RESEARCH STATEMENT

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My research area is at the intersection of Algebraic Topology, Algebraic Geometry, and Homological Algebra and Category Theory. In particular, I am interested in higher dimensional algebraic structures such as n-categories and n-stacks.

My thesis is motivated by Deligne's work about Picard stacks. I generalize his result to Picard 2-stacks. Picard 2-stacks were defined by Breen in [4]. In my thesis, I describe the 3-category of Picard 2-stacks and exhibit an algebraic model for it in terms of length 3 complexes of abelian sheaves with a correct amount of weakness. That is, I define a tricategory of length 3 complexes of abelian sheaves with weak morphisms which then I prove it to be triequivalent to the 3-category of Picard 2-stacks.

In the following, I first describe my doctoral research. Then, I explain my plans for future research.

1. Doctoral Research

1.1. **Origin of the Problem.** In SGA4 Exposé XVIII [6], Deligne studies the relation between the 2-category of Picard stacks and the bicategory of length 2 complexes of abelian sheaves. He shows that these two categories are equivalent. This equivalence implies an important fact about a certain derived category. As it is known, derived category of an abelian category has the correct setting to do cohomology and from this equivalence, we deduce a geometric interpretation of the derived category of category of complexes of abelian sheaves. To be more precise, $\mathrm{P{\scriptsize IC}}^\flat(\mathsf{S})$ be the category of Picard stacks over S with 1-morphisms isomorphism classes of additive functors and let $\mathrm{D}^{[-1,0]}(\mathsf{S})$ be the subcategory of the derived category of category of complexes of abelian sheaves $A^\bullet$ over a site S with $H^{-i}(A^\bullet) \neq 0$ only for $i = 0, 1$.

Proposition. *[6, Proposition 1.4.15] The functor*

$$\mathrm{D}^{[-1,0]}(\mathsf{S}) \longrightarrow \mathrm{P{\scriptsize IC}}^\flat(\mathsf{S})$$

defined by sending a complex of abelian sheaves $A^{-1} \to A^0$ to $\mathrm{T{\scriptsize ORS}}(A^{-1}, A^0)$ the Picard stack of A^{-1}-torsors that become trivial over A^0 is an equivalence.

1.2. **Description of My Work.** In my thesis, I study the question whether the above result can be generalized to the case of Picard 2-stacks and length 3 complexes of abelian sheaves. I adopt the methodology of Deligne and investigate the relation between the 3-category of Picard 2-stacks and the tricategory of length 3 complexes of abelian sheaves.

Let $A^\bullet : A^{-2} \to A^{-1} \to A^0$ be a length 3 complex of abelian sheaves. It is known that $\mathrm{T{\scriptsize ORS}}(\mathscr{A}, A^0)$ the 2-stack of $\mathscr{A}$-torsors trivial that become trivial over A^0 where $\mathscr{A} = \mathrm{T{\scriptsize ORS}}(A^{-1}, A^0)$ is the Picard 2-stack associated to $A^\bullet$. Moreover this construction is functorial, that is there exists a functor

$$2\wp : \mathrm{C}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{P\scriptsize{IC}}(\mathsf{S}),$$

where $\mathrm{C}^{[-2,0]}(\mathsf{S})$ denotes the 3-category of length 3 complexes of abelian sheaves with morphisms of complexes, homotopies, and morphisms of homotopies and $2\mathrm{P\scriptsize{IC}}(\mathsf{S})$ denotes the 3-category of Picard 2-stacks with additive 2-functors, natural 2-transformations, and modifications. A Picard 2-stack is defined by Breen in [4] as a fibered 2-category in 2-groupoids over a site S that satisfies effective 2-descent condition and whose fiberes are Picard 2-categories. A Picard 2-category is a monoidal 2-category that is braided and group-like and where braiding satisfies certain commutativity relations. Additive 2-functors are cartesian 2-functors between the underlying fibered 2-categories compatible with monoidal, braiding and group-like structures. We also assume that natural 2-transformations are invertible up to modifications and modifications are strictly invertible.

The functor $2\wp$ is not an equivalence with these definitions of $\mathrm{C}^{[-2,0]}(\mathsf{S})$ and $2\mathrm{P\scriptsize{IC}}(\mathsf{S})$, because not all additive 2-functors are obtained from morphisms of length 3 complexes of abelian sheaves. In this sense, the 1-morphisms of $\mathrm{C}^{[-2,0]}(\mathsf{S})$ are not geometric and the reason is their strictness. We resolve this problem by weakening these morphisms. We introduce a tricategory $\mathrm{T}^{[-2,0]}(\mathsf{S})$ with same objects as $\mathrm{C}^{[-2,0]}(\mathsf{S})$ and with 1-morphisms between any two complexes $A^\bullet$ and $B^\bullet$ that form a bigroupoid $\mathsf{Frac}(A^\bullet, B^\bullet)$ whose

- objects are ordered pairs $(q, M^\bullet, p)$ called fractions given by the diagram
$$A^\bullet \xleftarrow{\;q\;} M^\bullet \xrightarrow{\;p\;} B^\bullet$$
where q is a quasi-isomorphism and p is a morphism of complexes.
- 1-morphisms from $(q_1, M_1^\bullet, p_1)$ to $(q_2, M_2^\bullet, p_2)$ are given by the diagram
$$\begin{tikzcd} & M_1^\bullet \arrow[dl, "q_1"'] \arrow[dr, "p_1"] & \\ A^\bullet & K^\bullet \arrow[l, "q"] \arrow[r, "p"'] \arrow[u, dashed, "s"] \arrow[d, dashed, "r"'] & B^\bullet \\ & M_2^\bullet \arrow[ul, "q_2"] \arrow[ur, "p_2"'] & \end{tikzcd}$$
where each region commutes.
- 2-morphisms between two diagrams of the form 1.2 which are given by the commutative diagrams called "*diamonds*"(see [13]).

Then I prove in my dissertation the following theorem:

Theorem. *The trihomomorphism*

$$\mathrm{T}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{P\scriptsize{IC}}(\mathsf{S})$$

defined by sending $A^\bullet : A^{-2} \to A^{-1} \to A^0$ *to* $\mathrm{T\scriptsize{ORS}}(\mathscr{A}, A^0)$ *is a triequivalence.*

This theorem can be interpreted as giving a geometric description of complexes of abelian sheaves. The proof uses mostly the language and techniques developed in the paper of Aldrovandi and Noohi [1] such as butterflies of stacks, torsors, etc.

An immediate application of this theorem is the following corollary which generalizes Deligne's result [6, Proposition 1.4.15] from Picard stacks to Picard 2-stacks. Let $2\mathrm{P\scriptsize{IC}}^{\flat\flat}(\mathsf{S})$ denote the category of Picard 2-stacks obtained from $2\mathrm{P\scriptsize{IC}}(\mathsf{S})$ by ignoring the modifications and taking as morphisms the equivalence classes of additive 2-functors.

Let $\mathrm{D}^{[-2,0]}(\mathsf{S})$ be the subcategory of the derived category of category of complexes of abelian sheaves $A^\bullet$ over S with $H^{-i}(A^\bullet) \neq 0$ for $i = 0, 1, 2$. Then:

Corollary. *The functor*

$$\mathrm{D}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{P}\mathrm{IC}^{\flat\flat}(\mathsf{S})$$

given by sending a length 3 complex of abelian sheaves, $A^\bullet : A^{-2} \to A^{-1} \to A^0$ to its associated Picard 2-stack $\mathrm{T}\mathrm{ORS}(\mathscr{A}, A^0)$ and an equivalence class of fractions from $A^\bullet$ to $B^\bullet$ to an equivalence class of morphisms of associated Picard 2-stacks is an equivalence.

2. Plans for Future Research

Defining n-categories and n-stacks is one of the active research areas in mathematics. Various authors gave useful definitions of these notions (see [2], [12], [10], and [8]). However monoidal, group-like, and braiding structures on such objects are yet to be defined.

Beside this general question, in the first few years after completing my Ph.D, I plan to continue my research in the following directions:

2.1. Tensor Product of Picard 2-Stacks.

In SGA4 Exposé XVIII [6], Deligne also defines the tensor product of two Picard stacks. He defines it as the Picard stack associated to complex obtained by tensoring the complexes associated to each stack and truncating it. He then shows that this tensor product has a universal property similar to the tensor product on modules. Using the results in my thesis and following Deligne's procedure, the tensor product of Picard 2-stacks can be defined. This may be the Gray product of the 3-category of Picard 2-stacks. Moreover, the definition of tensor product given by Deligne is not geometric since he uses the derived category of complexes to define it. Therefore, it is an interesting direction to look for a geometric definition.

2.2. Postnikov Decomposition of Picard 2-Stacks.

In [4], Breen gives a classification of Picard stacks with homotopy groups π_1 and π_0 over a space X. This process is also known as Postnikov decomposition of a Picard stack. He shows that the cohomology groups $H^5(K(\pi_0, 3), \pi_1)$, $H^4(K(\pi_0, 2), \pi_1)$, $H^3(K(\pi_0, 1), \pi_1)$ classify respectively Picard, braided, and group-like stacks. The cohomology group $H^2(\pi_0, \pi_1)$ classifies stacks, that is abelian π_1-gerbes over π_0. Breen also intends in [4] to classify Picard 2-stacks. He proves that the cohomology group $H^2(\pi_0, \mathscr{B})$ classifies 2-stacks where $\mathscr{B} = \mathrm{Aut}(\pi_1)$, that is abelian $\mathscr{B}$-2-gerbes over π_0. However, he discusses informally that Picard, strongly braided, braided, and group-like 2-stacks are classified by cohomology groups $H^6(K(\pi_0, 4), \mathscr{B})$, $H^5(K(\pi_0, 3), \mathscr{B})$, $H^4(K(\pi_0, 2), \mathscr{B})$, and $H^3(K(\pi_0, 1), \mathscr{B})$ classifies respectively . He verifies this claim for 2-stacks with trivial π_1 in [5], but without this simplifying condition the claim is yet to be verified.

2.3. Extensions of Stacks.

Another problem that Breen studies is the classification of extensions of the form

$$1 \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow K \longrightarrow 1$$

where K is a discrete gr-category (i.e. a group) and $\mathscr{G}$ is a gr-category. This is generalization of O. Schreier's work on classification of group extensions when $\mathscr{G}$ and $\mathscr{H}$ are assumed to be groups and H. Sihn's work in her thesis [11] when $\mathscr{G}$ is assumed to be the gr-category associated to the crossed-module $G \to \mathrm{Aut}(G)$. Later, Alain Rousseau in his thesis [9] investigates the 2-category $\mathbf{Ext}(\mathscr{K},\mathscr{G})$ of extensions of a gr-category $\mathscr{K}$ by a gr-category $\mathscr{G}$.

$$1 \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow \mathscr{K} \longrightarrow 1$$

As the next step of the generalizations, one may try to define extension of gr-stacks by gr-stacks.

2.4. **Biextensions of length 3 complexes of abelian sheaves.** Let T be a topos and $\mathscr{C}$ be the category of abelian sheaves over T. In SGA7 Exposé VII [7], Grothendieck defines for P, Q, G any three objects in $\mathscr{C}$, $\mathbf{Biext}(P,Q;G)$ the Picard category of biextensions of (P,Q) by G and gives a homological interpretation to it. This result can be stackified and by Deligne's work in SGA4 Exposé XVIII, the Picard stack $\mathcal{B}iext(P,Q;G)$ can be identified with the stack associated to the complex $\tau_{\leq 0}\mathbb{R}\,\mathrm{Hom}(P\otimes Q, G[1])$. This work of Grothedieck has been generalized to length 2 complexes of abelian sheaves by Cristiana Bertolin in [3]. The question is, can one expect this generalization extend to length 3 complexes?

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