LORENTZIAN POLYNOMIALS, SEGRE CLASSES, AND ADJOINT POLYNOMIALS OF CONVEX POLYHEDRAL CONES

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Abstract. We consider polynomials expressing the cohomology classes of subvarieties of products of projective spaces, and limits of positive real multiples of such polynomials. We study the relation between these *covolume polynomials* and Lorentzian polynomials. While these are distinct notions, we prove that, like Lorentzian polynomials, covolume polynomials have M-convex support and generalize the notion of log-concave sequences. In fact, we prove that covolume polynomials are 'sectional log-concave', that is, the coefficients of suitable restrictions of these polynomials form log-concave sequences.

We observe that Chern classes of globally generated bundles give rise to covolume polynomials, and use this fact to prove that certain polynomials associated with *Segre classes* of subschemes of products of projective spaces are covolume polynomials. We conjecture that the same polynomials may be Lorentzian after a standard normalization operation.

Finally, we obtain a combinatorial application of a particular case of our Segre class result. We prove that the *adjoint polynomial* of a convex polyhedral cone contained in the nonnegative orthant, and sharing a face with it, is a covolume polynomials. This implies that these adjoint polynomials are M-convex and sectional log-concave, and in fact Lorentzian after a suitable change of variables.

1. Introduction

This paper consists of three parts. In the first part we consider 'covolume polynomials', that is, limits of positive multiples of polynomials whose coefficients are multidegrees of (irreducible) subvarieties of products of projective spaces. We are interested in studying covolume polynomials from the point of view of *Lorentzian* polynomials. Petter Brändén and June Huh defined and extensively studied Lorentzian polynomials in [BH20]. An equivalent notion, *completely log-concave* polynomials, was introduced at the same time by Nima Anari, Kuikui Liu, Shayan Oveis Gharan and Cynthia Vinzant ([ALGV18, AGV21]). Lorentzian polynomials generalize the notion of (ultra) log-concavity, in the sense that a homogeneous polynomial in two variables is Lorentzian if and only if its coefficients are nonnegative and form an ultra log-concave sequence with no internal zeros. In fact, the restriction of every Lorentzian polynomial to a plane spanned by two nonnegative vectors satisfies this ultra-log-concave property. Brändén and Huh prove that 'volume polynomials' of projective varieties are necessarily Lorentzian ([BH20, Theorem 4.6]). The covolume polynomials studied in this note (or, rather, their normalizations in the sense of Brändén and Huh, see [BH20, Corollary 3.7]) are in a sense a dual notion to volume polynomials. While they are not necessarily Lorentzian, we prove that they share several properties of note with Lorentzian polynomials. Some of these properties are due to their connection with volume polynomials; others are a consequence of Olivier Debarre's beautiful extension of the Fulton-Hansen connectedness theorem ([Deb96]).

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Covolume polynomials provide an alternative generalization of log-concavity, which we call 'sectional log-concavity' (Definition 2.5). Briefly, a homogeneous polynomial is sectional log-concave if and only if the coefficients of its restrictions to planes spanned by vectors with nonnegative components form a log-concave sequence with no internal zeros. (In the same sense, Lorentzian polynomials are 'sectional ultra-log-concave'.) Among other properties, we prove (Corollary 2.15):

Theorem I. *Covolume polynomials are sectional log-concave.*

In particular, the normalization of a covolume polynomial in two variables is necessarily Lorentzian. In higher dimension, the relation between covolume polynomials and Lorentzian polynomials is subtler. Examples show that covolume polynomials are not necessarily Lorentzian, even after normalization (Example 2.7), and Lorentzian polynomials are not necessarily covolume polynomials (Example 2.10). To each covolume polynomial there is associated a Lorentzian polynomial (Proposition 2.8), and it follows that the support of a covolume polynomial is M-convex in the sense of discrete convex analysis [Mur03], i.e., a 'polymatroid' (Corollary 2.11). In analogy with the behavior of Lorentzian polynomials, we prove that if $f(\underline{t})$ is a covolume polynomial, then so is $f(A\underline{t})$ for any matrix A with nonnegative entries (Theorem 2.12). Sectional log-concavity is a corollary of this result, which has other convenient consequences, such as the fact that the product of two covolume polynomials is a covolume polynomials. Lorentzian polynomials are similarly preserved by nonnegative changes of variables, by [BH20, Theorem 2.10].

The second part of the paper deals with *Segre zeta functions* of closed subschemes of products of projective spaces. Segre classes are a key ingredient in Fulton-MacPherson intersection theory. In previous work ([Alu17]) we constructed a univariate rational function $\zeta_I(t)$ encoding the information of the Segre class defined in projective spaces of arbitrarily large dimension by a set I of homogeneous polynomials. In this paper we extend the construction to subschemes of products of projective spaces; if Z is a closed subscheme of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$ defined by a set I of multihomogeneous polynomials, the corresponding 'Segre zeta function' $\zeta_I(t_1, \dots, t_\ell)$ of Z may be defined as a power series in ℓ variables. We point out (Theorem 3.2) that it is a rational function, with poles controlled by the multidegrees of elements of I. Thus, once a generating set is chosen, the interesting part of the information of a Segre zeta function is its numerator. Constraints on the numerators of Segre zeta functions yield nontrivial information on Segre classes, and this is our motivation in studying them. Our main result in this part is:

Theorem II. *With notation as above, the homogenization of the numerator of* $1 - \zeta_I$ *is a covolume polynomial. If the projective normal cone of* Z *is irreducible, then the homogenization of the numerator of* ζ_I *is a covolume polynomial.*

See Theorems 3.5 and 3.6 for more complete statements. This implies that e.g., the numerator of $1-\zeta_I$ is sectional log-concave and M-convex in the natural sense. In particular, the coefficients of the numerator of $1 - \zeta_I(t)$ in the univariate case (that is, the case of subschemes of $\mathbb{P}^n$ studied in [Alu17]) necessarily form a log-concave sequence of nonnegative integers with no internal zeros.

Analogous consequences hold for the numerator of ζ_I, under an irreducibility hypothesis on the normal cone. Some such hypothesis is necessary: examples show that the numerator of the Segre zeta function is *not* M-convex or sectional log-concave in general (Example 3.12), implying that its homogenization is not necessarily a covolume or Lorentzian polynomial. Whether the irreducibility hypothesis stated in Theorem II can be significantly weakened is an interesting question. Brandon Story ([Sto23]) has verified that in the univariate case, i.e., for $\ell = 1$, the coefficients of the numerator of the Segre zeta function form a log-concave sequence without internal zeros for several families of subschemes $Z \subseteq \mathbb{P}^{n_1}$, including many cases of reducible subschemes.

The situation with $1 - \zeta_I$, whose numerator is a covolume polynomial by Theorem II, is also intriguing. Experimental evidence suggests the following.

Conjecture. *Let Z be a closed subscheme of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$, and I a generating set for its ideal. Then the normalization of the homogenization of the numerator of $1 - \zeta_I(t_0, \ldots, t_\ell)$ is Lorentzian.*

Since there are covolume polynomials whose normalization is not Lorentzian, this conjecture does not follow formally from our result; if true, it appears to be a novel and unexpected phenomenon. The reader could compare this conjecture with Conjecture 15 in [HMMS22], to the effect that Schubert polynomials are expected to have Lorentzian normalization. Schubert polynomials also are covolume polynomials (Example 2.4); this fact alone does not explain why their normalizations should be Lorentzian.

This may be a good place to quote Karim Adiprasito, June Huh, and Eric Katz ([AHK17]): *"We believe that behind any log-concave sequence that appears in nature there is... a 'Hodge structure' responsible for the log-concavity."* We wonder whether the (sectional) log-concavity of the numerator of $1 - \zeta_I$ may signal the presence of a new structure in the sense meant in this reference. These polynomials (and their normalizations) are not directly expressed as volume polynomials of projective varieties.

Theorem II follows from a general statement that may have different applications. Let X be an irreducible variety, $q : X \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$ a proper map, and let $\mathscr{R}$ be a globally generated vector bundle on X. We can write

$$q_*(c(\mathscr{R}) \cap [X]) = \sum_{0 \le i_j \le n_j} a_{i_1 \ldots i_\ell} h_1^{i_1} \cdots h_\ell^{i_\ell} \cap [\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}] \,,$$

where h_j is the pull-back of the hyperplane class from the j-th factor of the product. Let

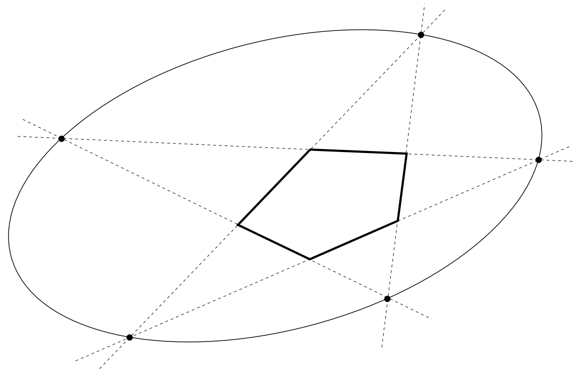
$$P(t_1, \ldots, t_\ell) = \sum_{i_k \le s_k} a_{i_1 \ldots i_\ell} t_1^{i_1} \cdots t_\ell^{i_\ell} \,.$$

Proposition. *The homogenization of $P(t_1, \ldots, t_\ell)$ is a covolume polynomial.*

(See Proposition 3.9.) In particular, this polynomial must be M-convex and sectional log-concave. For example, the degrees of the Chern classes of a globally generated vector bundle over projective space must form a log-concave sequence with no internal zeros. For remarks implying the same conclusion, see [BEST21, §1.3(B), §9], drawing on [Laz04, §1.6].

While our main motivation in establishing Theorem II is intrinsic to the theory of Segre classes, we offer a concrete combinatorial application of this result in the particular case of *monomial* schemes: in the third part of the paper we consider *adjoint polynomials* of convex polytopes (or, equivalently, convex polyhedral cones). These polynomials were introduced by Joe Warren ([War96]); the case of polygons had been considered earlier by Eugene Wachspress ([Wac75]). Adjoint polynomials are used in the definition of "Wachspress coordinates" of convex polytopes; we refer the reader to [KR20] for a discussion of the context and recent progress in the study of Wachspress coordinates. We also note that adjoint polynomials appear as numerators of canonical forms of certain 'positive geometries' introduced in the study of scattering amplitudes. This is nicely explained in unpublished notes by Christian Gaetz [Gae20]; see [KPR$^+$21] for recent work on adjoints from this perspective.

The adjoint curve of a polygon is the curve of minimal degree containing the points of intersection of pairs of lines extending non-adjacent edges of the polygon:



an analogous description holds in any dimension. In [KR20], Kathlén Kohn and Kristian Ranestad observe that the numerator of the Segre zeta function of a monomial scheme may be interpreted as an adjoint polynomial of a related polytope. We sharpen this result in Proposition 4.5, by proving that the adjoint polynomial of any convex polyhedral cone contained in the nonnegative orthant and containing a face of the nonnegative orthant is the numerator of $1-\zeta_I$ for a suitable choice of I. Together with Theorem II, this implies the following.

Theorem III. *Adjoint polynomials of convex polyhedral cones contained in the nonnegative orthant and sharing a face with it are covolume polynomials.*

See Theorem 4.3 for a more complete statement. In particular, these adjoint polynomials are necessarily M-convex and sectional log-concave

It is conceivable that this result may extend to all convex polyhedral cones contained in the nonnegative orthant. We prove that Theorem III does extend to a certain class of cones which we call 'orthantal'; see Definition 4.9 and Corollary 4.10.

It is also conceivable that adjoint polynomials of all convex polyedral cones contained in the nonnegative orthant are actually Lorentzian after normalization. The conjecture stated above would imply this fact for orthantal cones.

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2. Covolume polynomials

We work over an algebraically closed field k. *Varieties* are assumed to be irreducible, but as their irreducibility is key to the main objects studied in this paper, we occasionally remind the reader of this fact.

We consider (irreducible) subvarieties of products of projective spaces $\mathbb{P} := \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^{n_\ell}$ and their classes in the Chow ring of $\mathbb{P}$. We will let h_j denote the pull-back of the hyperplane class from the j-th factor. Every class in the codimension-d graded piece $A^d(\mathbb{P})$ of the Chow ring of $\mathbb{P}$ may be written uniquely as

$$\alpha = \sum_{\sum i_j = d} a_{i_0 \dots i_\ell}\, h_0^{i_0} \cdots h_\ell^{i_\ell} \cap [\mathbb{P}]\,,$$

where $i_j \geq 0$ and $a_{i_0\dots i_\ell}$ are nonnegative integers. For any choice of indeterminates $t_0, \dots, t_\ell$, we associate with α the polynomial

$$P_\alpha(t_0, \dots, t_\ell) := \sum_{0 \leq i_j, \sum i_j = d} a_{i_0\dots i_\ell}\, t_0^{i_0} \cdots t_\ell^{i_\ell}\,.$$

In other words, P_α is the (unique) polynomial with integer coefficients and of degree $\leq n_j$ in the j-th variable such that

$$\alpha = P_\alpha(h_0, \dots, h_\ell) \in A(\mathbb{P}) \cong \mathbb{Z}[h_0, \dots, h_\ell]/(h_0^{n_0+1}, \dots, h_\ell^{n_\ell+1})\,.$$

Definition 2.1. A polynomial P with nonnegative real coefficients is a *covolume* polynomial if it is a limit of polynomials of the form $cP_{[W]}$ for a positive real number c and a closed subvariety W of a product of projective spaces. ⌟

Remark 2.2. If W is a subvariety of $\mathbb{P} := \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^{n_\ell}$ with class $[W] = P(h_0, \dots, h_\ell) \cap [\mathbb{P}]$, then for all $m_j \geq n_j$, the 'cone' W' over W in $\mathbb{P}' := \mathbb{P}^{m_0} \times \cdots \times \mathbb{P}^{m_\ell}$ is a subvariety with class $[W'] = P(h'_0, \dots, h'_\ell) \cap [\mathbb{P}']$, where h'_j denotes the pull-back of the hyperplane class from the j-th factor of $\mathbb{P}'$.

Thus, if $P = P_{[W]}$ with $W \subseteq \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^{n_\ell}$, then we may in fact assume all $n_j \gg 0$, or even $n_0 = \cdots = n_\ell \gg 0$. ⌟

Example 2.3. Constant polynomials (including 0) trivially are covolume polynomials. Linear polynomials with nonnegative real coefficients are covolume polynomials. Indeed, by continuity we can assume that the coefficients are positive and rational, and up to a multiple we may then assume the coefficients to be integers. For positive integers $a_0, \dots, a_\ell$, general sections of $\mathscr{O}(a_0h_0 + \cdots + a_\ell h_\ell)$ on $(\mathbb{P}^n)^{\ell+1}$ are irreducible for $n \geq 2$ and any $\ell \geq 0$. The corresponding polynomial is $a_0t_0 + \cdots + a_\ell t_\ell$. ⌟

Example 2.4. For a nontrivial example, we note that the Schubert polynomial $\mathfrak{S}_w(t_0, \dots, t_\ell)$ associated with a permutation $w \in \mathcal{S}_{\ell+1}$ is a covolume polynomial. This follows from the proof of [HMMS22, Theorem 6]. ⌟

We are interested in log-concavity properties of covolume polynomials. We recall that a sequence $a_0, \dots, a_n$ of nonnegative real numbers is *log-concave* if $\forall i$, $a_i^2 \geq a_{i-1}a_{i+1}$. Further, the sequence 'has no internal zeros' if $\forall i \leq j \leq k$, $a_ia_k \neq 0 \implies a_j \neq 0$. We say that a polynomial is log-concave if its coefficients form a log-concave sequence with no internal zeros. Powerful generalizations of this notion to polynomials in more variables have been considered in the literature: among these *strongly* log-concave polynomials ([Gur09]), *completely* log-concave polynomials ([ALGV18]), *Lorentzian* polynomials ([BH20]). These notions are equivalent for homogeneous polynomials, as proved in [BH20, Theorem 2.30]. We will consistently use [BH20] as a reference for facts concerning this notion, and refer to it by the term *Lorentzian.*

We will establish that covolume polynomials share certain key properties with Lorentzian polynomials, and may also be viewed as a generalization of log-concave polynomials. Again, we say that a homogeneous bivariate polynomial

$$P(u, v) := \sum_{i+j=d} a_{ij}u^iv^j \in \mathbb{R}[u, v]$$

is *log-concave* if and only if its coefficients $a_{d0}, \dots, a_{0d}$ form a nonzero log-concave sequence of nonnegative real numbers with no internal zeros.

Definition 2.5. For $\ell \geq 1$, a homogeneous polynomial $P(\underline{t}) \in \mathbb{R}[t_0, \dots, t_\ell]$ is *sectional log-concave* if for all $(\ell+1) \times 2$ matrices A with nonnegative real entries, the polynomial $P(A\binom{u}{v})$ is log-concave or identically 0. ⌟

A bivariate homogeneous polynomial is sectional log-concave if it is log-concave (cf. Remark 2.17), and in general a homogeneous polynomial is sectional log-concave if all its restrictions to 'nonnegative' lower dimensional subspaces are sectional log-concave. Lorentzian polynomials are sectional log-concave; in fact, if $P(\underline{t})$ is Lorentzian and A is as above, then $P(A\binom{u}{v})$ is *ultra*-log-concave as a consequence of [BH20, Theorem 2.10 and Example 2.26]. One of our main goals in this section is to prove that covolume polynomials are sectional log-concave.

The following characterization of two-variable covolume polynomials is a simple corollary of another result of June Huh ([Huh12, Theorem 21]). Following [BH20], we consider the *normalization* operator on the polynomial ring, defined on monomials by $N(t_0^{i_0} \cdots t_\ell^{i_\ell}) := t_0^{i_0} \cdots t_\ell^{i_\ell}/i_0! \cdots i_\ell!$. By [BH20, Corollary 3.7], the normalization of a Lorentzian polynomial is Lorentzian.

Lemma 2.6. *A nonzero homogeneous bivariate polynomial*

$$P(u,v) := \sum_{i+j=d} a_{ij} u^i v^j \in \mathbb{R}[u,v]$$

is a covolume polynomial if and only if it is log-concave, that is, its coefficients $a_{d0}, \dots, a_{0d}$ *form a nonzero log-concave sequence of nonnegative real numbers with no internal zeros.*

Therefore, a homogeneous polynomial $P(u,v) \in \mathbb{R}[u,v]$ *is a covolume polynomial if and only if its normalization is Lorentzian.*

Proof. Assume that $P(u,v) \in \mathbb{R}[u,v]$ is a covolume polynomial. By continuity, it suffices to verify the statement when the coefficients of P are rational. Therefore, we may assume that there exists a positive rational c and an irreducible subvariety W of codimension d in $\mathbb{P}^n \times \mathbb{P}^n$, with $n \gg 0$ (cf. Remark 2.2), such that

$$[W] = \sum_{i+j=d} c a_{ij} h^i k^j \cap [\mathbb{P}^n \times \mathbb{P}^n]\,.$$

Here h, resp. k denote the pull-backs of the hyperplane classes from the first, resp. second factor. Thus,

$$[W] = \sum_{i+j=d} c a_{ij} [\mathbb{P}^{n-i} \times \mathbb{P}^{n-j}] = \sum_{i+j=\dim W} c a_{n-i,n-j} [\mathbb{P}^i \times \mathbb{P}^j]$$

is the class of an irreducible subvariety of $\mathbb{P}^n \times \mathbb{P}^n$. By [Huh12, Theorem 21], the coefficients $c a_{n-i,n-j}$ form a nonzero log-concave sequence with no internal zeros, and it follows that the same holds for $a_{d0}, \dots, a_{0d}$.

Conversely, assume that $a_{d0}, \dots, a_{0d}$ form a nonzero log-concave sequence of nonnegative real numbers with no internal zeros. Such sequences are limits of log-concave sequences of nonnegative rational numbers with no internal zeros, so we may assume $a_{d0}, \dots, a_{0d}$ are rational. Clearing denominators, there exists $c \in \mathbb{Q}_{>0}$ such that the coefficients $c a_{ij}$ are integers; so $c a_{d0}, \dots, c a_{0d}$ form a log-concave sequence of nonnegative integers with no internal zeros. Again by [Huh12, Theorem 21], a positive integer multiple of

$$\sum_{i+j=d} c a_{ij} [\mathbb{P}^j \times \mathbb{P}^i] = \sum_{i+j=d} c a_{ij} h^i k^j \cap [\mathbb{P}^d \times \mathbb{P}^d]$$

is the class of an irreducible subvariety of $\mathbb{P}^d \times \mathbb{P}^d$, and it follows that $P(u, v)$ is a covolume polynomial.

The coefficients of the polynomial $\sum_{i+j=d} a_{ij} u^i v^j$ form a log-concave sequence with no internal zeros if and only if the coefficients of the normalization $\sum_{i+j=d} a_{ij} u^i v^j / i! j!$ form an *ultra* log-concave sequence with no internal zeros, if and only if the normalization is Lorentzian, cf. [BH20, Example 2.26]. □

Lemma 2.6 raises the natural question of whether the normalization of a covolume polynomial may be Lorentzian in general. This is not the case.

Example 2.7. Let W be the image of the embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7 \times \mathbb{P}^3 \times \mathbb{P}^1$$

obtained from the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$, the Segre embedding of the first two factors $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, and the identity $\mathbb{P}^1 \to \mathbb{P}^1$ on the third factor. Let

$$[W] = \sum_{i_0+i_1+i_2=9} a_{i_0 i_1 i_2} h_0^{i_0} h_1^{i_1} h_2^{i_2} \cap [\mathbb{P}^7 \times \mathbb{P}^3 \times \mathbb{P}^1] \,.$$

Denoting by k_i the pull-back of the hyperplane classes from the i-th factor of the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we have

$$\begin{aligned} a_{i_0 i_1 i_2} &= \int h_0^{7-i_0} h_1^{3-i_1} h_2^{1-i_2} \cap [W] \\ &= \text{coeff. of } k_0 k_1 k_2 \text{ in } (k_0 + k_1 + k_2)^{7-i_0} (k_0 + k_1)^{3-i_1} k_2^{1-i_2} \,. \end{aligned}$$

It follows that

$$P_{[W]}(t_0, t_1, t_2) = 2t_0^7 t_1 + 2t_0^6 t_1^2 + 2t_0^6 t_1 t_2 + 2t_0^5 t_1^3 + 4t_0^5 t_1^2 t_2 + 6t_0^4 t_1^3 t_2$$

is a covolume polynomial. The normalization of $P_{[W]}$ is

$$N(P_{[W]}) = \frac{1}{2520} t_0^7 t_1 + \frac{1}{720} t_0^6 t_1^2 + \frac{1}{360} t_0^6 t_1 t_2 + \frac{1}{360} t_0^5 t_1^3 + \frac{1}{60} t_0^5 t_1^2 t_2 + \frac{1}{24} t_0^4 t_1^3 t_2 \,,$$

and this polynomial is *not* Lorentzian. Indeed,

$$\frac{\partial^5}{\partial t_0^5} \frac{\partial^2}{\partial t_1^2} N(P_{[W]}) = t_0^2 + 2t_0 t_1 + 2t_0 t_2 + t_1^2 + 4t_1 t_2 \,,$$

with Hessian

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 4 \\ 2 & 4 & 0 \end{pmatrix} .$$

This matrix has *two* positive eigenvalues, contrary to the requirement for Lorentzianity (cf. [BH20, §2.4]). ⌟

On the other hand, the normalization of a simple transformation of a covolume polynomial *is* Lorentzian.

Proposition 2.8. *Let* $P(t_0, \ldots, t_\ell)$ *be a covolume polynomial. Let* $n_j \gg 0$, $j = 0, \ldots, \ell$, *be any integers such that*

$$Q(u_0, \ldots, u_\ell) = u_0^{n_0} \cdots u_\ell^{n_\ell} P\left(\frac{1}{u_0}, \ldots, \frac{1}{u_\ell}\right)$$

is a polynomial. Then $N(Q)$ *is Lorentzian.*

In fact, we will prove that, possibly up to inessential factors, $N(Q)$ is a *volume polynomial* in the sense of [BH20, §4.2].

Proof. For all j, $N(Q)$ is Lorentzian if $N(u_jQ)$ is Lorentzian; indeed, the former is the derivative of the latter with respect to u_j, and derivatives of Lorentzian polynomials are Lorentzian by definition. Therefore, it suffices to show that the normalization of

$$Q(u_0,\dots,u_\ell)=u_0^d\cdots u_\ell^d P\left(\frac{1}{u_0},\dots,\frac{1}{u_\ell}\right)$$

is Lorentzian, where $d=\deg P$. By continuity and up to a constant multiple we may then assume that $P=P_{[W]}$ is the covolume polynomial associated with the class of an irreducible subvariety W of a product of projective spaces; and by the same argument used above, it suffices to show that there exist $n_0\ge d,\dots,n_\ell\ge d$ such that

$$Q_{[W]}(u_0,\dots,u_\ell)=u_0^{n_0}\cdots u_\ell^{n_\ell} P_{[W]}\left(\frac{1}{u_0},\dots,\frac{1}{u_\ell}\right)$$

has Lorentzian normalization. By Remark 2.2, we may choose $W\subseteq \mathbb{P}^{n_0}\times\cdots\times\mathbb{P}^{n_\ell}$ with $n_j\gg 0$. Then

$$P_{[W]}=\sum_{i_0+\cdots+i_\ell=d} a_{i_0\dots i_\ell} t_0^{i_0}\cdots t_\ell^{i_\ell}$$

where $0\le i_j\le n_j$ for all j and

$$[W]=\sum_{i_0+\cdots+i_\ell=d} a_{i_0\dots i_\ell} h_0^{i_0}\cdots h_\ell^{i_\ell}\cap[\mathbb{P}^{n_0}\times\cdots\times\mathbb{P}^{n_\ell}]$$

with the usual notation. We have

$$a_{i_0\dots i_\ell}=\int h_0^{n_0-i_0}\cdots h_\ell^{n_\ell-i_\ell}\cap[W]\,.$$

Therefore

$$\begin{aligned}
N(Q_{[W]}(u_0,\dots,u_\ell)) &= N\left(u_0^{n_0}\cdots u_\ell^{n_\ell}P_{[W]}\left(\frac{1}{u_0},\dots,\frac{1}{u_\ell}\right)\right)\\
&= \sum_{i_0+\cdots+i_\ell=d} a_{i_0\dots i_\ell}\frac{u_0^{n_0-i_0}}{(n_0-i_0)!}\cdots\frac{u_\ell^{n_\ell-i_\ell}}{(n_\ell-i_\ell)!}\\
&= \sum_{j_0+\cdots+j_\ell=\dim W} a_{n_0-j_0\dots n_\ell-j_\ell}\frac{u_0^{j_0}}{j_0!}\cdots\frac{u_\ell^{j_\ell}}{j_\ell!}\\
&= \sum_{j_0+\cdots+j_\ell=\dim W}\left(\int h_0^{j_0}\cdots h_\ell^{j_\ell}\cap[W]\right)\frac{u_0^{j_0}}{j_0!}\cdots\frac{u_\ell^{j_\ell}}{j_\ell!}\\
&= \frac{1}{(\dim W)!}\int\sum_{j_0+\cdots+j_\ell=\dim W}\binom{\dim W}{j_0\dots j_\ell}(u_0h_0)^{j_0}\cdots(u_\ell h_\ell)^{j_\ell}\cap[W]\\
&= \frac{1}{(\dim W)!}\int(u_0h_0+\cdots+u_\ell h_\ell)^{\dim W}\cap[W]\,.
\end{aligned}$$

This shows that, up to a scalar factor, $N(Q_{[W]})$ is a volume polynomial. It follows that $N(Q_{[W]})$ is Lorentzian by [BH20, Theorem 4.6], and this concludes the argument. □

Remark 2.9. A sharp version of this argument is given for the case of Schubert polynomials (cf. Example 2.4) in [HMMS22, Theorem 6]. ⌟

Example 2.10. As we verified in Example 2.7, the polynomial

$$P(t_0,t_1,t_2) = 2t_0^7t_1 + 2t_0^6t_1^2 + 2t_0^6t_1t_2 + 2t_0^5t_1^3 + 4t_0^5t_1^2t_2 + 6t_0^4t_1^3t_2$$

is a covolume polynomial. For $n_0 = 7, n_1 = 3, n_2 = 1$, we have

$$\begin{aligned} Q(u_0,u_1,u_2) &= u_0^7u_1^3u_2P\left(\frac{1}{u_0},\frac{1}{u_1},\frac{1}{u_2}\right) \\ &= 6u_0^3 + 4u_0^2u_1 + 2u_0^2u_2 + 2u_0u_1^2 + 2u_0u_1u_2 + 2u_1^2u_2\,. \end{aligned}$$

The normalization of this polynomial,

$$N(Q) = u_0^3 + 2u_0^2u_1 + u_0^2u_2 + u_0u_1^2 + 2u_0u_1u_2 + u_1^2u_2\,,$$

is Lorentzian as prescribed by Proposition 2.8.

This also shows that a polynomial whose normalization is Lorentzian is not necessarily a covolume polynomial. Indeed, if Q were a covolume polynomial, then by Proposition 2.8 it would follow that $N(P)$ is Lorentzian, and we have verified that this is not the case in Example 2.7.

The same argument shows that *Lorentzian* polynomials are not necessarily covolume polynomials. Indeed, the polynomial

$$A(u_0,u_1,u_2,u_3) = u_0^2u_1 + u_0^2u_2 + u_0^2u_3 + u_0u_1u_2 + u_0u_1u_3 + 4u_0u_2u_3 + u_1u_2u_3$$

is Lorentzian, but the normalization of

$$t_0^2t_1t_2t_3\cdot A\left(\frac{1}{t_0},\frac{1}{t_1},\frac{1}{t_2},\frac{1}{t_3}\right) = t_2t_3 + t_1t_3 + t_1t_2 + t_0t_3 + t_0t_2 + 4t_0t_1 + t_0^2$$

is not Lorentzian. ⌟

By Proposition 2.8, every covolume polynomial may be expressed in terms of a Lorentzian polynomial: if $P(t_0,\dots,t_\ell)$ is a covolume polynomial, then there exist nonnegative integers $n_0,\dots,n_\ell$ such that

$$P(t_0,\dots,t_\ell) = t_0^{n_0}\cdots t_\ell^{n_\ell}Q\left(\frac{1}{t_0},\dots,\frac{1}{t_\ell}\right) \tag{2.1}$$

where Q is a polynomial whose normalization is Lorentzian. The result that follows is a consequence of this observation.

Recall that a subset $S \subseteq \mathbb{N}^{\ell+1}$ is 'M-convex' if for all i and all $\alpha,\beta \in S$ such that $\alpha_i > \beta_i$, there exists j such that $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in S$, $\beta - e_j + e_i \in S$, where e_i is that i-th standard unit vector. (See [Mur03], [BH20, §2].) This (symmetric) *exchange property* generalizes the exchange property defining matroids; M-convex sets are *generalized polymatroids.*

Corollary 2.11. *The support of a covolume polynomial is an M-convex set.*

Proof. Let

$$P = \sum_{i_0+\cdots+i_\ell=d} a_{i_0\dots i_\ell}t_0^{i_0}\cdots t_\ell^{i_\ell}$$

be a covolume polynomial. The corresponding polynomial Q as in Proposition 2.8 or (2.1) is given by

$$Q = \sum_{j_0+\cdots+j_\ell=\sum_k n_k - d} a_{n_0-j_0\dots n_\ell-j_\ell}u_0^{j_0}\cdots u_\ell^{j_\ell}\,.$$

As this polynomial is Lorentzian, its support, that is, the set

$$\{(j_0,\dots,j_\ell)\,|\,a_{n_0-j_0\dots n_\ell-j_\ell}\neq 0\}$$

is M-convex ([BH20, Definition 2.6]). It is then straightforward to check that the support of P, that is,

$$\{(i_0,\dots,i_\ell)\,|\,a_{i_0\dots i_\ell}\neq 0\}$$

also satisfies the symmetric exchange property and is therefore M-convex. □

Lemma 2.6 and Corollary 2.11 indicate that while covolume polynomials (or even their normalizations) are not necessarily Lorentzian, they share some key properties with Lorentzian polynomials. The result that follows is possibly the most useful such analogue. Recall from [BH20, Theorem 2.10] that if $f(\underline{w})$ is a Lorentzian polynomial, then so is $f(A\underline{v})$ for any matrix A with nonnegative entries. Covolume polynomials are similarly preserved by nonnegative coordinate changes.

Theorem 2.12. *Let $P(\underline{t})\in\mathbb{R}[t_0,\dots,t_\ell]$ be a covolume polynomial, and let A be an $(\ell+1)\times(m+1)$ matrix with nonnegative real entries. Then $P(A\underline{u})\in\mathbb{R}[u_0,\dots,u_{m+1}]$ is a covolume polynomial.*

For example, the 'dilation' (replacing a variable by a nonnegative constant multiple of the same variable) and 'diagonalization' (setting two variables equal to each other) operators preserve the property of being a covolume polynomial. The following consequence is the analogue of [BH20, Corollary 2.32].

Corollary 2.13. *The product of two covolume polynomials is a covolume polynomial.*

Proof. By continuity and multiplication by scalar multiples, we are reduced to showing that $P_{[W']}(\underline{t})P_{[W'']}(\underline{t})$ is a covolume polynomial, where W' and W'' are irreducible subvarieties of $\mathbb{P}:=\mathbb{P}^{n_0}\times\cdots\times\mathbb{P}^{n_\ell}$. For this, note that $W'\times W''$ is an irreducible subvariety of $\mathbb{P}\times\mathbb{P}$; therefore, $P_{[W']}(\underline{t})P_{[W'']}(\underline{u})=P_{[W'\times W'']}(\underline{t},\underline{u})$ is a covolume polynomial, and by diagonalization (that is, by Theorem 2.12), so is $P_{[W']}(\underline{t})P_{[W'']}(\underline{t})$. □

Proof of Theorem 2.12. By continuity, we may assume that $P(\underline{t})$ has rational coefficients and A has positive rational entries; and up to a scalar multiple we can then assume that A has positive integer entries and $P(\underline{t})=P_{[W]}(\underline{t})$ is the polynomial associated with an irreducible subvariety W of $\mathbb{P}:=(\mathbb{P}^M)^{\ell+1}$, with $M\gg 0$. Explicitly,

$$P(\underline{t})=\sum_{f_0+\cdots+f_\ell=d}\beta_{f_0\dots f_\ell}t_0^{f_0}\cdots t_\ell^{f_\ell}$$

where

$$[W]=\sum_{f_0+\cdots+f_\ell=d}\beta_{f_0\dots f_\ell}h_0^{f_0}\cdots h_\ell^{f_\ell}\cap[(\mathbb{P}^M)^{\ell+1}]$$

and as usual h_i denotes the pull-back of the hyperplane class from the corresponding factor of $(\mathbb{P}^M)^{\ell+1}$.

The codimension of W is the degree d of $P_{[W]}$. We choose an integer $S>d$, implying in particular $(m+1)S>d$; and we may assume (cf. Remark 2.2) that M is sufficiently large to allow us to define Segre-Veronese embeddings

$$\sigma_i:\quad (\mathbb{P}^S)^{m+1}\to\mathbb{P}^M$$

for $i=0,\dots,\ell$, such that

$$\sigma_i^*(h)=a_{i0}k_0+\cdots+a_{im}k_m$$

where h is the hyperplane class in $\mathbb{P}^M$ and $k_0,\dots,k_m$ are the pull-backs of the hyperplane classes from the factors. These embeddings determine an embedding

$$\varphi:\quad (\mathbb{P}^S)^{m+1}\hookrightarrow(\mathbb{P}^M)^{\ell+1}$$

such that for $i = 0, \dots, \ell$,

$$\varphi^*(h_i) = a_{i0}k_0 + \cdots + a_{im}k_m\,.$$

Claim 2.14. *For a general $\gamma \in \mathrm{PGL}(M+1)^{\ell+1}$, the inverse image $\varphi^{-1}({}^\gamma W)$ of the γ-translate of W is irreducible.*

This claim follows from a result of Olivier Debarre, [Deb96, Théorème 2.2, 2)a))]. To verify this, let

$$[\varphi(\mathbb{P}^S)^{m+1}] = \sum_{e_0,\dots,e_\ell} \alpha_{e_0\dots e_\ell} h_0^{e_0}\cdots h_\ell^{e_\ell} \cap [(\mathbb{P}^M)^{\ell+1}]$$

where the sum is over all nonnegative $e_0, \dots, e_\ell$ such that $\sum_i (M - e_i) = (m+1)S$. The coefficients $\alpha_{e_0\dots e_\ell}$ equal the intersection numbers

$$\begin{aligned}\alpha_{e_0\dots e_\ell} &= \int h_0^{M-e_0}\cdots h_\ell^{M-e_\ell} \cap [\varphi(\mathbb{P}^S)^{m+1}] \\ &= \int \prod_{i=0}^{\ell} (a_{i1}k_1 + \cdots + a_{im}k_m)^{M-e_i} \cap [(\mathbb{P}^S)^{m+1}]\end{aligned}$$

and in particular $\alpha_{e_0\dots e_\ell} \neq 0$ for all nonnegative $e_0, \dots, e_\ell$ such that $\sum_i (M-e_i) = (m+1)S$ since the entries of A are assumed to be positive.

By Debarre's theorem, $\varphi^{-1}({}^\gamma W)$ is irreducible for a general γ if for all nonempty subsets $I \subseteq \{0, \dots, \ell\}$ there exist $\underline{e}$, $\underline{f}$ such that

$$\alpha_{e_0\dots e_\ell} \neq 0, \quad \beta_{f_0\dots f_\ell} \neq 0, \quad \text{and} \quad \sum_{i\in I} (e_i + f_i) < |I|M\,.$$

As observed, $\alpha_{e_0\dots e_\ell} \neq 0$ for all $\underline{e}$ such that $\sum_i (M-e_i) = (m+1)S$. As W has codimension d, there exist $\underline{f}$ with $\sum_j f_j = d$ such that $\beta_{f_0\dots f_\ell} \neq 0$; note that for all $I \subseteq \{0, \dots, \ell\}$, $\sum_{i\in I} f_i \le d$. Given a nonempty $I \subseteq \{0, \dots, \ell\}$, let $e_i = M$ for $i \notin I$ and choose any $e_i \ge 0$ for $i \in I$ s.t. $\sum_{i\in I} e_i = |I|M - (m+1)S$. (We can do this because I is not empty and $M \ge (m+1)S$.) Then we have

$$\sum_{i\in I} (e_i + f_i) \le |I|M - (m+1)S + d < |I|M$$

as we are assuming $(m+1)S > d$.

Thus the hypothesis of Debarre's theorem is satisfied and Claim 2.14 follows. ⌟

The class of $\varphi^{-1}({}^\gamma W)$ is

$$[\varphi^{-1}({}^\gamma W)] = \varphi^*([W]) = P_{[W]}(a_{00}k_0 + \cdots + a_{0m}k_m, \dots, a_{\ell 0}k_0 + \cdots + a_{\ell m}k_m)\,.$$

The conclusion is that

$$P_{[W]}(A\underline{u}) = P_{[\varphi^{-1}({}^\gamma W)]}(\underline{u})$$

is the polynomial associated with the irreducible $\varphi^{-1}({}^\gamma W)$, hence a covolume polynomial, as needed. □

Corollary 2.15. *Covolume polynomials are sectional log-concave.*

Proof. Given Definition 2.5, this is now an immediate consequence of Theorem 2.12 and Lemma 2.6. □

Example 2.16. (Cf. Example 2.7.) The polynomial

$$P(t_0, t_1, t_2) := 2t_0^7t_1^2 + 2t_0^6t_1^3 + 2t_0^6t_1^2t_2 + 2t_0^5t_1^4 + 4t_0^5t_1^3t_2 + 6t_0^4t_1^4t_2$$

as well as its normalization $N(P)$ are not Lorentzian, but P *is* sectional log-concave, as it is a covolume polynomial. ⌟

Remark 2.17. Let $P(u,v)$ be a homogeneous log-concave polynomial[1]. By Lemma 2.6, P is a covolume polynomial; by Theorem 2.12, $P(A\binom{u}{v})$ is a covolume polynomial for all 2×2 invertible matrices A with nonnegative entries; and then $P(A\binom{u}{v})$ must be a log-concave polynomial, again by Lemma 2.6. This particular case of Theorem 2.12 implies that log-concave polynomials are indeed sectional log-concave, as claimed earlier in this section.

One way to view this fact is as follows: for bivariate homogeneous polynomials, the property of having Lorentzian normalization is preserved by nonnegative coordinate changes. We don't know if this is also the case for homogeneous polynomials in more variables. It does not appear to be a direct consequence of the fact that the property of being Lorentzian *is* preserved by nonnegative coordinate changes (i.e., [BH20, Theorem 2.10]).

The fact that the class of log-concave polynomials is preserved by nonnegative changes of coordinates is easily seen to be equivalent to the assertion that if $f(t)$ is a (non-homogenous) log-concave univariate polynomial, then so is $f(t+1)$. For alternative arguments proving this fact, see [Hog74, Theorem 2] and Corollary 8.4 in the survey [Bre94]. ⌟

3. Segre classes

We work over an algebraically closed field k; *schemes* are assumed to be of finite type. The Segre class $s(Z,Y)$ of a closed embedding $Z\subseteq Y$ of schemes is a class in the Chow group $A_*(Z)$ recording important intersection-theoretic information about the embedding. For a thorough treatment of Segre classes, the reader is addressed to [Ful84, Chapter 4]. By definition, the Segre class $s(Z,Y)$ is the Segre class of the cone C_ZY; in particular, if $Z\subseteq Y$ is a regular embedding, then $s(Z,Y)=c(N_ZY)^{-1}\cap[Z]$ is the inverse Chern class of the normal bundle of Z in Y. Segre classes are preserved by birational morphisms ([Ful84, Proposition 4.2(a)]). These two properties characterize Segre classes: the birational invariance reduces the computation of $s(Z,Y)$ to the computation of $s(E,\widetilde{Y})$, where E is the exceptional divisor in the blow-up $\widetilde{Y}$ of Y along Z, and since E is regularly embedded,

$$s(E,\widetilde{Y})=c(N_E\widetilde{Y})^{-1}\cap[E]=\frac{[E]}{1+E}\,,$$

where

$$\frac{1}{1+E}=1-E+E^2-E^3+\cdots\,.$$

(Cf. [Ful84, Corollary 4.2.2].)

Segre classes are a key ingredient in the definition of the intersection product in Fulton-MacPherson intersection theory ([Ful84, Proposition 6.1(a)]), with direct applications to enumerative geometry. They can also be used to express important classical invariants such as multiplicity, Milnor numbers, local Euler obstructions, etc. For a survey of the role of Segre classes in the theory of singularities, see [Alu22].

Let Z be a closed subscheme of projective space $\mathbb{P}^n$, determined by a homogeneous ideal I of $k[x_0,\dots,x_n]$. For $N\geq n$, any generating set of I, viewed in $k[x_0,\dots,x_N]$, defines a subscheme Z_N of $\mathbb{P}^N$, which we may view as a cone over $Z=Z_n$. Let $\iota_N: Z_N\to\mathbb{P}^N$ be the embedding. We consider the push-forwards $\iota_{N*}s(Z,\mathbb{P}^N)$ of the corresponding Segre

[1] Recall that by this we mean that the coefficients of $P(u,v)$ form a log-concave sequence *with no internal zeros.*

classes. In previous work we have observed that these classes are organized by a power series

$$\zeta_I(t) := \sum_{i\geq 0} s^{(i)} t^i \tag{3.1}$$

with integer coefficients $s^{(i)}$, such that

$$\iota_{N*} s(Z_N, \mathbb{P}^N) = \sum_{i=0}^{N} s^{(i)} H^i \cap [\mathbb{P}^N]\,,$$

where H denotes the hyperplane class in $\mathbb{P}^N$. (The fact that ζ_I is well-defined is the content of [Alu17, Lemma 5.2].)

Theorem 3.1 ([Alu17], Theorem 5.8)**.** *The power series $\zeta_I(t)$ is rational. More precisely, let $d_0, \dots, d_r$ be the degrees of the elements in any homogeneous generating set for I; then*

$$\zeta_I(t) = \frac{P(t)}{\prod_{i=0}^r (1 + d_i t)}$$

where $P(t) \in \mathbb{Z}[t]$ is a polynomial with nonnegative coefficients, trailing term of degree $\operatorname{codim}(I)$*, and leading coefficient $\prod_{i=0}^r d_i$.*

As a consequence of this result, the polynomial

$$\prod_{i=0}^{r} (1 + d_i t) - P(t)$$

has degree $\leq r$. In this section we will prove that *the coefficients of this polynomial form a log-concave sequence of nonnegative integers with no internal zeros.* We will also prove that the coefficients of $P(t)$ also form a log-concave sequence of nonnegative integers with no internal zeros, provided that the normal cone of Z in $\mathbb{P}^n$ is irreducible.

We will view the set-up described above as a particular case of the following more general situation. Let $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$ be a closed subscheme, whose ideal I is generated by a finite set of multihomogeneous polynomials. We consider the subscheme $Z_{\underline{N}}$ defined by the same polynomials in $\mathbb{P}^{\underline{N}} := \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_\ell}$, for all $\underline{N} = (N_1, \dots, N_\ell)$ with $N_j \geq n_j$. The analogue of (3.1) is a power series

$$\zeta_I(t_1, \dots, t_\ell) := \sum_{i\geq 0} s^{(i_1 \dots i_\ell)} t_1^{i_1} \cdots t_\ell^{i_\ell} \tag{3.2}$$

such that for all $\underline{N}$ as above, and denoting by $\iota_{\underline{N}}$ the inclusion $Z_{\underline{N}} \hookrightarrow \mathbb{P}^{\underline{N}}$ and by h_j the pull-back of the hyperplane class from the j-th factor,

$$\iota_{\underline{N}*} s(Z_{\underline{N}}, \mathbb{P}^{\underline{N}}) = \sum_{i_j \leq N_j} s^{(i_1 \dots i_\ell)} h_1^{i_1} \cdots h_\ell^{i_\ell} \cap [\mathbb{P}^{\underline{N}}]\,. \tag{3.3}$$

The natural extension of Theorem 3.1 holds for this power series.

Theorem 3.2. *The power series $\zeta_I(t_1, \dots, t_\ell)$ is rational. More precisely, let $(e_{1k}, \dots, e_{\ell k})$, $k = 0, \dots, r$, be the multidegrees of the elements in any multihomogeneous generating set for I, and let*

$$Q(t_1, \dots, t_\ell) = \prod_{k=0}^{r} (1 + e_{1k} t_1 + \cdots + e_{\ell k} t_\ell)\,.$$

Then

$$\zeta_I(t_1,\dots,t_\ell)=\frac{P(t_1,\dots,t_\ell)}{Q(t_1,\dots,t_\ell)}$$

where $P(t_1,\dots,t_\ell)\in\mathbb{Z}[t_1,\dots,t_\ell]$ *is a polynomial with nonnegative coefficients, trailing term of total degree* $\operatorname{codim}(I)$*, and leading term equal to* $\prod_{k=0}^r(e_{1k}t_1+\cdots+e_{\ell k}t_\ell)$*.*

A proof of this result may be obtained by following the same blueprint as the proof of Theorem 3.1 given in [Alu17]; for the case $\ell=2$, also see [Jor20, §5.2].

We focus on the polynomial

$$R(t_1,\dots,t_\ell)=Q(t_1,\dots,t_\ell)-P(t_1,\dots,t_\ell)\,,$$

for which

$$1-\zeta_I(t_1,\dots,t_\ell)=\frac{R(t_1,\dots,t_\ell)}{Q(t_1,\dots,t_\ell)}\,.$$

As a consequence of Theorem 3.2, $R(t_1,\dots,t_\ell)$ is a polynomial with integer coefficients and total degree $\le r$. Our main goal is to establish log-concavity properties of these polynomials. As they are not necessarily homogeneous, we adapt Definition 2.5 accordingly.

Definition 3.3. A polynomial $f(t_1,\dots,t_\ell)\in\mathbb{R}[t_1,\dots,t_\ell]$ is *sectional log-concave* if for all $\ell\times 2$ matrices A with nonnegative real entries, the polynomial $p(A\binom{1}{v})$ is either identically 0 or its coefficients form a log-concave sequence of nonnegative real numbers with no internal zeros. ⌟

Remark 3.4. If f is homogeneous, then it is sectional log-concave in the sense of Definition 3.3 if and only if it is in the sense of Definition 2.5. For any $f(t_1,\dots,t_\ell)$, f is sectional log-concave if any homogenization $F(t_0,\dots,t_\ell)$ of f is sectional log-concave. Indeed, assume F is homogeneous and $f=F|_{t_0=1}$. Then $f(A\binom{1}{v})=F(u,A\binom{u}{v})|_{u=1}$, and $F(u,A\binom{u}{v})$ is log-concave or identically 0 if F is sectional log-concave.

If f is obtained by setting one of the variables of a homogeneous polynomial F to 1, we will say that f is a *de-homogenization* of F. ⌟

Similarly, we will say that $f(t_1,\dots,t_\ell)\in\mathbb{R}[t_1,\dots,t_\ell]$ is *M-convex* if it is the de-homogenization of an M-convex polynomial.

A precise result can be established for the polynomials $R(t_1,\dots,t_\ell)$ arising as numerators of $1-\zeta_i$.

Theorem 3.5. *Let* $Z\subseteq\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_\ell}$ *be a closed subscheme whose ideal* I *is generated by forms of multidegrees* $(e_{1k},\dots,e_{\ell k})$*,* $k=0,\dots,r$*. Define the polynomial* $R(t_1,\dots,t_\ell)$ *by the identity*

$$1-\zeta_I(t_1,\dots,t_\ell)=\frac{R(t_1,\dots,t_\ell)}{\prod_{k=0}^r(1+e_{1k}t_1+\cdots+e_{\ell k}t_\ell)}\,.$$

Then $R(t_1,\dots,t_\ell)$ *is the de-homogenization of a covolume polynomial of degree* r*. In particular:*

- $R(t_1,\dots,t_\ell)$ *has nonnegative coefficients and* M*-convex support;*
- $R(t_1,\dots,t_\ell)$ *is sectional log-concave;*
- *If* $R=\sum_{0\le i_j\le r_j}b_{i_1\dots i_\ell}t_1^{i_1}\cdots t_\ell^{i_\ell}$*, then the normalization of the polynomial*

$$\sum_{0\le i_j\le r_j}b_{i_1\dots i_\ell}u_0^{i_1+\cdots+i_\ell}u_1^{r_1-i_1}\cdots u_\ell^{r_\ell-i_\ell}$$

  is Lorentzian;

- *In the univariate case, i.e., $\ell = 1$, the coefficients of $R(t_1)$ form a log-concave sequence with no internal zeros.*

It is natural to ask whether the numerator of ζ_I itself satisfies the same constraints. We can prove that this is the case, but only subject to an irreducibility hypothesis.

Theorem 3.6. *Let $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$ be a closed subscheme whose ideal I is generated by forms of multidegrees $(e_{1k}, \dots, e_{\ell k})$, $k = 0, \dots, r$. Define the polynomial $P(t_1, \dots, t_\ell)$ by the identity*

$$\zeta_I(t_1, \dots, t_\ell) = \frac{P(t_1, \dots, t_\ell)}{\prod_{k=0}^r (1 + e_{1k}t_1 + \cdots + e_{\ell k}t_\ell)} .$$

Assume that the projectivized normal cone $\mathbb{P}(C_Z(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}))$ is irreducible. Then $P(t_1, \dots, t_\ell)$ is the de-homogenization of a covolume polynomial of degree $r+1$. In particular,

- *$P(t_1, \dots, t_\ell)$ has nonnegative coefficients and M-convex support;*
- *$P(t_1, \dots, t_\ell)$ is sectional log-concave;*
- *If $P = \sum_{0 \le i_j \le r_j} a_{i_1 \dots i_\ell} t_1^{i_1} \cdots t_\ell^{i_\ell}$, then the normalization of the polynomial*

$$\sum_{0 \le i_j \le r_j} a_{i_1 \dots i_\ell} u_0^{i_1 + \cdots + i_\ell} u_1^{r_1 - i_1} \cdots u_\ell^{r_\ell - i_\ell}$$

  is Lorentzian;
- *In the univariate case, i.e., $\ell = 1$, the coefficients of $P(t_1)$ form a log-concave sequence with no internal zeros.*

We will informally refer to the polynomials P, resp. R, in these statements as the 'numerators' of ζ_I, resp. $1 - \zeta_I$. This is an abuse of language since the polynomials depend on the multidegrees of the chosen generators; it is harmless in the sense that the statements hold for any choice of generators.

Example 3.7. Segre classes have applications in enumerative geometry; the prototypical example is the computation of the number 3264 of smooth plane conics that are tangent to five general smooth conics, cf. [Ful84, Examples 9.1.8, 9.1.9].

The *characteristic numbers* for the family of smooth plane curves of degree d are the numbers of such curves that contain a selection of general points and are tangent to a selection of general lines. For plane cubics, the characteristic numbers are

$$1, \quad 4, \quad 16, \quad 64, \quad 976, \quad 3424, \quad 9766, \quad 21004, \quad 33616$$

([Mai71, Alu88, KS91]): for example, there are 33,616 cubics tangent to 9 lines in general position. The information of the characteristic numbers is equivalent to the information of the push-forward to the $\mathbb{P}^9$ of plane cubics of the Segre class of a scheme naturally supported on the set of non-reduced curves:

$$48[\mathbb{P}^4] - 480[\mathbb{P}^3] + 3930[\mathbb{P}^2] - 38220[\mathbb{P}^1] + 372960[\mathbb{P}^0] .$$

This scheme is cut out by 10 quartic hypersurfaces, and it follows that the corresponding Segre zeta function is

$$\zeta_I(t) := \frac{48t^5 + 1440t^6 + 19290t^7 + 142020t^8 + 567840t^9 + 1048576t^{10}}{(1 + 4t)^{10}} . \tag{3.4}$$

The numerator of $1 - \zeta_I(t)$ equals

$$1 + 40t + 720t^2 + 7680t^3 + 53760t^4 + 258000t^5 + 858720t^6 + 1946790t^7 + 2807100t^8 + 2053600t^9$$

and as prescribed by Theorem 3.5 the coefficients of this polynomial form a log-concave sequence with no internal zeros. ⌟

Remark 3.8. The characteristic numbers may be interpreted as the multidegrees of the closure of the graph of the duality map (in the case of cubics, the map associating to a smooth cubic the corresponding dual sextic). It follows that they form a log-concave sequence of integers with no internal zeros, by [Huh12, Theorem 21]. ⌟

Our main tool in the proof of Theorems 3.5 and 3.6 will be a result providing a large supply of polynomials that are de-homogenizations of covolume polynomials.

Proposition 3.9. *Let X be an irreducible variety and let*

$$q : X \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$$

be a proper map. Let $\mathscr{R}$ be a globally generated vector bundle of rank r on X, and write

$$q_*(c(\mathscr{R}) \cap [X]) = \sum_{0 \le i_j \le n_j} a_{i_1 \dots i_\ell} h_1^{i_1} \cdots h_\ell^{i_\ell} \cap [\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}] . \tag{3.5}$$

Then the polynomial

$$\sum_{0 \le i_j \le n_j} a_{i_1 \dots i_\ell} t_1^{i_1} \cdots t_\ell^{i_\ell} \tag{3.6}$$

is the de-homogenization of a covolume polynomial of degree $r + (\sum n_j) - \dim X$.

Proof. Let V be a $(D+1)$-dimensional vector space of global sections generating $\mathscr{R}$. Consider the product

$$\begin{array}{ccc} & \mathbb{P}(V) \times X & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathbb{P}(V) & & X \end{array} ,$$

and denote by H the hyperplane class in $\mathbb{P}(V)$. Let $\mathscr{K}$ be the kernel of the surjection $\mathscr{V} = V \otimes \mathscr{O}_X \to \mathscr{R}$. Identify $\mathbb{P}(V) \times X$ with $\mathbb{P}(\mathscr{V})$ and view $\mathbb{P}(\mathscr{K})$ as a codimension-$r = \operatorname{rk} \mathscr{R}$ subscheme of $\mathbb{P}(V) \times X$; also note that $p_1^* H = c_1(\mathscr{O}_{\mathscr{V}}(1))$. By definition, $\mathbb{P}(\mathscr{K})$ is the zero-scheme of the composition

$$\mathscr{O}_{\mathscr{V}}(-1) \hookrightarrow p_2^* \mathscr{V} \twoheadrightarrow p_2^* \mathscr{R} ;$$

therefore, of the corresponding section $\mathscr{O} \to p_2^* \mathscr{R} \otimes \mathscr{O}_{\mathscr{V}}(1)$. This section is regular ([Ful84, B.5.6]), and it follows that the class $[\mathbb{P}(\mathscr{K})]$ in $A_*(\mathbb{P}(V) \times X)$ is the top Chern class of $p_2^* \mathscr{R} \otimes \mathscr{O}_{\mathscr{V}}(1)$:

$$[\mathbb{P}(\mathscr{K})] = \sum_{i=0}^{r} (p_1^* H^{r-i})(p_2^* c_i(\mathscr{R})) \cap [\mathbb{P}(V) \times X] \tag{3.7}$$

in $A_{D+\dim X - r}(\mathbb{P}(V) \times X)$. Next, consider the proper map

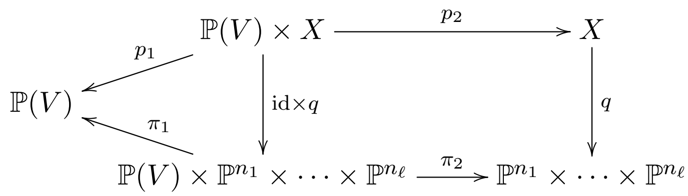
$$\mathbb{P}(V) \times X \xrightarrow{\ \mathrm{id} \times q\ } \mathbb{P}(V) \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell} .$$

We claim that

$$(\mathrm{id} \times q)_*([\mathbb{P}(\mathscr{K})]) = \sum_{i=0}^{r} \sum_{\sum i_j = i - \dim X + \sum n_j} a_{i_1 \dots i_\ell} H^{r-i} h_1^{i_1} \cdots h_\ell^{i_\ell} \cap [\mathbb{P}(V) \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}] \tag{3.8}$$

with $a_{i_1 \dots i_\ell}$ as in (3.5), and where H, h_j denote the pull-backs of the corresponding classes to the product $\mathbb{P}(V) \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$.

To prove (3.8), consider the diagram



We have

$$p_1^* = (\mathrm{id} \times q)^* \circ \pi_1^* \quad \text{and} \quad (\mathrm{id} \times q)_* \circ p_2^* = \pi_2^* \circ q_*$$

([Ful84, Proposition 1.7]). By (3.7), these identities and the projection formula give

$$\begin{aligned}(\mathrm{id} \times q)_*([\mathbb{P}(\mathscr{K})]) &= \sum_{i=0}^{r} (\mathrm{id} \times q)_*((p_1^* H^{r-i})(p_2^* c_i(\mathscr{R})) \cap [\mathbb{P}(V) \times X]) \\ &= \sum_{i=0}^{r} (\pi_1^* H^{r-i}) \cap \pi_2^* q_*(c_i(\mathscr{R}) \cap [X]) \,.\end{aligned}$$

Using (3.5), we see that this class equals

$$\sum_{i=0}^{r} (\pi_1^* H^{r-i}) \cap \pi_2^* \sum_{\sum i_j = i - \dim X + \sum n_j} a_{i_1 \dots i_\ell} h_1^{i_1} \cdots h_\ell^{i_\ell} \cap [\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}]$$

and (3.8) follows.

Since by definition $(\mathrm{id} \times q)_*([\mathbb{P}(\mathscr{K})])$ is a multiple of the class of the irreducible subvariety $(\mathrm{id} \times q)(\mathbb{P}(\mathscr{K}))$, this shows that

$$\sum_{i=0}^{r} \sum_{\sum i_j = i - \dim X + \sum n_j} a_{i_1 \dots i_\ell} t_0^{r-i} t_1^{i_1} \cdots t_\ell^{i_\ell}$$

is a covolume polynomial. The polynomial (3.6) is obtained from this polynomial by setting $t_0 = 1$, so this proves the statement. □

Example 3.10. The degrees of the Chern classes of a globally generated bundle over projective space form a log-concave sequence of nonnegative integers with no internal zeros.

This follows from Proposition 3.9 and Lemma 2.6 in the very particular case where $\ell = 1$ and q the identity map $\mathbb{P}^{n_1} \to \mathbb{P}^{n_1}$. ⌟

Theorems 3.5 and 3.6 are consequences of Proposition 3.9.

Proof of Theorems 3.5 and 3.6. In both statements, the four listed properties are formal consequence of the assertion that the polynomial is the de-homogenization of a covolume polynomial. Specifically, the first property follows from Corollary 2.11; the second from Corollary 2.15 (cf. Remark 3.4); the third from Proposition 2.8; and the last property from Lemma 2.6.

Therefore, it suffices to prove that $R(t_1, \dots, t_\ell)$ is a de-homogenization of a covolume polynomial, and so is $P(t_1, \dots, t_\ell)$ if $\mathbb{P}(C_Z(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}))$ is irreducible.

Let $\underline{N} = (N_1, \dots, N_\ell)$, with $N_j \geq n_j$ for all j. Denote by $\mathscr{O}_j(1)$ the pull-back of the hyperplane line bundle from the j-th factor of $\mathbb{P}^{\underline{N}}$. The subscheme $Z_{\underline{N}}$ of $\mathbb{P}^{\underline{N}}$ is cut out by hypersurfaces X_k, $k = 0, \dots, r$, with $\mathscr{O}(X_k) = \mathscr{O}_1(e_{1k}) \otimes \cdots \otimes \mathscr{O}_\ell(e_{\ell k})$. We have the fiber square

$$\begin{array}{ccc} Z_{\underline{N}} & \xhookrightarrow{\iota_{\underline{N}}} & \mathbb{P}^{\underline{N}} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ X_0 \times \cdots \times X_r & \hookrightarrow & \mathbb{P}^{\underline{N}} \times \cdots \times \mathbb{P}^{\underline{N}} \end{array} \tag{3.9}$$

where Δ is the diagonal embedding. This is an instance of the situation considered in [Ful84, §6.1]; the Fulton-MacPherson intersection product $(X_0 \times \cdots \times X_r) \cdot \mathbb{P}^{\underline{N}}$ is one term in the class

$$\delta^*(N_{X_0 \times \cdots \times X_r}(\mathbb{P}^{\underline{N}} \times \cdots \times \mathbb{P}^{\underline{N}})) \cap s(Z_{\underline{N}}, \mathbb{P}^{\underline{N}}) = c(\iota_{\underline{N}}^* \mathscr{N}) \cap s(Z_{\underline{N}}, \mathbb{P}^{\underline{N}}),$$

where

$$\mathscr{N} = \oplus_{k=0}^r \left(\mathscr{O}_1(e_{1k}) \otimes \cdots \otimes \mathscr{O}_\ell(e_{\ell k})\right).$$

By [Ful84, Example 6.1.6], this class only has terms in codimension $\le r+1$. Thus, its push-forward

$$\begin{aligned} \iota_{\underline{N}*}\big(c(\iota_{\underline{N}}^* \mathscr{N}) \cap s(Z_{\underline{N}}, \mathbb{P}^{\underline{N}})\big) &= c(\mathscr{N}) \cap \iota_{\underline{N}*} s(Z_{\underline{N}}, \mathbb{P}^{\underline{N}}) \\ &= \prod_{k=0}^r (1 + e_{1k} h_1 + \cdots + e_{\ell k} h_\ell) \cap \iota_{\underline{N}*} s(Z_{\underline{N}}, \mathbb{P}^{\underline{N}}) \end{aligned}$$

may be written as a polynomial

$$\sum_{0 \le i_j \le N_j} a_{i_1 \dots i_\ell} h_1^{i_1} \cdots h_\ell^{i_\ell} \cap [\mathbb{P}^{\underline{N}}]$$

of total degree $\le (r+1)$. By (3.2), (3.3), and Theorem 3.2,

$$P(h_1, \dots, h_\ell) = \prod_{k=0}^r (1 + e_{1k} h_1 + \cdots + e_{\ell k} h_\ell) \cap \iota_{\underline{N}*} s(Z_{\underline{N}}, \mathbb{P}^{\underline{N}})$$

in $A_*(\mathbb{P}^{\underline{N}})$, that is, modulo $h_j^{N_j+1}$ for all j. Taking $N_j \ge \max(n_j, r+1)$, we get the equality *of polynomials*

$$P(t_1, \dots, t_\ell) = \sum_{0 \le i_j \le N_j} a_{i_1 \dots i_\ell} t_1^{i_1} \cdots t_\ell^{i_\ell} \in \mathbb{Z}[t_1, \dots, t_\ell].$$

Summarizing, let $N_j \gg 0$ for all j; then the polynomial $P(t_1, \dots, t_\ell) \in \mathbb{Z}[t_1, \dots, t_\ell]$ is the unique lift of degree $\le N_j$ in t_j of the class

$$c(\mathscr{N}) \cap \iota_{\underline{N}*} s(Z_{\underline{N}}, \mathbb{P}^{\underline{N}}). \tag{3.10}$$

It also follows that $R(t_1, \dots, t_\ell)$ is the unique lift of degree $\le N_j$ in t_j of the class

$$c(\mathscr{N}) \cap (1 - \iota_{\underline{N}*} s(Z_{\underline{N}}, \mathbb{P}^{\underline{N}})). \tag{3.11}$$

Next, consider the blow-up $\widetilde{\mathbb{P}}^{\underline{N}}$ of $\mathbb{P}^{\underline{N}}$ along $Z_{\underline{N}}$, with exceptional divisor $E = \mathbb{P}(C_{Z_{\underline{N}}} \mathbb{P}^{\underline{N}})$ and notation as in the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{j} & \widetilde{\mathbb{P}}^{\underline{N}} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\nu} \\ Z_{\underline{N}} & \xrightarrow[\iota_{\underline{N}}]{} & \mathbb{P}^{\underline{N}} \end{array} \tag{3.12}$$

By the birational invariance of Segre classes,

$$\iota_{\underline{N}*}s(Z_{\underline{N}},\mathbb{P}^{\underline{N}})=\nu_*j_*s(E,\widetilde{\mathbb{P}}^{\underline{N}})=\nu_*j_*\left(c(N_E\widetilde{\mathbb{P}}^{\underline{N}})^{-1}\cap[E]\right)\,.$$

stacking diagrams (3.9) and (3.12) gives a fiber square

$$\begin{array}{ccc} E & \xrightarrow{\quad j\quad} & \widetilde{\mathbb{P}}^{\underline{N}} \\ {\scriptstyle \delta\circ\rho}\big\downarrow & & \big\downarrow{\scriptstyle \Delta\circ\nu} \\ X_0\times\cdots\times X_r & \hookrightarrow & \mathbb{P}^{\underline{N}}\times\cdots\times\mathbb{P}^{\underline{N}} \end{array}$$

whose excess intersection bundle (in the sense of [Ful84, §6.3]) is

$$(\delta\circ\rho)^*N_{X_0\times\cdots\times X_r}(\mathbb{P}^{\underline{N}}\times\cdots\times\mathbb{P}^{\underline{N}})/N_E\widetilde{\mathbb{P}}^{\underline{N}}=\rho^*\iota_{\underline{N}}^*\mathscr{N}/j^*\mathscr{O}(E)=j^*\mathscr{R}$$

with

$$\mathscr{R}=\nu^*(\oplus_{k=0}^r(\mathscr{O}_1(e_{1k})\otimes\cdots\otimes\mathscr{O}_\ell(e_{\ell k})))/\mathscr{O}(E)\,.$$

With this notation, and repeatedly using the projection formula, the class in (3.10) (represented by the polynomial $P(t_1,\dots,t_\ell)$) may be rewritten as

$$\begin{aligned} c(\mathscr{N})\cap\iota_{\underline{N}*}s(Z_{\underline{N}},\mathbb{P}^{\underline{N}}) &= c(\mathscr{N})\cap\nu_*j_*\left(c(N_E\widetilde{\mathbb{P}}^{\underline{N}})^{-1}\cap[E]\right)\\ &=(\nu\circ j)_*\left(c(j^*\nu^*\mathscr{N})c(N_E\widetilde{\mathbb{P}}^{\underline{N}})^{-1}\cap[E]\right)\\ &=(\nu\circ j)_*\left(c(\rho^*\iota_{\underline{N}}^*\mathscr{N})c(j^*\mathscr{O}(E))^{-1}\cap[E]\right)\\ &=(\nu\circ j)_*\left(c(j^*\mathscr{R})\cap[E]\right)\,. \end{aligned}$$

Since $\mathscr{R}$ is generated by global sections, Proposition 3.9 implies that *if $E=\mathbb{P}(C_{Z_{\underline{N}}}\mathbb{P}^{\underline{N}})$ is irreducible, then* $P(t_1,\dots,t_\ell)$ is the de-homogenization of a covolume polynomial. This establishes the first part of Theorem 3.6, since $\mathbb{P}(C_{Z_{\underline{N}}}\mathbb{P}^{\underline{N}})$ is irreducible if and only if $\mathbb{P}(C_Z(\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_\ell}))$ is (recall that $Z_{\underline{N}}$ is a cone over Z).

Concerning the class in (3.11), represented by the polynomial $R(t_1,\dots,t_\ell)$:

$$\begin{aligned} c(\mathscr{N})\cap(1-\iota_{\underline{N}*}s(Z_{\underline{N}},\mathbb{P}^{\underline{N}})) &= c(\mathscr{N})\cap\nu_*\left(1-\frac{[E]}{1+E}\right)\\ &=c(\mathscr{N})\cap\nu_*\frac{[\widetilde{\mathbb{P}}^{\underline{N}}]}{1+E}\\ &=\nu_*\left(c(\nu^*\mathscr{N})c(\mathscr{O}(E))^{-1}\cap[\widetilde{\mathbb{P}}^{\underline{N}}]\right)\\ &=\nu_*\left(c(\mathscr{R})\cap[\widetilde{\mathbb{P}}^{\underline{N}}]\right)\,. \end{aligned}$$

Since $\mathscr{R}$ is generated by global sections and $[\widetilde{\mathbb{P}}^{\underline{N}}]$ is irreducible, Proposition 3.9 implies (unconditionally) that $R(t_1,\dots,t_\ell)$ is the de-homogenization of a covolume polynomial, establishing the first part of Theorem 3.5 and completing the proof. □

As proved above, the polynomials P and R determined by a closed subscheme $Z\subseteq\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_\ell}$ are closely related to Lorentzian polynomials. It is natural to inquire whether their homogenizations may themselves be Lorentzian, perhaps after normalization.

For $\ell=1$, the *normalization* of the numerator R of $1-\zeta_I$ is Lorentzian, as a consequence of Theorem 3.5 and of [BH20, Example 2.26]. It is not necessarily Lorentzian itself.

Example 3.11. Let $Z = \mathbb{P}^{n-3}$ as a subscheme of $\mathbb{P}^n$, with ideal $I = (x_0, x_1, x_2) \subseteq k[x_0, \dots, x_n]$. If H denotes the hyperplane class in $\mathbb{P}^n$, the Segre class $s(Z, \mathbb{P}^n) = c(N_{\mathbb{P}^{n-3}}\mathbb{P}^n)^{-1} \cap [\mathbb{P}^{n-3}]$ pushes forward to $H^3/(1+H)^3 \cap [\mathbb{P}^n]$, and it follows that

$$1 - \zeta_I(t_1) = 1 - \frac{t_1^3}{(1+t_1)^3} = \frac{1 + 3t_1 + 3t_1^2}{(1+t_1)^3} \,.$$

The homogenization $t_0^2 + 3t_0t_1 + 3t_1^2$ is not Lorentzian, i.e., the sequence $1, 3, 3$ is not *ultra*-log-concave. Its normalization *is* Lorentzian, i.e., the sequence is log-concave, as prescribed by Theorem 3.5.

For another example, let $I = (x_0, y_0, z_0) \subseteq k[x_0, \dots, x_n; y_0, \dots, y_n; z_0, \dots, z_n)$, defining an inclusion of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ in $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, for all $n \geq 1$. This is also a complete intersection, and it follows that

$$1 - \zeta_I(t_1, t_2, t_3) = 1 - \frac{t_1t_2t_3}{(1+t_1)(1+t_2)(1+t_3)} = \frac{1 + t_1 + t_2 + t_3 + t_1t_2 + t_1t_3 + t_2t_3}{(1+t_1)(1+t_2)(1+t_3)} \,,$$

so that the homogenization of $R(t_1, t_2, t_3)$ is

$$t_0^2 + t_0t_1 + t_0t_2 + t_0t_3 + t_1t_2 + t_1t_3 + t_2t_3 \,. \tag{3.13}$$

This polynomial is sectional log-concave and M-convex (as prescribed by Theorem 3.5), but it is not Lorentzian. We note that its normalization

$$\frac{1}{2}t_0^2 + t_0t_1 + t_0t_2 + t_0t_3 + t_1t_2 + t_1t_3 + t_2t_3$$

is Lorentzian. ⌟

We know of no example of a closed subscheme $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$ for which the *normalization* of the homogenization of the numerator of $1 - \zeta_I$ is not Lorentzian. This does not appear to follow directly from Theorem 3.5.

Question 1. *Is the* normalization *of the homogenization of the numerator of* $1 - \zeta_I$ *always a Lorentzian polynomial?*

The conjecture stated in §1 proposes that the answer to this question should be affirmative. We have verified that this is the case for several hundred randomly chosen monomial ideals.

Concerning the polynomial P of Theorem 3.6, that is, the 'numerator of ζ_I', we note that the hypothesis of irreducibility of the normal cone cannot be removed.

Example 3.12. Let $I = (x_0y_0, x_0z_0) \subseteq k[x_0, \dots, x_n; y_0, \dots, y_n; z_0, \dots, z_n]$, defining a closed subscheme $Z \subseteq \mathbb{P} := \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, $n \gg 0$. The support of this subscheme is the union of an irreducible divisor and a codimension-two subvariety; Z is not irreducible, and therefore its normal cone is not irreducible. The function $\zeta_I(t_1, t_2, t_3)$ may be determined as follows: with evident notation, the Segre class of Z is $(h_1 + \text{h.o.t.}) \cap [\mathbb{P}]$, since this is the case away from the codimension-2 component; since I is generated by divisors with classes $h_1 + h_2$, $h_1 + h_3$, we must have

$$\zeta_I(t_1, t_2, t_3) = \frac{t_1 + (t_1 + t_2)(t_1 + t_3)}{(1 + t_1 + t_2)(1 + t_1 + t_3)}$$

by Theorem 3.2. (For an alternative argument, compute the Segre class of Z in $(\mathbb{P}^n)^3$ by 'residual intersection', [Ful84, Proposition 9.2] or [Alu94, Proposition 3]; and then let $n \to \infty$.) The homogenization of the numerator $P(t_1, t_2, t_3)$ is

$$t_0t_1 + t_1^2 + t_1t_2 + t_1t_3 + t_2t_3 \,; \tag{3.14}$$

the support of this polynomial consists of the points

$$(1,1,0,0) \quad , \quad (0,2,0,0) \quad , \quad (0,1,1,0) \quad , \quad (0,1,0,1) \quad , \quad (0,0,1,1)$$

and is *not* M-convex: for $\alpha = (1,1,0,0)$ and $\beta = (0,0,1,1)$ we have $\alpha_0 > \beta_0$ and there is no j such that $\beta_j > \alpha_j$ and $\alpha - e_0 + e_j$, $\beta - e_j + e_0$ are both in S. This polynomial is also not sectional log-concave: setting $t_0 = 4u, t_1 = u, t_2 = v, t_3 = v$ gives $5u^2 + 2uv + v^2$, and $2^2 \not\geq 5 \cdot 1$.

By Proposition 2.8, the numerator is not the de-homogenization of a covolume polynomial. ⌟

Since the homogenization of the polynomial P from Example 3.12 is not M-convex, it also follows that it is not Lorentzian and neither is its normalization. In fact, the following simple example shows that the numerator of ζ_I is not necessarily Lorentzian before normalization even in the univariate (i.e., $\ell = 1$) case. (The Segre zeta function (3.4) in Example 3.7 also provides an example.)

Example 3.13. For any n, let $Z \subseteq \mathbb{P}^n$ consist of a hyperplane $\mathbb{P}^{n-1}$ with an embedded component along the transversal intersection of two smooth quadric hypersurfaces in this hyperplane. More precisely, let Z be the subscheme of $\mathbb{P}^n$ defined by the ideal $I \subseteq k[x_0, \dots, x_n]$ generated by (x_0^2, x_0Q_1, x_0Q_2), where Q_1 and Q_2 are general homogeneous quadratic polynomials. The Segre class of Z may be computed by residual intersection, and this yields

$$\zeta_I(t_1) = \frac{t_1 + 7t_1^2 + 18t_1^3}{(1+2t_1)(1+3t_2)^2}\,.$$

The sequence $1, 7, 18$ is not ultra-log-concave, so the homogenization

$$t_0^2t_1 + 7t_0t_1^2 + 18t_1^3$$

is not Lorentzian. The same sequence is log-concave (that is, the *normalization* of the polynomial is Lorentzian). ⌟

It would be interesting to establish to what extent the irreducibility hypothesis in Theorem 3.6 can be weakened.

Question 2. *For what subschemes of a product of projective spaces is the numerator of ζ_I necessarily sectional log-concave?*

B. Story [Sto23] has verified that the numerator of ζ_I *is* log-concave for several families of subschemes $Z \subseteq \mathbb{P}^n$ (that is, the $\ell = 1$ case) not satisfying any *a priori* irreducibility condition.

4. Adjoint polynomials

Our main motivation in establishing Theorems 3.5 and 3.6 is the general study of Segre classes: constraints on the possible numerators of Segre zeta functions translate into constraints on what classes can be Segre classes of subschemes of e.g., projective space, thus may be an aid in their computation. In this section we provide an alternative motivation for this work, by interpreting Theorem 3.5 in the special case of *monomial* ideals. There is a connection between Segre zeta functions of monomial ideals and *adjoint polynomials,* first noted by Kathlén Kohn and Kristian Ranestad ([KR20, Proposition 1]). Kohn and Ranestad focused on the numerator of ζ_I. We recover an analogous result for the numerator of $1 - \zeta_I$, with the advantage that the corresponding polytopal object is convex. In short, we will prove that adjoint polynomials of certain convex polytopes are necessarily covolume polynomials; in particular, they are M-convex and sectionally log-concave.

Following Joe Warren ([War96]), we consider *polyhedral cones,* that is, convex hulls of finite sets of rays emanating from the origin in an $\mathbb{R}$-vector space V. For a set of nonzero vectors $S \in V$, we will denote by $\mathscr{P}_S$ the convex polyhedral cone obtained by taking the convex hull of the rays through the vectors $\underline{v} \in S$.

For example, one may consider $\mathscr{P}_S$ for S the set of vertices of a convex polytope embedded in a hyperplane $x_0 = 1$. It is convenient to think of polyhedral cones as a 'projective' version of polytopes.

We will denote by $V(\mathscr{P})$ a set of vectors spanning the vertex rays of $\mathscr{P}$. Each such vector is determined up to the choice of a positive real scalar. We may take $V(\mathscr{P}_S)$ to be a subset of S.

A *triangulation* of a polyhedral cone $\mathscr{P}$ of dimension d is a partition of $\mathscr{P}$ into a collection of d-dimensional simplicial cones whose vertex rays are subsets of the rays of $\mathscr{P}$ and such that the intersections of any two simplicial cones are faces of both.

Definition 4.1. ([War96]) Let $\mathscr{P}$ be a polyhedral cone in $\mathbb{R}^{\ell+1}$ and let $T(\mathscr{P})$ be a triangulation of $\mathscr{P}$. We define the *adjoint polynomial* of $\mathscr{P}$ to be

$$\mathbb{A}_{\mathscr{P}}(t_0, \dots, t_\ell) = \sum_{\sigma \in T(\mathscr{P})} \mathrm{Vol}(\sigma) \prod_{(v_0,\dots,v_\ell) \in V(\mathscr{P}) \smallsetminus V(\sigma)} (v_0 t_0 + \dots + v_\ell t_\ell)\,. \tag{4.1}$$

(This definition is independent of the chosen triangulation.) ⌟

The quantity $\mathrm{Vol}(\sigma)$ in (4.1) is the absolute value of the determinant of the matrix whose entries are the coordinates of the vertices of the simplicial cone σ. It should be viewed as the volume of σ, up to a normalization factor.

The adjoint polynomial in Definition 4.1 is determined by $\mathscr{P}$ up to a positive real scalar factor: replacing a vertex $\underline{v}$ of $\mathscr{P}$ by a multiple $\lambda \underline{v}$ with $\lambda \in \mathbb{R}_{>0}$ has the effect of multiplying each summand of $\mathbb{A}_{\mathscr{P}}(\underline{t})$ by λ, since this either multiplies by λ a column of the determinant computing $\mathrm{Vol}(\sigma)$ or exactly one of the other factors. We could fix this factor, for example by requiring all vertices to have length 1, or by requiring the non-coordinate vertices to lie on the hyperplane $\{a_0 = 1\}$, but no such choice is necessary for what follows.

The independence of the definition of $\mathbb{A}_{\mathscr{P}}(\underline{t})$ on the choice of a triangulation is proved in [War96, Theorem 4]; also see Remark 4.7.

The adjoint polynomial $\mathbb{A}_{\mathscr{P}}(\underline{t})$ of a polyhedral cone $\mathscr{P}$ in $\mathbb{R}^{\ell+1}$ is a homogeneous polynomial of degree $|V(\mathscr{P})| - \ell - 1$ ([War96, Theorem 1]). Note that the coefficients of an adjoint polynomial are not necessarily nonnegative.

Example 4.2. The adjoint polynomial for the cone $\mathscr{P}_S \subseteq \mathbb{R}^3$ for $S = \{(1,0,0), (1,-1,0), (1,0,-1), (1,-1,-1)\}$ (the vertices of a square in the hyperplane $x_0 = 1$) is $\mathbb{A}_{\mathscr{P}_S}(\underline{t}) = 2t_0 - t_1 - t_2$. ⌟

As we are interested in studying Lorentzian properties of adjoint polynomials, it is natural to impose a condition guaranteeing that the coefficients are nonnegative. The most natural such condition is that the spanning set S should be contained in the nonnegative orthant. We will further require that the cone should contain all but one coordinate rays, that is, that the cone shares a face with the nonnegative orthant. A mild generalization of this condition will be considered in Corollary 4.10.

Theorem 4.3. *Let $S \subseteq \mathbb{R}^{\ell+1}_{\geq 0}$ be a finite set including the coordinate vectors $\underline{e}_1, \dots, \underline{e}_\ell$. Then $\mathbb{A}_{\mathscr{P}_S}(t_0, \dots, t_\ell)$ is a covolume polynomial. In particular, $\mathbb{A}_{\mathscr{P}_S}$ is M-convex and sectional log-concave, and for all integers $n_0, \dots, n_\ell \geq 0$ such that*

$$u_0^{n_0} \cdots u_\ell^{n_\ell} \mathbb{A}_{\mathscr{P}_S}\left(\frac{1}{u_0}, \dots, \frac{1}{u_\ell}\right) \tag{4.2}$$

is a polynomial, the normalization of this polynomial is Lorentzian.

Example 4.4. Let $\mathscr{P}$ be the polyhedral cone with vertex rays spanned by the vectors

$$\underline{v}_1 := (1,1,\sqrt{2},0,0),\quad \underline{v}_2 := (1,\sqrt{2},1,0,0),\quad \underline{v}_3 := (1,0,0,1,0),\quad \underline{v}_4 := (1,0,0,0,1)$$

as well as the coordinate directions

$$\underline{e}_1 = (0,1,0,0,0),\quad \underline{e}_2 = (0,1,0,0,0),\quad \underline{e}_3 = (0,1,0,0,0),\quad \underline{e}_4 = (0,1,0,0,0)\,.$$

A triangulation of $\mathscr{P}$ consists of the four simplices

$$\langle \underline{v}_1,\underline{v}_2,\underline{v}_3,\underline{v}_4,\underline{e}_1\rangle,\quad \langle \underline{v}_1,\underline{v}_3,\underline{v}_4,\underline{e}_1,\underline{e}_2\rangle,\quad \langle \underline{v}_1,\underline{v}_2,\underline{e}_2,\underline{e}_1,\underline{e}_4\rangle,\quad \langle \underline{v}_3,\underline{e}_1,\underline{e}_2,\underline{e}_3,\underline{e}_4\rangle\,,$$

and the adjoint polynomial of $\mathscr{P}$ is

$$\begin{aligned}\mathbb{A}_{\mathscr{P}}(t_0,\dots,t_4) &= t_0^3 + (1+\sqrt{2})t_1t_0^2 + (1+\sqrt{2})t_2t_0^2 + t_3t_0^2 + t_4t_0^2 + \sqrt{2}t_1^2t_0 + 3t_0t_1t_2\\ &+ (1+\sqrt{2})t_1t_3t_0 + (1+\sqrt{2})t_4t_1t_0 + \sqrt{2}t_2^2t_0 + (1+\sqrt{2})t_2t_3t_0 + (1+\sqrt{2})t_4t_2t_0 + t_3t_4t_0\\ &+ \sqrt{2}t_1^2t_3 + \sqrt{2}t_1^2t_4 + 3t_1t_2t_3 + 3t_1t_2t_4 + \sqrt{2}t_1t_3t_4 + \sqrt{2}t_2^2t_3 + \sqrt{2}t_2^2t_4 + t_2t_3t_4\sqrt{2}\,.\end{aligned}$$

The 21 terms in its support (of 35 in the simplex of tuples $(a_0,\dots,a_4)$ of nonnegative integers with $\sum a_i = 3$) form an M-convex set, as prescribed by Theorem 4.3. In fact, the normalization of $\mathbb{A}_{\mathscr{P}}$ is Lorentzian (the polynomial itself is not Lorentzian). The reader may verify that the normalization of

$$u_0^3u_1^2u_2^2u_3u_4\cdot \mathbb{A}_{\mathscr{P}}\left(\frac{1}{u_0},\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_4}\right)\,,$$

a homogeneous degree-6 polynomial, is Lorentzian as stated in Theorem 4.3. ⌟

Theorem 4.3 is proved by relating the adjoint polynomial to a Segre zeta function. As mentioned at the beginning of this section, Kathlén Kohn and Kristian Ranestad express such a relation in [KR20, Proposition 1]; their result deals with ζ_I, and correspondingly with not necessarily convex regions. We adopt the context of convex polyhedral cones and focus on the function $1-\zeta_I$. For these considerations, I is an ideal generated by a set of *monomials.*

Precisely, let F be a finite set of vectors $(v_1,\dots,v_\ell)\in\mathbb{Z}^\ell_{\geq 0}$. We can associate with F two objects:

- The convex polyhedral cone $\mathscr{P}_S$ in $\mathbb{R}^{\ell+1}$ spanned by the set

$$S = \{\underline{v} := (1,v_1,\dots,v_\ell)\,|\,(v_1,\dots,v_\ell)\in F\}\cup\{\underline{e}_1,\dots,\underline{e}_\ell\}\,. \tag{4.3}$$

- The ideal $I\subseteq k[x_1,\dots,x_\ell]$ generated by the *monomials*

$$x_1^{v_1}\cdots x_\ell^{v_\ell}\quad,\quad \text{with } (v_1,\dots,v_\ell)\in F$$

  and the corresponding Segre zeta function $\zeta_I(t_1,\dots,t_\ell)$.

Proposition 4.5. *With notation as above, let $\underline{v}_0,\dots,\underline{v}_r\in V(\mathscr{P}_S)$ be the vertex ray vectors other than $\underline{e}_1,\dots,\underline{e}_\ell$. Then*

$$1-\zeta_I(t_1,\dots,t_\ell) = \left.\frac{\mathbb{A}_{\mathscr{P}_S}(t_0,\dots,t_\ell)}{\prod_{i=0}^r \underline{v}_i\cdot(t_0,\dots,t_\ell)}\right|_{t_0=1}\,.$$

Remark 4.6. In other words, the adjoint polynomial $\mathbb{A}_{\mathscr{P}_S}$ is the degree-r homogenization of the polynomial R appearing in Theorem 3.5. ⌟

Proof. The Segre class $s(Z,W)$ of a monomial subscheme of a variety W may be expressed as an integral. Explicitly, let $X_1,\dots,X_\ell$ be hypersurfaces of a variety W meeting with normal crossings; let Z be the closed subscheme cut out by monomials in the X_i with classes $v_1X_1+\cdots+v_\ell X_\ell$ as $(v_1,\dots,v_\ell)\in\mathbb{Z}^\ell_{\geq 0}$ ranges over a finite set F; and let $N\subseteq\mathbb{R}^\ell$ be the complement in the nonnegative orthant $\mathbb{R}^\ell_{\geq 0}$ of the convex hull of the translates of the nonnegative orthant by the points in F. Then, using coordinates $(a_1,\dots,a_\ell)$ for $\mathbb{R}^\ell$, we have ([Alu16, Theorem 1.1])

$$s(Z,W)=\int_N \frac{\ell!\,X_1\cdots X_\ell\,da_1\cdots da_\ell}{(1+X_1a_1+\cdots+X_\ell a_\ell)^{\ell+1}}\,.$$

(The result in [Alu16] holds in a somewhat more general setting, but this won't play a role here.) This should be interpreted by computing the integral on the right-hand side as a rational function in the parameters X_j, then expanding this function as a power series, and evaluating each monomial $X_1^{a_1}\cdots X_\ell^{a_\ell}$ as a class in $A_*(Z)$; see [Alu16] for a more extensive discussion. For instance, monomials $X_1^{a_1}\cdots X_\ell^{a_\ell}$ with $\sum_j a_j>\dim W$ evaluate to 0, so that the integral may be written as a sum of only finitely many terms for any given Z and W.

If N is the nonnegative orthant itself, the integral evaluates to 1. Therefore,

$$[W]-\iota_*s(Z,W)=\int_{\mathbb{R}^\ell_{\geq 0}\smallsetminus N}\frac{\ell!\,X_1\cdots X_\ell\,da_1\cdots da_\ell}{(1+X_1a_1+\cdots+X_\ell a_\ell)^{\ell+1}}\,, \tag{4.4}$$

where $\iota: Z\to W$ is the inclusion. Adopting the notation introduced in (4.3) and viewing the polytope $\mathscr{P}_S$ in $\mathbb{R}^{\ell+1}$, with coordinates $(a_0,\dots,a_\ell)$,

$$\mathbb{R}^\ell_{\geq 0}\smallsetminus N=\{a_0=1\}\cap\mathscr{P}_S\,.$$

Applying (4.4) with $W=\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_\ell}$, $n_j\gg 0$, and $X_j=$ the hypersurface in W obtained by restricting the j-th factor to a hyperplane, we get

$$1-\zeta_I(t_1,\dots,t_\ell)=\int_{\{a_0=1\}\cap\mathscr{P}_S}\frac{\ell!\,t_1\cdots t_\ell\,da_1\cdots da_\ell}{(1+t_1a_1+\cdots+t_\ell a_\ell)^{\ell+1}}\,. \tag{4.5}$$

Now let $T(\mathscr{P}_S)$ be a triangulation of $\mathscr{P}_S$, as in the Definition 4.1. According to [Alu16, §2], a simplex with vertices $\underline{v}_0,\dots,\underline{v}_d$ and $\underline{e}_{i_1},\dots,\underline{e}_{i_{\ell-d}}$, where $\underline{v}_j$ correspond to vectors in F, contributes

$$\frac{\mathrm{Vol}(\sigma)\,t_1\cdots t_\ell}{\prod_{j=0}^d(1+\underline{v}_{j1}t_1+\cdots+\underline{v}_{j\ell}t_\ell)\prod_{j=1}^{\ell-d}t_j}=\left.\frac{\mathrm{Vol}(\sigma)\,t_1\cdots t_\ell}{\prod_{j=0}^d(\underline{v}_j\cdot\underline{t})\prod_{j=1}^{\ell-d}(\underline{e}_j\cdot\underline{t})}\right|_{t_0=1}=\left.\frac{\mathrm{Vol}(\sigma)\,t_1\cdots t_\ell}{\prod_{\underline{v}\in V(\sigma)}(\underline{v}\cdot\underline{t})}\right|_{t_0=1}$$

to the integral in (4.5), and therefore

$$\begin{aligned}1-\zeta_I(t_1,\dots,t_\ell)&=\int_{\{a_0=1\}\cap\mathscr{P}_S}\frac{\ell!\,t_1\cdots t_\ell\,da_1\cdots da_\ell}{(1+t_1a_1+\cdots+t_\ell a_\ell)^{\ell+1}}\\&=\left.\frac{\sum_{\sigma\in T(\mathscr{P}_S)}\mathrm{Vol}(\sigma)\,\prod_{\underline{v}\in V(\mathscr{P}_S)\smallsetminus V(\sigma)}(\underline{v}\cdot\underline{t})\,t_1\cdots t_\ell}{\prod_{\underline{v}\in V(\mathscr{P}_S)}(\underline{v}\cdot\underline{t})}\right|_{t_0=1}\\&=\left.\frac{\mathbb{A}_{\mathscr{P}_S}(t_0,\dots,t_\ell)\,t_1\cdots t_\ell}{\prod_{\underline{v}\in V(\mathscr{P}_S)}(\underline{v}\cdot\underline{t})}\right|_{t_0=1}\\&=\left.\frac{\mathbb{A}_{\mathscr{P}_S}(t_0,\dots,t_\ell)}{\prod_{i=0}^r(\underline{v}_i\cdot\underline{t})}\right|_{t_0=1}\,.\end{aligned}\tag{4.6}$$

as stated. ∎

Remark 4.7. As an aside, the statement of Proposition 4.5 implies that the definition of adjoint polynomial is independent of the triangulation (cf. [War96, Theorem 4]), for the convex polyhedral cones considered here. In fact, the integral expression for the adjoint polynomial worked out in (4.6) extends to arbitrary polyhedral cones (but not, to our knowledge, the interpretation in terms of a Segre zeta function), and this implies the independence on the choice of triangulation in general, providing an alternative to [War96, Theorem 4]. ⌟

Proof of Theorem 4.3. The adjoint polynomial of a polyhedral cone depends continuously on the coordinates of vectors spanning its vertex rays, so we may express it as a limit of adjoint polynomials of polyhedral cones whose vertex rays contain vectors with rational components. By definitions, limits of covolume polynomials are covolume polynomials, so we are reduced to the case of polyhedral cones with rational vertex rays. In fact, by choosing a large enough integer d, we may assume that all non-coordinate vectors in S are of the form $(d, v_1, \dots, v_\ell) \in \mathbb{Z}^{\ell+1}_{\geq 0}$. We are then reduced to proving the assertion of Theorem 4.3 for polyhedral cones spanned by a set S consisting of $\underline{e}_1, \dots, \underline{e}_\ell$ and of vectors of this type.

For this, let

$$S' = \{\underline{e}_1, \dots, \underline{e}_\ell\} \cup \{(1, v_1, \dots, v_\ell) \,|\, (d, v_1, \dots, v_\ell) \in S\}\,.$$

The set S' is of the form considered in Proposition 4.5. It follows that $\mathbb{A}_{\mathscr{P}_{S'}}$ is the homogenization of the numerator of $1 - \zeta_I$ for a suitable (monomial) ideal I, cf. Remark 4.6. By Theorem 3.5, $\mathbb{A}_{\mathscr{P}_{S'}}$ is a covolume polynomial. Now Definition 4.1 implies that

$$\mathbb{A}_{\mathscr{P}_S}(t_0, t_1, \dots, t_\ell) = \mathbb{A}_{\mathscr{P}_{S'}}(dt_0, t_1, \dots, t_\ell)\,;$$

therefore, $\mathbb{A}_{\mathscr{P}_S}$ is obtained from a covolume polynomial by a nonnegative change of coordinates. By Theorem 2.12 we can conclude that $\mathbb{A}_{\mathscr{P}_S}$ is a covolume polynomial, as needed. □

We do not know whether the adjoint polynomials considered in Theorem 4.3 are also necessarily Lorentzian after normalization. An affirmative answer to Question 1 would imply that this is the case.

We also do not know whether Theorem 4.3 extends to all convex polyhedral cones contained in the nonnegative orthant. The hypothesis of convexity cannot be removed, in the sense that there are non-convex unions of polyhedral cones contained in the nonnegative orthant and whose adjoint polynomial is not M-convex, hence not a covolume polynomial.

Example 4.8. Let $\underline{v}_1 = (1,1,1,0)$, $\underline{v}_2 = (1,1,0,1)$, along with the coordinate vectors $\underline{e}_0 = (1,0,0,0), \dots, \underline{e}_3 = (0,0,0,1)$ in $\mathbb{R}^4$. The adjoint polynomial of the union of the three simplicial cones with vertex rays $\underline{e}_0\underline{e}_1\underline{v}_1\underline{v}_2$, $\underline{e}_0\underline{v}_1\underline{v}_2\underline{e}_3$, $\underline{e}_0\underline{v}_1\underline{e}_2\underline{e}_3$ is

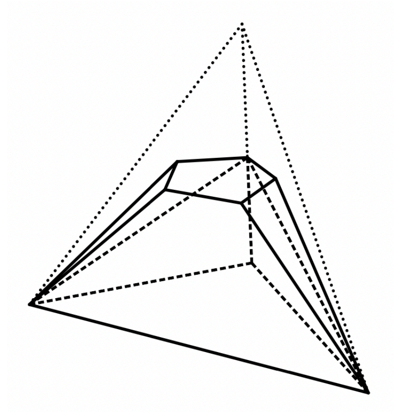
$$t_0t_1 + t_1^2 + t_1t_2 + t_1t_3 + t_2t_3$$

as the reader may verify. This polynomial is not M-convex or sectional log-concave. (This example is the polyhedral version of Example 3.12; see [KR20, Proposition 1].) ⌟

On the other hand, Theorem 4.3 extends easily to the following class of polyhedral cones.

Definition 4.9. We say that a convex polyhedral cone is *orthantal* if it shares a face with a simplex enclosing it and contained in the nonnegative orthant. ⌟

The intersection of an orthantal polyhedral cone with the hyperplane $a_0 = 1$ will be a convex polytope enclosed in a simplex with which it shares a face, and we are assuming that this simplex is contained in the nonnegative orthant.



The polyhedral cones considered in Theorem 4.3 are orthantal, with the simplex equal to the nonnegative orthant itself.

Corollary 4.10. *The adjoint polynomial of a convex orthantal polyhedral cone is a covolume polynomial. Therefore, it is M-convex, sectional log-concave, and the normalizations of the corresponding polynomials* (4.2) *are Lorentzian.*

We need the following elementary fact recording the effect of a linear transformation on an adjoint polynomial. Let $A = (a_{ij})$, $0 \le i, j \le \ell$ be a nonsingular matrix and let $\mathscr{P}$ be any polyhedral cone in $\mathbb{R}^{\ell+1}$. Denote by $A\mathscr{P}$ the polyhedral cone with vertices $A \cdot \underline{v}$ for $\underline{v} \in V(\mathscr{P})$. Denote by $A^{\rm t}$ the transpose of A.

Lemma 4.11.
$$\mathbb{A}_{A\mathscr{P}}(\underline{t}) = |\det(A)| \cdot \mathbb{A}_{\mathscr{P}}\left(A^{\rm t}(\underline{t})\right) .$$

Remark 4.12. Since the adjoint polynomial is only defined up to a positive factor, the term $|\det(A)|$ in Lemma 4.11 is actually superfluous. ⌟

Proof. By definition,
$$\mathbb{A}_{A\mathscr{P}}(\underline{t}) = \sum_{\sigma\in T(A\mathscr{P})} \mathrm{Vol}(\sigma) \prod_{(w_0,\dots,w_\ell)\in V(A\mathscr{P})\smallsetminus V(\sigma)} (w_0t_0 + \cdots + w_\ell t_\ell)$$
where $T(A\mathscr{P})$ denotes a triangulation of $A\mathscr{P}$. A triangulation of $A\mathscr{P}$ can be obtained by mapping by A the simplices of a triangulation for $\mathscr{P}$; the volume of the simplices in the triangulation is multiplied by $\det(A)$. Therefore
$$\begin{aligned}\mathbb{A}_{A\mathscr{P}}(\underline{X}) &= \sum_{\sigma\in T(\mathscr{P})} |\det(A)|\,\mathrm{Vol}(\sigma) \prod_{\underline{v}=(v_0,\dots,v_\ell)\in V(\mathscr{P})\smallsetminus V(\sigma)} ((A\underline{v})_0t_0 + \cdots + (A\underline{v})_\ell t_\ell)\\ &= |\det(A)| \sum_{\sigma\in T(\mathscr{P})} \mathrm{Vol}(\sigma) \prod_{(v_0,\dots,v_\ell)\in V(\mathscr{P})\smallsetminus V(\sigma)} \left(\sum_{i=0}^{\ell}\sum_{j=0}^{\ell} a_{ij}v_jt_i\right)\\ &= |\det(A)| \sum_{\sigma\in T(\mathscr{P})} \mathrm{Vol}(\sigma) \prod_{(v_0,\dots,v_\ell)\in V(\mathscr{P})\smallsetminus V(\sigma)} \left(v_0\sum_{i=0}^{\ell} a_{i0}t_i + \cdots + v_\ell\sum_{i=0}^{\ell} a_{in}t_i\right)\end{aligned}$$
with the stated consequence. □

Proof of Corollary 4.10. Let $\mathscr{P}$ be an orthantal polynomial. Let $(a_{0j}, \dots, a_{\ell j})$, $j = 0, \dots, \ell$, be the vectors in $V(\Sigma)$ for the simplex Σ containing $\mathscr{P}$ and contained in $\mathbb{R}^{\ell+1}_{\ge 0}$, with

$(a_{0j}, \dots, a_{\ell j})$, $j = 1, \dots, \ell$ the vertices of the simplicial face of $\mathscr{P}$ in common with Σ. Let

$$A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0\ell} \\ a_{10} & a_{11} & \cdots & a_{1\ell} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell 0} & a_{\ell 1} & \cdots & a_{\ell\ell} \end{pmatrix} .$$

Since Σ is a full dimensional simplex contained in $\mathbb{R}^{\ell+1}_{\geq 0}$, A has nonnegative entries and is nonsingular. By Lemma 4.11,

$$\mathbb{A}_{\mathscr{P}}(\underline{t}) = \mathbb{A}_{A^{-1}\mathscr{P}} \left(\sum_{i=0}^{\ell} a_{i0} t_i, \dots, \sum_{i=0}^{\ell} a_{i\ell} t_i \right) .$$

The chosen simplicial face of the polyhedral cone $A^{-1}\mathscr{P}$ has vertices along $\underline{e}_1, \dots, \underline{e}_\ell$, while the other vertex of the enclosing simplex is mapped to $\underline{e}_0$. Therefore $A^{-1}\mathscr{P}$ is enclosed in the simplex with vertices $\underline{e}_0, \dots, \underline{e}_\ell$, that is, the nonnegative orthant.

By Theorem 4.3, $\mathbb{A}_{A^{-1}\mathscr{P}}$ is a covolume polynomial. Since $\mathbb{A}_{\mathscr{P}}$ is obtained from $\mathbb{A}_{A^{-1}\mathscr{P}}$ by a nonnegative change of variables, Theorem 2.12 implies that $\mathbb{A}_{\mathscr{P}}$ is also a covolume polynomial, completing the proof. □

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