Tensor Product of Picard Stacks

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2 Structure Theorem

3 Tensor Product

A 2-stack $\mathbb C$ over a site S is a 2-category fibered in 2-groupoids such that:

- $\operatorname{Hom}_{\mathbb{C}_U}(X, Y)$ is a stack over S/U.
- Every 2-descent datum is effective.

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A Picard 2-stack ${\mathbb C}$ is a 2-stack associated with

- a group-like structure
 - operation: $\otimes : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ morphism of stacks.
 - associativity: $(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$.
- a commutativity-like structure: $X \otimes Y \xrightarrow{\sim} Y \otimes X$.

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Remark: A Picard stack is a Picard 2-stack $\mathbb C$ whose only 2-morphisms are identities.

For any $U \in \mathsf{S}$, define a 2-groupoid \mathbb{C}_U as

- objects: $a \in A^0(U)$,
- 1-morphisms: $(f, a) \in A^{-1}(U) \times A^0(U)$ such that $(f, a) : a \rightarrow a + \lambda(f)$,
- 2-morphisms: $(\sigma, f, a) \in A^{-2}(U) \times A^{-1}(U) \times A^{0}(U)$ such that $(\sigma, f, a) : (f, a) \Rightarrow (\delta(\sigma) + f, a).$

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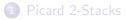
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 \mathbb{C} is a pre-pre- 2 stack $\xrightarrow{\text{stackify} \times 2}$ \mathbb{C}^{\sim} is a Picard 2-stack. <u>Notation</u>: $\mathbb{C}^{\sim} = [A^{\bullet}]^{\sim}$ (associated Picard 2-stack)

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 \mathbb{C} is a pre-pre- 2 stack $\xrightarrow{\text{stackify} \times 2}$ \mathbb{C}^{\sim} is a Picard 2-stack. <u>Notation</u>: $\mathbb{C}^{\sim} = [A^{\bullet}]^{\sim}$ (associated Picard 2-stack) **Special Case**: When $A^{-2} = 0$, $[A^{\bullet}]^{\sim}$ is a Picard stack.







Structure Theorem for Picard 2-Stacks

Theorem (T.)

The trihomomorphism

$$2ch: T^{[-2,0]}(S) \longrightarrow 2PIC(S)$$

defined by sending A^{\bullet} to $[A^{\bullet}]^{\sim}$ is a triequivalence where $T^{[-2,0]}(S)$ is the tricategory of A^{\bullet} and $2P_{IC}(S)$ is the 3-category of Picard 2-stacks.

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Corollary (Deligne, SGA4 Exposé XVIII)

For any Picard stack \mathscr{C} , there exists there exists a morphism of abelian sheaves $A^{-1} \rightarrow A^0$ whose associated Picard stack $[A^{-1} \rightarrow A^0]^{\sim}$ is equivalent to \mathscr{C} .

Picard 2-Stacks

2 Structure Theorem



Tensor Product of Picard Stacks

The tensor product of two Picard stacks \mathscr{C}_1 and \mathscr{C}_2 is

$$\mathscr{C}_1 \otimes \mathscr{C}_2 := [\tau_{\geq -1} (A_1^{\bullet} \otimes^{\mathbb{L}} A_2^{\bullet})]^{\sim}$$

where $\mathscr{C}_1 \simeq [A_1^{\bullet}]^{\sim}$ and $\mathscr{C}_2 \simeq [A_2^{\bullet}]^{\sim}$ and $- \otimes^{\mathbb{L}} -$ is the derived tensor product.

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Theorem

Let \mathscr{C}_1 and \mathscr{C}_2 be two Picard stacks. There exists a biadditive functor

$$\bigotimes : \mathscr{C}_1 \times \mathscr{C}_2 {\rightarrow} \mathscr{C}_1 \otimes \mathscr{C}_2$$

such that for any Picard stack ${\mathscr C}$ the functor

 $\mathsf{HOM}(\mathscr{C}_1\otimes\mathscr{C}_2,\mathscr{C}) \longrightarrow \mathsf{HOM}(\mathscr{C}_1,\mathscr{C}_2;\mathscr{C}),$

defined by sending $F : \mathscr{C}_1 \otimes \mathscr{C}_2 \rightarrow \mathscr{C}$ to $F \circ \bigotimes : \mathscr{C}_1 \times \mathscr{C}_2 \rightarrow \mathscr{C}$ is an equivalence of Picard stacks.

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Theorem (in progress)

Let \mathbb{C}_1 and \mathbb{C}_2 be two Picard 2-stacks. There exists a biadditive 2-functor

$$\bigotimes: \mathbb{C}_1 \times \mathbb{C}_2 \longrightarrow \mathbb{C}_1 \otimes \mathbb{C}_2$$

such that for any Picard 2-stack \mathbb{C} , the morphism

$$\mathbb{H}\mathsf{OM}(\mathbb{C}_1\otimes\mathbb{C}_2,\mathbb{C}) \longrightarrow \mathbb{H}\mathsf{OM}(\mathbb{C}_1,\mathbb{C}_2;\mathbb{C})$$

defined by composing \bigotimes is an equivalence of Picard 2-stacks.

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