Linear orbits of d-tuples of points in $\mathbb{P}^1$

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§0. INTRODUCTION

The group $\mathrm{PGL}(2)$ of linear transformations of the projective line $\mathbb{P}^1$ acts naturally on the set of configurations of points on the line. We call each configuration of d points (some of which may coincide in the same point on the line) a 'd-tuple' of points; for a given d, the set of d-tuples of points in $\mathbb{P}^1$ forms a dimension-d projective space $\mathbb{P}^d$. In this note we are concerned with the orbits of this action of $\mathrm{PGL}(2)$ on $\mathbb{P}^d$. The closure of each orbit is a projective subvariety of $\mathbb{P}^d$ of which we determine the degree (§1), the 'boundary'–i.e., the complement of an orbit in its closure–(§2), and the multiplicity at points of the boundary (§3). These results are used to provide a complete classification of the non-singular orbit closures, and criteria for an orbit closure to be non-singular in codimension 1 (§4).

Although seemingly natural objects of study, we didn't find a lot of work on these orbits in the literature. Some of the results presented here appear also in [**Mukai-Umemura**], in one form or another; and the 'combinatorial' computation of the degree that we will sketch in this introduction goes back to [**Enriques-Fano**]. But for example Mukai and Umemura establish the non-singularity of the orbit closures of a specific 6-tuple and a specific 12-tuple by an ad-hoc coordinate computation. We hope to provide here a more unifying approach. Lucy Moser-Jauslin has developed techniques for the study of embeddings of $\mathrm{SL}(2)$ and $\mathrm{PGL}(2)$, and the degree of the orbits can be computed within her framework ([**Moser**], §8).

Our main motivation in this study is to prepare the ground for the much richer case of the action of $\mathrm{PGL}(3)$ on spaces parametrizing plane curves. The approach we use in this note is susceptible to be employed in higher dimensions, although the technical difficulties mount very rapidly. The reader wishing to approach the $\mathrm{PGL}(3)$ case (see [**Aluffi-Faber**]) will find here a sample of the essential techniques.

The main idea for the degree and multiplicity computations is the following: for each given d-tuple of points on $\mathbb{P}^1$, build a smooth variety $\widetilde{V}$ and a proper map from this to the closure of the orbit of the d-tuple. In fact this $\widetilde{V}$ will be a compactification of $\mathrm{PGL}(2)$, determined by the d-tuple, which we obtain by a suitable blow-up of the $\mathbb{P}^3$ of 2×2 (homogeneous) matrices. After the construction, we reduce the calculations to calculations on $\widetilde{V}$, where some intersection calculus (particularly, the formalism of Segre classes of [**Fulton**]) allows us to perform them. The blow-up construction also allows us to determine explicitly the boundary of the orbit.

The classification of smooth orbit closures follows from the multiplicity computations of §3; we use the classification of finite subgroups of $\mathrm{PGL}(2)$, which can be found for example in [**Weber**].

We now sketch here the easy 'combinatorial' computation of the degree of the orbit closure of a d-tuple consisting of $d \geq 3$ *distinct* points. In this case the orbit closure is 3-dimensional, so its degree may be computed as the intersection product with three hyperplanes of $\mathbb{P}^d$.

For the hyperplanes, take 3 distinct 'point-conditions', i.e., hyperplanes in $\mathbb{P}^d$ consisting of the d-tuples that contain a certain given point. One checks easily that the intersection multiplicity of the orbit closure and three point-conditions (determined by three distinct points p_1, p_2, p_3) at a d-tuple equals the product of the multiplicities of p_1, p_2 and p_3 in the d-tuple: so the intersection is automatically *transversal* if the d-tuple consists of d *distinct* points. Therefore, in this case the degree is just the number of points of intersection: the computation then comes down to counting the number of elements of PGL(2) that send a given d-tuple (consisting of d distinct points) to a d-tuple that contains 3 (distinct) given points. Since an element of PGL(2) is uniquely determined by prescribing the images of 3 distinct points, one sees that the answer must be

$$d(d-1)(d-2).$$

To get the degree of the orbit closure, we have to divide this number by the number of elements of PGL(2) sending a d-tuple to itself: i.e., the order of the stabilizer of the d-tuple. For example:

(1) The stabilizer of a 3-tuple consisting of 3 distinct points is S_3, so the degree of the orbit closure is 1 (the orbit closure is $\mathbb{P}^3$).

(2) A general 4-tuple has stabilizer $C_2 \times C_2$, so the degree of the orbit closure is $\frac{4 \cdot 3 \cdot 2}{4} = 6$. The 4-tuples with $j = 0$ (resp. 1728) have stabilizers A_4 (resp. D_4), so that the orbit closure has degree 2 (resp. 3).

(3) For $d \geq 5$, a general d-tuple has trivial stabilizer, so the degree of the orbit closure is $d(d-1)(d-2)$.

It would be easy to apply the same procedure to examine the case in which some points of the d-tuples appear with multiplicity. However, we don't see how to obtain by this approach a unified treatment of all cases; more importantly, this approach wouldn't help us to study the singularity of these orbit closures, and more important still we don't see how this kind of computations could be interpreted to attack higher dimensional cases such as the one dealt with in [**Aluffi-Faber**].

§1. The predegree of the orbit closure.

We work over an algebraically closed field of characteristic 0.

The first question we consider is the computation of the degree of the closure (in $\mathbb{P}^d$) of the orbit of a d-tuple under the action of PGL(2). Here we think of $\mathbb{P}^d$ as the space parametrizing homogeneous forms of degree d on $\mathbb{P}^1$, and each point of this space is identified with the d-tuple of zeros of the form corresponding to it. Also, we will denote by s the number of *distinct* points in the d-tuple. As mentioned in the introduction, the main ingredient in the computation is the construction for each d-tuple of a non-singular variety dominating the orbit closure.

First we observe this is not necessary if the whole d-tuple is concentrated in one point (that is, if $s = 1$). We'll refer to this particular d-tuple as to the 'd-fold point', and the reader should have no difficulties in checking that the orbit of the d-fold point (that is, the set of all such d-tuples) is simply the degree-d rational normal curve in $\mathbb{P}^d$.

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Next, let's consider the case when the d-tuple is distributed among 2 distinct points, that is one r-fold point and one distinct $(d-r)$-fold point. Again, in this case the reader will see immediately that the orbit consists of all d-tuples with the same multiplicity data.

PROPOSITION 1.1. *The orbit closures of d-tuples consisting of an r-fold point and a $(d-r)$-fold (distinct) point are surfaces in $\mathbb{P}^d$, of degree: $2r(d-r)$ if $r \neq d/2$, $r(d-r) = r^2$ if $r = d/2$.*

PROOF: For this, we dominate the orbit closure with $\mathbb{P}^1 \times \mathbb{P}^1$, using the map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^d$ defined by

$$((a_0 : a_1), (b_0 : b_1)) \mapsto (a_1 x - a_0 y)^r (b_1 x - b_0 y)^{d-r} \quad :$$

it is clear that this map is finite, and that the complement of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ maps onto the orbit we are considering. Also, it is clear that the degree of this map is 1 if $d \neq 2r$, and 2 if $d = 2r$: so to get the statement we just need to check that the self-intersection of the pull-back of the hyperplane class from $\mathbb{P}^d$ to $\mathbb{P}^1 \times \mathbb{P}^1$ via the above map is $2r(d-r)$. This is straightforward: if h_1, h_2 denote the hyperplane class of the factors, the pull-back of the hyperplane class from $\mathbb{P}^d$ is $(rh_1 + (d-r)h_2)$, and

$$\int_{\mathbb{P}^1 \times \mathbb{P}^1} (rh_1 + (d-r)h_2)^2 = \int_{\mathbb{P}^1 \times \mathbb{P}^1} 2r(d-r)h_1 h_2 = 2r(d-r) \quad .$$

(Here and in the following $\int$ will denote 'degree' in the sense of [**Fulton**]) ∎

It's worth observing that if $r = d/2$, then the orbit closure is a (regular) projection to $\mathbb{P}^d$ of the r-th Veronese embedding of $\mathbb{P}^2$—the degree is indeed r^2 in this case, as it should be. For example, for $r = 2$ this is the (non-singular) projection of the Veronese surface in $\mathbb{P}^5$ to $\mathbb{P}^4$.

Now we move to the most interesting case, that of a d-tuple distributed in $s \geq 3$ points. In this case the orbit and its closure have dimension 3. In order to construct a non-singular threefold dominating the orbit closure of a given d-tuple, we resolve the indeterminacies of a rational map associated naturally to the given d-tuple.

Choose coordinates $(x : y)$ in $\mathbb{P}^1$, and let C stand for a homogeneous form in $(x : y)$ of degree $d \geq 3$, and for the d-tuple of points on $\mathbb{P}^1$ corresponding to it. The PGL(2)-orbit of C in $\mathbb{P}^d$ is the image of the map

$$c \colon \mathrm{PGL}(2) \to \mathbb{P}^d$$

sending $\alpha \in \mathrm{PGL}(2)$ to the form $C \circ \alpha$. Observe that this map is finite (if at least three points of the d-tuple are distinct), and its degree equals the order of the stabilizer of C. This map determines a rational map from the $\mathbb{P}^3$ of 2×2 matrices to $\mathbb{P}^d$, which we also denote by c.

Now we will resolve this rational map: i.e., we will construct a variety $\widetilde{V}$ filling a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{PGL}(2) & \subset & \widetilde{V} & \xrightarrow{\ \tilde{c}\ } & \mathbb{P}^d \\[2pt]
\| & & \pi \downarrow & & \| \\[2pt]
\mathrm{PGL}(2) & \subset & \mathbb{P}^3 & \dashrightarrow^{c} & \mathbb{P}^d
\end{array}
$$

The image of $\tilde{c}$ in $\mathbb{P}^d$ is precisely the orbit closure. Thus the degree of the orbit closure can be found by computing the third power of the pull-back of the hyperplane class of $\mathbb{P}^d$ to $\widetilde{V}$, and dividing by the order of the stabilizer of C. We call 'predegree' the product of the degree by the order of the stabilizer: since the d-tuple is supported on at least 3 points, this term will be synonymous for the 3-fold self-intersection of the pull-back of the hyperplane from $\mathbb{P}^d$.

The base locus of $c \colon \mathbb{P}^3 \text{-}\,\text{-}\,\text{-}\!> \mathbb{P}^d$ consists of the matrices α for which the form $C \circ \alpha$ is identically zero. This happens exactly when α is a rank-1 matrix with image a point of the d-tuple C. The base locus of c is therefore supported on a finite number of 'parallel' lines in the (non-singular) quadric of rank-1 matrices. There are as many distinct lines as there are distinct points in the d-tuple C.

PROPOSITION 1.2. *A variety $\widetilde{V}$ as above can be obtained by blowing up $\mathbb{P}^3$ along the support of the base locus of c.*

PROOF: To see this, call 'point-conditions in $\mathbb{P}^3$' the inverse image of the point-conditions of $\mathbb{P}^d$ (defined above). The map c is then the map defined by the linear system generated by the point-conditions in $\mathbb{P}^3$, and therefore the base locus of c is actually cut out by the point-conditions. Now we argue that a point-condition in $\mathbb{P}^3$ is a degree-d hypersurface consisting of nothing but a collection of hyperplanes, one for each point in the d-tuple C, each appearing with the same multiplicity as the corresponding point appears in C. This is immediate: give coordinates

$$\begin{pmatrix} p_0 & p_1 \\ p_2 & p_3 \end{pmatrix}$$

to the $\mathbb{P}^3$ of matrices; and suppose C is given by the equation

$$F(x : y) = 0 \quad .$$

Then the point-condition corresponding to e.g. the point $(1 : 0)$ has equation

$$F(p_0 : p_2) = 0 \quad ,$$

so is indeed a union of hyperplanes as argued.

Let $\widetilde{V}$ be the blow-up of $\mathbb{P}^3$ along the lines supporting the base locus of c. The (a priori rational) map $\tilde{c}$ making the above diagram commute is then defined by the linear system on $\widetilde{V}$ generated by the proper transforms of the point-conditions: so the base locus of $\tilde{c}$ is cut out by the proper transforms in $\widetilde{V}$ of the point-conditions. But since the point-conditions are supported on unions of *hyperplanes*, they necessarily intersect transversally in $\mathbb{P}^3$ along the base locus of c: therefore their intersection in $\widetilde{V}$ is empty, and we can conclude that the map $\tilde{c} \colon \widetilde{V} \to \mathbb{P}^d$ is indeed a morphism. ∎

Now computing the 3-fold self-intersection of the class of the proper transform of a point-condition (i.e., the predegree of the orbit closure) is a straightforward intersection calculus exercise. We use [**Aluffi-Faber**], Proposition 3.2: the self-intersection is computed as the self-intersection of the point-condition in $\mathbb{P}^3$ (i.e.,

d^3) *minus* contributions coming from each component of the base locus of c. The formula gives

$$\text{predegree} = d^3 - \sum_{i=1}^{s} \int_{L_i} \frac{(m_i + dh)^3}{1 + 2h} \quad,$$

where the summation runs over the *distinct* points $p_1, \ldots, p_s$ of the d-tuple, L_i is the line in the base locus corresponding to p_i, m_i is the multiplicity of p_i in the d-tuple (thus the multiplicity of the point-conditions along L_i), and h denotes the hyperplane class in L_i. The degree is computed by taking the coefficient of h in the expression under $\int$. Doing this gives:

PROPOSITION 1.3. *For $d \geq 3$, the predegree of the orbit closure of a d-tuple is*

$$d^3 - 3d(\sum_{i=1}^{s} m_i^2) + 2(\sum_{i=1}^{s} m_i^3) \quad.$$

So the predegree of a d-tuple C can be written in terms of just d and *two* numbers, each of which is a sum of 'local contributions' given by each point of C. For example, if the d-tuple consists of $d - r$ simple points and one r-fold point, then

$$\sum_{i=1}^{s} m_i^2 = r^2 + d - r, \qquad \sum_{i=1}^{s} m_i^3 = r^3 + d - r,$$

so

$$\text{predegree} = d^3 - 3d(r^2 + d - r) + 2(r^3 + d - r)$$
$$= (d - r)(d - r - 1)(d + 2r - 2) \quad.$$

As seen in [**Aluffi-Faber**], this general feature of the predegree (being determined by a few numbers recording local data) is preserved in the PGL(3) case, at least for smooth curves.

For $s = 1$ or 2, the formula of this proposition gives 0: which reflects the fact that in these cases the orbits have dimension < 3. We also remark that the $\mathbb{P}^1 \times \mathbb{P}^1$ used to dominate the orbit closure in the case $s = 2$ in Proposition (1.1) can also be seen as one component of the exceptional divisor of the same blow-up construction used for the case $s \geq 3$.

§2. THE BOUNDARY OF AN ORBIT CLOSURE

We turn now to the question of determining the 'boundary' of the orbit of a d-tuple C, by which we mean the complement of the orbit in its closure. Observe that the boundary of an orbit is necessarily itself the union of orbits, and has dimension ≤ 2. Since the orbit of a d-tuple has dimension 3 as soon as the d-tuple consists of at least 3 distinct points, we can conclude right away that the boundary of the orbit of a given d-tuple must consist of a union of orbits of d-tuples concentrated in at most *two* points. We will show:

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PROPOSITION 2.1. *The boundary of the (3-dimensional) orbit of C is the union of the 1-dimensional orbit of x^d and of those 2-dimensional orbits of $x^r y^{d-r}$ for which r is the multiplicity of a point of C.*

PROOF: We use again the variety $\widetilde{V}$ constructed in §1. The rank-1 matrices *not* in the base locus have image in the orbit of x^d; so we only have to determine the image in $\mathbb{P}^d$ of the components of the exceptional divisor in $\widetilde{V}$. Give coordinates

$$\begin{pmatrix} p_0 & p_1 \\ p_2 & p_3 \end{pmatrix}$$

to the $\mathbb{P}^3$ of matrices; the locus of rank-1 matrices is given by $p_0 p_3 - p_1 p_2 = 0$. Suppose the d-tuple C has equation $a_0 x^d + a_1 x^{d-1} y + \cdots + a_d y^d = 0$, corresponding to the point $(a_0 : a_1 : \cdots : a_d) \in \mathbb{P}^d$ (with obvious choice of coordinates there). Assume that $(1 : 0)$ is a point of multiplicity $r \geq 1$ in C, i.e., $a_0 = a_1 = \cdots = a_{r-1} = 0, a_r \neq 0$. Then $p_2 = p_3 = 0$ is a component of the base locus of c and we can study $\widetilde{V}$ locally by blowing up $\mathbb{P}^3$ along $p_2 = p_3 = 0$.

On the affine piece $p_0 = 1$ we have coordinates (p_1, p_2, p_3). On an affine piece of the blow-up, coordinates (q_1, q_2, q_3) are given by

$$\begin{cases} p_1 = q_1 \\ p_2 = q_2 \\ p_3 = q_2 q_3 \end{cases}$$

The map induced by c is then given by

$$(q_1, q_2, q_3) \mapsto (b_0 : b_1 : \cdots : b_d)$$

with

$$b_0 x^d + \cdots + b_d y^d \sim a_r(x + q_1 y)^{d-r}(q_2 x + q_2 q_3 y)^r + \cdots + a_d(q_2 x + q_2 q_3 y)^d.$$

Note that we can factor out $q_2^{\,r}$ from the last expression, so that

$$b_0 x^d + \cdots + b_d y^d \sim a_r(x + q_1 y)^{d-r}(x + q_3 y)^r$$
$$+ a_{r+1} q_2 (x + q_1 y)^{d-r-1}(x + q_3 y)^{r+1} + \cdots + a_d q_2^{d-r}(x + q_3 y)^d.$$

The exceptional divisor is given here by $q_2 = 0$. The restriction of the map $\tilde{c} : \widetilde{V} \to \mathbb{P}^d$ to the component of the exceptional divisor of $\widetilde{V}$ corresponding to the r-fold point is then given by restricting the last expression to $q_2 = 0$: we get d-tuples corresponding to points

$$(*) \qquad\qquad b_0 x^d + \cdots + b_d y^d \sim a_r(x + q_1 y)^{d-r}(x + q_3 y)^r :$$

we conclude that the image of the exceptional divisor corresponding to a point in C of multiplicity r is the closure of the PGL(2)-orbit of $x^{d-r} y^r$. (The boundary of *this* orbit is the orbit of x^d.) The statement follows. ∎

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§3. MULTIPLICITIES.

We will now use the blow-up construction described in §1 to compute the multiplicity of the closure of an orbit along the orbits making up its boundary. For $s = 1$ and $s = 2$, $r = d/2$ (notations as in §1) we have remarked that the orbit closure is essentially a Veronese, so it is non-singular. To analyze the situation for $s = 2$, $r \neq d/2$ and $s \geq 3$, we first need the following fact.

Identify $\mathbb{P}^d$ with the space of d-tuples of points on $\mathbb{P}^1$, by giving it coordinates $(a_0 : \cdots : a_d)$ and associating with every $A = (a_0 : \cdots : a_d)$ the d-tuple of zeros of $F_A(x : y) = a_0 x^d + a_1 x^{d-1} y + \cdots + a_d y^d$. Then let $H_A(x : y)$ denote the Hessian of this form with respect to x, y, a form itself of degree $2d - 4$ in $(x : y)$ for each given A. For a given $(\xi : \eta)$ in $\mathbb{P}^1$, the equation $H_A(\xi : \eta) = 0$ determines the quadric of all d-tuples A whose Hessian vanishes at $(\xi : \eta)$. We'll use freely a few facts about the Hessians, whose verification will generally be left to the reader; the most important is the following, which we want to highlight:

LEMMA 3.1. *The orbit of the d-fold point in $\mathbb{P}^d$ is cut out scheme-theoretically by the equations $H_A(\xi : \eta) = 0$, $(\xi : \eta) \in \mathbb{P}^1$.*

PROOF: Clearly the Hessian of x^d is identically zero. On the other hand, if the Hessian of a form is identically zero, then after a change of coordinates a column in the matrix of second derivatives vanishes. Since the characteristic of the ground field is zero, the form is in the orbit of x^d. To finish the proof it suffices to show that the quadrics $H_A(\xi : \eta)$ cut out the orbit at the d-tuple $x^d = 0$. Now the tangent space to $H_A(\xi : \eta)$ at $(1 : 0 : \cdots : 0)$ is

$$\sum_{i=0}^{d} i(i-1) a_i \xi^{2d-i-2} \eta^{i-2} = 0 \quad,$$

so the intersection of the tangent spaces at $(1 : \cdots : 0)$ is given by $a_2 = \cdots = a_d = 0$, the tangent space to the orbit. ∎

To evaluate the multiplicity of the orbit closure of a d-tuple at points of its boundary, we use the techniques of [**Fulton**], Chapter 4: the multiplicity of a variety Y along an irreducible subvariety X is the coefficient of $[X]$ in the Segre class $s(X, Y)$ of X in Y ([**Fulton**], §4.3), and Segre classes behave well with respect to proper maps ([**Fulton**], §4.2). For each component of the boundary of an orbit closure, we'll pull-back equations for the component (essentially provided by the above lemma) to the varieties constructed in the degree computations. Computing the relevant term in the Segre class will be manageable on these varieties as they are non-singular. A push-forward will then give the Segre class in the orbit closure, and compute the multiplicity.

The boundary of the orbit closure of a d-tuple supported on a pair of points consists just of the orbit of a d-fold point.

PROPOSITION 3.2. *(s = 2) If $r \neq d/2$, the orbit closure of a d-tuple consisting of one r-fold point and one $(d - r)$-fold point has multiplicity 2 along its boundary. If $r = d/2$, this orbit closure is non-singular.*

PROOF: Pull back all equations $H_A(\xi : \eta) = 0$ via the map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^d$ considered in Proposition (1.1). With the notations of §1, $H_A(\xi : \eta)$ pulls back to

$$(a_1 b_0 - a_0 b_1)^2 (d-1)(d-r) r (a_1 \xi - a_0 \eta)^{2r-2} (b_1 \xi - b_0 \eta)^{2(d-r)-2} \quad;$$

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as $(\xi : \eta)$ varies in $\mathbb{P}^1$ we see that the equations of the orbit of the d-tuple pull back to the square of the equation of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. The diagonal maps isomorphically onto the orbit of the d-fold point, and the map from $\mathbb{P}^1 \times \mathbb{P}^1$ to the orbit closure has degree 1 if $r \neq d/2$: thus, pushing forward to $\mathbb{P}^d$, it follows that the first term in the Segre class of the orbit of the d-fold point in the orbit closure is *twice* the class of the orbit. The first assertion follows. If $r = d/2$, the map from $\mathbb{P}^1 \times \mathbb{P}^1$ to the orbit closure has degree 2: thus the first term in the Segre class is the orbit of the d-fold point, with coefficient $2/2 = 1$. So the orbit closure is non-singular in this case, as already observed earlier. ∎

$s \geq 3$. If the d-tuple consists of at least 3 distinct points, then its stabilizer in $\mathrm{PGL}(2)$ is finite, so its orbit closure is a *threefold* in $\mathbb{P}^d$. We have seen in §2 that the boundary of the orbit of a d-tuple consists of the union of the 1-dimensional orbit of x^d and the 2-dimensional orbits of $x^r y^{d-r}$, for all r that appear as the multiplicity of a point in the d-tuple.

We call 'premultiplicity' the product of the multiplicity of the orbit closure of a d-tuple C (with $s \geq 3$) and the order of its stabilizer. Given C, consider its Hessian H_C, this time specifically as a degree-$(2d - 4)$ form on $\mathbb{P}^1$, and thus as a $(2d-4)$-tuple determined by C. An important role is going to be played by the points of this $(2d-4)$-tuple that lie *away* from C. We state the results first:

PROPOSITION 3.3. *The premultiplicity of the orbit closure of C along the orbit of the d-fold point is*

$$\sum_i k_i^2 + 4s - 8 \quad ,$$

where the summation runs over all zeros of the Hessian H_C external to the d-tuple, and the k_i denote the multiplicity of H_C at such points.

For example, suppose the Hessian is simple at all points external to C; since the Hessian has degree $2d - 4$, and each point with multiplicity r on C contributes precisely a $(2r - 2)$-fold point to the Hessian, we find that in this case H_C has exactly $2s - 4$ simple points outside of C, so the premultiplicity along the orbit of the d-fold point must be

$$(2s - 4) + (4s - 8) = 6(s - 2) \quad .$$

In particular, the orbit closure of the general d-tuple, $d \geq 5$, has multiplicity $6(d-2)$ along this orbit.

Next for the 2-dimensional components of the boundary. For every point p of C of multiplicity r, denote by C_p the residual $(d - r)$-tuple to p in C. In this case it matters whether the point p of C is a point of the Hessian *of its residual C_p in C* (thus automatically external to C_p !).

As seen in §2, p contributes to the boundary of the orbit closure of C by the orbit of $x^r y^{d-r}$. The next result may be seen as a refinement of that statement:

PROPOSITION 3.4. *Each r-fold point p of the d-tuple contributes to the premultiplicity of the orbit closure along the orbit of $x^r y^{d-r}$ by*

$$2 + \textit{mult. of } p \textit{ in } H_{C_p}$$

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if $r \neq d/2$, and

$$4 + 2 \, (\text{mult. of } p \text{ in } H_{C_p})$$

if $r = d/2$.

So the orbit closure of the general d-tuple has multiplicity $2d$ along its only boundary component (i.e., the orbit of xy^{d-1}), for $d \geq 5$.

PROOFS: For the first computation (multiplicity along orbit of the d-fold point), every point $(\xi : \eta)$ in $\mathbb{P}^1$ gives one equation for the orbit of the d-fold point in $\mathbb{P}^d$, i.e. $H_A(\xi : \eta) = 0$ (see Lemma (3.1)). Now if $\varphi \in \mathbb{P}^3$, the Hessian of the translate by φ is given by

$$H_{A \circ \varphi} = (\det \varphi)^2 \, H_A \circ \varphi \quad :$$

therefore each of the above equations for the orbit of x^d pulls-back in $\mathbb{P}^3$ to the square of the equation of the locus D of rank-1 matrices, times the equation of the point-condition in $\mathbb{P}^3$ *relative to the Hessian of the d-tuple.* As seen in §1, point-conditions are separated above the base locus by the blow-up resolving the rational map determined by the d-tuple, and as shown in the proof of Proposition 2.1, the exceptional divisors are mapped onto 2-dimensional boundary components. Equations for the inverse image of the orbit of x^d in the blow-up are therefore

$$\widetilde{D}^2 \, \widetilde{H}(\xi : \eta) \quad , \quad (\xi : \eta) \in \mathbb{P}^1$$

where $\widetilde{D}$ is the equation for the proper transform of D, and $\widetilde{H}(\xi : \eta)$ is the point-condition in the blow-up *relative to the points in the Hessian not contained in the d-tuple.* The scheme-theoretic inverse image consists then of a non-reduced scheme supported on the proper transform of the locus of rank-1 matrices, with length 2 over the support, and embedded components along pencils of matrices whose image is a point of the Hessian *not* contained in the d-tuple; each of these pencils maps isomorphically to the 1-dimensional orbit of x^d. To examine the situation along these pencils, observe that every point of the Hessian (say of multiplicity k), determines a component of every $\widetilde{H}(\xi : \eta)$, in fact a k-fold plane containing the pencil. As $(\xi : \eta)$ moves in $\mathbb{P}^1$, these components define a scheme supported on the pencil. The defining ideal is the k-th power of that of the pencil and its *algebraic* multiplicity ([**Fulton**], §4.3) is equal to k^2. By [**Fulton**], Proposition 9.2, applied to

$$\widetilde{D} \subset \tilde{c}^{-1}(\text{orbit of } x^d) \subset \widetilde{V} \quad ,$$

the contribution of each embedded pencil to the Segre class is then k^2 times its class, and this gives the term $\sum k_i^2$ in the formula. It remains therefore to be seen that the proper transform $\widetilde{D}$ of the locus of rank-1 matrices accounts for the term $4s - 8$ in the premultiplicity. Now we claim that all we have to check is that $\widetilde{D}^2$ pushes forward to $(2-s)$ times the class of the orbit of x^d: indeed, it will follow that the contribution of $\widetilde{D}$ to the 1-dimensional term of the Segre class (i.e., $-(2\widetilde{D})^2$) pushes forward in $\mathbb{P}^d$ to $(4s - 8)$ times the class of the orbit of x^d, and we will be done. Now a straightforward computation shows that the push-forward of $\widetilde{D}^2$ is the push-forward *from $\mathbb{P}^3$ of D^2 minus* the s lines of the base locus (which map isomorphically to the orbit of x^d). Finally, D^2 consists, as a class on the quadric D, of 2 lines of each ruling, and the ruling parametrizing matrices with given kernel

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pushes forward to 0 in $\mathbb{P}^d$; so the push-forward is indeed $2 - s$ times the orbit, as needed.

For the second statement (the multiplicity along the orbit of $x^r y^{d-r}$), suppose p is a point of multiplicity r in the d-tuple, and factor the map $\mathbb{P}^3$ - - -> $\mathbb{P}^d$ through

$$\mathbb{P}^3 \text{ - - ->} \mathbb{P}^1 \times \mathbb{P}^{d-r} \to \mathbb{P}^d \quad ,$$

where $\mathbb{P}^3$ maps to each factor $\mathbb{P}^1$ and $\mathbb{P}^{d-r}$ as usual, by extending the action of $\mathrm{PGL}(2)$ on the r-fold point p and its residual $(d-r)$-tuple C_p respectively; the orbit closure of this point (p, C_p) in $\mathbb{P}^1 \times \mathbb{P}^{d-r}$ maps surjectively to the orbit closure of the d-tuple in $\mathbb{P}^d$. The point is that the map $\mathbb{P}^1 \times \mathbb{P}^{d-r} \to \mathbb{P}^d$ is an immersion at every point $(p, (d-r)q)$ if $p \neq q$; moreover, in this case the inverse image of $rp + (d-r)q$ consists of precisely $(p, (d-r)q)$ if $r \neq d/2$, and of the two points $(p, (d-r)q)$ and (q, rp) if $r = d/2$. Thus we only have to show that the premultiplicity of the orbit closure of (p, C_p) in $\mathbb{P}^1 \times \mathbb{P}^{d-r}$ is $2 +$ mult. of p in the Hessian of C_p.

For this, we observe that equations for the set of points in $\mathbb{P}^1 \times \mathbb{P}^{d-r}$ of type $(p, (d-r)q)$ are (again by Lemma (3.1)) given by $H_A(\xi : \eta) = 0$, where now the Hessian is taken for $A \in \mathbb{P}^{d-r}$. Pulling back to $\mathbb{P}^3$, and recalling again that the Hessian of a translate is the translate of the Hessian multiplied by the square of the determinant of the translation, we find that equations in $\mathbb{P}^3$ for the inverse image of the locus of pairs $(p, (d-r)q)$ are

$$(det\varphi)^2 \, H_{C_p}(\varphi(\xi : \eta)) = 0.$$

Now blow-up $\mathbb{P}^3$ as usual, and study it over the pencil of all φ whose image is the r-fold point p of the d-tuple. By arguing as in §1, one sees that the blow-up resolves the map $\mathbb{P}^3$ - - -> $\mathbb{P}^1 \times \mathbb{P}^{d-r}$; pulling back the above equation to the blow-up, we find that (near the pencil) the inverse image of the locus of pairs $(p, (d-r)q)$ is supported on the proper transform of the determinant hypersurface (with length 2), and on the component of the exceptional divisor over the pencil (with length $2 +$ mult. of p in H_{C_p}). Now pairs $(p, (d-r)q)$ with $p \neq q$ don't come from the determinant hypersurface (which maps to d-fold points only), so the premultiplicity equals the length of the part supported on the exceptional divisor, and this concludes the proof of the last claim. ∎

§4. Smooth orbit closures and more.

The results of §3, together with a description of the finite subgroups of $\mathrm{PGL}(2)$ (see [**Weber**], §§67-77), allow us to give an immediate classification of the smooth $\mathrm{PGL}(2)$-orbit closures.

First we present the following lemma, some instances of which appeared already above. Its proof may be left to the reader.

LEMMA 4.1. *The map* $\mathbb{P}^d \to \mathbb{P}^{md}$, $f \mapsto f^m$ *is an embedding.*

If the d-tuple corresponding to f is supported on $s \geq 3$ points, the orbit closure of f^m has degree equal to m^3 times the degree of the orbit closure of f (for example by Proposition 1.3), whereas the multiplicities along corresponding boundary components are equal.

Because of the lemma, in the remainder of this section we will only consider d-tuples for which the g.c.d. of the multiplicities of the s points equals one. We will also assume that $s \geq 3$; recall that the orbit (closure) of x^d is smooth and that the orbit closure of $x^r y^{d-r}$ is smooth if and only if $d = 2r$.

With these assumptions, we have:

PROPOSITION 4.2. *The smooth 3-dimensional PGL(2)-orbit closures are:*

(1) *the orbit closure of $x^3 + y^3$, with stabilizer $D_3 = S_3$;*
(2) *the orbit closure of $x^4 + xy^3$, with stabilizer A_4;*
(3) *the orbit closure of $x^5 y - xy^5$, with stabilizer S_4;*
(4) *the orbit closure of $x^{11}y + 11x^6 y^6 - xy^{11}$, with stabilizer A_5.*

PROOF: The orbit closure of a d-tuple f is smooth if and only if its multiplicity along the orbit of x^d equals one, i.e., the premultiplicity along that orbit equals the order of the stabilizer of f. From Proposition (3.3), this premultiplicity equals $\sum k_i^2 + 4s - 8$, where the k_i are the multiplicities of the points of the Hessian of f external to f. Counted with multiplicity, there are $2s - 4$ such points (i.e., $\sum k_i = 2s - 4$), so the premultiplicity is $\geq 6(s - 2)$.

Assuming that f has smooth orbit closure, it follows that the order of its stabilizer is $\geq 6(s - 2)$. In particular, its stabilizer is non-trivial. It now suffices to consider the action of the finite subgroups G of $\mathrm{PGL}(2)$ on $\mathbb{P}^1$ and the orbits of points with non-trivial stabilizer. Following [**Weber**], §68, we list these groups and the lengths of the special orbits:

(0) $G = C_n$; lengths 1, 1;
(1) $G = D_n$; lengths 2, n, n;
(2) $G = A_4$; lengths 4, 4, 6;
(3) $G = S_4$; lengths 6, 8, 12;
(4) $G = A_5$; lengths 12, 20, 30.

Determining the d-tuples f with smooth orbit closure is now an easy matter:

(0) Assume $\mathrm{Stab}(f) = C_n$. Then $n \geq 6(s-2) > s$. It follows that f is supported on one or two points, a contradiction.
(1) Assume $\mathrm{Stab}(f) = D_n$. Then $2n \geq 6(s - 2)$ so $n \geq 3(s - 2) \geq s$. Again, if $n > s$ it follows that $s = 2$, a contradiction; so we get $n = s = 3$ and $\mathrm{Stab}(f) = D_3 = S_3$. Clearly the multiplicities of the 3 points are all equal, thus by our assumption they are all one. So this is the orbit closure of $x^3 + y^3$, which is $\mathbb{P}^3$. Of course smoothness also follows from considering the Hessian of f.
(2) Assume $\mathrm{Stab}(f) = A_4$. Then $12 \geq 6(s - 2)$ so $s \leq 4$. It follows that $s = 4$ and that all multiplicities are equal (to one). This is the orbit closure of $x^4 + xy^3$; computing the Hessian, we see that it is indeed smooth.
(3) Assume $\mathrm{Stab}(f) = S_4$. Then $24 \geq 6(s-2)$ so $s \leq 6$. It follows that $s = 6$ and that all multiplicities are equal to one. This is the orbit closure of $x^5 y - xy^5$, which is indeed smooth, as its Hessian has simple zeros.
(4) Assume $\mathrm{Stab}(f) = A_5$. Then $60 \geq 6(s - 2)$ so $s \leq 12$. It follows that $s = 12$ and that all multiplicities are equal to one. This is the orbit closure of $x^{11}y + 11x^6 y^6 - xy^{11}$ ([**Weber**], §74). It is smooth as its Hessian has 20 simple zeros. ∎

It turns out that it is also possible to classify the orbit closures that are smooth in codimension one. The answer is particularly pretty in case the multiplicities of the s points of the d-tuple are all equal. In that case we may and will assume that they are all equal to one, so that $d = s$; call such a d-tuple *simple*. Note that the orbit closure of a simple d-tuple has at most one boundary component.

PROPOSITION 4.3. *The orbit closure of a simple d-tuple f is smooth in codimension one if and only if f is a special orbit for the action of a finite subgroup G of $PGL(2)$ on $\mathbb{P}^1$ (i.e., f is an orbit of length smaller than the order of G).*

PROOF: Let f be a simple d-tuple (so $d = s$). If $d = 1$ (resp. 2) the orbit closure of f is smooth; take $G = C_n$ (resp. D_n) for an $n \geq 2$. So we assume $d \geq 3$. From Proposition (3.4), the premultiplicity of the orbit closure of f along its only boundary component equals $\sum(2 + \text{mult. of } p \text{ in } H_{C_p})$, where the summation runs over the d points p of f. Assuming that the orbit closure of f is smooth in codimension one, it follows that the stabilizer of f has order $\geq 2d$. The "only if" part of the proposition follows. It remains to check that the orbit closures of the special orbits are indeed smooth in codimension one. This is an easy verification (see below). ∎

It is perhaps worthwhile to remark that the proposition above seems to constitute an answer to the question raised in [**Mukai-Umemura**], Remark (3.6): the $PGL(2)$-orbit closures of special G-orbits ($G \subset PGL(2)$ finite) may be characterized as the orbit closures of simple d-tuples that are smooth in codimension one.

The general case is somewhat harder. Let f be a d-tuple supported on $s \geq 3$ points, and assume that the orbit closure of f is smooth in codimension one. Suppose that there are s_a points with multiplicity a. Then the stabilizer of f has order at least $2s_a$. We conclude that f is supported on the special orbits for the action of its stabilizer G on $\mathbb{P}^1$. Clearly G is not cyclic, so there are 3 such orbits. Call them A, B and C, and write $f = A^a B^b C^c$ with a, b and c positive integers. Call A-multiplicity the contribution of the points of A to the multiplicity of the orbit closure of f along the orbit of $x^a y^{d-a}$. By Proposition (3.4), this equals

$$\frac{d_A(2 + \text{mult. of } p \text{ in the Hessian of } A_p^a B^b C^c)}{\text{order of } G}$$

where d_A is the degree of A, p a point of A and A_p the residual $(d_A - 1)$-tuple.

Similarly we define the B-multiplicity and the C-multiplicity. The following result is an immediate consequence.

PROPOSITION 4.4. *Let G be a finite, non-cyclic subgroup of $PGL(2)$. Denote by A, B and C the three special orbits for the action of G on $\mathbb{P}^1$. Let $f = A^a B^b C^c$, with a, b and c positive integers. Assume that G is the $PGL(2)$-stabilizer of f. The $PGL(2)$-orbit closure of f is smooth in codimension one if and only if a, b and c are mutually distinct and the A-multiplicity, the B-multiplicity and the C-multiplicity are equal to one.*

When one or two of a, b and c are zero, the proposition remains true, *mutatis mutandis*.

Computing the multiplicity of the Hessian at p becomes simpler when one chooses the right coordinates. Namely, p is one of the two fixed points of an element of G

(of order $m = $ (order of G)$/d_A$). Choose coordinates x, y so that p and the other fixed point are given by $x = 0$ and $y = 0$ respectively. Writing out A_p, B and C in these coordinates, we see that only powers of x^m occur:

$$A_p = y^{d_A - 1} + A_1 y^{d_A - 1 - m} x^m + A_2 y^{d_A - 1 - 2m} x^{2m} + \dots,$$
$$B = y^{d_B} + B_1 y^{d_B - m} x^m + B_2 y^{d_B - 2m} x^{2m} + \dots,$$
$$C = y^{d_C} + C_1 y^{d_C - m} x^m + C_2 y^{d_C - 2m} x^{2m} + \dots.$$

Now one immediately checks that the multiplicity of the Hessian of $A_p^a B^b C^c$ at p is $m - 2$ when

$$A_1 a + B_1 b + C_1 c \neq 0;$$

that it is $2m - 2$ when

$$A_1 a + B_1 b + C_1 c = 0 \qquad \text{and}$$
$$(A_1^2 - 2A_2)a + (B_1^2 - 2B_2)b + (C_1^2 - 2C_2)c \neq 0,$$

etc. Thus the A-multiplicity is 1, 2, ..., correspondingly.

Finally we list for each of the finite, non-cyclic subgroups G of $\mathrm{PGL}(2)$ the special orbits and the relevant equations. (Some of these results were obtained using Maple.)

(1) $G = D_n$: $A = xy$, $B = x^n + y^n$, $C = x^n - y^n$; the A-multiplicity is 1 iff

$$b \neq c;$$

the B-multiplicity is 1 iff

$$-a + \frac{(n-1)(n-2)}{6} b + \frac{n(n-1)}{2} c \neq 0;$$

the C-multiplicity is 1 iff

$$-a + \frac{n(n-1)}{2} b + \frac{(n-1)(n-2)}{6} c \neq 0.$$

(2) $G = A_4$: $A = x^4 + 2\sqrt{-3}x^2 y^2 + y^4$, $B = x^4 - 2\sqrt{-3}x^2 y^2 + y^4$, $C = x^5 y - xy^5$; the A-multiplicity is 1 iff

$$a - 8b + 20c \neq 0,$$

otherwise it is 2; the B-multiplicity is 1 iff

$$8a - b - 20c \neq 0,$$

otherwise it is 2; the C-multiplicity is 1 if

$$a \neq b;$$

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it is 2 when $a = b$ (unless $c = 14a = 14b$, in which case it is 4); note however that when $a = b$ the stabilizer is S_4, so the actual multiplicities are 1, respectively 2 (see also below).

(3) $G = S_4$: $A = x^5y - xy^5$, $B = x^8 + 14x^4y^4 + y^8$, $C = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$; the A-multiplicity is 1 iff

$$a - 14b + 33c \neq 0,$$

otherwise it is 2; the B-multiplicity is 1 iff

$$20a - 7b - 88c \neq 0,$$

otherwise it is 2; the C-multiplicity is 1 iff

$$45a - 84b - 11c \neq 0,$$

it is 2 when $45a - 84b - 11c = 0$, unless $(a, b, c) \sim (5852, 561, 19656)$, in which case it is 3.

(4) $G = A_5$:

$$A = x^{11}y + 11x^6y^6 - xy^{11},$$
$$B = x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20},$$
$$C = x^{30} + 522x^{25}y^5 - 10005x^{20}y^{10} - 10005x^{10}y^{20} - 522x^5y^{25} + y^{30};$$

the A-multiplicity is 1 iff

$$11a - 228b + 522c \neq 0,$$

otherwise it is 2; the B-multiplicity is 1 iff

$$88a - 57b - 580c \neq 0,$$

otherwise it is 2; the C-multiplicity is 1 iff

$$99a - 285b - 58c \neq 0,$$

it is 2 otherwise, unless $(a, b, c) \sim (26864005, 431607, 43733250)$, in which case it is 3.

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