j=0}^{\dim X} (-\hbar)^j \left((-1)^j c_*^T(\varphi)_j\right) = \sum_{j=0}^{\dim X} \hbar^j c_*^T(\varphi)_j = c_*^T(\varphi)^\hbar .$$

(Note that here the class is indexed by dimension, while in the definition given in (5) it is indexed by codimension.) □

Example 4.3. Let $X = \mathbb{P}^1$ and consider the constructible function $1\!\!1_{\mathbb{P}^1}$. For simplicity, we will only work $\mathbb{C}^*$-equivariantly. Then

$$c_*(1\!\!1_{\mathbb{P}^1}) = [\mathbb{P}^1] + 2[\mathrm{pt}] = c(T\mathbb{P}^1) \cap [\mathbb{P}^1].$$

By definition of homogenization,

$$c_*(1\!\!1_{\mathbb{P}^1})^\hbar = \hbar[\mathbb{P}^1] + 2[\mathrm{pt}].$$

On the other hand, by the self-intersection formula,

$$\iota^*(\iota_*[\mathbb{P}^1]_{\mathbb{C}^*}) = c_1^{\mathbb{C}^*}(T^*(\mathbb{P}^1)) \cap [\mathbb{P}^1]_{\mathbb{C}^*} = (c_1(T^*(\mathbb{P}^1)) - \hbar) \cap [\mathbb{P}^1] = -\hbar[\mathbb{P}^1] - 2[\mathrm{pt}].$$

Together with the fact that $\mathrm{CC}(1\!\!1_{\mathbb{P}^1}) = -[T^*_{\mathbb{P}^1}\mathbb{P}^1] = -\iota_*[\mathbb{P}^1]_{\mathbb{C}^*}$, this implies that

$$\iota^*[\mathrm{CC}(1\!\!1_{\mathbb{P}^1})]_{\mathbb{C}^*} = c_*(1\!\!1_{\mathbb{P}^1})^\hbar$$

as claimed. ⌟

Specializing Theorem 4.2 to the constructible functions $\varphi = 1\!\!1_Z$ and $\varphi = \mathrm{Eu}_Z$ gives the following.

Corollary 4.4. *Let $Z \subseteq X$ be a T-stable constructible subset, and let $\mathbb{C}^*$ act on the fibers of $T^*(X)$ by the character $\hbar^{-1}$. Then the homogenized CSM class satisfies*

$$c^T_{\mathrm{SM}}(Z)^\hbar = \iota^*[\mathrm{CC}(1\!\!1_Z)]_{T\times\mathbb{C}^*} \ \in H_0^{T\times\mathbb{C}^*}(X).$$

If $Z \subseteq X$ is a T-stable subvariety then the homogenized Chern-Mather class satisfies

$$c^T_{\mathrm{Ma}}(Z)^\hbar = (-1)^{\dim Z}\iota^*[T^*_Z X]_{T\times\mathbb{C}^*} \ \in H_0^{T\times\mathbb{C}^*}(X).$$

Remark 4.5. If one further specializes Theorem 4.2 to the characteristic function $\varphi = 1\!\!1_X$ and forgets the T-action, then one obtains the classical index formula for a nonsingular compact variety X:

$$(-1)^{\dim X}\int_X \iota^*[T^*_X X] = \int_X c(1\!\!1_X)_0 = \chi(X),$$

where $\chi(X)$ is the Euler characteristic. (Note that as X is nonsingular, $\mathrm{CC}(1\!\!1_X) = \mathrm{CC}(\mathrm{Eu}_X) = (-1)^{\dim X} T^*_X X$, cf. (16).) ⌟

4.2. **Characteristic classes of Lagrangian cycles are pull-backs.** Next, we recall a commutative diagram considered also by Ginzburg in [Gin86, Appendix], which is largely based on results from [BB81, BK81, KT84]. In the specific case of flag manifolds, this will be used in §8.1 below.

$$\begin{tikzcd}
\mathrm{Perv}(X) \arrow[d, "\chi_{\mathrm{stalk}}"'] & \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X) \arrow[l, "\mathrm{DR}"', "\sim"] \arrow[d, "\mathrm{Char}"] & \\
\mathcal{F}(X) \arrow[r, "\mathrm{CC}", "\sim"'] & L(X) \arrow[r, "c_*^{\vee,\mathrm{Gi}}"] & H_*(X).
\end{tikzcd} \tag{24}$$

Here $\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$, resp., $\mathrm{Perv}(X)$ denote the (sets of objects of the) Abelian categories of algebraic holonomic $\mathcal{D}_X$-modules with regular singularities, resp., perverse (algebraically) constructible complexes of sheaves of $\mathbb{C}$-vector spaces on X. The functor DR is defined on $M^{\cdot} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$ by

$$\mathrm{DR}(M^{\cdot}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathscr{O}_X, M^{\cdot})[\dim X],$$

that is, it computes the DeRham complex of a holonomic module (up to a shift), viewed as an *analytic* $\mathcal{D}_X$-module. This functor realizes the Riemann-Hilbert correspondence, and is an equivalence. We refer to e.g., [KT84, Gin86] for details. The left map χ_{stalk} computes the stalkwise Euler characteristic of a constructible complex, and the right map Char gives the characteristic cycle of a holonomic $\mathcal{D}_X$-module. The map CC is the characteristic cycle map for constructible functions from diagram (15). The commutativity of diagram (24) is shown in [Gin86] using deep $\mathcal{D}$-module techniques; it also follows from [Sch03, Ex. 5.3.4 on pp. 359-360]. Also note that DR in (24) factors through the corresponding Grothendieck group, so it may also be applied to complexes of regular holonomic $\mathcal{D}$-modules.

We define the map $c_*^{\vee,\mathrm{Gi}}$ by

$$c_*^{\vee,\mathrm{Gi}} := \iota^*_{\hbar=-1}, \tag{25}$$

i.e., by the specialization at $\hbar = -1$ of the pull-back via the zero-section map $\iota : X \to T^*(X)$, with $\mathbb{C}^*$ acting on the fibers of $T^*(X)$ by dilation with character $\hbar^{-1}$. The motivation for the notation is that this map $c_*^{\vee,\mathrm{Gi}}$ agrees with a map defined by Ginzburg in [Gin86, §A.3] in a different way (and just denoted there by the notation c_*). Indeed, Ginzburg's class is characterized by the requirement that its value at $[T^*_Z X]$ agrees with the (signed) Chern-Mather class of Z; this identity is given in [Gin86, Lemma A3.2]. The class defined in (25) likewise satisfies

$$c_*^{\vee,\mathrm{Gi}}([T^*_Z X]) = c_{\mathrm{Ma}}(Z)^\vee \tag{26}$$

for all subvarieties $Z \subseteq X$, where $c_{\mathrm{Ma}}(Z)^\vee := \breve{c}_*(\breve{\mathrm{E}}\mathrm{u}_Z)$ is the class obtained by changing the sign of the components of $c_{\mathrm{Ma}}(Z)$ of odd codimension in Z, cf. (14). Identity (26) follows from Theorem 4.2, by forgetting the T-action. Therefore Ginzburg's natural transformation and ours agree in the non-equivariant setting, and the composition $c_*^{\vee,\mathrm{Gi}} \circ \mathrm{CC} = \breve{c}_*$ coincides with the signed version of MacPherson's natural transformation from constructible functions to homology (see also the work of Sabbah [Sab85]). Theorem 4.2 generalizes this observation to the equivariant setting.

5. Preliminaries on cohomology of flag manifolds

In what follows, we set up notation and recall basic facts about the flag manifolds and their cohomology. We will use the notation from [AM16] and we refer the reader to [Bri05] for further details.

5.1. **Schubert cells and varieties.** Let G/B be the complete flag manifold, where G is a complex semisimple Lie group and B is a Borel subgroup. Let B^- be the opposite Borel group and $T := B \cap B^-$ the maximal torus. The Weyl group $N_G(T)/T$ is denoted by W, $\ell : W \to \mathbb{N}$ is the length function, and w_0 denotes the longest element. Notice that $B^- = w_0 B w_0$. There is a root system R associated to (G,T) with simple roots $\Delta := \{\alpha_i\}_{1\le i\le r}$ such that α_i is positive with respect to B. The Weyl group W is generated by the simple reflections $s_i := s_{\alpha_i}$. A root $\alpha \in R$ is positive if it can be written as a nonnegative combination of simple roots; this will be denoted by $\alpha > 0$.

For $w \in W$, define the Schubert cell $X(w)^\circ := BwB/B \cong \mathbb{C}^{\ell(w)}$ and the opposite Schubert cell $Y(w)^\circ := B^- wB/B \cong \mathbb{C}^{\dim X - \ell(w)}$. Their closures give the Schubert variety $X(w) = \overline{BwB/B}$ and the opposite Schubert variety $Y(w) = \overline{B^- wB/B}$. These are complex projective algebraic varieties such that $\dim_{\mathbb{C}} X(w) = \operatorname{codim}_{\mathbb{C}} Y(w) = \ell(w)$. The Bruhat order $\le$ is a partial order on the Weyl group W; it may be defined by declaring that $u \le v$ if and only if $X(u) \subseteq X(v)$.

More generally, let $P \subseteq G$ be a parabolic subgroup containing B and let G/P be the corresponding partial flag manifold. Let W_P be the subgroup of W generated by the simple reflections in P and denote by W^P the set of minimal length representatives for the cosets of W_P in W. For each $w \in W$, $\ell(wW_P)$ denotes the length of the minimal length representative for the coset wW_P. Let $w_P \in W_P$ be the longest element. For each $w \in W^P$ there are Schubert cells $X(wW_P)^\circ := BwP/P$ and $Y(wW_P)^\circ := B^- wP/P$ in G/P, whose closures are the Schubert varieties $X(wW_P)$ and $Y(wW_P)$. Let $f : G/B \to G/P$ be the natural projection. If $w \in W^P$ then f restricts to isomorphisms $X(w)^\circ \to X(wW_P)^\circ$, and $f^{-1}(Y(wW_P)) = Y(ww_P)$. It follows that $\dim_{\mathbb{C}} X(wW_P) = \operatorname{codim}_{\mathbb{C}} Y(wW_P) = \ell(w)$. The Bruhat order on W restricts to a partial ordering on W/W_P such that for $u, v \in W$, $uW_P \le vW_P$ iff $X(uW_P) \subseteq X(vW_P)$. In particular,

$$X(wW_P) = \bigsqcup_{wW_P \ge vW_P} X(vW_P)^\circ \quad \text{and} \quad Y(wW_P) = \bigsqcup_{wW_P \le vW_P} Y(vW_P)^\circ, \tag{27}$$

thus the Schubert cells form a stratification of the corresponding Schubert varieties.

5.2. **Schubert classes.** Since the varieties G/P are smooth and projective, throughout this paper we will identify the equivariant homology and cohomology of G/P. In particular, any T-stable, subvariety $Y \subseteq G/P$ of complex codimension c determines a fundamental class $[Y]_T \in H_T^{2c}(G/P)$. We will omit the subscript T for non-equivariant classes. By [Gra01, Proposition 2.1], the identities in (27) imply that the (equivariant) fundamental classes $\{[X(wW_P)]_T\}_{w\in W^P}$ and $\{[Y(wW_P)]_T\}_{w\in W^P}$ form $H_T^*(\mathrm{pt})$-bases for the equivariant cohomology $H_T^*(G/P)$, i.e.,

$$H_T^*(G/P) = \bigoplus_{w \in W^P} H_T^*(\mathrm{pt})[X(wW_P)]_T = \bigoplus_{w \in W^P} H_T^*(\mathrm{pt})[Y(wW_P)]_T.$$

The opposite Schubert classes $[Y(wW_P)]_T$ are Poincaré dual to the Schubert classes $[X(wW_P)]_T$, in the sense that

(28)
$$\langle [X(uW_P)]_T, [Y(vW_P)]_T \rangle := \int_{G/P} [X(uW_P)]_T \cdot [Y(vW_P)]_T = \delta_{uW_P, vW_P}$$

with respect to the usual intersection pairing (see e.g., [Bri05, Proposition 1.3.6]). This holds since opposite Schubert cells intersect generically transversally [Ric92, Corollary 1.5].

Occasionally we will need to switch between B and B^- Schubert data. This is done by utilizing the left multiplication by w_0. We briefly recall the salient facts, and we refer the reader e.g., to [Knu, MNS22] for further details. Let $n_{w_0} \in G$ be a representative of $w_0 \in W = N_G(T)/T$. Left-multiplication by n_{w_0} induces an automorphism $\varphi_{w_0} : G/P \to G/P$, $gP \mapsto n_{w_0} gP$. This is *not* T-equivariant, but it is equivariant with respect to the group automorphism $\chi_0 : T \to T$ defined by $\chi_0(t) = n_{w_0} t n_{w_0}^{-1}$. This means that $\varphi_{w_0}(t.gP) = \chi_0(t).\varphi_{w_0}(gP)$. From functoriality of equivariant cohomology, it follows that φ_{w_0} induces an automorphism $\varphi_{w_0}^* : H_T^*(G/P) \to H_T^*(G/P)$, which 'twists' the coefficients in the base ring $H_T^*(\mathrm{pt})$ according to the automorphism χ_0. Non-equivariantly, $\varphi_{w_0}^*$ is the identity map. Observe that since $H_G^*(G/P) = H_T^*(G/P)^W$, $\varphi_{w_0}^*$ is a $H_G^*(G/P)$-algebra homomorphism, i.e., for any $a \in H_G^*(G/P), b \in H_T^*(G/P)$, $\varphi_{w_0}^*(a \cdot b) = a \cdot \varphi_{w_0}^*(b)$. Our main examples for classes in $H_G^*(G/P)$ will be the Chern classes of homogeneous vector bundles on G/P. Since $w_0^2 = \mathrm{id}$, it follows that $\varphi_{w_0}^*$ is an involution. For $\mathbb{T} = T \times \mathbb{C}^*$, with $\mathbb{C}^*$ acting trivially, the automorphism $\varphi_{w_0}^*$ can be extended to one of $H_{\mathbb{T}}^*(G/P)$ by letting $\varphi_{w_0}^*(\hbar) = \hbar$.

Since $\varphi_{w_0}^{-1}(Y(w)^\circ) = X(w_0 w)^\circ$, it follows that

$$\varphi_{w_0}^*[Y(w)]_T = [X(w_0 w)]_T; \quad \varphi_{w_0}^*(c_{\mathrm{SM}}^T(Y(w)^\circ)) = c_{\mathrm{SM}}^T(X(w_0 w)^\circ). \tag{29}$$

Finally, we observe that φ_{w_0} commutes with the G-equivariant projection $f : G/B \to G/P$, therefore $\varphi_{w_0}^*$ commutes with the pull-back f^* and the push-forward f_*.

6. Demazure-Lusztig operators

In this section we recall the definition of the geometric version of the Demazure-Lusztig (DL) operators which appear in the degenerate Hecke algebra (cf. [Gin98]). We also recall how these operators determine the equivariant CSM classes $c_{\mathrm{SM}}^T(X(w)^\circ)$ and their duals (cf. [AM16]).

6.1. **Definition and basic properties.** Recall the datum of $G \supset B \supset T$. Let also $P_i \supset B$ be the (standard) minimal parabolic subgroup associated to the simple root α_i and $\pi_i : G/B \to G/P_i$ the projection.

For each simple reflection $s_i \in W$ one can associate two operators on $H_T^*(G/B)$. The first is the Bernsteĭn-Gelfand-Gelfand (BGG) operator $\partial_i : H_T^*(G/B) \to H_T^{*-2}(G/B)$, defined by $\partial_i = \pi_i^*(\pi_i)_*$; cf. [BGG73]. Since π_i is G-equivariant, ∂_i is $H_T^*(\mathrm{pt})$-linear. It satisfies

$$\partial_i[X(w)]_T = \begin{cases} [X(ws_i)]_T & ws_i > w; \\ 0 & \text{otherwise} \end{cases}; \; \partial_i[Y(w)]_T = \begin{cases} [Y(ws_i)]_T & ws_i < w; \\ 0 & \text{otherwise.} \end{cases}$$

See e.g., [Bri05]. (The second equality follows from the first by applying $\varphi_{w_0}^*$.)

The second operator is the algebra automorphism $\mathfrak{s}_i : H_T^*(G/B) \to H_T^*(G/B)$ obtained by the *right* Weyl group multiplication by (a representative of) $s_i \in W$ on G/T. The projection $G/T \to G/B$ is a B/T-bundle, and since B/T (which is the unipotent group of B) is equivariantly contractible, G/T and G/B have the same cohomology groups. From this definition it follows that $\mathfrak{s}_i$ is homogeneous and $H_T^*(\mathrm{pt})$-linear. One may show that

$$\mathfrak{s}_i = \mathrm{id} + c_1^T(\mathcal{L}_{\alpha_i})\partial_i,$$

where $\mathcal{L}_\alpha := G \times^B \mathbb{C}_\alpha$ is the homogeneous line bundle over G/B with fiber over $1.B$ the T-module $\mathbb{C}_\alpha$ of weight α. We refer to [AM16, §2] (where a different sign convention is used for $\mathcal{L}_\alpha$) for more details about $\mathfrak{s}_i$.

For each simple reflection $s_i \in W$ define two non-homogeneous operators:

$$\mathcal{T}_i := \partial_i - \mathfrak{s}_i; \quad \mathcal{T}_i^\vee := \partial_i + \mathfrak{s}_i. \tag{30}$$

These are $H_T^*(\mathrm{pt})$-linear operators acting on $H_T^*(G/B)$. The 'dual' operator $\mathcal{T}_i^\vee$ is precisely the Demazure-Lusztig operator discussed by Ginzburg in [Gin98, (47)] in relation with the degenerate Hecke algebra; see also [LLT96, LLT97, Lus85]. The operators $\mathcal{T}_i, \mathcal{T}_i^\vee$ satisfy the braid relations for W and $\mathcal{T}_i^2 = \mathrm{id}$ ([AM16, Proposition 4.1]). Thus, we may define operators $\mathcal{T}_w, \mathcal{T}_w^\vee$ for any element w of the Weyl group. From this it follows that

$$\mathcal{T}_u\mathcal{T}_v = \mathcal{T}_{uv}; \quad \mathcal{T}_u^\vee\mathcal{T}_v^\vee = \mathcal{T}_{uv}^\vee \quad \forall v, w \in W, \tag{31}$$

therefore these operators give a 'twisted representation' of W on $H_T^*(G/B)$; cf. [LLT96].

Using the formulae for the action of $\partial_i, \mathfrak{s}_i$ on Schubert classes one can also write formulae for the action of $\mathcal{T}_i, \mathcal{T}_i^\vee$ (see [AM16, §6.3]). We recall these, as they will be used in the proof of the orthogonality properties. The action of $\mathcal{T}_k$ is given by

$$\mathcal{T}_k([X(w)]_T) = \begin{cases} -[X(w)]_T & \text{if } \ell(ws_k) < \ell(w) \\ \begin{aligned}&(1 + w(\alpha_k))[X(ws_k)]_T + [X(w)]_T \\ &+ \sum \langle \alpha_k, \beta^\vee \rangle [X(ws_k s_\beta)]_T\end{aligned} & \text{if } \ell(ws_k) > \ell(w) \end{cases} \tag{32}$$

where the sum is over all positive roots $\beta \neq \alpha_k$ such that $\ell(w) = \ell(ws_k s_\beta)$, and $w(\alpha_k)$ denotes the natural W action $w \cdot \alpha_k$ on T-weights; note that $w(\alpha_k) > 0$ since $ws_k > w$. The action of the dual operator $\mathcal{T}_k^\vee$ is given by

$$\mathcal{T}_k^\vee([X(w)]_T) = \begin{cases} [X(w)]_T & \text{if } \ell(ws_k) < \ell(w) \\ \begin{array}{l}(1 - w(\alpha_k))[X(ws_k)]_T - [X(w)]_T \\ - \sum \langle \alpha_k, \beta^\vee \rangle [X(ws_k s_\beta)]_T\end{array} & \text{if } \ell(ws_k) > \ell(w) \end{cases} \tag{33}$$

where the sum is as before. One may obtain similar formulae for the actions on the opposite classes $[Y(w)]_T$ by using the automorphism $\varphi_{w_0}^*$ from Equation (29).

The relevance of the DL operators comes from the following result, proved in [AM16, Theorem 6.4].

Theorem 6.1. *Let $w \in W$ be an element of the Weyl group. Then*

$$\mathcal{T}_i(c_{\mathrm{SM}}^T(X(w)^\circ)) = c_{\mathrm{SM}}^T(X(ws_i)^\circ).$$

Therefore, for every $w \in W$, we have

$$c_{\mathrm{SM}}^T(X(w)^\circ) = \mathcal{T}_{w^{-1}}([X(\mathrm{id})]_T).$$

Recall also (cf. (14)) the definition of the *dual* CSM classes $c_{\mathrm{SM}}^{T,\vee}(X(w)^\circ)$, obtained from $c_{\mathrm{SM}}^T(X(w)^\circ)$ by changing signs of each homogeneous component according to *co*dimension. Then Theorem 6.1 together with the definition of the dual DL operators $\mathcal{T}_i^\vee$ implies that

$$c_{\mathrm{SM}}^{T,\vee}(X(w)^\circ) = \mathcal{T}_{w^{-1}}^\vee([X(\mathrm{id})]_T). \tag{34}$$

Remark 6.2. Similar statements hold for homogenized classes. For instance, the homogenization of the operator $\mathcal{T}_i$ is $\mathcal{T}_i^\hbar := \hbar\partial_i - \mathfrak{s}_i$. Recursive application of these operators yield the homogenization of the class $c_{\mathrm{SM}}^T(X(w)^\circ)$. The homogenization of the dual class $c_{\mathrm{SM}}^{T,\vee}(X(w)^\circ)$ is obtained by applying $\mathcal{T}_i^{\hbar,\vee} := \hbar\partial_i + \mathfrak{s}_i$. We leave the details to the reader. ⌟

6.2. **Adjointness.** Next we prove the key property in the proof of the 'Hecke orthogonality' in §7.2. Recall that for $a, b \in H_T^*(G/B)$, $\langle a, b\rangle$ denotes $\int_{G/B} a \cdot b$.

Proposition 6.3. *The operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are adjoint to each other, i.e., for any $a, b \in H_T^*(G/B)$ there is an identity in $H_T^*(\mathrm{pt})$:*

$$\langle \mathcal{T}_i(a), b\rangle = \langle a, \mathcal{T}_i^\vee(b)\rangle.$$

Therefore, $\langle \mathcal{T}_w(a), b\rangle = \langle a, \mathcal{T}_{w^{-1}}^\vee(b)\rangle$ for all $w \in W$.

Proof. It suffices to show that the BGG operator ∂_i is self-adjoint and that the adjoint of $\mathfrak{s}_i$ is $-\mathfrak{s}_i$. We first verify that ∂_i is self-adjoint; while this is well-known, we include a proof for completeness. Let P_i be the minimal parabolic group and $\pi_i : G/B \to G/P_i$ the natural projection. Recall that $\partial_i = \pi_i^*(\pi_i)_*$. Then by the projection formula

$$\langle \partial_i(a), b\rangle = \int_{G/B} \pi_i^*(\pi_i)_*(a)\cdot b = \int_{G/P_i} (\pi_i)_*(a)\cdot(\pi_i)_*(b) = \langle a, \partial_i(b)\rangle,$$

where the last equality follows by symmetry.

In order to verify that $\mathfrak{s}_i$ and $-\mathfrak{s}_i$ are adjoint, let $e_w := wB \in G/B$ denote the T-fixed point in G/B corresponding to w (so $e_{\mathrm{id}} = 1.B$ is the B-fixed point). Then $\int_{G/B} a\cdot b$ is the coefficient of $[X(\mathrm{id})]_T = [e_{\mathrm{id}}]_T$ in the expression for $a\cdot b$ with respect to the Schubert basis. Recall also that

$$\mathfrak{s}_i[e_{\mathrm{id}}]_T = -[e_{s_i}]_T = P(t)[X(s_i)]_T - [e_{\mathrm{id}}]_T$$

where $P(t) \in H_T^2(\mathrm{pt})$. (Cf. e.g., [AM16, (4) and §6.3].) Then

$$\begin{aligned}\langle \mathfrak{s}_i(a), b\rangle &= \int_X \mathfrak{s}_i(a)\cdot b = \int_X \mathfrak{s}_i(a)\cdot \mathfrak{s}_i\mathfrak{s}_i(b) = \int_X \mathfrak{s}_i(a\cdot \mathfrak{s}_i(b)) \\ &= -\int_X a\cdot \mathfrak{s}_i(b) = -\langle a, \mathfrak{s}_i(b)\rangle,\end{aligned}$$

using the fact that $\mathfrak{s}_i$ is an $H_T^*(\mathrm{pt})$-algebra homomorphism and squares to the identity. □

7. Orthogonality properties of CSM classes of Schubert cells

In this section we prove two orthogonality results for equivariant CSM classes of Schubert cells in flag manifolds.

The first, 'geometric orthogonality', states that the CSM classes of Schubert cells are orthogonal to the Segre-MacPherson (SM) classes of opposite Schubert cells. It will follow from the equivariant versions of general transversality results from [Sch17], particularly Theorem 3.6.

The second is the 'Hecke orthogonality' mentioned in the introduction. This holds only for complete flag manifolds G/B and it states that the CSM class are orthogonal to the dual/signed CSM classes. It is a consequence of the adjointness property from Proposition 6.3 and the fact that the CSM classes of Schubert cells may be calculated by using DL operators.

As a consequence of these orthogonalites we obtain an identity among the SM classes and signed CSM classes of Schubert cells in G/B; cf. Theorem 7.5. This is a key ingredient used in §8.2 in the proof of a positivity property of CSM classes conjectured in [AM16]. In addition, in §7.4 we use Hecke orthogonality to prove two interesting properties about the Schubert expansion of CSM classes.

7.1. **The geometric orthogonality.** Recall (cf. (23)) that the Segre-MacPherson (SM) class of a constructible function $\varphi \in \mathcal{F}^T_{\mathrm{inv}}(G/P)$ is defined by

$$s^T_{\mathrm{SM}}(\varphi) := \frac{c_*^T(\varphi)}{c^T(T(G/P))}.$$

Theorem 7.1 (Geometric orthogonality). *Let $u, v \in W^P$. Then*

$$\big\langle c^T_{\mathrm{SM}}(X(uW_P)^\circ), s^T_{\mathrm{SM}}(Y(vW_P)^\circ)\big\rangle = \delta_{u,v}.$$

Proof. The stratification by the B-orbits, and that by the B^--orbits, are Whitney stratifications of G/P. Indeed, the Whitney conditions hold generically on the B, respectively B^- strata, and then by equivariance also everywhere along each orbit. Further, by [Ric92, Corollary 1.5], the intersections of of B and B^--orbits are transversal to each other. Therefore we may apply Theorem 3.6 to calculate

$$\begin{aligned}\big\langle c^T_{\mathrm{SM}}(X(uW_P)^\circ), s^T_{\mathrm{SM}}(Y(vW_P)^\circ)\big\rangle &= \int_{G/P} c^T_{\mathrm{SM}}(X(uW_P)^\circ)\cdot s^T_{\mathrm{SM}}(Y(vW_P)^\circ)\\ &= \int_{G/P} c^T_{\mathrm{SM}}(X(uW_P)^\circ\cap Y(vW_P)^\circ)\\ &= \chi(X(uW_P)^\circ\cap Y(vW_P)^\circ)\\ &= \delta_{u,v}.\end{aligned}$$

The last equality follows because if $u = v$ then the intersection $X(uW_P)^\circ \cap Y(vW_P)^\circ$ is the single (T-fixed) point e_{uW_P}, and if $u \neq v$ then the intersection is either empty or a T-stable variety with no T-fixed points (see e.g., [Bri05, §1.3]), therefore its Euler characteristic is equal to 0. □

7.2. **The Hecke orthogonality.** The goal of this subsection is to prove the following theorem:

Theorem 7.2 (Hecke orthogonality). *The equivariant CSM classes of Schubert cells in $H^*_T(G/B)$ satisfy the following orthogonality property:*

$$\langle c^T_{\mathrm{SM}}(X(u)^\circ), c^{T,\vee}_{\mathrm{SM}}(Y(v)^\circ)\rangle = \delta_{u,v}\prod_{\alpha>0}(1+\alpha).$$

An analogous orthogonality property holds for opposite Schubert cells:

$$\langle c^T_{\mathrm{SM}}(Y(u)^\circ), c^{T,\vee}_{\mathrm{SM}}(X(v)^\circ)\rangle = \delta_{u,v}\prod_{\alpha>0}(1-\alpha).$$

In order to prove Theorem 7.2 we need the following lemma, which is a consequence of Theorem 6.1 and the formulae in (32) and (33); the proof is left to the reader.

Lemma 7.3. *For $w \in W$, let $e(w) := \prod_{\alpha>0, w^{-1}(\alpha)<0}(1+\alpha)$ and $\breve{e}(w) := \prod_{\alpha>0, w^{-1}(\alpha)<0}(1-\alpha)$. Then*

$$c^T_{\mathrm{SM}}(X(w)^\circ) = e(w)[X(w)]_T + \textit{terms involving } [X(v)]_T \textit{ for } v < w;$$
$$c^{T,\vee}_{\mathrm{SM}}(X(w)^\circ) = \breve{e}(w)[X(w)]_T + \textit{terms involving } [X(v)]_T \textit{ for } v < w.$$

Proof of Theorem 7.2. To prove the first equality, observe that $c_{\mathrm{SM}}^{T,\vee}(Y(v)^\circ) = \mathcal{T}_{v^{-1}w_0}^\vee[Y(w_0)]_T$; this follows from identity (34) by applying the automorphism φ_{w_0}. By Proposition 6.3 and identities (31) and (34), we have

$$\begin{aligned}\langle c_{\mathrm{SM}}^T(X(u)^\circ), c_{\mathrm{SM}}^{T,\vee}(Y(v)^\circ)\rangle &= \langle \mathcal{T}_{u^{-1}}[X(\mathrm{id})]_T, \mathcal{T}_{v^{-1}w_0}^\vee[Y(w_0)]_T\rangle \\ &= \langle \mathcal{T}_{w_0 v}\mathcal{T}_{u^{-1}}[X(\mathrm{id})]_T, [Y(w_0)]_T\rangle \\ &= \langle \mathcal{T}_{w_0 v u^{-1}}[X(\mathrm{id})]_T, [Y(w_0)]_T\rangle \\ &= \text{coefficient of } [X(w_0)]_T \text{ in } c_{\mathrm{SM}}^T(X(uv^{-1}w_0)^\circ).\end{aligned}$$

By Lemma 7.3, this coefficient is 0 unless $u = v$, and it equals $\prod_{\alpha>0}(1+\alpha)$ if $u = v$. This verifies the first equality. The second equality follows from the first, by applying the automorphism $\varphi_{w_0}^*$. □

Corollary 7.4 (CSM Poincaré duality)**.** *Ordinary CSM classes are Poincaré dual to dual CSM classes of opposite cells. That is:*

$$\langle c_{\mathrm{SM}}(X(u)^\circ), c_{\mathrm{SM}}^\vee(Y(v)^\circ)\rangle = \delta_{u,v}. \tag{35}$$

Proof. This follows from the previous theorem by specializing $\alpha \mapsto 0$. □

In ordinary homology, the leading terms of $c_{\mathrm{SM}}(X(u)^\circ)$ and $c_{\mathrm{SM}}^\vee(Y(v)^\circ)$ are $[X(u)]$, $[Y(v)]$, respectively: we may view these CSM classes as 'deformations' of the fundamental classes by lower dimensional terms. Corollary 7.4 states that these deformations preserve the intersection pairing: cf. (28) and (35).

7.3. **Consequences of orthogonality I: equality of SM and dual CSM classes.** Combining the geometric and Hecke orthogonalities from Theorem 7.1 and Theorem 7.2, together with the fact that the Poincaré pairing is non-degenerate, we obtain the main result of this section:

Theorem 7.5. *Let $v \in W$. Then the following equality holds in $H_T^*(G/B)$:*

$$c_{\mathrm{SM}}^{T,\vee}(X(v)^\circ) = \big(\prod_{\alpha\in R_+}(1-\alpha)\big)s_{\mathrm{SM}}^T(X(v)^\circ).$$

In particular, this yields the following identity in $H^*(G/B)$:

$$c_{\mathrm{SM}}^\vee(X(v)^\circ) = s_{\mathrm{SM}}(X(v)^\circ). \tag{36}$$

This identity is one of the two key ingredients in the proof of the positivity of CSM classes of Schubert cells. The other is provided by the theory of $\mathcal{D}$-modules, which we will use in §8 to prove that the signed SM class is effective.

Example 7.6. For $X = \mathrm{Fl}(2) = \mathbb{P}^1$ and $v = w_0$ the longest Weyl group element, $c_{\mathrm{SM}}^\vee(X(v)^\circ) = [\mathbb{P}^1] - [\mathrm{pt}]$, $c_{\mathrm{SM}}(X(v)^\circ) = [\mathbb{P}^1] + [\mathrm{pt}]$ and $c(T\mathbb{P}^1) = [\mathbb{P}^1] + 2[\mathrm{pt}] = 1 + 2[\mathrm{pt}]$. Then

$$s_{\mathrm{SM}}(X(v)^\circ) = \frac{[\mathbb{P}^1]+[\mathrm{pt}]}{1+2[\mathrm{pt}]} = (1-2[\mathrm{pt}])([\mathbb{P}^1]+[\mathrm{pt}]) = [\mathbb{P}^1] - [\mathrm{pt}],$$

verifying identity (36) in this case.

More generally, an algorithm calculating CSM classes of Schubert cells $X(v)^\circ$ (and therefore their duals as well) was obtained in [AM16], and this may be used to verify the identities from Theorem 7.5 explicitly in many concrete cases. For instance, the following are the (non-equivariant) CSM classes of the Schubert cells in $\mathrm{Fl}(3)$, the variety parametrizing flags in $\mathbb{C}^3$:

$$\begin{aligned}
c_{\mathrm{SM}}(X(w_0)^\circ) &= [\mathrm{Fl}(3)] + [X(s_1s_2)] + [X(s_2s_1)] + 2[X(s_1)] + 2[X(s_2)] + [\mathrm{pt}];\\
c_{\mathrm{SM}}(X(s_1s_2)^\circ) &= [X(s_1s_2)] + [X(s_1)] + 2[X(s_2)] + [\mathrm{pt}];\\
c_{\mathrm{SM}}(X(s_2s_1)^\circ) &= [X(s_2s_1)] + 2[X(s_1)] + [X(s_2)] + [\mathrm{pt}];\\
c_{\mathrm{SM}}(X(s_1)^\circ) &= [X(s_1)] + [\mathrm{pt}];\\
c_{\mathrm{SM}}(X(s_2)^\circ) &= [X(s_2)] + [\mathrm{pt}];\\
c_{\mathrm{SM}}(X(\mathrm{id})^\circ) &= [\mathrm{pt}].
\end{aligned}$$

The total Chern class of $\mathrm{Fl}(3)$ is

$$\begin{aligned}
c(T\,\mathrm{Fl}(3)) = \sum_v c_{\mathrm{SM}}(X(v)^\circ) &= [\mathrm{Fl}(3)] + 2[X(s_1s_2)] + 2[X(s_2s_1)] + 6[X(s_1)]\\
&\quad + 6[X(s_2)] + 6[\mathrm{pt}]
\end{aligned}$$

Again one can check identity (36), by using the multiplication table in $H^*(\mathrm{Fl}(3))$. ⌟

Example 7.7. View $\mathbb{P}^2$ as a partial flag manifold. The Schubert cells are isomorphic to $\mathbb{A}^i$, $i = 0, 1, 2$, and we have

$$c_{\mathrm{SM}}(\mathbb{A}^2) = c_{\mathrm{SM}}(\mathbb{P}^2) - c_{\mathrm{SM}}(\mathbb{P}^1) = [\mathbb{P}^2] + 2[\mathbb{P}^1] + [\mathbb{P}^0] \quad .$$

The cell $\mathbb{A}^2$ is *not* its own opposite, yet

$$\begin{aligned}
\langle c_{\mathrm{SM}}(\mathbb{A}^2), c^\vee_{\mathrm{SM}}(\mathbb{A}^2)\rangle_{\mathbb{P}^2} &= \int ([\mathbb{P}^2] + 2[\mathbb{P}^1] + [\mathbb{P}^0]) \cdot ([\mathbb{P}^2] - 2[\mathbb{P}^1] + [\mathbb{P}^0])\\
&= 1 - 4 + 1 = -2 \neq 0 \quad .
\end{aligned}$$

Further, note that identity (36) does not extend to the parabolic case. Indeed,

$$\frac{c_{\mathrm{SM}}(\mathbb{A}^2)}{c(T\mathbb{P}^2)} = [\mathbb{P}^2] - [\mathbb{P}^1] + [\mathbb{P}^0] \neq c^\vee_{\mathrm{SM}}(\mathbb{A}^2).$$

Finally, note that

$$\begin{aligned}
\left\langle c_{\mathrm{SM}}(\mathbb{A}^2), \frac{c_{\mathrm{SM}}(\mathbb{A}^2)}{c(T\mathbb{P}^2)} \right\rangle_{\mathbb{P}^2} &= \int ([\mathbb{P}^2] + 2[\mathbb{P}^1] + [\mathbb{P}^0]) \cdot ([\mathbb{P}^2] - [\mathbb{P}^1] + [\mathbb{P}^0])\\
&= 1 - 2 + 1 = 0
\end{aligned}$$

as we would expect from the geometric orthogonality in Theorem 7.1. ⌟

7.4. **Consequences of orthogonality II: the transition matrix between Schubert and CSM classes.** We present next two consequences of the Hecke orthogonality property, in the non-equivariant setting.

The CSM class of each Schubert cell may be written in terms of the Schubert basis:

$$c_{\mathrm{SM}}(X(v)^\circ) = \sum_{u\in W} c(u;v)[X(u)] \quad . \tag{37}$$

with $c(u;v) \in \mathbb{Z}$. A natural question is to find the inverse of the matrix $(c(u;v))_{u,v\in W}$.

Proposition 7.8. *The inverse of the matrix* $\big(c(u;v)\big)_{u,v}$ *is the matrix*

$$\big((-1)^{\ell(u)-\ell(v)}c(w_0v;w_0u)\big)_{u,v} \quad .$$

Proof. Let $(d(u;v))_{u,v}$ be the inverse matrix. In other words,

$$[X(v)] = \sum_{u\in W} d(u;v)c_{\mathrm{SM}}(X(u)^\circ) \quad .$$

By Corollary 7.4, $d(u;v) = \langle [X(v)], c_{\mathrm{SM}}^\vee(Y(u)^\circ)\rangle$. This is the coefficient of $[Y(v)]$ in the expansion of $c_{\mathrm{SM}}^\vee(Y(u)^\circ)$ in the basis of (opposite) Schubert classes. The statement follows from the definition of dual CSM classes and the fact that $[Y(w)] = [X(w_0w)]$ for every $w \in W$. □

Example 7.9. For Fl(3) we consider the matrix A whose (i,j) entry is the coefficient $c(w_i,w_j)$, where we list the permutations $w_i \in S_3$, $i = 1,\dots,6$ in the order

$$\mathrm{id}, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1.$$

From Example 7.6 (see also the 'non-equivariant' part of the matrix shown in [AM16, Example 6.7]), the matrix A and its inverse are given by:

$$A = \begin{pmatrix} 1&1&1&1&1&1\\ 0&1&0&1&2&2\\ 0&0&1&2&1&2\\ 0&0&0&1&0&1\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1&-1&-1&2&2&-1\\ 0&1&0&-1&-2&1\\ 0&0&1&-2&-1&1\\ 0&0&0&1&0&-1\\ 0&0&0&0&1&-1\\ 0&0&0&0&0&1 \end{pmatrix}.$$

The second matrix is (up to signs) the anti-transpose of the first one. This is the content of Proposition 7.8 in this example.

Larger examples may be also computed explicitly, making use of [AM16, Corollary 4.2]. A previous version of this article, available on the arχiv, presented the two 24×24 matrices for the case of Fl(4). ⌟

Consider now the problem of defining a constructible function Θ_V on a given variety V, such that $c_*(\Theta_V) = [V]$. Such a function is of course not uniquely defined, but there are interesting situations in which a particularly natural function satisfies this property. For example, if V is a toric variety compactifying a torus V°, then $\Theta_V = 1\!\!1_{V^\circ}$ is such a function ([Alu06, Théorème 4.2]). Proposition 7.8 implies that Schubert varieties offer another class of examples.

Corollary 7.10. *Let* $\Theta_v = \sum_u (-1)^{\ell(v)-\ell(u)} c(w_0 v; w_0 u) 1\!\!1_{X(u)^\circ}$*.*
Then $c_*(\Theta_v) = [X(v)]$*.*

Proof. This follows immediately from (37) and Proposition 7.8. ☐

To illustrate, let $w = w_0$, the maximum length element in the Weyl group. Then according to Corollary 7.10

$$[G/B] = c_* \left(\sum_v (-1)^{\ell(w_0)-\ell(v)} 1\!\!1_{X(v)^\circ} \right) \quad .$$

This identity is independently proven in [AM16, Proposition 5.5].

Remark 7.11. Consider the Schubert expansion of the equivariant CSM class:

$$c^T_{\mathrm{SM}}(X(w)^\circ) = \sum c^T(v;w)[X(v)]_T,$$

where $c^T(v;w) \in H^*_T(\mathrm{pt}) = Sym_{\mathbb{Z}}\mathfrak{X}(T)$. There is an interpretation of the coefficients $c^T(u;v)$ in the affine nil-Hecke algebra, found by S.J. Lee in [Lee18]. We briefly recall this interpretation, and refer to *loc. cit.* for all the details. The affine nil-Hecke algebra $\mathcal{H}_{nil}$ is generated by the elements $\bar{\partial}_w$ and $\lambda \in \mathfrak{X}(T)$, subject to the following relations:

(1) $\lambda\mu = \mu\lambda$, for any $\lambda, \mu \in \mathfrak{X}(T)$;
(2) $\bar{\partial}_w \bar{\partial}_y = \delta_{\ell(wy),\ell(w)+\ell(y)} \bar{\partial}_{wy}$;
(3) $\bar{\partial}_i \lambda = s_i\lambda \cdot \bar{\partial}_i - \langle \lambda, \alpha_i^\vee \rangle$.

For each simple root α_i, define the element $\bar{s}_i := 1 + \alpha_i \bar{\partial}_i \in \mathcal{H}_{nil}$. The elements $\bar{\partial}_i$ and $\bar{s}_i$ satisfy the braid relations in the Weyl group W. Let $w = s_{i_1} \cdot \ldots \cdot s_{i_k} \in W$ be a reduced decomposition. Then according to [Lee18, Theorem 6.2] there is an identity

$$(\bar{s}_{i_1} + \bar{\partial}_{i_1}) \cdot \ldots \cdot (\bar{s}_{i_k} + \bar{\partial}_{i_k}) = \sum_v c^T(v;w)\bar{\partial}_v.$$

Geometrically, $\bar{\partial}_i$ corresponds to the BGG operator ∂_i, $\bar{s}_i$ to $-\mathfrak{s}_i$, and the weight λ to the Chevalley multiplication by the equivariant Chern class $c_1^T(G \times^B \mathbb{C}_{-\lambda})$ (the class of a G-equivariant line bundle over G/B). In the related work [SZZ20], Su, Zhao and Zhong observe a relation between the affine Hecke algebra and a K-theoretic version of stable envelopes/CSM classes. (Also see §9.) ⌟

8. CSM classes and characteristic cycles for flag manifolds

In the classical results of Beilinson-Bernstein [BB81], Brylinski-Kashiwara [BK81], and Kashiwara-Tanisaki [KT84], the theory of characteristic cycles associated to holonomic $\mathcal{D}$-modules on flag manifolds becomes a powerful geometric tool to study the representation theory related to the Kazhdan-Lusztig polynomials. Characteristic cycles of $\mathcal{D}$-modules and those of constructible functions are closely related. There are multiple sign conventions in the literature, and in this section we carefully spell out the relation and conventions used in this paper; essentially, we require that characteristic cycles of holonomic $\mathcal{D}$-modules are effective. Then we utilize Theorem 4.2 to find the precise relation between CSM classes of Schubert cells and the characteristic cycles of Verma modules. The main result is Theorem 8.3, which is also the main ingredient in the proof of the positivity of CSM classes. As an application, we define certain 'Kazhdan-Lusztig' classes in §8.3, and show they are also positive.

8.1. **CSM classes and characteristic cycles of Verma modules.** Let $X = G/B$ be the generalized flag manifold. We recall some results from [KT84], and in order to satisfy the hypotheses from *loc.cit.* we assume in addition that G is simply connected.[2] Let $\rho \in \mathfrak{X}(T)$ denote half the sum of positive roots. For $w \in W$ let M_w be the Verma module of highest weight $-w\rho - \rho$, a module over the universal enveloping algebra $U(\mathfrak{g})$; see [HTT08, p. 291]. Let $\mathcal{M}_w$ denote the holonomic $\mathcal{D}_X$-module

$$\mathcal{M}_w = \mathcal{D}_X \otimes_{U(\mathfrak{g})} M_w.$$

Consider the constructible complex $\mathrm{DR}(\mathcal{M}_w)$. According to [KT84, Theorem 3] (where it is attributed to Brylinski-Kashiwara [BK81] and Beilinson-Bernstein [BB81]) there is an identity

$$\mathrm{DR}(\mathcal{M}_w) = \mathbb{C}_{X(w)^\circ}[\ell(w)]\,;$$

where the right-hand side is the shifted constant sheaf on $X(w)^\circ$, also cf. [HTT08, Corollary 12.3.3(i)]. (Note that the definition of DR from [KT84] differs from the one from [Gin86] and [HTT08] by a shift of $\dim X$.) It follows that the constructible function associated to the Verma module M_w is

$$\chi_{\mathrm{stalk}}(\mathrm{DR}(\mathcal{M}_w)) = (-1)^{\ell(w)}\, 1\!\!1_{X(w)^\circ}.$$

By the commutativity of diagram (24),

$$\mathrm{Char}(\mathcal{M}_w) = (-1)^{\ell(w)}\, \mathrm{CC}(1\!\!1_{X(w)^\circ}); \tag{38}$$

therefore, Corollary 4.4 implies the following result. Let $c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ) := c_{\mathrm{SM}}^{T}(X(w)^\circ)^\hbar$ denote the $\hbar$-homogenization of degree $\dim G/B$.

Corollary 8.1. *Let $w \in W$. Then*

$$c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ) = (-1)^{\ell(w)} \iota^*[\mathrm{Char}(\mathcal{M}_w)]_{T\times\mathbb{C}^*}$$

where $\iota : G/B \to T^*(G/B)$ is the zero-section and $\mathbb{C}^*$ acts on the fibers of $T^*(G/B)$ by the character $\hbar^{-1}$.

[2] Note that for complex semisimple G, the flag varieties G/P, and the Schubert cells, only depend on the Lie algebra of G (see e.g., [CG97, §3.1]), so this assumption is harmless for our (co)homological calculations.

8.2. **The CSM class as a Segre class: proof of the main theorem.** In this section we prove Theorem 1.3 from the introduction. Recall that this is the main ingredient to prove that in the non-equivariant case the CSM classes are effective, thus proving the positivity conjecture stated in [AM16]. We start by proving the following lemma, which is known among experts, but for which we could not find a reference.

Lemma 8.2. *The following equality holds in $H_T^*(G/B)$:*

$$c^T(T(G/B)) \cdot c^T(T^*(G/B)) = \prod_{\alpha>0}(1-\alpha^2).$$

Proof. The Chern class of G/B is given by $c^T(T(G/B)) = \prod_{\alpha>0}(1+c_1^T(\mathcal{L}_{-\alpha}))$ where (as in §6.1) $\mathcal{L}_{-\alpha} = G \times^B \mathbb{C}_{-\alpha}$. Since the localization of $\mathcal{L}_{-\alpha}$ at the T-fixed point e_w is $w(-\alpha)$, it follows that $c^T(T(G/B))_{|w} = \prod_{\alpha>0}(1-w(\alpha))$. From this we obtain that

$$(c^T(T(G/B)) \cdot c^T(T^*(G/B)))_{|w} = \prod_{\alpha>0}(1-w(\alpha))(1+w(\alpha)) = \prod_{\alpha>0}(1-\alpha)(1+\alpha),$$

because w permutes the set of roots. □

Recall the following construction from §2.4. Consider the projection $\overline{q} : \mathbb{P}(T^*(G/B)\oplus \mathbb{1}) \to G/B$ and the tautological subbundle $\mathscr{O}_{T^*(G/B)\oplus\mathbf{1}}(-1) \subseteq T^*(G/B)\oplus\mathbb{1}$; this is an inclusion of T-equivariant bundles. If C is a T-stable cycle in $T^*(G/B)$, then the Segre operator acts on C by

$$\mathrm{s}^T(C) = \overline{q}_*\left(\frac{[\overline{C}]}{c^T(\mathscr{O}_{T^*(G/B)\oplus\mathbf{1}}(-1))}\right)$$

where $\overline{C}$ denotes the closure of C in $\mathbb{P}(T^*(G/B)\oplus\mathbb{1})$; cf. (11). This operator takes values in the completion $\hat{H}_*^T(G/B)$. It follows from the next result that in the cases of interest here it in fact takes values in the localization at the non-zero elements in $H_T^*(\mathrm{pt})$.

Theorem 8.3. *Let $w \in W$. The following equality holds:*

$$c_{\mathrm{SM}}^T(X(w)^\circ) = \Big(\prod_{\alpha>0}(1+\alpha)\Big)\ \mathrm{s}^T(\mathrm{Char}(\mathcal{M}_w)).$$

Proof. We have

$$
\begin{aligned}
c^T_{\mathrm{SM}}(X(w)^\circ) &\overset{(23)}{=} c^T(T(G/B)) \cap s^T_{\mathrm{SM}}(X(w)^\circ) \\
&\overset{\text{Thm.7.5}}{=} \frac{c^T(T(G/B))}{\prod_{\alpha>0}(1-\alpha)} \cap c^{T,\vee}_{\mathrm{SM}}(X(w)^\circ) \\
&\overset{(14)}{=} (-1)^{\dim X(w)^\circ} \frac{c^T(T(G/B))}{\prod_{\alpha>0}(1-\alpha)} \cap \check{c}^T_*(\mathbb{1}_{X(w)^\circ}) \\
&\overset{\text{Cor.3.5}}{=} (-1)^{\ell(w)} \frac{c^T(T(G/B)) \cdot c^T(T^*(G/B))}{\prod_{\alpha>0}(1-\alpha)} \cap \mathrm{s}^T(\mathrm{CC}(\mathbb{1}_{X(w)^\circ})) \\
&\overset{\text{Lem.8.2}}{=} (-1)^{\ell(w)} \Big(\prod_{\alpha>0}(1+\alpha)\Big) \mathrm{s}^T(\mathrm{CC}(\mathbb{1}_{X(w)^\circ})) \\
&\overset{(38)}{=} \Big(\prod_{\alpha>0}(1+\alpha)\Big) \mathrm{s}^T(\mathrm{Char}(\mathcal{M}_w))
\end{aligned}
$$

as stated. □

As explained in introduction, Theorem 8.3 implies that the non-equivariant CSM classes of Schubert cells are effective. For the convenience of the reader we recall the statement, adding also a positivity statement of Segre-MacPherson classes which follows immediately.

Corollary 8.4 (Positivity of CSM classes)**.** *(a) Let $X = G/P$ be a generalized flag manifold and $w \in W$. Then the non-equivariant CSM class $c_{\mathrm{SM}}(X(wW_P)^\circ)$ is effective, i.e., in the Schubert expansion*

$$c_{\mathrm{SM}}(X(wW_P)^\circ) = \sum_{vW_P \le wW_P} c(vW_P; wW_P)[X(vW_P)] \in H_*(X),$$

the coefficients $c(vW_P; wW_P)$ are non-negative.

(b) Let $X = G/B$ and $w \in W$. Then the Segre-MacPherson class $s_{\mathrm{SM}}(X(w)^\circ)$ is Schubert alternating, i.e., in the Schubert expansion

$$s_{\mathrm{SM}}(X(w)^\circ) = \sum_{v \le w} d(v; w)[X(v)] \in H_*(X),$$

the coefficients $d(v; w)$ satisfy $(-1)^{\ell(w)-\ell(v)} d(v; w) \ge 0$.

Proof. Part (a) follows from Theorem 8.3, as explained in the introduction. Part (b) follows from (a) and identity (36). □

Remark 8.5. In [AMSS22] we utilize the methods in this paper to extend the statement of part (b) to any flag manifold G/P. If G/P is a Grassmannian, this was conjectured in [FR18], see §1.5 and Conjecture 8.4. ⌟

8.3. **Kazhdan-Lusztig classes.** In analogy to Corollary 8.1, in this section we define *Kazhdan-Lusztig (KL) classes* associated to the intersection cohomology (IC) complex. We show that the KL classes of Schubert varieties are positive. In some important situations, the KL classes equal to the Mather classes.

Let X be a smooth complex algebraic variety, and $Y \subseteq X$ a subvariety. Denote by $\mathrm{IC}(Y) \in \mathrm{Perv}(X)$ the intersection cohomology (IC) complex of the subvariety Y, with the convention that the restriction to the regular part Y^{reg} is the shifted constant sheaf $\mathbb{C}_{Y^{\mathrm{reg}}}[\dim Y]$.

Definition 8.6. The *Kazhdan-Lusztig (KL) class* of Y, denoted by $\mathrm{KL}(Y)$, is defined by

$$\mathrm{KL}(Y) := (-1)^{\dim Y} c_*^T(\chi_{\mathrm{stalk}}(\mathrm{IC}(Y))) \in H_*^T(X).$$

Now let $X = G/B$ and $Y = X(w) \subseteq G/B$ a Schubert variety. The Riemann-Hilbert correspondence gives an equality $\mathrm{DR}(\mathcal{L}_w) = \mathrm{IC}(X(w))$, where $\mathcal{L}_w := \mathcal{D}_X \otimes_{U(\mathfrak{g})} L_w$ is the holonomic $\mathcal{D}_X$-module associated to L_w, the quotient of the Verma module M_w by its maximal proper submodule; see e.g., [KT84, Theorem 3] (or [HTT08, (12.2.13) and Corollary 12.3.3(ii)]). By the commutativity of diagram (24), $\mathrm{Char}(\mathcal{L}_w) = \mathrm{CC}(\chi_{stalk}(\mathrm{IC}(X(w))))$; then by Theorem 4.2 it follows that

$$\mathrm{KL}(X(w))^{\hbar} = (-1)^{\ell(w)} \iota^*[\mathrm{Char}(\mathcal{L}_w)].$$

By the proof of the Kazhdan-Lusztig conjectures [BB81, BK81] (see also [HTT08, Chapter 12]) we have

$$\mathrm{Char}(\mathcal{L}_w) = \sum_{u \leq w} (-1)^{\ell(w)-\ell(u)} P_{u,w}(1)\,\mathrm{Char}(\mathcal{M}_u)$$

where $P_{u,w}(q)$ is the Kazhdan-Lusztig polynomial. By Corollary 8.1, this proves:

Proposition 8.7. *The KL class of $X(w)$ is given by:*

$$\mathrm{KL}(X(w)) = \sum_{u \leq w} P_{u,w}(1)\, c_{\mathrm{SM}}^T(X(u)^\circ) \quad \in H_*^T(G/B).$$

Equivalently,

$$(-1)^{\ell(w)} \chi_{\mathrm{stalk}}(\mathrm{IC}(X(w))) = \sum_{u \leq w} P_{u,w}(1) \mathbb{1}_{X(u)^\circ}. \tag{39}$$

Corollary 8.8. *The non-equivariant KL class is strongly Schubert effective. That is, the coefficients $k(u;w)$ from the Schubert expansion*

$$\mathrm{KL}(X(w)) = \sum_{u \leq w} k(u;w)[X(u)],$$

satisfy $k(u;w) > 0$.

Proof. This follows from the positivity of CSM classes (Corollary 1.4), observing that $[X(u)]$ is the initial term of the CSM class $c_{\mathrm{SM}}(X(u)^\circ)$, combined with the fact that the KL polynomials have non-negative coefficients and $P_{u,w}(1) \geq 1$ (see e.g., [HTT08, Theorem 13.2.11] and [Hum90, §7.11]). $\square$

Remark 8.9. Proposition 8.7 and Corollary 8.8 hold more generally for Schubert varieties in partial flag manifolds G/P. Indeed, the projection $f : G/B \to G/P$ induces a pull-back $f^* : \mathcal{F}(G/P) \to \mathcal{F}(G/B)$ defined by $f^*(\varphi)(g.B) = \varphi(g.P)$. Since f is a smooth morphism,

$$\begin{aligned} f^*\chi_{\text{stalk}}(\mathrm{IC}(X(wW_P))) &= \chi_{\text{stalk}}(\mathrm{IC}(f^{-1}(X(wW_P))[\dim P/B]) \\ &= (-1)^{\dim P/B}\chi_{\text{stalk}}(\mathrm{IC}(X(ww_P))), \end{aligned}$$

where w_P is the longest element in W_P. One combines this with the fact that the parabolic Kazhdan-Lusztig polynomials coincide with those from G/B; see Deodhar's results [Deo87, Proposition 3.4 and Theorem 4.1]. Details are left to the reader. ⌟

For general flag manifolds G/P, there are situations in which the characteristic cycle of the IC sheaf $\mathrm{IC}(X(wW_P))$ is known to be irreducible, and hence equal to the conormal cycle $T^*_{X(wW_P)}(G/P)$. For example, this is the case for the IC sheaf of Schubert varieties in the ordinary Grassmannians, by a result of Bressler, Finkelberg, and Lunts [BFL90]; see [BF97] for generalizations. In this case, by Corollary 4.4,

$$\mathrm{KL}(X(wW_P))^\hbar = (-1)^{\ell(wW_P)}\iota^*[\mathrm{CC}(\chi_{\text{stalk}}(\mathrm{IC}(X(wW_P)))] = c^T_{\mathrm{Ma}}(X(wW_P))^\hbar.$$

In such cases, the following interesting equality holds:

$$P_{uW_P,wW_P}(1) = \mathrm{Eu}_{X(wW_P)}(p) \tag{40}$$

for $p \in X(uW_P)^\circ$, $uW_P \leq wW_P$. Here (recall) $\mathrm{Eu}_{X(wW_P)}$ is MacPherson's local Euler obstruction. Indeed, if the characteristic cycle of the IC sheaf is irreducible, then it must equal the conormal cycle, so this follows from (16) and the extension of (39) to G/P. The equality in (40) may also be deduced from the microlocal index formula ([Kas83, Theorem 6.3.1], [Dub84, Théorème 3]); see also [Sch03, Remark 5.0.4 on pp. 294-295] and [Sch05, Theorem 3.9]. We refer to [Jon10, BF97, MS20] for calculations of the local Euler obstruction for Schubert varieties in (cominuscule) Grassmannians.

9. CSM classes and stable envelopes

Stable envelopes were introduced by Maulik and Okounkov [MO19] in their study of symplectic resolutions, and in relation to integrable systems; see also the series of papers by Rimányi, Tarasov and Varchenko [RTV14, RTV15a, RTV15b].

Maulik and Okounkov [MO19] remark that in the case of $T^*(G/B)$, the stable envelopes are given by classes of certain conic Lagrangian cycles. We give an outline of the proof of this fact, by identifying them to characteristic cycles for Verma modules (up to sign); cf. Lemma 9.4. With this, we deduce that the pullback of the stable envelopes to the zero section G/B coincide with the CSM classes of the Schubert cells (Proposition 9.5). This allows us to create a fruitful dictionary between the stable envelopes theory and that of CSM classes, which we use to deduce a localization formula for the

CSM classes (Corollary 9.8) and a Chevalley formula (Theorem 9.10). We also generalize these results to the case of partial flag manifolds.

Remark 9.1. The dictionary between stable envelopes and characteristic classes may also be used to give another proof of the geometric orthogonality from Theorem 7.5. This proof was given in an initial version of this paper on the arχiv. ⌟

9.1. **Definition of stable envelopes.** We recall the definition of stable envelopes for $T^*(G/B)$; see [MO19, Chapter 3] or [Su17a] for more details.

The action of the torus T on G/B extends to one on $T^*(G/B)$. There is an additional dilation action by $\mathbb{C}^*$ on the fibers of $T^*(G/B)$ with a character χ, which we choose to be $\chi = \hbar^{-1}$, coinciding with the conventions in [MO19, Su17a]. Explicitly, $\mathbb{C}^*$ acts on the fibers of $T^*(G/B)$ by $z.(x,\xi) := (x, z^{-1}\xi)$, where $z \in \mathbb{C}^*$, $x \in G/B$ and $\xi \in T^*_x(G/B)$. The $T \times \mathbb{C}^*$ fixed points in $T^*(G/B)$ are $\{(e_w, 0)\}$ for $w \in W$. For any $w \in W$ and $\gamma \in H^*_{T\times\mathbb{C}^*}(T^*(G/B))$, we denote by $\gamma|_w$ the restriction of γ to the fixed point $(e_w, 0)$. As seen from Theorem 9.2 below, the definition of stable envelopes depends on the choices of the character χ and of a Weyl chamber in the Lie algebra of the maximal torus. We let $+$ denote the positive chamber determined by the Borel subgroup B, and $-$ denote the opposite chamber. Our choice for χ also coincides with that from Corollary 4.4 and Corollary 8.1 above, and it leads to natural identities between localizations of CSM classes and stable envelopes; cf. Proposition 9.5.

The $+$ version of stable envelopes is characterized by the following theorem.

Theorem 9.2 ([MO19, Su17a])**.** *There exist unique classes*

$$\{\mathrm{stab}_+(w) \in H^{2\dim G/B}_{T\times\mathbb{C}^*}(T^*(G/B)) \,|\, w \in W\}$$

which satisfy the following properties:

(1) $\mathrm{stab}_+(w)$ *is supported on* $\bigcup_{u\leq w} T^*_{X(u)}(G/B)$*, i.e.,* $\mathrm{stab}_+(w)|_u = 0$ *unless* $u \leq w$*;*
(2) $\mathrm{stab}_+(w)|_w = \prod_{\alpha>0, w\alpha<0} (w\alpha - \hbar) \prod_{\alpha>0, w\alpha>0} w\alpha$*;*
(3) $\mathrm{stab}_+(w)|_u$ *is divisible by* $\hbar$*, for any* $u < w$ *in the Bruhat order.*

Remark 9.3. (1) By the first two properties, the transition matrix between $\{\mathrm{stab}_+(w) \,|\, w \in W\}$ and the fixed point basis in the localized cohomology

$$H^*_{T\times\mathbb{C}^*}(T^*(G/B))_{\mathrm{loc}} := H^*_{T\times\mathbb{C}^*}(T^*(G/B)) \otimes_{H^*_{T\times\mathbb{C}^*}(\mathrm{pt})} \mathrm{Frac}\, H^*_{T\times\mathbb{C}^*}(\mathrm{pt})$$

is triangular with nontrivial diagonal terms. Hence the stable envelopes $\{\mathrm{stab}_+(w) \,|\, w \in W\}$ form a basis in $H^*_{T\times\mathbb{C}^*}(T^*(G/B))_{\mathrm{loc}}$, called the *stable basis* for $T^*(G/B)$.

(2) Similarly, there are stable envelopes

$$\{\mathrm{stab}_-(w) \in H_{T\times\mathbb{C}^*}(T^*(G/B)) \,|\, w \in W\}$$

for the negative chamber, satisfying the following analogous properties:
(a) $\mathrm{stab}_-(w)$ is supported on $\bigcup_{u\geq w} T^*_{Y(u)}(G/B)$;
(b) $\mathrm{stab}_-(w)|_w = \prod_{\alpha>0, w\alpha>0}(w\alpha-\hbar)\prod_{\alpha>0,w\alpha<0} w\alpha$; and
(c) $\mathrm{stab}_-(w)|_u$ is divisible by $\hbar$, for any $u>w$ in the Bruhat order. ⌟

The following was observed by Maulik and Okounkov [MO19, p. 69, Remark 3.5.3], but for completeness we include a sketch of the proof, using Corollary 8.1.

Lemma 9.4. *For any $w\in W$,*

$$[\mathrm{Char}(\mathcal{M}_w)] = (-1)^{\dim X-\ell(w)}\,\mathrm{stab}_+(w)\in H^*_{T\times\mathbb{C}^*}(T^*(G/B)).$$

Sketch of the proof. We need to check that conditions (1)–(3) in Theorem 9.2 are satisfied. The support condition (1) follows from the definition of the characteristic cycle. To check (2) and (3) we notice first that $\iota : X\to T^*(X)$ is $T\times\mathbb{C}^*$-equivariant, and that the fixed loci satisfy $(T^*(X))^{T\times\mathbb{C}^*} = X^T$, since $\mathbb{C}^*$ acts trivially on X. By Corollary 8.1, for every $u\leq w$ the localization of $[\mathrm{Char}(\mathcal{M}_w)]$ is given by

$$[\mathrm{Char}(\mathcal{M}_w)]|_u = (-1)^{\ell(w)} c^{T,\hbar}_{\mathrm{SM}}(X(w)^\circ)|_u.$$

The homogenized CSM class can be written as

$$c^{T,\hbar}_{\mathrm{SM}}(X(w)^\circ) = \sum_{u\leq w, 0\leq k}\hbar^k(c^T(u;w)[X(u)])_k$$

where $c^T(u;w)$ are polynomials in $H^*_T(\mathrm{pt})$ of degree $\leq \ell(u)$ (cf. [AM16, Proposition 6.5(a)]) and $(c^T(u;w)[X(u)])_k$ is the component of $c^T(u;w)[X(u)]$ in $H^T_{2k}(X)$. Since $wB\notin X(u)$ for $u<w$, the localization $[X(u)]|_w$ is equal to 0. It follows that the localization $\mathrm{Char}(\mathcal{M}_w)|_w$ equals the homogenization

$$\mathrm{Char}(\mathcal{M}_w)|_w = (-1)^{\ell(w)}c^{T,\hbar}_{\mathrm{SM}}(X(w)^\circ)|_w = (-1)^{\ell(w)}(c^T(w,w)[X(w)]|_w)^\hbar.$$

The coefficient $c(w;w)$ is calculated in Lemma 7.3, and the localization $[X(w)]|_w$ is the Euler class of the normal bundle of $X(w)$ at the smooth point w; see e.g., [Knu, §2] for a combinatorial formula for this. We leave it as an exercise to use these formulae in order to check the correct normalization from condition (2). (Similar results were also obtained by Rimányi and Varchenko [RV18] using Weber's localization formulae [Web12].)

Let $c_0 := c^T_{\mathrm{SM}}(X(w)^\circ)_{\deg 0}\in H^T_0(X)$ be the degree 0 part of the non-homogenized class $c^T_{\mathrm{SM}}(X(w)^\circ)$. To check condition (3) it suffices to show $(c_0)|_u = 0$ for any $u<w$. By [AM16, Proposition 6.5(d)] $c_0 = [e_w]$, the equivariant class of the T-fixed point w. Clearly $[e_w]|_u = 0$ for $u\neq w$ and this finishes the proof. □

9.2. **CSM classes, stable envelopes and localization formulae.** From Corollary 4.4 and Lemma 9.4, we obtain immediately the following formula; a different proof may be found in [RV18].

Proposition 9.5. *Let $w \in W$ be a Weyl group element. Then*

$$\iota^*(\mathrm{stab}_+(w)) = (-1)^{\dim X} c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ).$$

We can also relate the stable basis associated to the negative chamber $\{\mathrm{stab}_-(w) \mid w \in W\}$ and CSM classes associated to the opposite Schubert cells. This uses the extension of the automorphism $\varphi_{w_0}^*$ from Equation (29) to $\varphi_{w_0}^* : H^*_{T\times\mathbb{C}^*}(T^*(G/B)) \to H^*_{T\times\mathbb{C}^*}(T^*(G/B))$. The extended automorphism preserves $\hbar$ (since $\mathbb{C}^*$ acts trivially on G/B), and it acts on $H_T^*(\mathrm{pt})$ by w_0. The following follows immediately from the definition of the stable envelopes, see Remark 9.3(2).

Lemma 9.6. *The automorphism $\varphi_{w_0}^*$ satisfies*

$$\varphi_{w_0}^*(\mathrm{stab}_+(w)) = \mathrm{stab}_-(w_0 w).$$

We then obtain a parallel to Proposition 9.5. Denote by $c_{\mathrm{SM}}^{T,\hbar,\vee}(Y(w)^\circ)$ the $\hbar$-homogenization of the dual class $c_{\mathrm{SM}}^{T,\vee}(Y(w)^\circ)$.

Proposition 9.7. *The following equalities hold:*

(i) $\iota^*(\mathrm{stab}_-(w))|_{\hbar\mapsto-\hbar} = (-1)^{\ell(w)} c_{\mathrm{SM}}^{T,\hbar,\vee}(Y(w)^\circ)$;
(ii) $\iota^*(\mathrm{stab}_-(w)) = (-1)^{\dim X} c_{\mathrm{SM}}^{T,\hbar}(Y(w)^\circ)$.

Proof. This is a standard calculation, using Proposition 9.5 and Lemma 9.6. □

In [Su17a], the last-named author found localization formulae for the stable envelopes at any torus fixed point, in the process generalizing formulae of Anderson-Jantzen-Soergel/Billey [AJS94, Bil99] for the localization of Schubert classes. Proposition 9.7 implies similar localization formulae for the homogenized CSM classes. We record this next. Recall that α_i denote the simple roots for (G, B, T).

Corollary 9.8. *Fix $u, w \in W$ two elements such that $w \leq u$ in Bruhat ordering, and fix a reduced decomposition $u = s_{i_1} \cdot \ldots \cdot s_{i_\ell}$. Then the localization $c_{\mathrm{SM}}^{T,\hbar}(Y(w)^\circ)|_u$ equals*

$$c_{\mathrm{SM}}^{T,\hbar}(Y(w)^\circ)_{|u} = (-1)^{\dim G/B - \ell(u)} \prod_{\alpha\in R^+\setminus R(u)} (\alpha - \hbar) \sum \hbar^{\ell-k} \prod_{t=1}^{k} \beta_{j_t}, \tag{41}$$

where the sum is over all subwords $s_{i_{j_1}} s_{i_{j_2}} \ldots s_{i_{j_k}}$ of $u = s_{i_1} \ldots s_{i_\ell}$ such that $w = s_{i_{j_1}} s_{i_{j_2}} \ldots s_{i_{j_k}}$; for $1 \leq t \leq l$, $\beta_t := s_{i_1} s_{i_2} \ldots s_{i_{t-1}} \alpha_{i_t}$ with $\beta_1 = \alpha_{i_1}$; and $R(u) = \{\beta_i | 1 \leq i \leq \ell\}$.

Note that the set $R(u)$ coincides with the set of *inversions* of u^{-1}, i.e., the set of those positive roots α such that $u^{-1}(\alpha) < 0$; cf. [Hum90, p. 14]. Moreover, the sum in the equation (41) does not depend on the reduced expression for u, see [Su17a]. A similar formula for the localization of the CSM class $c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ)$ can be obtained by applying the automorphism $\varphi_{w_0}^*$ to (41) and using Equation (29).

9.3. **Partial flag manifolds.** In this section we generalize the above relation between CSM classes and stable envelopes in the case of partial flag manifolds. Our main result is Proposition 9.9. We also use this and a Chevalley formula for stable envelopes to deduce a Chevalley formula for CSM classes (Theorem 9.10).

There is an analogue of the Existence Theorem 9.2 which yields the set of stable envelopes $\{\mathrm{stab}^P_\pm(u) : u \in W^P\}$, with the $\pm$ sign denotes the positive/negative Weyl chamber. For each sign choice, the corresponding set forms a basis for the cohomology ring $H^*_{T\times\mathbb{C}^*}(T^*(G/P))$ localized at $H^*_{T\times\mathbb{C}^*}(\mathrm{pt})$. See e.g., [Su17a] for more details. We can consider diagram (24) for G/P; there are well defined push-forwards for each of the (associated Grothendieck) groups in this diagram, and as in §4.2 the given maps commute with (proper) push-forward. The next proposition states that the relation between stable envelopes and CSM classes extends to partial flag manifolds. Let $f : G/B \to G/P$ be the natural map.

Proposition 9.9. *Let $w \in W^P$ be a minimal length representative. The following hold:*

(a) *The constructible function associated to the direct image complex $f_*(\mathcal{M}_w)$ of the holonomic module $\mathcal{M}_w$ is $(-1)^{\ell(w)}1\!\!1_{X(wW_P)^\circ}$.*
(b) *There is an identity*

$$\iota^*(\mathrm{stab}^P_+(w)) = (-1)^{\dim(G/P)} c^{T,\hbar}_{\mathrm{SM}}(X(wW_P)^\circ).$$

Proof. Using that DR and χ_{stalk} commute with push-forward we obtain

$$f_*((\chi_{\mathrm{stalk}} \circ \mathrm{DR})\mathcal{M}_w) = (-1)^{\ell(w)} f_*(1\!\!1_{X(w)^\circ}).$$

Then (a) follows because for $w \in W^P$ the restriction $f : X(w)^\circ \to X(wW_P)^\circ$ is an isomorphism. Part (b) follows because

$$\mathrm{stab}^P_+(w) = (-1)^{\dim(G/P)-\ell(w)}[\mathrm{Char}(f_*(\mathcal{M}_w))],$$

(with a proof similar to that from Lemma 9.4), and from Corollary 4.4. $\square$

We now turn to the Chevalley formula. This formula gives the Schubert expansion of a product of a divisor Schubert class $[Y(s_\beta)]$ by another class $[Y(w)]$, in an appropriate cohomology ring of G/P; here $s_\beta \in W \setminus W_P$ is a simple reflection and $w \in W^P$. We refer to [FW04], in the non-equivariant setting, and e.g., to [BM15, §8] for the formula in the equivariant ring $H^*_T(G/P)$. For stable envelopes, a Chevalley formula was found by the last-named author in [Su16, Theorem 3.7]. Then Proposition 9.9 determines a formula to multiply the CSM class $c^{T,\hbar}_{\mathrm{SM}}(Y(w)^\circ)$ by any divisor class. We record the result next. Let ϖ_β be the fundamental weight corresponding to the simple root β, and let R^+_P denote the set of positive roots in P.

Theorem 9.10. *Let $w \in W^P$, and β be a simple root not in P. Then the following identity holds in $H^*_{T\times\mathbb{C}^*}(G/P)$:*

$$\begin{aligned}[Y(s_\beta)]_T \cup c^{T,\hbar}_{\mathrm{SM}}(Y(w)^\circ) =&(\varpi_\beta - w(\varpi_\beta))c^{T,\hbar}_{\mathrm{SM}}(Y(w)^\circ)\\ &+ \hbar \sum (\varpi_\beta, \alpha^\vee) c^{T,\hbar}_{\mathrm{SM}}(Y(ws_\alpha W_P)^\circ),\end{aligned}$$

where the sum is over roots $\alpha \in R^+ \setminus R^+_P$ such that $\ell(ws_\alpha W_P) > \ell(w)$, $\alpha^\vee$ is the coroot of α, and $(\cdot,\cdot)$ is the evaluation pairing.

The classical Chevalley formula Theorem can be deduced from Theorem 9.10 via a limiting process as follows. Write

$$c^{T,\hbar}_{\mathrm{SM}}(Y(w)^\circ) = \sum_{u \geq w} c^\hbar(u;w)[Y(u)]_{T\times\mathbb{C}^*},$$

where $u \in W^P$ and the coefficients $c^\hbar(u;w) \in H^{2(\dim G/P - \ell(u))}_{T\times\mathbb{C}^*}(\mathrm{pt})$. using this and Lemma 7.3, we deduce that

$$\lim_{\hbar\to\infty} \frac{c^{T,\hbar}_{\mathrm{SM}}(Y(w)^\circ)}{(\hbar)^{\dim G/P - \ell(w)}} = [Y(w)]_T.$$

For any root $\alpha \in R^+ \setminus R^+_P$, such that $\ell(ws_\alpha W_P) > \ell(w)$ we have

$$\lim_{\hbar\to\infty} \frac{\hbar c^{T,\hbar}_{\mathrm{SM}}(Y(ws_\alpha W_P)^\circ)}{(\hbar)^{\dim G/P - \ell(w)}} = [Y(ws_\alpha W_P)]_T$$

if and only if $\ell(ws_\alpha W_P) = \ell(w) + 1$. Otherwise, the limit is 0. Hence, if we divide both sides of the equation in Theorem 9.10 by $(\hbar)^{\dim G/P - \ell(w)}$, and let $\hbar$ go to ∞, we obtain

$$[Y(s_\beta)]_T \cup [Y(w)]_T = (\varpi_\beta - w(\varpi_\beta))[Y(w)]_T + \sum (\varpi_\beta, \alpha^\vee)[Y(ws_\alpha W_P)]_T,$$

where the sum is over those roots $\alpha \in R^+ \setminus R^+_P$ such that $\ell(ws_\alpha W_P) = \ell(w) + 1$. This is the classical Chevalley formula; see e.g., [BM15, Theorem 8.1].

10. Appendix: Non-characteristic pullback results

In this Appendix we explain how Corollary 3.5 allows us to extend a 'non-characteristic pullback formula' of [Sch17, Theorem 1.4, (3.12), (3.16)] for the (signed) Segre-MacPherson classes to the T-equivariant context. Let $f : X \to Y$ be a T-equivariant morphism of smooth complex algebraic varieties with a T-action. In order to recall the definition of *non-characteristic*, we need to introduce the following commutative diagram (whose right square is cartesian, see [Sch17, Diagram (2.10)]):

$$\begin{tikzcd} T^*(X) \arrow[d, "\pi_X"] & f^*(T^*(Y)) \arrow[l, "t"'] \arrow[r, "f'"] \arrow[d, "\pi'"] & T^*(Y) \arrow[d, "\pi_Y"] \\ X \arrow[r, equal] & X \arrow[r, "f"] & Y\,. \end{tikzcd}$$

Here f' is the map induced by base change, whereas t is the dual of the differential of f. Then f is by definition *non-characteristic* with respect to a closed conic subset $C \subseteq T^*(Y)$ (i.e., a closed algebraic subset stable under the $\mathbb{C}^*$-action given by multiplication on the fibers of the vector bundle $T^*(Y)$) if

$$f'^{-1}(C) \cap \ker(t) \subseteq \iota'(X)\,,$$

with $\iota' : X \to f^*(T^*(Y))$ the zero-section of the vector bundle $f^*(T^*(Y))$. By [Sch17, Lemma 3.2] this is %marginpar(84) equivalent to requiring that $t : f'^{-1}(C) \to T^*(X)$ is proper and therefore finite. If C is moreover T-stable, then C, $f'^{-1}(C)$ and $C' := t(f'^{-1}(C)) \subseteq T^*(X)$ are $\mathbb{T}$-stable for $\mathbb{T} = T \times \mathbb{C}^*$ resp., $\mathbb{T} = T$ and one gets an induced group homomorphism

$$t_* \circ f'^* : H_*^{\mathbb{T}}(C) \to H_*^{\mathbb{T}}(C')\,. \tag{42}$$

Here we use the $\mathbb{T}$-equivariant map $f' : f^*T^*(Y) \to T^*(Y)$ of ambient smooth complex algebraic varieties for the refined Gysin map

$$f'^* : H_*^{\mathbb{T}}(C) \to H_*^{\mathbb{T}}(f^{-1}(C)).$$

We will use the group homomorphism (42) for suitable characteristic cycles in the following theorem, which holds for both choices $\mathbb{T} = T \times \mathbb{C}^*$ and $\mathbb{T} = T$. The case $\mathbb{T} = T$ is used in Theorem 10.2 and Corollary 10.3, whereas the case $\mathbb{T} = T \times \mathbb{C}^*$ is used in Theorem 10.5.

Theorem 10.1. *Let $f : X \to Y$ be a T-equivariant morphism of smooth complex algebraic varieties of dimension $m = \dim X, n = \dim Y$. Assume that f is non-characteristic with respect to the support $C := \mathrm{supp}(\mathrm{CC}(\gamma)) \subseteq T^*(Y)$ of the characteristic cycle $\mathrm{CC}(\gamma)$ of a T-invariant constructible function $\gamma \in \mathcal{F}^T_{\mathrm{inv}}(Y)$. Then $C' := t(f'^{-1}(C))$ is pure m-dimensional, with*

$$t_* f'^*(\mathrm{CC}(\gamma)) = (-1)^{m-n} \cdot \mathrm{CC}(f^*(\gamma))\,.$$

In particular, the left hand side is a Lagrangian cycle in $T^*(X)$, i.e., belongs to $L_T(X)$.

Proof. Forgetting the T-action, this is [Sch17, Theorem 3.3]. If $\gamma \in \mathcal{F}^T_{\mathrm{inv}}(Y)$, then this is in fact an equality of $\mathbb{T}$-stable cycles for $\mathbb{T} = T \times \mathbb{C}^*$ resp., $\mathbb{T} = T$. $\square$

Consider now the corresponding commutative diagram of T-equivariant *projective completions*:

$$\begin{tikzcd}
\mathbb{P}(T^*X \oplus \mathbb{1}) \arrow[d, "\overline{\pi}_X"] & U \subseteq \mathbb{P}(f^*(T^*Y) \oplus \mathbb{1}) \arrow[l, "\overline{t}"'] \arrow[r, "\overline{f}"] \arrow[d, "\overline{\pi}'"] & \mathbb{P}(T^*Y \oplus \mathbb{1}) \arrow[d, "\overline{\pi}_Y"] \\
X \arrow[r, equal] & X \arrow[r, "f"] & Y.
\end{tikzcd}$$

The right square is cartesian, but the map $\bar{t}$ is only defined on the complement U of $\mathbb{P}(\ker(t)\oplus\{0\})$. For the application to Segre classes it is important to note that

$$\bar{t}^*(\mathscr{O}_{\mathbb{P}(T^*X\oplus\mathbf{1})}(-1)) \cong (\overline{f}^*(\mathscr{O}_{\mathbb{P}(T^*Y\oplus\mathbf{1})}(-1)))|U\,.$$

In the context of Theorem 10.1 one has $\overline{f}^{-1}(\overline{C}) \subseteq U$ and $\bar{t}\left(\overline{f}^{-1}(\overline{C})\right) = \overline{C'}$. Then a simple calculation gives

$$\begin{aligned} f^*\left(\mathrm{s}^T(\mathrm{CC}(\gamma))\right) &= \mathrm{s}^T\left(t_* f'^*(\mathrm{CC}(\gamma))\right) \\ &= (-1)^{m-n}\cdot \mathrm{s}^T\left(\mathrm{CC}(f^*(\gamma))\right) \in \hat{H}^T_*(X) \end{aligned} \tag{43}$$

exactly as in the non-equivariant context in [Sch17, (3.12)]. These classes correspond to *signed* equivariant Segre-MacPherson (SM) classes (as in (22)). In terms of the 'unsigned' classes (23), this proves the following result.

Theorem 10.2. *Let $f: X\to Y$ be a T-equivariant morphism of smooth complex algebraic varieties. Assume that f is non-characteristic with respect to the support* $\mathrm{supp}(\mathrm{CC}(\gamma)) \subseteq T^*(Y)$ *of the characteristic cycle* $\mathrm{CC}(\gamma)$ *of a T-invariant constructible function $\gamma \in \mathcal{F}^T_{\mathrm{inv}}(Y)$. Then*

$$f^*\left(s^T_{\mathrm{SM}}(\gamma)\right) = s^T_{\mathrm{SM}}(f^*(\gamma)) \in \hat{H}^T_*(X)\,.$$

The equivariant intersection formula used in this paper, Theorem 3.6, is an equivariant version of [Sch17, Theorem 1.2]. It may be proved by applying Theorem 10.2 to the diagonal inclusion $d: X\to X\times X$ of a smooth complex algebraic variety X with a T-action. Then d is T-equivariant for the diagonal T-action on $X\times X$. Let $\alpha,\beta\in\mathcal{F}^T_{\mathrm{inv}}(X)$. Then $\alpha\cdot\beta = d^*(\alpha\boxtimes\beta)$ for $\alpha\boxtimes\beta\in\mathcal{F}^T_{\mathrm{inv}}(X\times X)$ defined by $\alpha\boxtimes\beta(x,x') := \alpha(x)\cdot\beta(x')$. Moreover

$$\mathrm{CC}(\alpha\boxtimes\beta) = \mathrm{CC}(\alpha)\boxtimes\mathrm{CC}(\beta)\,,$$

since $\mathrm{Eu}_{Z\times Z'} = \mathrm{Eu}_Z\boxtimes\mathrm{Eu}_{Z'}$ [Mac74, Eq. 3 on p. 426]. Similarly,

$$c^T_*(\alpha\boxtimes\beta) = c^T_*(\alpha)\boxtimes c^T_*(\beta)\,, \tag{44}$$

e.g., by the corresponding multiplicativity of the T-equivariant Chern-Mather classes defined via the T-equivariant Nash blow-up [Ohm06, §4.3]. This yields:

Corollary 10.3. *Assume the T-equivariant diagonal embedding $d: X\to X\times X$ of a smooth complex algebraic variety X is non-characteristic with respect to the support* $\mathrm{supp}(\mathrm{CC}(\alpha\boxtimes\beta))$ *of the characteristic cycle* $\mathrm{CC}(\alpha\boxtimes\beta)$ *for two T-invariant constructible functions $\alpha,\beta\in\mathcal{F}^T_{\mathrm{inv}}(X)$. Then*

$$c^T_*(\alpha\cdot\beta) = c^T_*(\alpha)\cdot s^T_{\mathrm{SM}}(\beta) \in H^T_*(X) \subseteq \hat{H}^T_*(X)\,.$$

If X is moreover compact, then

$$\begin{aligned} \langle c^T_*(\alpha), s^T_{\mathrm{SM}}(\beta)\rangle &:= \int_X c^T_*(\alpha)\cdot s^T_{\mathrm{SM}}(\beta) \\ &= \int_X c^T_*(\alpha\cdot\beta) = \int_X c^T_0(\alpha\cdot\beta) = \chi(X;\alpha\cdot\beta). \end{aligned}$$

Remark 10.4. Let $\alpha \in \mathcal{F}^T_{\mathrm{inv}}(X)$, resp., $\beta \in \mathcal{F}^T_{\mathrm{inv}}(X)$ be constructible with respect to algebraic Whitney stratifications $\mathcal{S} := \{S \subseteq X\}$, resp., $\mathcal{S}' := \{S' \subseteq X\}$ of X, i.e., $\alpha|S$ and $\beta|S'$ are constant for all strata $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$. Assume that all strata $S \in \mathcal{S}$ are *transversal* to all strata $S' \in \mathcal{S}'$, i.e., for all $x \in S \cap S'$ one has the equality $T_x(S) + T_x(S') = T_x(X)$. This is also equivalent to requiring that the diagonal $d(X)$ is transversal in $X \times X$ to all product strata $S \boxtimes S'$. Then d is non-characteristic with respect to the support $\mathrm{supp}(\mathrm{CC}(\alpha \boxtimes \beta))$ of the characteristic cycle $\mathrm{CC}(\alpha \boxtimes \beta)$ (see [Sch17]). $\lrcorner$

As a final application we get the following intersection formula for characteristic cycles fitting with the corresponding orthogonality relation for stable envelopes (and see [Sch17, Corollary 3.5] for the corresponding non-equivariant result).

Theorem 10.5. *Let X be a compact smooh complex algebraic variety with a T-action and $\mathbb{C}^*$ acting on the fibers of $T^*(X)$ by the character $\hbar^{-1}$. Assume that the diagonal embedding $d : X \to X \times X$ is non-characteristic with respect to* $\mathrm{supp}(\mathrm{CC}(\alpha \times \beta))$ *for some given $\alpha, \beta \in \mathcal{F}^T_{\mathrm{inv}}(X)$. Then $\mathrm{CC}(\alpha) \cdot \mathrm{CC}(\beta)$ is supported on the zero-section $\iota(X) \subseteq T^*(X)$, with*

$$\begin{aligned}\langle [\mathrm{CC}(\alpha)]_{T\times\mathbb{C}^*}, [\mathrm{CC}(\beta)]_{T\times\mathbb{C}^*}\rangle &:= \int_{T^*(X)} [\mathrm{CC}(\alpha)]_{T\times\mathbb{C}^*} \cdot [\mathrm{CC}(\beta)]_{T\times\mathbb{C}^*} \\ &= (-1)^{\dim X} \cdot \chi(X; \alpha \cdot \beta)\,.\end{aligned}$$

Proof. Consider the cartesian diagram

$$\begin{tikzcd} T^*(X) \arrow[r, "(\mathrm{id},a)"] \arrow[d, "\pi"'] & T^*(X) \times_X T^*(X) \arrow[r, "d'"] \arrow[d, "t"] & T^*(X \times X) \\ X \arrow[r, "\iota"'] & T^*(X)\,, & \end{tikzcd}$$

with $\iota : X \to T^*(X)$ the zero-section, $a : T^*(X) \to T^*(X)$ the antipodal map and $f : X \to pt$ the proper constant map. By the non-characteristic assumption,

$$t : d'^{-1}(\mathrm{supp}(\mathrm{CC}(\alpha) \times \mathrm{CC}(\beta))) \to T^*(X)$$

is proper. Then, by base change, the restriction of π

$$\mathrm{supp}(\mathrm{CC}(\alpha)) \cap \mathrm{supp}(a_* \mathrm{CC}(\beta)) \subseteq (\mathrm{id}, a)^{-1} d'^{-1}(\mathrm{supp}(\mathrm{CC}(\alpha) \times \mathrm{CC}(\beta))) \xrightarrow{\pi} X$$

is proper. Since $a_* \mathrm{CC}(\beta) = \mathrm{CC}(\beta)$, the $\mathbb{C}^*$-stable subset

$$\mathrm{supp}(\mathrm{CC}(\alpha)) \cap \mathrm{supp}(\mathrm{CC}(\beta))$$

has to be contained in the zero-section $\iota(X) \subseteq T^*(X)$. By Theorem 10.1 and base-change one gets:

$$\begin{aligned}\langle [\mathrm{CC}(\alpha)]_{T\times\mathbb{C}^*}, [\mathrm{CC}(\beta)]_{T\times\mathbb{C}^*}\rangle &= f_*\pi_*(\mathrm{id},a)^* d'^* \left([\mathrm{CC}(\alpha)]_{T\times\mathbb{C}^*} \boxtimes [\mathrm{CC}(\beta)]_{T\times\mathbb{C}^*}\right)\\ &= f_*\pi_*(\mathrm{id},a)^* d'^* [\mathrm{CC}(\alpha\boxtimes\beta)]_{T\times\mathbb{C}^*}\\ &= f_*\iota^* t_* d'^* [\mathrm{CC}(\alpha\boxtimes\beta)]_{T\times\mathbb{C}^*}\\ &= (-1)^{\dim X}\cdot f_*\iota^*[\mathrm{CC}(\alpha\cdot\beta)]_{T\times\mathbb{C}^*}\,.\end{aligned}$$

Since

$$\iota^*[\mathrm{CC}(\alpha\cdot\beta)]_{T\times\mathbb{C}^*} = c_*^T(\alpha\boxtimes\beta)^\hbar$$

by Theorem 4.2, and

$$f_*\iota^*[\mathrm{CC}(\alpha\cdot\beta)]_{T\times\mathbb{C}^*} = \int_X c_*^T(\alpha\boxtimes\beta)^\hbar = \chi(X;\alpha\cdot\beta).$$

by functoriality. $\square$

References

[AF23] David Anderson and William Fulton. *Equivariant Cohomology in Algebraic Geometry*. Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, Cambridge, 2023.

[AJS94] Henning Haahr Andersen, Jens Carsten Jantzen, and Wolfgang Soergel. Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p: independence of p. *Astérisque*, 220, 1994.

[Alu03] Paolo Aluffi. Inclusion-exclusion and Segre classes. *Comm. Algebra*, 31(8):3619–3630, 2003. Special issue in honor of Steven L. Kleiman.

[Alu04] Paolo Aluffi. Shadows of blow-up algebras. *Tohoku Math. J. (2)*, 56(4):593–619, 2004.

[Alu06] Paolo Aluffi. Classes de Chern des variétés singulières, revisitées. *C. R. Math. Acad. Sci. Paris*, 342(6):405–410, 2006.

[AM09] Paolo Aluffi and Leonardo Constantin Mihalcea. Chern classes of Schubert cells and varieties. *J. Algebraic Geom.*, 18(1):63–100, 2009.

[AM16] Paolo Aluffi and Leonardo Constantin Mihalcea. Chern–Schwartz–MacPherson classes for Schubert cells in flag manifolds. *Compos. Math.*, 152(12):2603–2625, 2016.

[AMSS19] Paolo Aluffi, Leonardo C. Mihalcea, Jörg Schürmann, and Changjian Su. Motivic Chern classes of Schubert cells, Hecke algebras, and applications to Casselman's problem. *to appear in Ann. Sci. Éc. Norm. Supér., arXiv preprint arXiv:1902.10101*, 2019.

[AMSS22] Paolo Aluffi, Leonardo C. Mihalcea, Jörg Schürmann, and Changjian Su. Positivity of Segre-MacPherson classes. In *Facets of Algebraic Geometry. A Collection in Honor of William Fulton's 80th Birthday. Vol. I*, volume 472 of *London Math. Soc. Lecture Note Ser.*, pages 1–28. Cambridge Univ. Press, Cambridge, 2022.

[And12] Dave Anderson. Introduction to equivariant cohomology in algebraic geometry. In *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., pages 71–92. Eur. Math. Soc., Zürich, 2012.

[BB81] Alexandre Beĭlinson and Joseph Bernstein. Localisation de g-modules. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(1):15–18, 1981.

[BF97] Brian D. Boe and Joseph H. G. Fu. Characteristic cycles in Hermitian symmetric spaces. *Canad. J. Math.*, 49(3):417–467, 1997.

[BFL90] P. Bressler, M. Finkelberg, and V. Lunts. Vanishing cycles on Grassmannians. *Duke Math. J.*, 61(3):763–777, 1990.

[BGG73] I. N. Bernsteĭn, I. M. Gelfand, and S. I. Gelfand. Schubert cells, and the cohomology of the spaces G/P. *Uspehi Mat. Nauk*, 28(3(171)):3–26, 1973.

[Bil99] Sara C. Billey. Kostant polynomials and the cohomology ring for G/B. *Duke Math. J.*, 96(1):205–224, 1999.

[BK81] J.-L. Brylinski and M. Kashiwara. Kazhdan-Lusztig conjecture and holonomic systems. *Invent. Math.*, 64(3):387–410, 1981.

[BM15] Anders S. Buch and Leonardo C. Mihalcea. Curve neighborhoods of Schubert varieties. *J. Differential Geom.*, 99(2):255–283, 2015.

[Bri05] Michel Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, Trends Math., pages 33–85. Birkhäuser, Basel, 2005.

[BS81] Jean-Paul Brasselet and Marie-Hélène Schwartz. Sur les classes de Chern d'un ensemble analytique complexe. In *The Euler-Poincaré characteristic (French)*, volume 83 of *Astérisque*, pages 93–147. Soc. Math. France, Paris, 1981.

[CG97] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry.* Birkhäuser Boston, Inc., Boston, MA, 1997.

[Deo87] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan-Lusztig polynomials. *J. Algebra*, 111(2):483–506, 1987.

[Dub84] Alberto S. Dubson. Formule pour l'indice des complexes constructibles et des Modules holonomes. *C. R. Acad. Sci. Paris Sér. I Math.*, 298(6):113–116, 1984.

[EG98] Dan Edidin and William Graham. Equivariant intersection theory. *Invent. Math.*, 131(3):595–634, 1998.

[EG05] Dan Edidin and William Graham. Nonabelian localization in equivariant K-theory and Riemann-Roch for quotients. *Adv. Math.*, 198(2):547–582, 2005.

[FK99] Sergey Fomin and Anatol N. Kirillov. Quadratic algebras, Dunkl elements, and Schubert calculus. In *Advances in geometry*, volume 172 of *Progr. Math.*, pages 147–182. Birkhäuser Boston, Boston, MA, 1999.

[FR18] L. Fehér and R. Rimányi. Chern-Schwartz-MacPherson classes of degeneracy loci. *Geom. Topol.*, 22(6):3575–3622, 2018.

[Ful84] William Fulton. *Intersection theory.* Springer-Verlag, Berlin, 1984.

[Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1993.

[FW04] W. Fulton and C. Woodward. On the quantum product of Schubert classes. *J. of Alg. Geom.*, 13(4):641–661, 2004.

[Gin86] Victor Ginsburg. Characteristic varieties and vanishing cycles. *Invent. Math.*, 84(2):327–402, 1986.

[Gin98] Victor Ginzburg. Geometric methods in the representation theory of Hecke algebras and quantum groups. In *Representation theories and algebraic geometry (Montreal, PQ, 1997)*, volume 514 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 127–183. Kluwer Acad. Publ., Dordrecht, 1998.

[Gra01] William Graham. Positivity in equivariant Schubert calculus. *Duke Math. J.*, 109(3):599–614, 2001.

[GRTV13] V. Gorbounov, R. Rimányi, V. Tarasov, and A. Varchenko. Quantum cohomology of the cotangent bundle of a flag variety as a Yangian Bethe algebra. *J. Geom. Phys.*, 74:56–86, 2013.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory*, volume 236 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 2008.

[Huh16] June Huh. Positivity of Chern classes of Schubert cells and varieties. *J. Algebraic Geom.*, 25(1):177–199, 2016.

[Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[Jon10] Benjamin F. Jones. Singular Chern classes of Schubert varieties via small resolution. *Int. Math. Res. Not. IMRN*, (8):1371–1416, 2010.

[Kas83] Masaki Kashiwara. *Systems of microdifferential equations*, volume 34 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1983.

[Ken90] Gary Kennedy. MacPherson's Chern classes of singular algebraic varieties. *Comm. Algebra*, 18(9):2821–2839, 1990.

[Knu] Allen Knutson. A Schubert Calculus recurrence from the noncomplex W-action on G/B. arXiv:math/0306304v1.

[KT84] M. Kashiwara and T. Tanisaki. The characteristic cycles of holonomic systems on a flag manifold related to the Weyl group algebra. *Invent. Math.*, 77(1):185–198, 1984.

[KT96] Steven Kleiman and Anders Thorup. Mixed Buchsbaum-Rim multiplicities. *Amer. J. Math.*, 118(3):529–569, 1996.

[Kum02] Shrawan Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2002.

[Lee18] Seung Jin Lee. Chern class of Schubert cells in the flag manifold and related algebras. *J. Algebraic Combin.*, 47(2):213–231, 2018.

[LLT96] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. Twisted action of the symmetric group on the cohomology of a flag manifold. In *Parameter spaces (Warsaw, 1994)*, volume 36 of *Banach Center Publ.*, pages 111–124. Polish Acad. Sci. Inst. Math., Warsaw, 1996.

[LLT97] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. Flag varieties and the Yang-Baxter equation. *Lett. Math. Phys.*, 40(1):75–90, 1997.

[Lus85] George Lusztig. Equivariant K-theory and representations of Hecke algebras. *Proc. Amer. Math. Soc.*, 94(2):337–342, 1985.

[Mac74] R. D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.

[Mih15] Leonardo Constantin Mihalcea. Binomial determinants and positivity of Chern-Schwartz-MacPherson classes. *Australas. J. Combin.*, 62:155–171, 2015.

[MNS22] Leonardo C. Mihalcea, Hiroshi Naruse, and Changjian Su. Left Demazure-Lusztig operators on equivariant (quantum) cohomology and K-theory. *Int. Math. Res. Not. IMRN*, (16):12096–12147, 2022.

[MO19] Davesh Maulik and Andrei Okounkov. Quantum groups and quantum cohomology. *Astérisque*, (408):ix+209, 2019.

[MS15] Laurenţiu G. Maxim and Jörg Schürmann. Characteristic classes of singular toric varieties. *Comm. Pure Appl. Math.*, 68(12):2177–2236, 2015.

[MS20] L. C. Mihalcea and R. Singh. Mather classes and conormal spaces of Schubert varieties in cominuscule spaces. *available on arχiv:2006.04842*, 2020.

[Ohm06] Toru Ohmoto. Equivariant Chern classes of singular algebraic varieties with group actions. *Math. Proc. Cambridge Philos. Soc.*, 140(1):115–134, 2006.

[Ohm12] Toru Ohmoto. A note on the Chern-Schwartz-MacPherson class. In *Singularities in geometry and topology*, volume 20 of *IRMA Lect. Math. Theor. Phys.*, pages 117–131. Eur. Math. Soc., Zürich, 2012.

[PP01] Adam Parusiński and Piotr Pragacz. Characteristic classes of hypersurfaces and characteristic cycles. *J. Algebraic Geom.*, 10(1):63–79, 2001.

[Ric92] R. W. Richardson. Intersections of double cosets in algebraic groups. *Indag. Math. (N.S.)*, 3(1):69–77, 1992.

[RTV14] R. Rimányi, V. Tarasov, and A. Varchenko. Cohomology classes of conormal bundles of Schubert varieties and Yangian weight functions. *Math. Z.*, 277(3-4):1085–1104, 2014.

[RTV15a] R. Rimányi, V. Tarasov, and A. Varchenko. Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety. *J. Geom. Phys.*, 94:81–119, 2015.

[RTV15b] Richárd Rimányi, Vitaly Tarasov, and Alexander Varchenko. Partial flag varieties, stable envelopes, and weight functions. *Quantum Topol.*, 6(2):333–364, 2015.

[RV18] Richárd Rimányi and Alexander Varchenko. Equivariant Chern-Schwartz-MacPherson classes in partial flag varieties: interpolation and formulae. In *Schubert varieties, equivariant cohomology and characteristic classes—IMPANGA 15*, EMS Ser. Congr. Rep., pages 225–235. Eur. Math. Soc., Zürich, 2018.

[Sab85] Claude Sabbah. Quelques remarques sur la géométrie des espaces conormaux. *Astérisque*, (130):161–192, 1985.

[Sch65a] Marie-Hélène Schwartz. Classes caractéristiques définies par une stratification d'une variété analytique complexe. I. *C. R. Acad. Sci. Paris*, 260:3262–3264, 1965.

[Sch65b] Marie-Hélène Schwartz. Classes caractéristiques définies par une stratification d'une variété analytique complexe. II. *C. R. Acad. Sci. Paris*, 260:3535–3537, 1965.

[Sch03] Jörg Schürmann. *Topology of singular spaces and constructible sheaves*, volume 63 of *Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)]*. Birkhäuser Verlag, Basel, 2003.

[Sch05] Jörg Schürmann. Lectures on characteristic classes of constructible functions. In *Topics in cohomological studies of algebraic varieties*, Trends Math., pages 175–201. Birkhäuser, Basel, 2005. Notes by Piotr Pragacz and Andrzej Weber.

[Sch17] Jörg Schürmann. Chern classes and transversality for singular spaces. In *Singularities in Geometry, Topology, Foliations and Dynamics*, Trends in Mathematics, pages 207–231. Birkhäuser, Basel, 2017.

[Str11] Judson P. Stryker, III. *Chern-Schwartz-MacPherson classes of graph hypersurfaces and Schubert varieties*. 2011. Ph.D. Dissertation, The Florida State University.

[Su16] Changjian Su. Equivariant quantum cohomology of cotangent bundle of G/P. *Adv. Math.*, 289:362–383, 2016.

[Su17a] Changjian Su. Restriction formula for stable basis of the Springer resolution. *Selecta Math. (N.S.)*, 23(1):497–518, 2017.

[Su17b] Changjian Su. *Stable Basis and Quantum Cohomology of Cotangent Bundles of Flag Varieties*. ProQuest LLC, Ann Arbor, MI, 2017. Thesis (Ph.D.)–Columbia University.

[SZZ20] Changjian Su, Gufang Zhao, and Changlong Zhong. On the K-theory stable bases of the Springer resolution. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(3):663–711, 2020.

[Web12] Andrzej Weber. Equivariant Chern classes and localization theorem. *J. Singul.*, 5:153–176, 2012.

Mathematics Department, Florida State University, Tallahassee FL 32306 USA
E-mail address: aluffi@math.fsu.edu

Department of Mathematics, Virginia Tech University, Blacksburg, VA 24061 USA
E-mail address: lmihalce@vt.edu

Mathematisches Institut, Universität Münster, Germany
E-mail address: jschuerm@uni-muenster.de

Yau Mathematical Sciences Center, Tsinghua University, Beijing, China
E-mail address: changjiansu@mail.tsinghua.edu.cn