# Complexes of Abelian Sheaves and Picard 2-Stacks

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## Introduction

In SGA4 Exposé XVIII, Deligne studies the relation between Picard stacks and length 2 complexes of abelian sheaves, as well as the relation between the morphisms of such objects. He proves ([3], Proposition 1.4.15) that the functor

$$\mathrm{D}^{[-1,0]}(\mathsf{S}) \longrightarrow \mathrm{P{\small IC}}^{\flat}(\mathsf{S})$$

is an equivalence where $\mathrm{D}^{[-1,0]}(\mathsf{S})$ is the subcategory of the derived category of category of complexes of abelian sheaves $A^\bullet$ over a site $\mathsf{S}$ with $H^{-i}(A^\bullet) \neq 0$ only for $i = 0, 1$ and $\mathrm{P{\small IC}}^{\flat}(\mathsf{S})$ is the category of Picard stacks over $\mathsf{S}$ with 1-morphisms isomorphism classes of additive functors.

## Goal

Our purpose is to generalize the above result to Picard 2-stacks.

## Method

1) Define the 3-category of Picard 2-stacks $2\mathrm{P{\small IC}}(\mathsf{S})$.

2) Define the tricategory of length 3 complexes of abelian sheaves $\mathrm{T}^{[-2,0]}(\mathsf{S})$.

3) Construct a trihomorphism $2\wp$ from $\mathrm{T}^{[-2,0]}(\mathsf{S})$ to $2\mathrm{P{\small IC}}(\mathsf{S})$.

4) Prove that the trihomomorphism $2\wp$ is a triequivalence.

5) Deduce a generalization of Deligne's result for Picard stacks to Picard 2-stacks.

## 3-category of Picard 2-Stacks

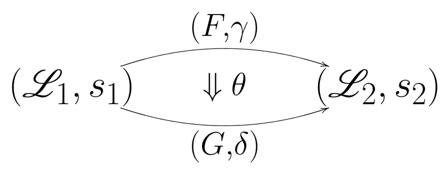
The detailed definition of *Picard 2-stack* over a site $\mathsf{S}$ as a fibered 2-category in 2-groupoids equipped with monoidal, braiding, group-like, and Picard structures can be found in Breen ([2],§8). For our purposes, we will define it as follows:

Let $A^\bullet = [A^{-2} \to A^{-1} \to A^0]$ be a complex of abelian sheaves where $\mathscr{A}$ is the Picard stack associated to $A^{-2} \to A^{-1}$, that is $\mathrm{T{\small ORS}}(A^{-2}, A^{-1})$. We define $\mathrm{T{\small ORS}}(\mathscr{A}, A^0)$ as Picard 2-stack associated to the $A^\bullet$. It consists of objects, 1-morphisms, and 2-mophisms defined as:

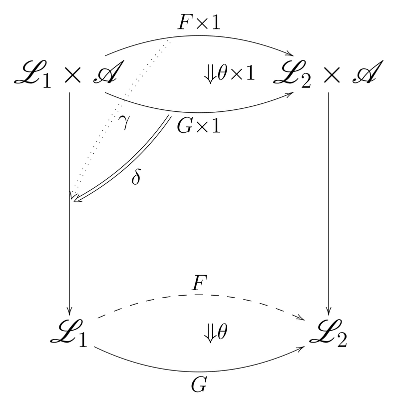
- An object is pair a $(\mathscr{L}, s)$ where $\mathscr{L}$ is an $\mathscr{A}$-torsor and $s : \mathscr{L} \to A^0$ is an $\mathscr{A}$-equivariant map.
- A 1-morphism from $(\mathscr{L}_1, s_1)$ to $(\mathscr{L}_2, s_2)$ is a pair $(F, \gamma)$

$$(F, \gamma) : (\mathscr{L}_1, s_1) \longrightarrow (\mathscr{L}_2, s_2),$$

  where $F$ is a $\mathscr{A}$-torsor morphism compatible with the torsor structure up to $\gamma$ and $s_2 \circ F = s_1$.
- A 2-morphism from $(F, \gamma)$ to $(G, \beta)$ is a natural 2-transformation $\theta$



  that makes the diagram commute.



We will see that $\mathrm{T{\small ORS}}(\mathscr{A}, A^0)$ is in a sense the only example of Picard 2-stacks.

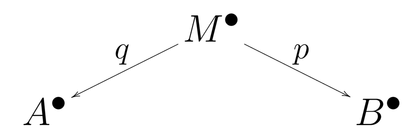
An *additive 2-functor* is a cartesian 2-functor between the underlying fibered 2-categories compatible with the monoidal, braided, and Picard structures carried by the fibers.

Picard 2-stacks over $\mathsf{S}$ form an obvious 3-category which we denote by $2\mathrm{P{\small IC}}(\mathsf{S})$. $2\mathrm{P{\small IC}}(\mathsf{S})$ has a hom-2-groupoid consisting of additive 2-functors, weakly invertible natural 2-transformations, and strict modifications. For any two Picard 2-stacks $\mathbb{P}$ and $\mathbb{Q}$, associated respectively to complexes $A^\bullet$ and $B^\bullet$, we denote this hom-2-groupoid by $\mathsf{Hom}(A^\bullet, B^\bullet)$.

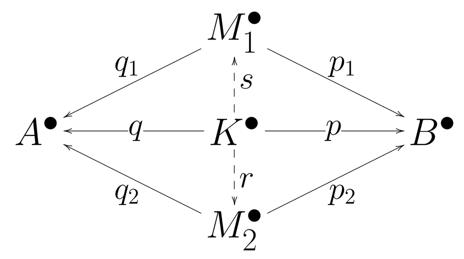
## Tricategory of Complexes of Abelian Sheaves $\mathrm{T}^{[-2,0]}(\mathsf{S})$

$\mathrm{T}^{[-2,0]}(\mathsf{S})$ is a tricategory of length 3 complexes of abelian sheaves placed in degrees $[-2, 0]$. For any two such complexes $A^\bullet$ and $B^\bullet$, its hom-bicategory $\mathsf{Frac}(A^\bullet, B^\bullet)$ is the bigroupoid that consists of objects, 1-morphisms, and 2-morphisms where

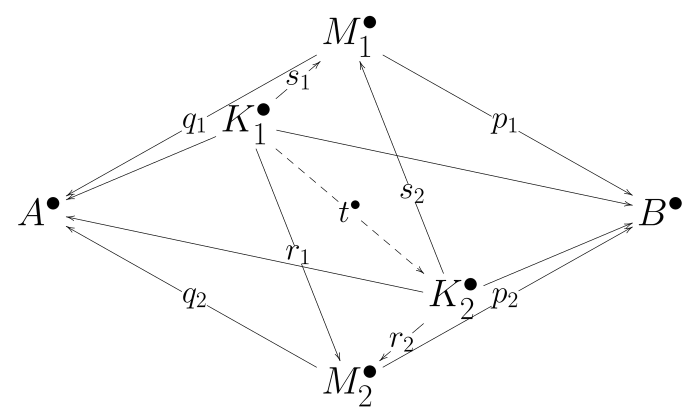
- An object is an ordered triple $(q, M^\bullet, p)$ called fraction



  with $M^\bullet$ a complex of abelian sheaves, $p$ a morphism of complexes, and $q$ a quasi-isomorphism.
- A 1-morphism from the fraction $(q_1, M_1^\bullet, p_1)$ to the fraction $(q_2, M_2^\bullet, p_2)$ is an ordered triple $(r, K^\bullet, s)$ with $K^\bullet$ a complex of abelian sheaves, $r$ and $s$ quasi-isomorphisms making the diagram



  commutative.
- A 2-morphism from the 1-morphism $(r_1, K_1^\bullet, s_1)$ to the 1-morphism $(r_2, K_2^\bullet, s_2)$ is an isomorphism $t^\bullet : K_1^\bullet \to K_2^\bullet$ of complexes of abelian sheaves such that the diagram that we will call "*diamond*"



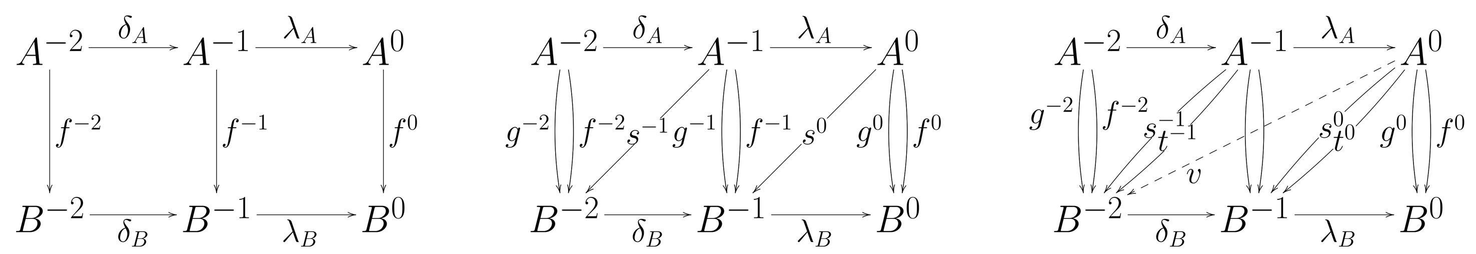
  commutes.

## Subtricategory of $\mathrm{T}^{[-2,0]}(\mathsf{S})$

$\mathrm{T}^{[-2,0]}(\mathsf{S})$ has a well known subtricategory $\mathrm{C}^{[-2,0]}(\mathsf{S})$. It has same objects as $\mathrm{T}^{[-2,0]}(\mathsf{S})$. For a pair of complexes of abelian sheaves $A^\bullet$ ,$B^\bullet$, its hom-2-groupoid $\mathrm{Hom}_{\mathrm{C}^{[-2,0]}(\mathsf{S})}(A^\bullet, B^\bullet)$ is the 2-groupoid associated to the complex

$$\mathrm{Hom}^{-2}(A^\bullet, B^\bullet) \longrightarrow \mathrm{Hom}^{-1}(A^\bullet, B^\bullet) \longrightarrow Z^0(\mathrm{Hom}^0(A^\bullet, B^\bullet))$$

of abelian groups. Explicitly $\mathrm{C}^{[-2,0]}(\mathsf{S})$ has same objects as $\mathrm{T}^{[-2,0]}(\mathsf{S})$ and for any two complexes of abelian sheaves $A^\bullet$ ,$B^\bullet$ its hom-2-groupoid has objects, 1-morphisms, and 2-morphisms defined respectively as:



with relations

$$\begin{aligned}
&g^0 - f^0 = \lambda_B \circ s^0,\\
&g^{-2} - f^{-2} = s^{-1} \circ \delta_A,\\
&g^{-1} - f^{-1} = \delta_B \circ s^{-1} + s^0 \circ \lambda_A,\\
&s^0 - t^0 = \delta_B \circ v,\\
&s^{-1} - t^{-1} = -v \circ \lambda_A.
\end{aligned}$$

It is easy to observe that $\mathrm{C}^{[-2,0]}(\mathsf{S})$ is a 3-category.

## Main Theorem

**Theorem.** *([4], Theorem 6.4) There is a triequivalence*

$$2\wp : \mathrm{T}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{P{\small IC}}(\mathsf{S}).$$

*defined by sending $A^\bullet$ to $\mathrm{T{\small ORS}}(\mathscr{A}, A^0)$.*

*Proof.* (Outline) The method that we adopt to prove our results is going to use mostly the language and techniques developed in [1] the paper of Aldrovandi and Noohi such as butterflies, torsors, etc. The main steps of the proof are:

- Construct the trihomomorphism $2\wp$ on $\mathrm{C}^{[-2,0]}(\mathsf{S})$.
- For any two complexes of abelian sheaves $A^\bullet$ and $B^\bullet$, show that the hom-bigroupoid $\mathsf{Frac}(A^\bullet, B^\bullet)$ is biequivalent to the hom-2-groupoid $\mathsf{Hom}(A^\bullet, B^\bullet)$. In particular, this means that for any morphism $F : \mathrm{T{\small ORS}}(\mathscr{A}, A^0) \to \mathrm{T{\small ORS}}(\mathscr{B}, B^0)$, there exists a fraction $(q, M^\bullet, p)$ such that $F \circ 2\wp(q) \simeq 2\wp(p)$.
- Use the 2nd step and the observation that $2\wp$ sends quasi-isomorphisms to equivalences, to extend $2\wp$ onto $\mathrm{T}^{[-2,0]}(\mathsf{S})$.
- Verify that $2\wp$ is essentially surjective, that is for any Picard 2-stack $\mathbb{P}$, there exists a complex of abelian sheaves $A^\bullet$ such that $\mathbb{P}$ is equivalent to $\mathrm{T{\small ORS}}(\mathscr{A}, A^0)$.

□

## Remark

The trihomomorphism $2\wp$ on $\mathrm{C}^{[-2,0]}(\mathsf{S})$ is not a triequivalence. A morphism of complexes of abelian sheaves $f \in Z^0(\mathrm{Hom}^0(A^\bullet, B^\bullet))$ is sent to a morphism $2\wp(f) : \mathrm{T{\small ORS}}(\mathscr{A}, A^0) \to \mathrm{T{\small ORS}}(\mathscr{B}, B^0)$ between associated Picard 2-stacks, but not all morphisms of Picard 2-stacks are obtained in this way. This means 2nd step of the proof does not hold with the hom-2-groupoid $\mathrm{Hom}_{\mathrm{C}^{[-2,0]}(\mathsf{S})}(A^\bullet, B^\bullet)$. The reason is the strictness of the 1-morphisms in $\mathrm{C}^{[-2,0]}(\mathsf{S})$ and in this sense, they are not geometric.

## Consequence of the Main Theorem

From the theorem, we deduce a generalization of Deligne's analogous result about Picard stacks in SGA4, Exposé XVIII to Picard 2-stacks.

**Corollary.** *([4],Corollary 6.5) The functor $2\wp$ induces an equivalence*

$$2\wp^{\flat\flat} : \mathrm{D}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{P{\small IC}}^{\flat\flat}(\mathsf{S})$$

*of categories.*

*Proof.* Denote by,

$2\mathrm{P{\small IC}}^{\flat\flat}(\mathsf{S})$: the category of Picard 2-stacks obtained from $2\mathrm{P{\small IC}}(\mathsf{S})$ by ignoring the modifications and taking as morphisms the equivalence classes of additive 2-functors.

$\mathrm{D}^{[-2,0]}(\mathsf{S})$: the subcategory of the derived category of category of complexes of abelian sheaves $A^\bullet$ over $\mathsf{S}$ with $H^{-i}(A^\bullet) \neq 0$ for $i = 0, 1, 2$.

Now, it is enough to observe from the definition of $\mathsf{Frac}(A^\bullet, B^\bullet)$ that

$$\pi_0(\mathsf{Frac}(A^\bullet, B^\bullet)) \simeq \mathrm{Hom}_{\mathrm{D}^{[-2,0]}(\mathsf{S})}(A^\bullet, B^\bullet),$$

where $\pi_0$ denotes the isomorphism classes of objects. Since the objects of $\mathrm{D}^{[-2,0]}(\mathsf{S})$ are same as the objects of $\mathrm{T}^{[-2,0]}(\mathsf{S})$, the essential surjectivity follows from the fact that $2\wp$ is essentially surjective. □

## Acknowledgements

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## References

[1] Ettore Aldrovandi and Behrang Noohi. Butterflies I: Morphisms of 2-group stacks. *Advances in Mathematics*, 221(3):687-773, 2009.

[2] Lawrence Breen. On the classification of 2-gerbes and 2-stacks. *Astérisque*, (225):160,1994.

[3] Pierre Deligne. La formule de dualité globale, 1973. SGA 4 III, Exposé XVIII.

[4] A. Emin Tatar. Length 3 complexes of abelian sheaves and picard 2-stacks. ArXiv:0906.2393v1