# The characteristic numbers of smooth plane cubics

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Abstract. The characteristic numbers for the family of smooth plane cubics are computed, verifying results of Maillard and Zeuthen

§1 **Introduction.** The last few years have witnessed a revived interest in the search for the 'characteristic numbers' of families, i.e. the numbers of elements in a family which are tangent to assortments of linear subspaces in general position in the ambient projective space. By the 'contact Theorem' of Fulton-Kleiman-MacPherson, these numbers determine the numbers of varieties in the family that satisfy tangency conditions to arbitrary configurations of projective varieties: this justifies the central role of the computation of the characteristic numbers in the field of enumerative geometry.

The problem received much attention in the last century, when in fact it contributed significantly to the development of algebraic geometry. Schubert's "Kalkül der abzälenden Geometrie" ([**S**]), published in 1879, is a compendium of the results obtained in a span of some decades by Schubert himself, Chasles, Halphen, Zeuthen and others. The validity of these achievements was soon questioned: in requesting rigorous foundations for algebraic geometry, Hilbert's 15th problem (1900) explicitly asked for a justification of the results in Schubert's book. Algebraic geometry found its foundations in the fifties; the challenge of justifying enumerative geometry had to wait somewhat longer to be accepted.

By now, most of the results in the "Kalkül der abzälenden Geometrie" have been verified or corrected, but the enterprise is not yet completed. While rich satisfactory theories are now available for quadrics (Van der Waerden, Vainsencher, Demazure, De Concini-Procesi, Laksov, Thorup-Kleiman, Tyrell, etc.) and triangles (Collino-Fulton, Roberts-Speiser), and much is known about twisted cubics (Kleiman-Strømme-Xambó), the families of plane curves still offer results which were 'known' in the last century and cannot be claimed such now.

The achievements of the classic school are here quite impressive. By 1864 Chasles (and others) had settled conics; already in 1871 a student of his, M.S. Maillard, computed in his thesis ([**M**]) the characteristic numbers for many families of plane cubic curves, including cuspidal, nodal, and smooth ones. One year later H.G. Zeuthen published a series of three amazingly short papers ([**Z1**]) again computing the numbers for cuspidal, nodal and smooth cubics; his results agree with Maillard's. Zeuthen finally published in 1873 a long analysis for plane curves of *any* degree ([**Z2**]), giving as an application the computation of the characteristic numbers for families of plane quartics. Apparently, noone ever tried to explicitly work out higher degree cases.

The problem for cubics or higher degree curves remained untouched - and therefore eventually unsettled- for at least one century. Then Sacchiero (1984) and Kleiman-Speiser (1985) verified Zeuthen and Maillard's results for *cuspidal* and *nodal* plane cubics. Kleiman and Speiser's approach replicates and advances Zeuthen and Maillard's, so it is expected to lead eventually to the verification of the numbers for the family of *smooth* cubics; but that program is not completed yet. Also, Sterz (1983) constructed a variety of 'complete cubics', by a sequence of 5 blow-ups over the $\mathbb{P}^9$ of plane cubics, giving some intersection relations ([**St**]).

Later, I independently constructed the same variety, by the same sequence of blow-ups. My approach was in a sense more 'geometric' than Sterz's, and I was able to use this variety to actually compute the characteristic numbers for the family of smooth plane cubics. The result once more agrees with Zeuthen and Maillard's.

There is an important difference between this approach and the classical one. Maillard and Zeuthen were computing the numbers by relating them to characteristic numbers of other more special families (e.g. cuspidal and nodal cubics); here, one aims directly to solving the specific problem for smooth cubics, and other families don't enter into play. This makes the problem more accessible in a sense, but it may on the other hand sacrifice the 'general picture' to the specific result.

In this note I describe the blow-up construction and the computation of the numbers. Full details appear, together with partial results for curves of higher degree, in my Ph.D. thesis ([**A**]), written at Brown under the supervision of W. Fulton.

**Aknowledgements.** I wish to thank A. Collino and W. Fulton for suggesting the problem, and for constant guidance and encouragement.

**§2 The problem and the approach.** Let $n_p, n_\ell$ be integers, with $n_p + n_\ell = 9$. The question to be answered is:

*How many smooth plane cubics contain $n_p$ points and are tangent to $n_\ell$ lines in general position?*

The set of smooth plane cubics is given a structure of variety by identifying it with an open subvariety $U$ of the $\mathbb{P}^9$ parametrizing all plane cubics. The conditions 'containing a point' and 'tangent to a line' determine divisors in $U$; call them 'point-conditions' and 'line-conditions' respectively. The question then translates into one of cardinality of intersection of $n_p$ point-conditions and $n_\ell$ line-conditions in $U$.

One verifies that for general choice of points and lines the conditions intersect transversally in $U$, so that actually the cardinality of the intersection can be computed as intersection number of the divisors.

The first impulse is of course to work in the $\mathbb{P}^9$ that compactifies $U$: closing the conditions to divisors of $\mathbb{P}^9$ (one obtains hyperplanes from point-conditions, hypersurfaces of degree 4 from line-conditions), and using Bézout's Theorem to compute the intersection numbers. This works if $n_p \geq 5$: in this case the intersection of the divisors in $\mathbb{P}^9$ is in fact contained in $U$, and the result given by Bézout's Theorem is correct. If $n_p \leq 4$, non-reduced cubics appear in the intersection of the divisors in $\mathbb{P}^9$, since a curve containing a multiple component is 'tangent' to any line and clearly one can always find non-reduced cubics containing any 4 or less given points.

The conclusion is that $\mathbb{P}^9$ is not the 'right' compactification of the variety $U$ of smooth cubics for this problem, because all line-conditions in $\mathbb{P}^9$ contain the locus of non-reduced cubics.

The intersection of all line-conditions is in fact a subscheme of $\mathbb{P}^9$ supported over the locus of non-reduced cubics. If we could blow-up $\mathbb{P}^9$ along this subscheme, this would provide us with a compactification of $U$ in which the proper transforms of the point- and line-conditions don't intersect outside $U$, and taking their intersection product would answer the original question. But performing such a task requires much non-trivial information about the subscheme, and we are not able to proceed directly.

What we can perform without losing control of the situation is the blow-up of $\mathbb{P}^9$ along a certain smooth subvariety of the locus of non-reduced cubics. The blow-up creates another compactification of $U$, in which one can again find the support of the intersection of the 'line-conditions' (i.e., of the closure of the line-conditions of $U$). Again, a smooth subvariety -in fact, a component- of this locus can be chosen as a center of a new blow-up, creating a new compactification. The process can be repeated, under the heuristic principle that at each step, blowing-up the 'largest' possible non-singular subvariety/component of the intersection of all line-conditions should somehow simplify the situation.

In fact, 5 blow-ups do the job in this case: a non-singular compactification of $U$ is produced in which 9 conditions intersect only inside $U$. The knowledge of the Chern classes of the normal bundles of the centers of the blow-ups is then the essential ingredient needed to compute the intersections and obtain the characteristic numbers. An intersection formula (see §4) that explicitly relates intersections under blow-ups can be used to reach the result.

Apparently, this step (the computation of the Chern classes of the normal bundles and their utilization to get the characteristic numbers) is the only one missing in Sterz's work.

Alternatively, one can use the same information to compute the Segre class of the scheme-theoretic intersection of all line-conditions in $\mathbb{P}^9$, and apply Fulton's intersection formula ([**F**, Proposition 9.1.1]). This Segre class has interesting symmetries, which may shed some light on the internal structure of this scheme.

**§3 The blow-ups.** In this section I will briefly describe the varieties obtained via the 5 blow-ups. Details are provided in [**A**, Chapter 2].

The diagram

$$\begin{tikzcd}
 & \widetilde{V} = V_5 \arrow[d] & & \\
 & V_4 & B_4 = \mathbb{P}(\mathcal{L}) \arrow[l] \arrow[d] & \\
 & V_3 \arrow[u, leftarrow] \arrow[d] & B_3 = S_3 \arrow[l] \arrow[d] & \mathrm{B}\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \arrow[l, "\sim"'] \\
B_2 \arrow[r] \arrow[d, "\mathbb{P}^3-bundle"'] & V_2 \arrow[d] & S_2 \arrow[l] \arrow[d] & \mathrm{B}\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \arrow[l, "\sim"'] \\
B_1 \arrow[r] \arrow[d, "\mathbb{P}^2-bundle"'] & V_1 \arrow[d] & S_1 \arrow[l] \arrow[d] & \mathrm{B}\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \arrow[l, "\sim"'] \arrow[d] \\
v_3(\check{\mathbb{P}}^2) = B_0 \arrow[r] & \mathbb{P}^9 = V_0 & S = S_0 \arrow[l] & \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \arrow[l]
\end{tikzcd}$$

contains most of the notations that will be explained in this section.

$S_0$ is the locus of non-reduced cubics, $B_0 = v_3(\check{\mathbb{P}}^2) \hookrightarrow \mathbb{P}^9$ is the Veronese of triple lines. $B_i$ will be the centers of the blow-ups, $V_i$ will be the blow-up $\mathrm{B}\ell_{B_{i-1}} V_{i-1}$ of $V_{i-1}$ along $B_{i-1}$, $S_i$ will be the proper transforms of $S_{i-1}$ under the i-th blow-up.

$\mathcal{L}$ is a certain sub-line bundle of the normal bundle $N_{B_3} V_3$ of $B_3$ in $V_3$. $\Delta$ is the diagonal in $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.

Also, $E_i$ will be the exceptional divisor of the i-th blow-up, and 'line-conditions in $V_i$' will be the closure in $V_i$ of the line-conditions of $U$: i.e., the line-conditions in $V_i$ will be the proper transforms of the line-conditions in $V_{i-1}$.

For each blow-up I will describe the intersection of all line-conditions and indicate the choice of the center of the next blow-up. The basic strategy is to blow-up along the 'largest possible' non-singular subvariety/component of the intersection of all line-conditions. In fact, the first three blow-ups *desingularize* the support of this intersection, the last two separate the conditions.

§**3.0 The $\mathbb{P}^9$ of plane cubics.** We noticed already that the intersection of all line-conditions in $\mathbb{P}^9$ is supported on the locus $S_0$ of non-reduced cubics. This locus is the image of a map

$$\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \xrightarrow{\phi} \mathbb{P}^9$$

sending the pair of lines $(\lambda, \mu)$ to the cubic consisting of the line $\lambda$ and of a double line supported on $\mu$.

The map $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \xrightarrow{\phi} S_0$ is an isomorphism off the diagonal $\Delta$ in $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$; therefore $S_0$ is non-singular off the (smooth) locus $B_0 = \phi(\Delta)$ of triple lines. In fact $S_0$ is singular along $B_0$.

$B_0$ is the center of the first blow-up.

§**3.1 The first blow-up.** Let $V_1$ be the blow-up of $\mathbb{P}^9$ along $B_0$, $E_1$ the exceptional divisor, $S_1$ the proper transform of $S_0$.

$S_1$ is isomorphic to the blow-up $\mathrm{B}\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ of $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ along the diagonal (call $e$ the exceptional divisor of this blow-up); in particular, it is non-singular.

The line-conditions in $V_1$ intersect along the smooth 4-dimensional $S_1$ and along a smooth 4-dimensional subvariety of $E_1$.

To see this, notice that the line-condition in $\mathbb{P}^9$ corresponding to a line $\ell$ has multiplicity 2 along $B_0$, and tangent cone at a triple line $\lambda^3$ supported on the hyperplane of cubics containing $\lambda \cap \ell$. Thus, the tangent cones at $\lambda^3$ to all line-conditions in $\mathbb{P}^9$ intersect along the 5-dimensional space of cubics containing $\lambda$. It follows that the normal cones to $B_0$ in the line-conditions intersect in a rank-3 vector subbundle of $N_{B_0}\mathbb{P}^9$, and correspondingly that the line-conditions in $V_1$ intersect also along a $\mathbb{P}^2$-bundle over $B_0$ contained in $E_1$.

Call this subvariety $B_1$, and choose it as the center for the next blow-up. $B_1$ intersects $S_1 \cong \mathrm{B}\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ along the exceptional divisor $e$.

§**3.2 The second blow-up.** Let $V_2$ be the blow-up of $V_1$ along $B_1$, $E_2$ the exceptional divisor, $\widetilde{E}_1, S_2$ the proper transforms of $E_1, S_1$ respectively.

$S_2$ is the blow-up of $S_1$ along a divisor, thus it is isomorphic to $S_1$ and hence to $\mathrm{B}\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.

A coordinate computation shows that the line-conditions in $V_1$ are generically smooth along $B_1$, and tangent to $E_1$. As a consequence, their proper transforms intersect in $E_2$ along $\widetilde{E}_1 \cap E_2$, which is a $\mathbb{P}^3$-bundle over $B_1$ contained in $E_2$.

Therefore the line-conditions in $V_2$ intersect along the smooth 4-dimensional $S_2$ and along a smooth 7-dimensional subvariety of $E_2$.

Choose this subvariety as the new center, call it $B_2$.

**§3.3 The third blow-up.** Let $V_3$ be the blow-up of $V_2$ along $B_2$, $E_3$ the exceptional divisor, $S_3$ the proper transform of $S_2$.

Again, $S_3$ is isomorphic to $\mathrm{B}\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.

$E_3$ is a $\mathbb{P}^1$-bundle over $B_2$. In each fiber of this bundle there are two distinguished distinct points $r_1, r_2$: namely the intersections with the proper transforms of $\widetilde{E}_1$ and $E_2$. Now, over any point in $B_2$ away from $S_3 \cap E_3$, one can find line-conditions that hit the fiber precisely at $r_1$ or precisely at $r_2$. This implies that over such points the line-conditions in $V_3$ cannot intersect.

Thus the line-conditions in $V_3$ intersect only along the smooth 4-dimensional $S_3$.

This completes the 'desingularization of the support' of the intersection of all line-conditions, and we are ready to choose $B_3 = S_3$ as the next center.

**§3.4 The fourth blow-up.** Let $V_4$ be the blow-up of $V_3$ along $B_3$, $E_4$ the exceptional divisor.

The line-conditions in $V_4$ meet along a subvariety of the exceptional divisor $E_4 = \mathbb{P}(N_{B_3}V_3)$. Notice that above $B_3 - E_3 \cong S_0 - B_0$, $E_4$ restricts to $\mathbb{P}(N_{S_0 - B_0}\mathbb{P}^9)$. Now, the tangent hyperplanes to the line-conditions in $\mathbb{P}^9$ at a non-reduced cubic $\lambda\mu^2 \in S_0 - B_0$ intersect in the 5-dimensional space of cubics containing $\mu$. It follows that the line-conditions in $V_4$ meet above $B_3 - E_3$ along the projectivization of a line-subbundle of $\mathbb{P}(N_{B_3 - E_3}V_3)$. This fact holds on the whole of $B_3$: the line-conditions in $V_4$ intersect along a smooth 4-dimensional subvariety of $E_4 = \mathbb{P}(N_{B_3}V_3)$, which is the projectivization $\mathbb{P}(\mathcal{L})$ of a line-subbundle of $N_{B_3}V_3$.

Choose $\mathbb{P}(\mathcal{L})$ to be the next center $B_4$.

**§3.5 The fifth blow-up.** Let $V_5$ be the blow-up of $V_4$ along $B_4$, $E_5$ the exceptional divisor, $\widetilde{E}_4$ the proper transform of $E_4$.

Finally, the intersection of all line-conditions is empty in $V_5$.

The verification of this fact is similar to the one in 3.3. Here, each fiber of $E_5$ over a point of $B_4$ is a 4-dimensional projective space; in this $\mathbb{P}^4$ lies a distinguished $\mathbb{P}^3$, namely the intersection of the fiber with $\widetilde{E}_4$. Now, one can produce line-conditions whose intersection is disjoint from this $\mathbb{P}^3$, and a line-condition which intersects the fiber precisely along this $\mathbb{P}^3$. Thus the intersection of the line-conditions must be empty.

$V_5$ is the compactification of $U$ we were looking for.

By slightly refining the arguments, one sees that the intersection of 9 point/line-conditions in general position in $V_5$ must be contained in $U$. The characteristic numbers are then the intersection numbers of the conditions in $V_5$, and one can proceed with the actual computation.

§**4 The numbers.** The essential ingredients to obtain the characteristic numbers from the construction in §3 are the Chern classes of the normal bundles of the centers of the blow-ups. In fact this information would be enough to determine the whole Chow ring of the blow-ups; but we don't need that much. We have 9 divisors in $\mathbb{P}^9$, and we wish to compute the intersection numbers of their proper transforms in some blow-up of $\mathbb{P}^9$, once the Chern classes of the normal bundles of the centers are known.

This task can be accomplished directly, by repeatedly applying the

PROPOSITION. *Let $V$ be a non-singular n-dimensional variety, $B \overset{i}{\hookrightarrow} V$ a non-singular closed subvariety of $V$, $X_1, \dots, X_n$ divisors on $V$. Let $\widetilde{V} = B\ell_B V$, and $\widetilde{X}_1, \dots \widetilde{X}_n$ the proper transforms of $X_1, \dots X_n$. Moreover, let $e_i = e_B X_i$ be the multiplicity of $X_i$ along $B$. Then*

$$\int_{\widetilde{V}} \widetilde{X}_1 \cdots \widetilde{X}_n = \int_V X_1 \cdots X_n - \int_B \frac{(e_1[B] + i^*[X_1]) \cdots (e_n[B] + i^*[X_n])}{c(N_B V)}.$$

This specializes to well-known formulas when $B$ is a point, and is itself a specialization of a more general relation among Segre classes (see [**A**, Chapter 1]). An elementary proof of the form stated here can be obtained by expanding

$$\int_V X_1 \cdots X_n = \int_{\widetilde{V}} ([\widetilde{X}_1] + e_1[E]) \cdots ([\widetilde{X}_n] + e_n[E])$$

($E$ is the exceptional divisor) and recalling that $\sum_{i \geq 0} [E]^i$ pushes forward to $c(N_B V)^{-1}$ by Corollary 4.2 and Proposition 4.1(a) in [**F**].

What we need to compute the intersection numbers of the conditions in $V_5$ is then, for each $V_i$:

(1) The Chern classes of $N_{B_i} V_i$;
(2) The multiplicities of the conditions in $V_i$ along $B_i$;
(3) The Chow ring of $B_i$.

We will now indicate how this information can be obtained.

As for the multiplicities, they are obtained along the construction: the line-conditions in $\mathbb{P}^9$ have multiplicity 2 along the locus $B_0$ of triple lines, while line-conditions in $V_i$, $i > 0$, are generically smooth (hence have multiplicity 1) along $B_i$. Also, point-conditions never contain $B_i$, so their multiplicities along the centers are always 0.

The Chow rings and the normal bundles of the centers can be obtained as follows.

$B_0$ is the locus of cubics consisting of 'triple lines', hence it is isomorphic to $\mathbb{P}^2$; call $h$ the hyperplane class in $B_0$. In fact $B_0$ is the third Veronese imbedding of $\mathbb{P}^2$ in $\mathbb{P}^9$: it follows that

$$c(N_{B_0} \mathbb{P}^9) = \frac{(1 + 3h)^{10}}{(1 + h)^3}.$$

$B_1$ is a $\mathbb{P}^2$-bundle over $B_0$, thus its Chow ring is generated by the pull-back $h$ of $h$ from $B_0$ and the class $\epsilon$ of the universal line bundle $\mathcal{O}_{B_1}(-1)$. A closer analysis of the situation (see §3.1) reveals that $B_1$ is actually isomorphic to the projectivization of the normal bundle to the locus of double lines in the $\mathbb{P}^5$ of conics. This determines the relations between $h$ and $\epsilon$, and gives substantial information about the imbedding $B_1 \hookrightarrow E_1$. $N_{B_1}V_1$ is an extension of $N_{B_1}E_1$ and $N_{E_1}V_1$, and one obtaines

$$c(N_{B_1}V_1) = (1+\epsilon)\frac{(1+3h-\epsilon)^{10}}{(1+2h-\epsilon)^6}.$$

$B_2$ is a $\mathbb{P}^3$-bundle over $B_1$: its Chow ring is generated by the pull-backs $h, \epsilon$ of $h, \epsilon$ from $B_1$ and by the class $\varphi$ of $\mathcal{O}_{B_2}(-1)$. Recall from 3.2 that $B_2 = \widetilde{E}_1 \cap E_2$: i.e., $B_2$ is the exceptional divisor in the blow-up of $E_1$ along $B_1$, and hence it is isomorphic to $\mathbb{P}(N_{B_1}E_1)$. This observation gives relation among $h, \epsilon, \varphi$. Also, $B_2 = \widetilde{E}_1 \cap E_2$ gives at once

$$c(N_{B_2}V_2) = (1+\varphi)(1+\epsilon-\varphi).$$

$B_3 = S_3$ is isomorphic to the blow-up $\mathrm{B}\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ of $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ along the diagonal. Its Chow ring is then generated by the pull-backs $\ell, m$ of the hyperplanes from the factors, and by the exceptional divisor $e$. One obtaines the relations

$$\int_{B_3} \ell^2 m^2 = 1, \quad \int_{B_3} e^2\ell^2 = -1, \quad \int_{B_3} e^2 m^2 = -1,$$
$$\int_{B_3} e^3\ell = -3, \quad \int_{B_3} e^3 m = -3, \quad \int_{B_3} e^4 = -6.$$

The total Chern class of $N_{B_3}V_3$ can be obtained as $\frac{c(TV_3)}{c(TB_3)}$: both $c(TV_3)$ and $c(TB_3)$ can be computed using the formula for Chern classes of blow-ups (Theorem 15.4 in [**F**]). The result is

$$\begin{aligned} c(N_{B_3}V_3) &= 1+7\ell+17m-16e+126m^2+99\ell m+21\ell^2-315e\ell+105e^2+582\ell m^2 \\ &+237\ell^2 m-2517e\ell^2+1611e^2\ell-358e^3+1026\ell^2m^2+9174e^2\ell^2-3912e^3\ell+652e^4. \end{aligned}$$

Finally, $B_4 = \mathbb{P}(\mathcal{L})$ is also isomorphic to $\mathrm{B}\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$; the Chern classes of $N_{B_4}V_4$ are easily obtained from $c_1(\mathcal{L})$, which can be computed directly as $3\ell+3m-4\epsilon$. One gets

$$\begin{aligned} c(N_{B_4}V_4) = 1-5\ell+5m+18m^2-27\ell m+3\ell^2+21e\ell-7e^2-30\ell m^2+75\ell^2 m \\ -225e\ell^2+135e^2\ell-30e^3+75\ell^2m^2. \end{aligned}$$

Once this information is obtained, 5 applications of the proposition for each number $n_p$ of points and $n_\ell$ of lines give the corresponding characteristic number. For example, the reader may now enjoy checking by hand that

$$\text{numbers of smooth cubics through 4 points and tangent to 5 lines} = \\ = 4^5 - 0 - 0 - 0 - 24 - 24 = 976,$$

or that

$$\text{numbers of smooth cubics through 3 points and tangent to 6 lines} = \\ = 4^5 - 0 - 0 - 0 - 390 - 282 = 3424.$$

The final result is the list

$$\begin{array}{rl}
1 & n_p = 9, n_\ell = 0 \\
4 & n_p = 8, n_\ell = 1 \\
16 & n_p = 7, n_\ell = 2 \\
64 & n_p = 6, n_\ell = 3 \\
256 & n_p = 5, n_\ell = 4 \\
976 & n_p = 4, n_\ell = 5 \\
3424 & n_p = 3, n_\ell = 6 \\
9766 & n_p = 2, n_\ell = 7 \\
21004 & n_p = 1, n_\ell = 8 \\
33616 & n_p = 0, n_\ell = 9
\end{array}$$

for the number of curves containing $n_p$ points and tangent to $n_\ell$ lines, agreeing with Maillard and Zeuthen.

**§5 Concluding remarks.** It seems plausible that the same procedure worked out here for cubics could in principle be executed to get the characteristic numbers for smooth quartics or for higher degree plane curves, but the usefulness of such an endeavor is questionable at this point. Until these 'blow-up constructions' are part of a general theory, the complication of the technical details is bound to keep the work at the level of brute force computation. Part of the construction (essentially the last two blow-ups) can in fact be carried out, giving the first 'non-trivial' characteristic number for smooth plane curves of any degree (see [**A**, Chapter 3]), but this seems to be in some sense a special case. The next 'non-trivial' number can still be computed for quartics (the results agree with Zeuthen's!), but not via a straightforward generalization from the computation for cubics ([**A**, Chapter 4]).

Perhaps Kleiman and Speiser's approach, pointing in the direction of Zeuthen's monumental 'general theory', will strike more deeply into the heart of the problem.

## References


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