Tensor Product of Picard Stacks

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Outline

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Picard 2-Stacks

A 2-stack $\mathbb{C}$ over a site S is a 2-category fibered in 2-groupoids such that:

- $\mathrm{Hom}_{\mathbb{C}_U}(X, Y)$ is a stack over S/U.
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A Picard 2-stack $\mathbb{C}$ is a 2-stack associated with

- a group-like structure
  - operation: $\otimes : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ morphism of stacks.
  - associativity: $(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$.
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Remark: A Picard stack is a Picard 2-stack $\mathbb{C}$ whose only 2-morphisms are identities.

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- *objects*: $a \in A^0(U)$,
- *1-morphisms*: $(f, a) \in A^{-1}(U) \times A^0(U)$ such that $(f, a) : a \rightarrow a + \lambda(f)$,
- *2-morphisms*: $(\sigma, f, a) \in A^{-2}(U) \times A^{-1}(U) \times A^0(U)$ such that $(\sigma, f, a) : (f, a) \Rightarrow (\delta(\sigma) + f, a)$.

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Special Case: When $A^{-2} = 0$, $[A^\bullet]^\sim$ is a Picard stack.

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Structure Theorem for Picard 2-Stacks

Theorem (T.)

The trihomomorphism

$$2\mathrm{ch} : \mathrm{T}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{P{\scriptsize IC}}(\mathsf{S})$$

defined by sending $A^\bullet$ to $[A^\bullet]^\sim$ is a triequivalence where $\mathrm{T}^{[-2,0]}(\mathsf{S})$ is the tricategory of $A^\bullet$ and $2\mathrm{P{\scriptsize IC}}(\mathsf{S})$ is the 3-category of Picard 2-stacks.

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Corollary (Deligne, SGA4 Exposé XVIII)

For any Picard stack $\mathscr{C}$*, there exists there exists a morphism of abelian sheaves* $A^{-1} \to A^0$ *whose associated Picard stack* $[A^{-1} \to A^0]^\sim$ *is equivalent to* $\mathscr{C}$*.*

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Tensor Product of Picard Stacks

The tensor product of two Picard stacks $\mathscr{C}_1$ and $\mathscr{C}_2$ is

$$\mathscr{C}_1 \otimes \mathscr{C}_2 := [\tau_{\geq -1}(A_1^\bullet \otimes^{\mathbb{L}} A_2^\bullet)]^\sim$$

where $\mathscr{C}_1 \simeq [A_1^\bullet]^\sim$ and $\mathscr{C}_2 \simeq [A_2^\bullet]^\sim$ and $- \otimes^{\mathbb{L}} -$ is the derived tensor product.

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Theorem

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two Picard stacks. There exists a biadditive functor

$$\bigotimes : \mathscr{C}_1 \times \mathscr{C}_2 \to \mathscr{C}_1 \otimes \mathscr{C}_2$$

such that for any Picard stack $\mathscr{C}$ the functor

$$\mathrm{HOM}(\mathscr{C}_1 \otimes \mathscr{C}_2, \mathscr{C}) \longrightarrow \mathrm{HOM}(\mathscr{C}_1, \mathscr{C}_2; \mathscr{C}),$$

defined by sending $F : \mathscr{C}_1 \otimes \mathscr{C}_2 \to \mathscr{C}$ to $F \circ \bigotimes : \mathscr{C}_1 \times \mathscr{C}_2 \to \mathscr{C}$ is an equivalence of Picard stacks.

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Theorem (in progress)

Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be two Picard 2-stacks. There exists a biadditive 2-functor

$$\bigotimes : \mathbb{C}_1 \times \mathbb{C}_2 \longrightarrow \mathbb{C}_1 \otimes \mathbb{C}_2$$

such that for any Picard 2-stack $\mathbb{C}$, the morphism

$$\mathbb{H}\mathrm{OM}(\mathbb{C}_1 \otimes \mathbb{C}_2, \mathbb{C}) \longrightarrow \mathbb{H}\mathrm{OM}(\mathbb{C}_1, \mathbb{C}_2; \mathbb{C})$$

defined by composing $\bigotimes$ is an equivalence of Picard 2-stacks.