THE FLORIDA STATE UNIVERSITY

COLLEGE OF ARTS AND SCIENCES

ON PICARD 2-STACKS AND LENGTH 3 COMPLEXES OF ABELIAN SHEAVES

By

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En değerli destekçilerim Bıdım, Kızım, ve rahmetli Dedem'e

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ABSTRACT

In Séminaire de Géometrie Algébrique 4 (SGA4), Exposé XVIII, Pierre Deligne proves that to any Picard stack one can associate a complex of abelian sheaves of length 2. He also studies the morphisms between such stacks and shows that such a morphism defines a class of fractions in the derived category of complexes of abelian sheaves of length 2. From these two preliminary results, he finally deduces that the derived category of complexes of abelian sheaves of length 2 is equivalent to the category of Picard stacks with morphisms being the isomorphism classes.

In this dissertation, we generalize his work, following closely his steps in SGA4, to the case of Picard 2-stacks. But this generalization requires first a clear description of a Picard 2-category as well as of a 2-functor between such 2-categories that respects Picard structure. Once this has been done, we can talk about category of Picard 2-stacks and prove that the derived category of complexes of abelian sheaves of length 3 is equivalent to the category of Picard 2-stacks.

CHAPTER 1

INTRODUCTION

Let $\mathrm{D}^{[-1,0]}(\mathsf{S})$ be the subcategory of the derived category of category of complexes of abelian sheaves $A^\bullet$ over a site S with $H^{-i}(A^\bullet) \neq 0$ only for $i = 0, 1$. Let $\mathrm{Pic}(\mathsf{S})^\flat(\mathsf{S})$ denote the category of Picard stacks over S with 1-morphisms isomorphism classes of additive functors. In SGA4 Exposé XVIII, Deligne shows the following.

Proposition. [9, Proposition 1.4.15] The functor

$$\wp^\flat : \mathrm{D}^{[-1,0]}(\mathsf{S}) \longrightarrow \mathrm{Pic}(\mathsf{S})^\flat(\mathsf{S})$$

given by sending a length 2 complex of abelian sheaves, $A^\bullet : A^{-1} \to A^0$ over S to its associated Picard stack $[A^{-1} \to A^0]^\sim$, an isomorphism class of fractions from $A^\bullet$ to $B^\bullet$ to an isomorphism class of morphisms of associated Picard stacks is an equivalence.

The purpose of this thesis is to generalize the above result to Picard 2-stacks over S. Let $2\mathrm{Pic}(\mathsf{S})^{\flat\flat}(\mathsf{S})$ denote the category of Picard 2-stacks, whose morphisms are equivalence classes of additive 2-functors. Let $\mathrm{D}^{[-2,0]}(\mathsf{S})$ be the subcategory of the derived category of category of complexes of abelian sheaves $A^\bullet$ over S with $H^{-i}(A^\bullet) \neq 0$ for $i = 0, 1, 2$.

Theorem I. The functor

$$2\wp^{\flat\flat} : \mathrm{D}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{Pic}(\mathsf{S})^{\flat\flat}(\mathsf{S})$$

given by sending a length 3 complex of abelian sheaves, $A^\bullet : A^{-2} \to A^{-1} \to A^0$ over S to its associated Picard 2-stack $[A^{-2} \to A^{-1} \to A^0]^\sim$, an equivalence class of fractions from $A^\bullet$ to $B^\bullet$ to an equivalence class of morphisms of associated Picard 2-stacks is an equivalence.

Basically, it gives a geometric description of the derived category of length 3 complexes of abelian sheaves. It states that any Picard 2-stack over a site S is biequivalent to a Picard 2-stack associated to a length 3 complex of abelian sheaves and that any morphism of Picard 2-stacks comes from a fraction of such complexes. A complex of abelian sheaves, whose only non-zero cohomology groups are placed at degrees -2,-1, and 0 can be thought as a length 3 complex of abelian sheaves, and therefore a morphism in $\mathrm{D}^{[-2,0]}(\mathsf{S})$ between any two complexes $A^\bullet$ and $B^\bullet$ is given by an equivalence class of fraction

$$(q, M^\bullet, p) : A^\bullet \xleftarrow{q} M^\bullet \xrightarrow{p} B^\bullet$$

with q being a quasi-isomorphism.

However, we prove a much stronger statement, so that the latter theorem becomes an immediate consequence of it. Let $2\mathrm{PIC}(\mathsf{S})(\mathsf{S})$ be the 3-category of Picard 2-stacks where 1-morphisms are additive 2-functors, 2-morphisms are natural 2-transformations, and 3-morphisms are modifications. Length 3 complexes of abelian sheaves over S placed in degrees $[-2,0]$ form a 3-category $\mathrm{C}^{[-2,0]}(\mathsf{S})$ by adding to the regular morphisms of complexes, the degree -1 and -2 morphisms. Then we easily construct an explicit trihomomorphism

$$2\wp : \mathrm{C}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{PIC}(\mathsf{S})(\mathsf{S}),$$

that is a 3-functor between 3-categories. Under this construction, length 3 complexes of abelian sheaves correspond to Picard 2-stacks. Although morphisms of such complexes induce morphisms between associated Picard 2-stacks, not all of them are obtained in this way. In this sense, the 1-morphisms of $\mathrm{C}^{[-2,0]}(\mathsf{S})$ are not geometric and the reason is their strictness. We resolve this problem by weakening $\mathrm{C}^{[-2,0]}(\mathsf{S})$ as follows: We introduce a tricategory $\mathrm{T}^{[-2,0]}(\mathsf{S})$ (a tricategory is a weak version of a 3-category in the sense of [13]) with same objects as $\mathrm{C}^{[-2,0]}(\mathsf{S})$. For any two complexes of abelian sheaves $A^\bullet$ and $B^\bullet$, morphisms between $A^\bullet$ and $B^\bullet$ in $\mathrm{T}^{[-2,0]}(\mathsf{S})$ is the bigroupoid $\mathsf{Frac}(A^\bullet, B^\bullet)$, whose main property is that it satisfies $\pi_0(\mathsf{Frac}(A^\bullet, B^\bullet)) \simeq \mathrm{Hom}_{\mathrm{D}^{[-2,0]}(\mathsf{S})}(A^\bullet, B^\bullet)$, where π_0 denotes the isomorphism classes of objects. Roughly speaking, objects of $\mathsf{Frac}(A^\bullet, B^\bullet)$ are fractions from $A^\bullet$ to $B^\bullet$ in the ordinary sense and its 2-morphisms are certain commutative diagrams (5.2.2) called "*diamonds*". Then we prove:

Theorem II. The trihomomorphism

$$2\wp : \mathrm{T}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{PIC}(\mathsf{S})(\mathsf{S})$$

defined by sending $A^\bullet$ a length 3 complex of abelian sheaves to its associated Picard 2-stack is a triequivalence.

Since in particular a triequivalence is essentially surjective, every Picard 2-stack is biequivalent to a Picard 2-stack associated to a complex of abelian sheaves. Then by ignoring the 3-morphisms and passing to the equivalence class of morphisms in the triequivalence of Theorem II, we deduce Theorem I.

The Chapters in this dissertation are organized as follows:

In Chapter 2, we recall the language of 2-categories and 3-categories that is commonly used through out the thesis.

In Chapter 3, we explain Deligne's work in SGA4 Exposé XVIII. We start with a detailed description of the 2-category $\mathrm{PIC}(\mathsf{S})$ of Picard stacks over a site S. We define the bicategory $\mathrm{T}^{[-1,0]}(\mathsf{S})$ of morphisms of abelian sheaves over the site S whose 1-morphism are called butterflies. We later construct a bifunctor from $\mathrm{PIC}(\mathsf{S})$ to $\mathrm{T}^{[-1,0]}(\mathsf{S})$ by sending a morphism of abelian sheaves $A^{-1} \to A^0$ to its associated Picard stack $\mathrm{TORS}(A^{-1}, A^0)$. We finish this Chapter by enouncing Deligne's characterization theorem for Picard stacks (3.8) from which (1) follows.

In Chapter 4, we construct the 3-category $2\mathrm{PIC}(\mathsf{S})$ of Picard 2-stacks. We give an example of a Picard 2-stack, namely $\mathrm{TORS}(\mathscr{A}, A^0)$, where $\mathscr{A}$ is a Picard stack and A^0 is an abelian sheaf. This example is of great importance for the rest since it is equivalent to the

Picard 2-stack associated to $A^\bullet : [A^{-1} \to A^{-1} \to A^0]$ a length 3 complex of abelian sheaves with $\mathscr{A}$ the Picard stack associated to the morphism $A^{-2} \to A^{-1}$. We call $\textsc{Tors}(\mathscr{A}, A^0)$ the Picard 2-stack associated to $A^\bullet$. For any two complexes $A^\bullet$ and $B^\bullet$, we denote by $\mathsf{Hom}(A^\bullet, B^\bullet)$ the hom-2-category of the morphisms between the associated Picard 2-stacks.

In Chapter 5, we first construct another 3-category, namely $\mathrm{C}^{[-2,0]}(\mathsf{S})$ of length 3 complexes of abelian sheaves. By weakening the morphisms of $\mathrm{C}^{[-2,0]}(\mathsf{S})$, we construct a tricategory $\mathrm{T}^{[-2,0]}(\mathsf{S})$ that has same objects as $\mathrm{C}^{[-2,0]}(\mathsf{S})$ and for any two length 3 complexes $A^\bullet$ and $B^\bullet$ of abelian sheaves, $\mathsf{Frac}(A^\bullet, B^\bullet)$ as the hom-bigroupoid.

Chapter 6 is the main Chapter of this thesis. Here, we prove the generalization of Deligne's characterization theorem. We first construct an explicit trihomomorphism $2\wp$ from the 3-category $\mathrm{C}^{[-2,0]}(\mathsf{S})$ to the 3-category $2\textsc{Pic}(\mathsf{S})$ of Picard 2-stacks. We also show that for any two length 3 complexes of abelian sheaves $A^\bullet$ and $B^\bullet$, there exists a biequivalence of bigroupoids between $\mathsf{Frac}(A^\bullet, B^\bullet)$ and the 2-category $\mathsf{Hom}(A^\bullet, B^\bullet)$. Using this biequivalence, we extend the trihomomorphism $2\wp$ constructed on $\mathrm{C}^{[-2,0]}(\mathsf{S})$ to a trihomomorphism on $\mathrm{T}^{[-2,0]}(\mathsf{S})$. We end this Chapter by proving that the latter trihomomorphism is a triequivalence (see Theorem II).

CHAPTER 2

PRELIMINARY

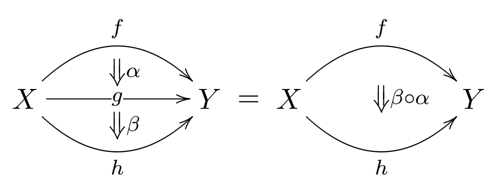
In this chapter, we are going to recall 2-categories and 3-categories from a perspective that is needed through out the thesis. We start with a short review of 2-categories. Then we explain by analogies 3-categories and give list of references for detailed treatment of the subject.

2.1 Language of 2-Categories

In this section, we revisit 2-categories. We assume familiarity with the category theory. We ignore any set theoretic problems which can be overcome by standard arguments using universes . For detailed treatment of 2-categories, we refer to [16], [24], [25], and [26].

Definition 2.1.1. A *bicategory* $\mathbb{C}$ is the collection of the following data:

1. a set of objects $\mathrm{Ob}\,\mathbb{C}$.
2. for any two objects X and Y, a category $\mathrm{Hom}_{\mathbb{C}}(X,Y)$ or $\mathrm{Hom}(X,Y)$ if there is no confusion, whose objects are called 1-morphisms and designated by $f : X \to Y$ and whose morphisms are called 2-morphisms and designated by $\alpha : f \Rightarrow g$ and whose composition law, designated by $\circ$, is called vertical composition and defined as



3. for any three objects X, Y, Z, a functor

$$\tau_{X,Y,Z} : \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \longrightarrow \mathrm{Hom}(X,Z).$$

We write $g \circ f$ and $\beta * \alpha$ for $\tau_{X,Y,Z}(f,g)$ and $\tau_{X,Y,Z}(\alpha,\beta)$, respectively. We call $\beta * \alpha$ the horizontal composition.

4. for any object X, a functor

$$\mathrm{I}_X : \mathbf{1} \longrightarrow \mathrm{Hom}(X,X),$$

defined on the category $\mathbf{1}$ with one object $*$ and one morphism id_* by sending $*$ to id_X.

5. for all X, Y, Z, W objects, there exists a natural isomorphism θ

$$\begin{array}{ccc} \mathrm{Hom}(X,Y)\times\mathrm{Hom}(Y,Z)\times\mathrm{Hom}(Z,W) & \xrightarrow{\tau_{X,Y,Z}\times 1} & \mathrm{Hom}(X,Z)\times\mathrm{Hom}(Z,W) \\ {\scriptstyle 1\times\tau_{Y,Z,W}}\downarrow & \Downarrow\theta & \downarrow{\scriptstyle \tau_{X,Z,W}} \\ \mathrm{Hom}(X,Y)\times\mathrm{Hom}(Y,W) & \xrightarrow[\tau_{X,Y,W}]{} & \mathrm{Hom}(X,W) \end{array}.$$

6. for all X, Y objects, there exits two natural isomorphisms $\mathfrak{l}$ and $\mathfrak{r}$

$$\begin{array}{ccc} \mathrm{Hom}(X,Y)\times\mathbf{1} & \longrightarrow & \mathrm{Hom}(X,Y) \\ & {\scriptstyle \mathfrak{r}_{X,Y}}\Uparrow & \\ {\scriptstyle 1\times\mathrm{I}_Y}\searrow & & \nearrow{\scriptstyle \tau_{X,Y,Y}} \\ & \mathrm{Hom}(X,Y)\times\mathrm{Hom}(Y,Y) & \end{array} \qquad \begin{array}{ccc} \mathbf{1}\times\mathrm{Hom}(X,Y) & \longrightarrow & \mathrm{Hom}(X,Y) \\ & {\scriptstyle \mathfrak{l}_{X,Y}}\Uparrow & \\ {\scriptstyle \mathrm{I}_X\times 1}\searrow & & \nearrow{\scriptstyle \tau_{X,X,Y}} \\ & \mathrm{Hom}(X,X)\times\mathrm{Hom}(X,Y) & \end{array}$$

These data must satisfy [1]:

(i) for every four composable 1-morphisms f, g, h, k, the diagram of 2-morphisms

$$\begin{array}{ccccc} & & (((kh)g)f) & & \\ & {\scriptstyle \tau}\swarrow\!\!\swarrow & & \searrow\!\!\searrow{\scriptstyle \tau*1} & \\ ((kh)(gf)) & & \circlearrowleft & & ((k(hg))f) \\ {\scriptstyle \tau}\Downarrow & & & & \Downarrow{\scriptstyle \tau} \\ (k(h(gf))) & & \xLeftarrow[1*\tau]{} & & (k((hg)f)) \end{array} \tag{2.1.1}$$

commutes.

(ii) for every two composable 1-morphisms $X \xrightarrow{g} Y \xrightarrow{f} Z$ the diagram of 2-morphisms

$$\begin{array}{ccc} (g\mathrm{I}_Y)f & \xRightarrow{\tau} & g(\mathrm{I}_Y f) \\ {\scriptstyle \mathfrak{r}*\mathbf{1}}\searrow\!\!\searrow & \circlearrowleft & \swarrow\!\!\swarrow{\scriptstyle \mathbf{1}*\mathfrak{l}} \\ & gf & \end{array} \tag{2.1.2}$$

commute.

[1]In diagrams (2.1.1) and (2.1.2), $\circ$ is omitted for compactness.

The stronger version of bicategories in which the composition of 1-morphisms is strictly associative is called 2-category. Formally,

Definition 2.1.2. A *2-category* $\mathbb{C}$ is a bicategory in which the natural transformations θ, $\mathfrak{r}$, and $\mathfrak{l}$ are identities.

Definition 2.1.3. Let $\mathbb{C}$ and $\mathbb{D}$ be two 2-categories. A *2-functor* $(F, \varepsilon) : \mathbb{C} \longrightarrow \mathbb{D}$ is given by the data

1. for all X object of $\mathbb{C}$, $F(X)$ is an object of $\mathbb{D}$,

2. for all X, Y objects of $\mathbb{C}$, there exists a functor $F_{X,Y}$

$$F_{X,Y} : \operatorname{Hom}_{\mathbb{C}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathbb{D}}(F(X), F(Y)) ,$$

3. for all X, Y, Z objects of $\mathbb{C}$, there exists a functorial 2-isomorphism $\varepsilon_{X,Y,Z}$

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathbb{C}}(X,Y) \times \operatorname{Hom}_{\mathbb{C}}(Y,Z) & \xrightarrow{\tau_{X,Y,Z}} & \operatorname{Hom}_{\mathbb{C}}(X,Z) \\
\Big\downarrow {\scriptstyle F_{X,Y} \times F_{Y,Z}} & \varepsilon_{A,B,C} \Uparrow & \Big\downarrow {\scriptstyle F_{X,Z}} \\
\operatorname{Hom}_{\mathbb{D}}(F(X),F(Y)) \times \operatorname{Hom}_{\mathbb{D}}(F(Y),F(Z)) & \xrightarrow[\tau_{FX,FY,FZ}]{} & \operatorname{Hom}_{\mathbb{D}}(F(X),F(Z))
\end{array}$$

satisfying that

(i) $F_{X,X}(\mathrm{id}_X) = \mathrm{id}_{F(X)}$

(ii) the diagram expressing the associativity of composition

$$\begin{array}{ccc}
F_{Z,W}(h) \circ F_{Y,Z}(g) \circ F_{X,Y}(f) & \xRightarrow{1 \times \varepsilon_{X,Y,Z}(f,g)} & F_{Z,W}(h) \circ F_{X,Z}(g \circ f) \\
\Big\Downarrow {\scriptstyle \varepsilon_{Y,Z,W}(g,h) \times 1} & & \Big\Downarrow {\scriptstyle \varepsilon_{X,Z,W}} \\
F_{Y,W}(h \circ g) \circ F_{X,Y}(f) & \xRightarrow[\varepsilon_{X,Y,W}]{} & F_{X,W}(h \circ g \circ f)
\end{array}$$

commutes for any $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$.

(iii) the diagrams expressing the composition with identity

$$\begin{array}{ccc}
F_{X,Y}(f) \circ F_{X,X}(\mathrm{id}_X) & \xRightarrow{\varepsilon_{X,X,Y}(\mathrm{id}_X,f)} & F_{X,Y}(f \circ \mathrm{id}_X) \\
{\scriptstyle \mathrm{id}_{F_{X,Y}(f)}} \Big\Downarrow & & \Big\| \\
F_{X,Y}(f) \circ \mathrm{id}_{F(X)} & =\!=\!=\!=\!= & F_{X,Y}(f)
\end{array}$$

$$\begin{array}{ccc}
F_{Y,Y}(\mathrm{id}_Y) \circ F_{X,Y}(f) & \xRightarrow{\varepsilon_{X,Y,Y}(f,\mathrm{id}_Y)} & F_{X,Y}(\mathrm{id}_X \circ f) \\
{\scriptstyle \mathrm{id}_{F_{X,Y}(f)}} \Big\Downarrow & & \Big\| \\
\mathrm{id}_{F(Y)} \circ F_{X,Y}(f) & =\!=\!=\!=\!= & F_{X,Y}(f)
\end{array}$$

commute for any $X \xrightarrow{f} Y$.

Remark 2.1.4. According to the common terminology of the weak 2-functor, we should have assumed that there exists an isomorphism $F_{X,X}(\mathrm{id}_x) \longrightarrow \mathrm{id}_{F(X)}$. However due to the Lemma 2.5 in [11], we can assume that the latter isomorphism is an identity.

Definition 2.1.5. A 2-functor is a *biequivalence* $F : \mathbb{C} \to \mathbb{D}$ if

1. F is essentially surjective. That is if for any object Y in $\mathbb{D}$, there exists an object X in $\mathbb{D}$ such that there exists a weakly invertible morphism $FX \to Y$.
2. for every object X, X' in $\mathbb{C}$, the category $\mathrm{Hom}_{\mathbb{C}}(X, X')$ is equivalent to the category $\mathrm{Hom}_{\mathbb{D}}(FX, FX')$.

Definition 2.1.6. Let $\mathbb{C}$ and $\mathbb{D}$ be two 2-categories and $F, G :\ \mathbb{C} \longrightarrow \mathbb{D}$ be two 2-functors.

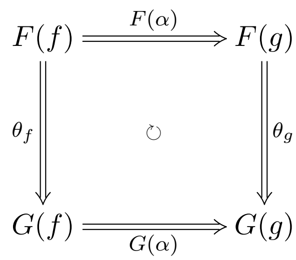
A *natural 2-transformation* $\mathbb{C} \underset{G}{\overset{F}{\underset{\Downarrow \theta}{\rightrightarrows}}} \mathbb{D}$ is the collection of the datum,

1. for all X an object in $\mathbb{C}$, $\theta_X : F(X) \to G(X)$ is a 1-morphism in $\mathbb{D}$,
2. for all $f : X \to Y$ 1-morphism in $\mathbb{C}$, there exists a 2-morphism θ_f in $\mathbb{D}$

$$\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
{\scriptstyle \theta_X} \Big\downarrow & {\scriptstyle \theta_f} \Nearrow & \Big\downarrow {\scriptstyle \theta_Y} \\
G(X) & \xrightarrow[G(f)]{} & G(Y)
\end{array}$$

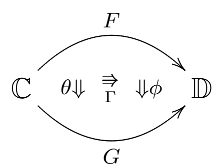
satisfying the condition,

for all $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ ($\Downarrow\alpha$) 2-morphism in $\mathbb{C}$, the diagram

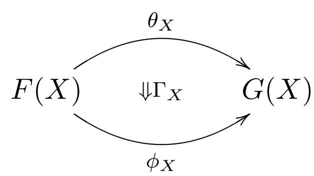


is commutative.

Definition 2.1.7. Let $\mathbb{C}$ and $\mathbb{D}$ be two 2-categories and $F, G : \mathbb{C} \to \mathbb{D}$ be two 2-functors and $\theta, \phi : F \Rightarrow G$ be two natural 2-transformations. A *modification*

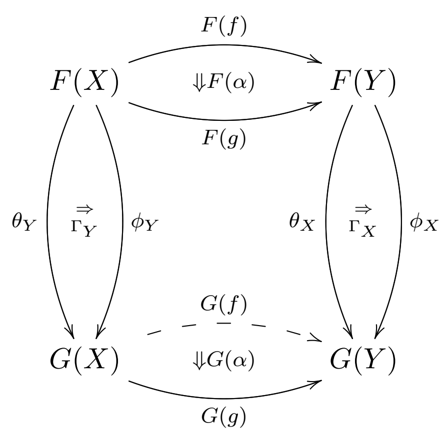


is the collection of

for all X an object in $\mathbb{C}$, 

is a 2-morphism in $\mathbb{D}$

satisfying that for any morphism $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ ($\Downarrow\alpha$) in $\mathbb{C}$, the diagram



commutes.

Definition 2.1.8. A *2-groupoid* $\mathbb{C}$ is a 2-category such that,

- all 2-morphisms of $\mathbb{C}$ are isomorphisms,
- all 1-morphisms of $\mathbb{C}$ are invertible up to a 2-isomorphism, that is for any 1-morphism $f : X \to Y$, there exists a 1-morphism $g : Y \to X$ and two 2-isomorphisms α and β such that $\alpha : g \circ f \Rightarrow \mathrm{id}_X$ and $\beta : f \circ g \Rightarrow \mathrm{id}_Y$.

2.2 Language of 3-Categories

Even though the language of tricategories is going to be extensively used, we are not going to remind here in full detail tricategories. Just for motivation, a 3-category can be thought as the category of 2-categories with 2-functors or weak 2-functors in the sense of Bénabou [4] and a tricategory as a weakened version of a 3-category. For more about tricategories, we refer the reader to [4], [13], [15], and [22].

Here we only recall the definition of triequivelence since it is the key ingredient of the main theorem (6.4.1).

Definition 2.2.1. [22] A trihomomorphism of tricategories $T : \mathfrak{C} \to \mathfrak{D}$ is called a *triequivalence* if it induces biequivalences $T_{X,Y} : \mathfrak{C}(X, Y) \to \mathfrak{D}(TX, TY)$ of hom-bicategories for all objects X, Y in $\mathfrak{C}$ (T is locally a biequivalence), and every object in $\mathfrak{D}$ is biequivalent in $\mathfrak{D}$ to an object of the form TX where X is an object in $\mathfrak{C}$.

CHAPTER 3

2-CATEGORY OF PICARD STACKS AND DELIGNE'S CHARACTERIZATION THEOREM FOR PICARD STACKS

3.1 Picard Categories

Definition 3.1.1. A category $\mathscr{C}$ with data

1. a functor $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$

2. a functorial isomorphism a

$$\begin{array}{ccc} \mathscr{C} \times \mathscr{C} \times \mathscr{C} & \xrightarrow{\otimes \times 1} & \mathscr{C} \otimes \mathscr{C} \\ {\scriptstyle 1\times\otimes}\downarrow & \not\Downarrow_{\mathsf{a}} & \downarrow{\scriptstyle \otimes} \\ \mathscr{C} \otimes \mathscr{C} & \xrightarrow[\otimes]{} & \mathscr{C} \end{array}$$

expressing an associativity constraint.

3. a functorial isomorphism c

$$\begin{array}{ccc} \mathscr{C} \times \mathscr{C} & \xrightarrow{\mathsf{s}} & \mathscr{C} \times \mathscr{C} \\ & {\scriptstyle \otimes}\searrow \quad \overset{\mathsf{c}}{\Rightarrow} \quad \swarrow{\scriptstyle \otimes} & \\ & \mathscr{C} & \end{array}$$

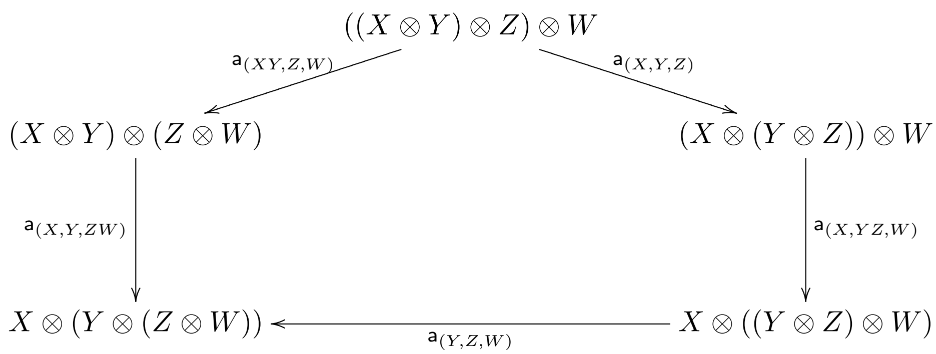
expressing a commutativity constraint where s is the functor

$$\begin{array}{rcl} \mathsf{s} : \mathscr{C} \times \mathscr{C} & \to & \mathscr{C} \times \mathscr{C} \\ (X, Y) & \mapsto & (Y, X) \end{array}$$

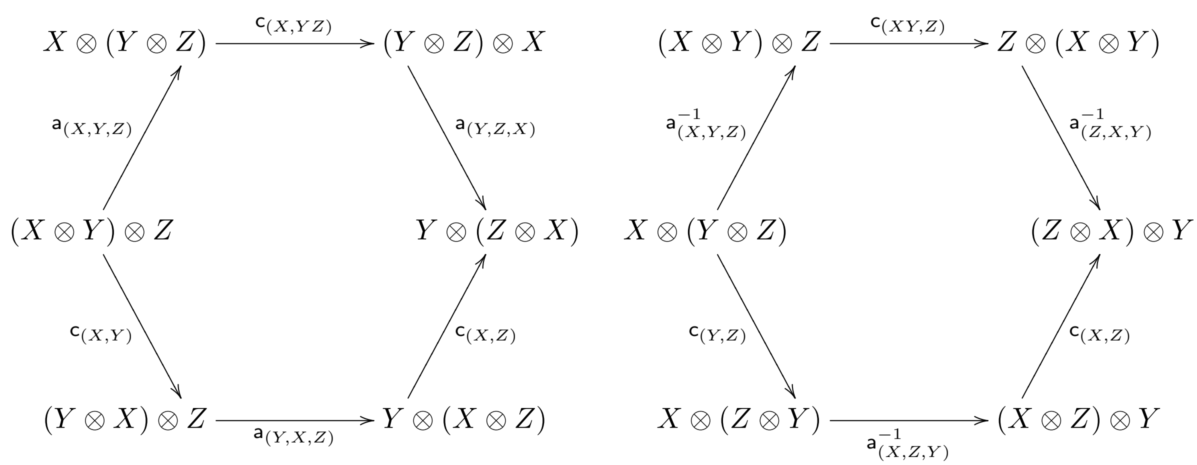
is called Picard if the above data satisfies the following conditions.

(i) for any object X in $\mathscr{C}$ the functor $X \otimes - : \mathscr{C} \longrightarrow \mathscr{C}$ is an equivalence.

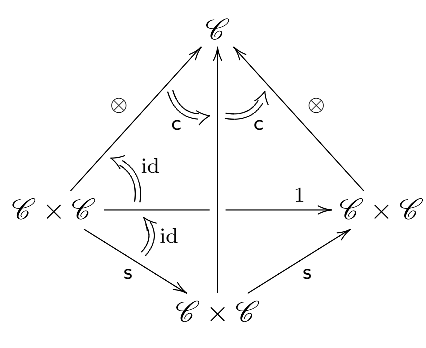
(ii) for all objects X, Y, Z, W in $\mathscr{C}$ the pentagon below commutes.



(iii) for all objects X, Y, Z in $\mathscr{C}$ the hexagones below



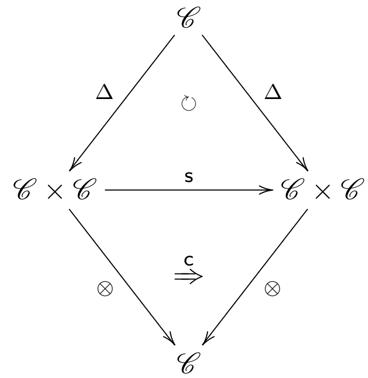
(iv)



is a commutative tetrahedron, that is for all X, Y objects in $\mathscr{C}$,

$$\mathsf{c}_{Y,X} \circ \mathsf{c}_{X,Y} = \mathrm{id}_{X \otimes Y}.$$

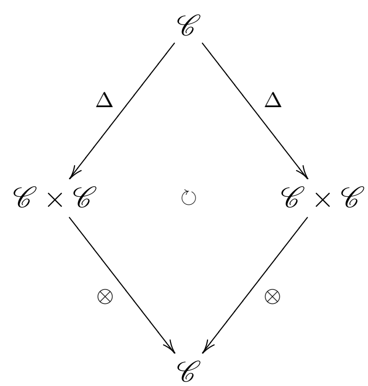
(v)



where

$$\begin{array}{rcl} \Delta : \mathscr{C} & \to & \mathscr{C} \times \mathscr{C} \\ X & \mapsto & (X, X) \end{array}$$

pastes to a commutative diagram



that is for all X object in $\mathscr{C}$,

$$\mathsf{c}_{X,X} = \mathrm{id}_{X \otimes X}.$$

Notation 3.1.2. We denote a Picard category by $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathsf{a}, \mathsf{c})$. In case there is no risk of confusion we are going to denote a Picard category by $(\mathscr{C}, \otimes)$.

3.2 Units in Picard Categories

Definition 3.2.1. Let $(\mathscr{C}, \otimes)$ be a Picard category. A pair (e, φ) is called a unit element where e is an object in $\mathscr{C}$ and $\varphi : e \otimes e \to e$ is an isomorphism.

Definition 3.2.2. Let $(\mathscr{C}, \otimes)$ be a Picard category and let (e_1, φ_1) and (e_2, φ_2) be two unit elements. A morphism $(e_1, \varphi_1) \to (e_2, \varphi_2)$ is given by an isomorphism $f : e_1 \to e_2$ in $\mathscr{C}$ such that the diagram

$$\begin{array}{ccc} e_1 \otimes e_1 & \xrightarrow{f \otimes f} & e_2 \otimes e_2 \\ {\scriptstyle \varphi_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \varphi_2} \\ e_1 & \xrightarrow[f]{} & e_2 \end{array} \tag{3.2.1}$$

commutes. We call such an isomorphism *unit morphism.*

This defines $\mathfrak{U}(\mathscr{C})$ the category of units of the Picard category $(\mathscr{C}, \otimes)$. In fact, $\mathfrak{U}(\mathscr{C})$ is a groupoid since a unit morphism is assumed to be an isomorphism. We are going to call these unit elements Saavedra units following the terminology by J.Kock. In [21], Kock shows that defining a unit element in a monoidal category as a cancellable-idempotent element - a definition due to Saavedra [30], is equivalent to the classical definition of a unit - also known as Left-Right unit. When the category has Picard structure defining unit element and unit morphism in the sense of Saavedra is even simpler since in this case every object and every morphism is cancellable. Restricted to the underlying strict monoidal category of $\mathscr{C}$, the definition (3.2.1) coincides with Kock's definition in [21]. There are immediate propositions which follow from the definition of Picard category and the unit element. They are far from being original. They can be found in the paper by Kock, Saavedra, and Deligne.

Proposition 3.2.3. *Let (e, φ) be a unit element in the Picard category $(\mathscr{C}, \otimes)$. Then for all $X \in \mathrm{Ob}\,\mathscr{C}$ there exists a unique functorial isomorphism $\alpha_X : e \otimes X \to X$ compatible with φ. In other words, for any morphism $f : X \to Y$ in $\mathscr{C}$, the following diagrams commute*

$$\begin{array}{ccc} e \otimes X & \xrightarrow{e \otimes f} & e \otimes Y \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} (e \otimes e) \otimes X & \xrightarrow{\mathsf{a}_{e,e,X}} & e \otimes (e \otimes X) \\ {\scriptstyle \varphi \otimes \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle e \otimes \alpha_X} \\ e \otimes X & \xrightarrow[=]{} & e \otimes X \end{array}$$

Proof. Let (e, φ) be a unit element and let X be an object in $\mathscr{C}$. The morphism $\varphi \otimes \mathrm{id}_X$ is in the set $\mathrm{Hom}_{\mathscr{C}}((e \otimes e) \otimes X, e \otimes X)$. We pre compose this morphism with the isomorphism $\mathsf{a}_{e,e,X}^{-1}$ to get a morphism $(\varphi \otimes \mathrm{id}_X) \circ \mathsf{a}_{e,e,X}^{-1}$ in the set $\mathrm{Hom}_{\mathscr{C}}(e \otimes (e \otimes X), e \otimes X)$. Since $e \otimes -$ is an equivalence, there exists a bijection between the sets $\mathrm{Hom}_{\mathscr{C}}(e \otimes X, X)$ and $\mathrm{Hom}_{\mathscr{C}}(e \otimes (e \otimes X), e \otimes X)$. We let α_X be the image of $(\varphi \otimes \mathrm{id}_X) \circ \mathsf{a}_{e,e,X}^{-1}$ under this bijection. By definition, it is clear that it is functorial and compatible with φ. □

Proposition 3.2.4. *A Picard category $(\mathscr{C}, \otimes)$ has a unit element.*

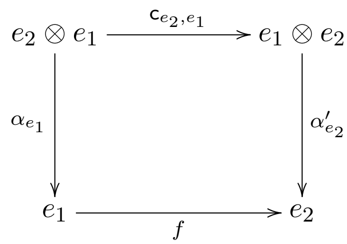
Proof. Let $X \in \mathscr{C}$. Since $X \otimes -$ is an equivalence, for all $Z \in \mathscr{C}$ there exists $Y \in \mathscr{C}$ such that $X \otimes Y \simeq Z$. In particular when $X = Z$ there exists $e_X \in \mathscr{C}$ such that $f : X \otimes e_X \simeq e_X$. Therefore the composition

$$X \otimes (e_X \otimes e_X) \xrightarrow{\mathsf{a}_{X,e_X,e_X}} (X \otimes e_X) \otimes e_X \xrightarrow{f \otimes e_X} X \otimes e_X\,, \tag{3.2.2}$$

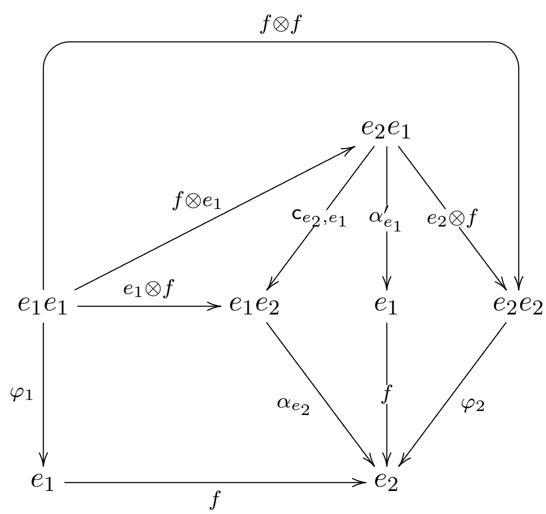
is an isomorphism in $\mathrm{Hom}_{\mathscr{C}}(X \otimes (e_X \otimes e_X), X \otimes e_X)$. By the equivalence of $e_X \otimes -$, we also know that $\mathrm{Hom}_{\mathscr{C}}(X \otimes (e_X \otimes e_X), X \otimes e_X)$ is in bijection with $\mathrm{Hom}_{\mathscr{C}}(e_X \otimes e_X, e_X)$. We define φ as the image of the composition (3.2.2) under this bijection. φ is an isomorphism since f is.Hence, the pair (e_X, φ) is a unit element in $\mathscr{C}$. □

Proposition 3.2.5. *Let $(\mathscr{C}, \otimes)$ be a Picard category. The groupoid of units $\mathfrak{U}(\mathscr{C})$ is contractible. That is between any two unit elements (e_1, φ_1) and (e_2, φ_2) there exists a unique unit morphism.*

Proof. We define $f : e_1 \to e_2$ as the isomorphism that makes the diagram



commutative. We need to verify that f satisfies the diagram (3.2.1).

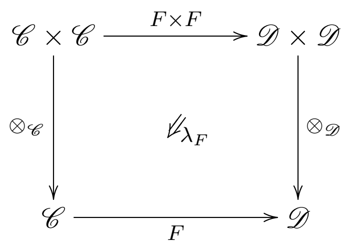


The outer cell is the diagram (3.2.1). The top cell and the triangular cell commute by naturality. The cell at the bottom left corner and the cell at the right most commute by the functorialiy of α and α', respectively. The remaining rectangular cell commutes by definition of f. □

Definition 3.2.6. Let $(\mathscr{C}, \otimes)$ be a Picard category. For any object X in $\mathscr{C}$ the pair (X^*, α_X) is called inverse element of X where $\alpha_X : X \otimes X^* \to e$ and (e, φ) is a unit element with $\varphi = \alpha_e$.

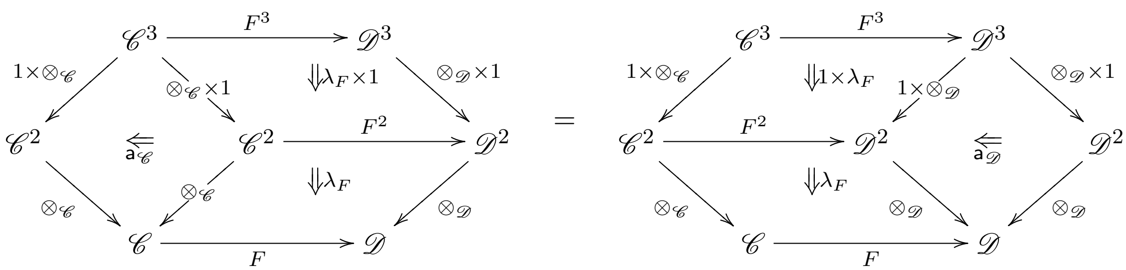
3.3 Morphisms of Picard Categories

Definition 3.3.1. Let $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathsf{a}_{\mathscr{C}}, \mathsf{c}_{\mathscr{C}})$ and $(\mathscr{D}, \otimes_{\mathscr{D}}, \mathsf{a}_{\mathscr{D}}, \mathsf{c}_{\mathscr{D}})$ be two Picard categories. An additive functor (F, λ_F) is given by a functor $F : \mathscr{C} \to \mathscr{D}$ and a natural isomorphism λ_F



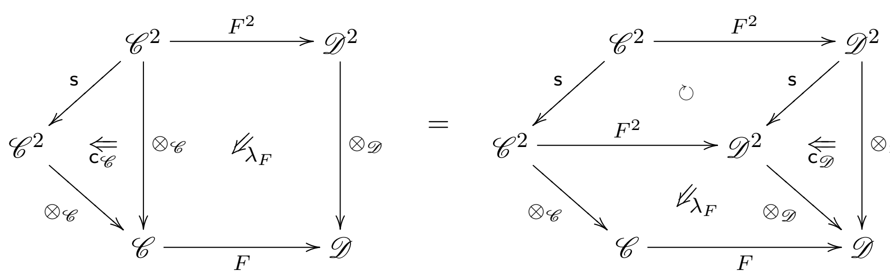
that satisfy the following conditions

(i) The pastings of the natural isomorphisms in the below diagrams are equal.

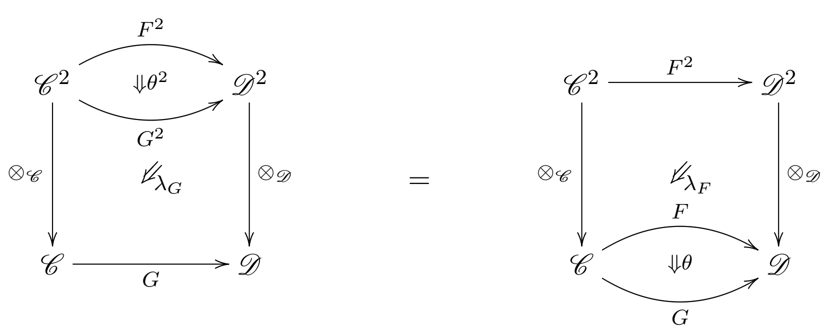


where $\mathscr{C}^3$ is abbreviation for $\mathscr{C} \times \mathscr{C} \times \mathscr{C}$.

(ii) The pastings of the natural isomorphisms in the below diagrams are equal.



Definition 3.3.2. A morphism of additive functors $\theta : (F, \lambda_F) \Rightarrow (G, \lambda_G)$ is a natural transformation $\theta : F \Rightarrow G$ that satisfy the following equation of natural transformations.



(3.3.1)

3.4 Picard Stacks

The concept of a stack over a site S is categorical analogue of sheaves. That is a stack is naively a sheaf of categories. In this section other than stacks and morphisms of stacks, we define fibered categories over S and Picard stacks over S which are categorical analogues of presheaves and abelian sheaves. Our main references for this section are [8], [14], [33].

3.4.1 Fibered Categories

In this section, we study the categories over a fixed site S, that is categories $\mathscr{C}$ equipped with a functor

$$p_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathsf{S} .$$

Definition 3.4.1. Let $\mathscr{C}$ be a category over S and let U be an object of S. A *fiber* of $\mathscr{C}$ over U, denoted by $\mathscr{C}_U$, is a subcategory of $\mathscr{C}$ such that $p_{\mathscr{C}}$ maps its objects and morphisms to U and id_U, respectively.

Definition 3.4.2. Let $\mathscr{C}$ be a category over S and let $f : X \to Y$ be a morphism in $\mathscr{C}$ such that

$$p_{\mathscr{C}}(X) = U \qquad p_{\mathscr{C}}(Y) = V \qquad p_{\mathscr{C}}(f) = i.$$

f is called *cartesian* if for any object X' and for any morphism $f' : X' \to Y$ in $\mathscr{C}$ such that $p_{\mathscr{C}}(X') = U$ and $p_{\mathscr{C}}(f') = i$, there exists a unique morphism $g : X' \to X$ in $\mathscr{C}$ satisfying $p_{\mathscr{C}}(g) = \mathrm{id}_U$ and $f \circ g = f'$.

Remark 3.4.3. The definition (3.4.2) can be equivalently expressed as follows. A morphism $f : X \to Y$ in $\mathscr{C}$ is cartesian if for any object X' in $\mathscr{C}$ the map $\mathrm{Hom}_{\mathscr{C}_U}(X', X) \to \mathrm{Hom}_i(X, Y)$ defined by $g \mapsto f \circ g$ is a bijection where $\mathrm{Hom}_i(X, Y)$ denotes the set of morphisms in $\mathscr{C}$ from X to Y that are mapped to i by $p_{\mathscr{C}}$.

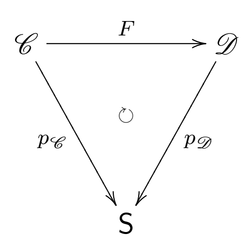
Definition 3.4.4. Let $\mathscr{C}$ be a category over S. We say that $\mathscr{C}$ is *fibered* over S if

(i) for every $i : U \to V$ morphism in S and for every object Y in $\mathscr{C}_V$, there exists an object X in $\mathscr{C}_U$ and a cartesian morphism $f : X \to Y$ in $\mathscr{C}$ such that $p_{\mathscr{C}}(f) = i$.

(ii) composition of cartesian morphisms is cartesian.

Definition 3.4.5. Let $\mathscr{C}$ and $\mathscr{D}$ be two fibered categories over S. A functor $F : \mathscr{C} \to \mathscr{D}$ is called a *morphism of fibered categories* or a *cartesian functor* if

(i) F preserves the base, that is if the diagram



commutes.

(ii) F maps cartesian morphisms to cartesian morphisms.

Even though we have defined fibered categories and functors between them in general, in the rest of the thesis we will only deal with fibered categories in groupoids, that is fibered categories where each fiber is a groupoid.

3.4.2 Sheaf Axiom for Fibered Categories

In this section, we define the analog of the sheaf axiom for fibered categories which were introduced as the categorical analogues of presheaves. In general, sheaf axiom describes how to obtain a global information from local data. In case of fibered categories, local data are objects and morphisms of fibers. Therefore sheaf axiom for fibered categories consists of the following conditions.

(i) Axiom on Morphisms: for any two objects X, Y in $\mathscr{C}_U$, the presheaf $\mathrm{Hom}_{\mathscr{C}_U}(X,Y)$ is a sheaf on S/U.

(ii) Axiom on Objects: every decent datum is effective.

A *decent datum* is a collection $(V_\bullet \to U, X, \varphi)$ where

- $\ldots V_2 \overset{d_0}{\underset{d_2}{\xrightarrow{d_1}}} V_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} V_0 \xrightarrow{\delta} U$ is a hypercover over U,
- X is an object in $\mathscr{C}_{V_0}$,
- $\varphi : d_1^*X \to d_0^*X$ is a morphism in $\mathscr{C}_{V_1}$,

satisfying the cocycle condition in $\mathscr{C}_{V_2}$

$$d_1^*\varphi = d_2^*\varphi \circ d_0^*\varphi.$$

A decent datum $(V_\bullet \to U, X, \varphi)$ is *effective* if there exists an object $Y \in \mathscr{C}_U$ together with isomorphism $\psi : \delta^*Y \to X$ in $\mathscr{C}_{V_0}$ compatible with φ.

3.4.3 Picard Stacks

Finally, we define the analog of a sheaf in a categorical context. A *stack* is a fibered category $\mathscr{C}$ over S that satisfies both axioms (3.4.2). If $\mathscr{C}$ satisfies only the first axiom (3.4.2), then $\mathscr{C}$ is called *prestack*. A (pre)stack $\mathscr{C}$ is *fibered in groupoids* over S if for every object $U \in \mathsf{S}$ $\mathscr{C}_U$ is a groupoid, that is a category whose morphisms are isomorphisms. In this thesis, we assume that all (pre)stacks are fibered in groupoids.

Definition 3.4.6. A *Picard stack* $\mathscr{P}$ over the site S is a stack equipped with a morphism of stacks

$$\otimes : \mathscr{P} \times \mathscr{P} \longrightarrow \mathscr{P}$$

inducing a Picard structure (3.1.1) on $\mathscr{P}$.

Definition 3.4.7. A *morphism of Picard stacks* $F : \mathscr{P}_1 \to \mathscr{P}_2$ is an additive (3.3.1) and cartesian (3.4.5) functor. By abuse of language, we call F additive functor.

Definition 3.4.8. Picard stacks over S form a 2-category, denoted by $\textsc{Pic}(\mathsf{S})$, whose

- 1-morphisms are additive functors (3.3.1),
- 2-morphisms are natural transformations compatible with the additive structure (3.3.2).

3.5 Associated Picard Stack

In this section, we define the Picard stack associated to a morphism of abelian sheaves. We also give an equivalent but more geometric realization of the associated stack in terms of torsors.

Let $\lambda : A^{-1} \to A^0$ be a morphism of abelian sheaves. For any $U \in \mathsf{S}$, we define a groupoid $\mathscr{P}_U$ as

- *objects*: $a \in A^0(U)$
- *morphisms*: $(f, a) \in A^{-1}(U) \times A^0(U)$ such that $(f, a) : a \to a + \lambda(f)$.

Proposition 3.5.1. *$\mathscr{P}$ is a pre-stack.*

Proof. Let $U \in \mathsf{S}$ and a_1, a_2 be two objects in $\mathscr{P}_U$. We want to show that $\mathrm{Hom}_{\mathscr{P}_U}(a_1, a_2)$ is a sheaf over S/U. $\mathrm{Hom}_{\mathscr{P}_U}(a_1, a_2)$ is a pre-sheaf (i.e a fibered category over S/U) defined by

- for any object $\alpha : V \to U$ in S/U,

  $$\mathrm{Hom}_{\mathscr{P}_U}(a_1, a_2)(\alpha) := \mathrm{Hom}_{\mathscr{P}_V}(\alpha^*(a_1), \alpha^*(a_2))$$

  where α^* is the restriction functor onto $\mathscr{P}_V$,

- for any morphism

  $$\begin{array}{ccc} V_1 & \xrightarrow{\ \beta\ } & V_2 \\ & {\scriptstyle\alpha_1}\searrow \quad \swarrow{\scriptstyle\alpha_2} & \\ & U & \end{array}$$

  in S/U, $\mathrm{Hom}_{\mathscr{P}_U}(a_1, a_2)(\beta)$ is the set map

  $$\mathrm{Hom}_{\mathscr{P}_{V_2}}(\alpha_2^*(a_1), \alpha_2^*(a_2)) \to \mathrm{Hom}_{\mathscr{P}_{V_1}}(\alpha_1^*(a_1), \alpha_1^*(a_2))$$

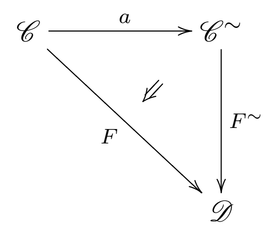
  defined by post composing $\beta^* : \mathscr{P}_{V_2} \to \mathscr{P}_{V_1}$.

Let $(W_\bullet \to V, (f, \delta^* \circ \alpha^*(a)))$ be a collection where $\alpha : V \to U$ is an object in S/U, $f \in A^{-1}(W_0)$, $W_\bullet \to V$ is a hyper-cover, and $\delta : W_0 \to V$ is an augmentation map that satisfies $\delta^* \circ \alpha^*(a_1) + \lambda_{W_0}(f) = \delta^* \circ \alpha^*(a_2)$. Since A^{-1} is a sheaf, there exists $g \in A^{-1}(V)$ such that $\delta^*(g) = f$. By the facts that λ is a functor and δ is a local epimorphism, g satisfies the relation

$$\alpha^*(a_1) + \lambda_V(g) = \alpha^*(a_2).$$

That is $(g, \alpha^*(a_1))$ is a morphism in $\mathscr{P}_V$ from $\alpha^*(a_1)$ to $\alpha^*(a_2)$ such that $\delta^*(g, \alpha^*(a_1)) = (f, \delta^* \circ \alpha^*(a_1))$. This shows that $\mathrm{Hom}_{\mathscr{P}_U}(a_1, a_2)$ is a sheaf. □

We recall the stack associated to a prestack. Let $\mathscr{C}$ be a prestack. There exits a stack $\mathscr{C}^{\sim}$ with a morphism of prestacks $a : \mathscr{C} \to \mathscr{C}^{\sim}$ which satisfy the following universal property: For any stack $\mathscr{D}$ and prestack morphism $F : \mathscr{C} \to \mathscr{D}$ there exists a unique prestack morphism $F^{\sim} : \mathscr{C}^{\sim} \to \mathscr{D}$ such that



There is also an explicit way of constructing the associated stack which involves "adding the descent data". This method is explained in [3] and [23].

We denote the stack associated to the morphism $A^{-1} \to A^0$ by $[A^{-1} \to A^0]^{\sim}$. An object of $[A^{-1} \to A^0]^{\sim}$ over U is a decent datum $(V_\bullet \to U, a, (f, a))$. We remark that in fact $[A^{-1} \to A^0]^{\sim}$ is a Picard stack where the Picard structure is defined by

$$(V_\bullet \to U, a, (f, a)) \otimes (V'_\bullet \to U, a', (f', a')) = (V_\bullet \times_U V'_\bullet \to U, a + a', (f + f', a + a'))$$

3.6 (A, B)-torsors

In this section, we give a geometric description of the associated Picard stack. Let A be a sheaf over the site S, not necessarily abelian. A (right) *A-torsor* is a space $L \to \mathsf{S}$ over S with a right group action

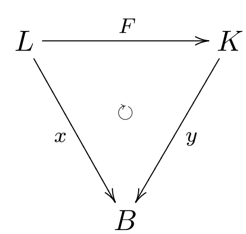
$$\mathrm{m} : L \times A \longrightarrow A$$

such that the morphism

$$(\mathrm{pr}, \mathrm{m}) : L \times A \longrightarrow L \times L$$

defined by $(l, a) \mapsto (l, m(l, a))$ is an equivalence. Moreover we require that there exists a collection of local sections $s_i : U_i \to L$ for an open cover $\{U_i\}$ of S.

Let $A \to B$ be a morphism of, not necessarily abelian, sheaves. An (A, B)-torsor is a pair (L, x), where L is an A-torsor and $x : L \to B$ is an A-equivariant morphism of sheaves (see [10]). A morphism between two pairs (L, x) and (K, y) is a morphism of sheaves $F : L \to K$ compatible with the action of A such that the diagram



commutes.

(A,B)-torsors form a stack denoted by $\textsc{Tors}(A,B)$. If A and B are abelian sheaves, we can also define a group-like structure on $\textsc{Tors}(A,B)$ as follows:

$$(L,x)\otimes(K,y) := (L\wedge^A K, x\wedge y)$$

where $L\wedge^A K$ is the contracted product and $x\wedge y$ is the A-equivariant morphism from $L\wedge^A K$ to B given by $x(l)y(k)$ where (l,k) is in $L\wedge^A K$. This group-like structure on $\textsc{Tors}(A,B)$ is Picard, that is $\textsc{Tors}(A,B)$ is a Picard stack. It provides a geometric realization of the associated Picard stack.

Theorem 3.6.1. *[5, Théorème 4.6] There is an equivalence of Picard stacks*

$$\textsc{Tors}(A^{-1},A^0) \xrightarrow{\sim} [A^{-1}\to A^0]^{\sim}$$

3.7 Length 2 Complexes of Abelian Sheaves

In the section (3.5) we looked at the relation between morphism of abelian sheaves and Picard stacks. We have already defined morphisms of Picard stacks. In this section, we define the morphisms between the morphisms of abelian sheaves which are called butterflies (see [3], [27], and [28]). Moreover, we show that morphisms of abelian sheaves, that is length 2 complexes of abelian sheaves form a bicategory $\mathrm{T}^{[-1,0]}(\mathsf{S})$ whose 1-morphisms are butterflies (3.7.1) and 2-morphisms are morphisms of butterflies 3.7.2.

Let $A^\bullet = [\lambda_A : A^{-1}\to A^0]$ and $B^\bullet = [\lambda_B : B^{-1}\to B^0]$ be two length 2 complexes of abelian sheaves.

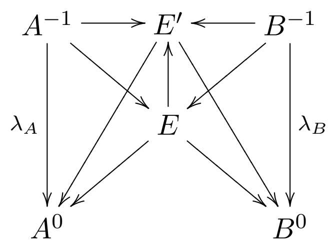
Definition 3.7.1. A *butterfly* from $A^\bullet$ to $B^\bullet$ is a commutative diagram of abelian sheaf morphisms of the form

$$\begin{array}{ccccc} A^{-1} & & & & B^{-1} \\ & \searrow^{\kappa} & & \swarrow^{\imath} & \\ \Big\downarrow{\scriptstyle \lambda_A} & & E & & \Big\downarrow{\scriptstyle \lambda_B} \\ & \swarrow_{\rho} & & \searrow_{\jmath} & \\ A^0 & & & & B^0 \end{array} \qquad (3.7.1)$$

where E is an abelian sheaf, the NW-SE sequence is a complex, and the NE-SW sequence is an extension. $[A^\bullet, E, B^\bullet]$ will denote a butterfly from $A^\bullet$ to $B^\bullet$.

A butterfly is called *flippable* or *reversible* if both diagonals of (3.7.1) are extensions. A *strong butterfly* is a butterfly equipped with a global section $s : A^0\to E$ such that $\rho\circ s = \mathrm{id}$.

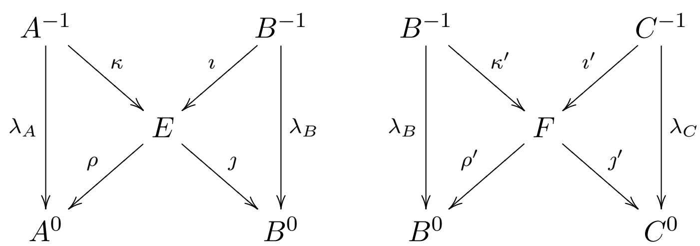
Definition 3.7.2. A *morphism of butterflies* $\varphi : [A^\bullet, E, B^\bullet]\to[A^\bullet, E', B^\bullet]$ is an abelian sheaf isomorphism $E\to E'$ satisfying the commutative diagrams below.



(3.7.2)

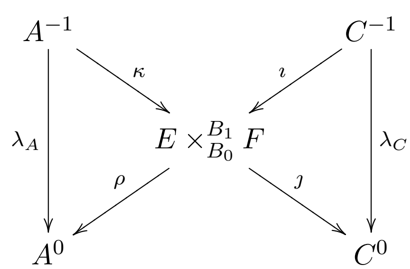
Two such morphisms (3.7.2) compose in an obvious way. Therefore butterflies from $A^\bullet$ to $B^\bullet$ form a groupoid denoted by $\mathsf{B}(A^\bullet, B^\bullet)$.

We can also compose butterflies. Given two butterflies



(3.7.3)

their composition is the butterfly



(3.7.4)

where the center $E \times^{B_1}_{B_0} F$ is given by the pushout-pullback construction (see [3]). From the definitions 3.7.1, 3.7.2, and the definition of composition of butterflies, it follows

Theorem 3.7.3. *[3, Theorem 5.1.4] Length 2 complexes of abelian sheaves equipped with butterflies as 1-morphisms and morphisms of butterflies as 2-morphisms form a bicategory* $\mathrm{T}^{[-1,0]}(\mathsf{S})$.

$\mathrm{T}^{[-1,0]}(\mathsf{S})$ has a full sub 2-category $\mathrm{C}^{[-1,0]}(\mathsf{S})$ that has same objects as $\mathrm{T}^{[-1,0]}(\mathsf{S})$ but whose 1-morphisms are strong butterflies. To be precise,

- objects of $\mathrm{C}^{[-1,0]}(\mathsf{S})$ are same as objects of $\mathrm{T}^{[-1,0]}(\mathsf{S})$, that are morphisms of abelian sheaves $A^\bullet = [\lambda_A : A^{-1} \to A^0]$.

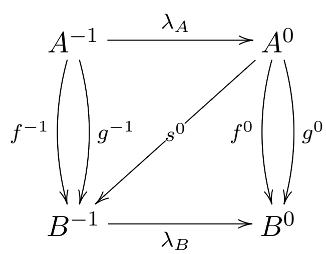
- for any two objects $A^\bullet$ and $B^\bullet$, a 1-morphism $f^\bullet : A^\bullet \to B^\bullet$ in $\mathrm{C}^{[-1,0]}(\mathsf{S})$ is a complex morphism from $A^\bullet$ to $B^\bullet$. That is $f^\bullet$ consists of two morphisms of abelain sheaves

f^{-1} and f^0 such that the diagram

$$\begin{array}{ccc} A^{-1} & \xrightarrow{\lambda_A} & A^0 \\ {\scriptstyle f^{-1}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle f^0} \\ B^{-1} & \xrightarrow[\lambda_B]{} & B^0 \end{array} \tag{3.7.5}$$

commutes.

- for any two parallel 1-morphisms $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$, a 2-morphism $s^\bullet : f^\bullet \Rightarrow g^\bullet$ in $\mathrm{C}^{[-1,0]}(\mathsf{S})$ is a homotopy between $f^\bullet$ and $g^\bullet$. That is $s^\bullet$ consists of a degree -1 morphism of abelian sheaves s^0 given by the diagrams



(3.7.6)

satisfying $g^0 - f^0 = \lambda_B \circ s^0$ and $g^{-1} - f^{-1} = s^0 \circ \lambda_A$.

Said differently, $\mathrm{C}^{[-1,0]}(\mathsf{S})$ is a 2-category of morphisms of abelian sheaves whose hom-category $\mathrm{Hom}_{\mathrm{T}^{[-1,0]}(\mathsf{S})}(A^\bullet, B^\bullet)$ for any two morphisms $A^\bullet$ and $B^\bullet$ is the groupoid associated to the complex

$$\mathrm{Hom}^{-1}(A^\bullet, B^\bullet) \xrightarrow{\partial} Z^0(\mathrm{Hom}^0(A^\bullet, B^\bullet))$$

of abelian groups, where elements of $\mathrm{Hom}^{-1}(A^\bullet, B^\bullet)$ are morphisms of complexes from $A^\bullet$ to $B^\bullet$ of degree -1 and where $Z^0(\mathrm{Hom}^0(A^\bullet, B^\bullet))$ is the abelian group of cocycles of degree 0 morphisms. The differential ∂ is defined as

$$(\partial(s^\bullet))^{-p} = \lambda_B^{-p-1} \circ s^{-p} + s^{-p+1} \circ \lambda_A^{-p}$$

for any $s^\bullet \in \mathrm{Hom}^{-1}(A^\bullet, B^\bullet)$ and $p = 0, 1$.

3.8 Characterization Theorem for Picard Stacks

We finish this chapter by recalling Deligne's characterization theorem for Picard stacks [9, §1.4]. This result is also revisited by Aldrovandi and Noohi in [3]. In order to be consistent with the rest of the thesis, we recall them as they are enounced in [3].

The characterization theorem states the following:

Theorem. *[3, Proposition 8.4.3] The functor*

$$\mathrm{T}^{[-1,0]}(\mathsf{S}) \longrightarrow \mathrm{P\scriptsize{IC}}(\mathsf{S}) \tag{3.8.1}$$

defined by sending a morphism of abelian sheaves $A^\bullet = [\lambda_A : A^{-1} \to A^0]$ *to* $\mathrm{T\scriptsize{ORS}}(A^{-1}, A^0)$ *is a biequivalence of bicategories.*

The bifunctor (3.8.1) in this theorem is first constructed on the strict sub 2-category $\mathrm{C}^{[-1,0]}(\mathsf{S})$ as follows.

- An object in $\mathrm{C}^{[-1,0]}(\mathsf{S})$ that is a morphism of abelian sheaves $A^\bullet = [\lambda_A : A^{-1} \to A^0]$ is sent to its associated Picard stack which is equivalent to $\mathrm{T\scriptsize{ORS}}(A^{-1}, A^0)$. (see Section (3.5)).
- A 1-morphism $f^\bullet : A^\bullet \to B^\bullet$ (3.7.5) in $\mathrm{C}^{[-1,0]}(\mathsf{S})$ is sent to a morphism of torsors

  $$\mathrm{T\scriptsize{ORS}}(A^{-1}, A^0) \longrightarrow \mathrm{T\scriptsize{ORS}}(B^{-1}, B^0)$$

  which sends an (A^{-1}, A^0)-torsor (L, x) to $(L \wedge^{A^{-1}}_{f^{-1}} B^{-1}, f^0 \circ x + \lambda_B)$ where $L \wedge^{A^{-1}}_{f^{-1}} B^{-1}$ denotes the contracted product of the A^{-1}-torsors L and B^{-1} such that the A^{-1}-torsor structure of B^{-1} is induced by f^{-1}.
- A 2-morphism $s^\bullet : f^\bullet \Rightarrow g^\bullet$ (3.7.6) in $\mathrm{C}^{[-1,0]}(\mathsf{S})$ is sent to a 2-morphism of torsors θ

  $$\mathrm{T\scriptsize{ORS}}(A^{-1}, A^0) \underset{\Downarrow\theta}{\rightrightarrows} \mathrm{T\scriptsize{ORS}}(B^{-1}, B^0)$$

  that assigns to any object (L, x) in $\mathrm{T\scriptsize{ORS}}(A^{-1}, A^0)$ a 1-morphism $\theta_{(L,x)}$

  $$\theta_{(L,x)} : (L \wedge^{A^{-1}}_{f^{-1}} B^{-1}, f^0 \circ x + \lambda_B) \longrightarrow (L \wedge^{A^{-1}}_{g^{-1}} B^{-1}, g^0 \circ x + \lambda_B),$$

  defined by sending (l, b) to $(l, b - s^0 \circ x(l))$.

This construction extends to $\mathrm{T}^{[-1,0]}(\mathsf{S})$ by the following theorem

Theorem. *[3, Theorem 8.3.1] For any two length 2 complexes of abelian sheaves* $A^\bullet$ *and* $B^\bullet$*, there is an equivalence of groupoids*

$$\mathsf{Hom}(A^\bullet, B^\bullet) \xrightarrow{\sim} \mathsf{B}(A^\bullet, B^\bullet),$$

where $\mathsf{Hom}(A^\bullet, B^\bullet)$ *is the groupoid of additive functors between the Picard stacks associated to* $A^\bullet$ *and* $B^\bullet$*.*

which shows that the bifunctor (3.8.1) is fully-faithful. To show that it is a biequivalence, one needs to show that it is essentially surjective which is the following statement.

Proposition. *[3, Proposition 8.3.2] Let* $\mathscr{A}$ *be a Picard stack. Then there exists a length 2 complex of abelian sheaves* $A^\bullet : A^{-1} \to A^0$ *such that* $\mathscr{A}$ *is equivalent to Picard stack* $[A^{-1} \to A^0]^\sim$*.*

Remark 3.8.1. In the paper [3], the authors also generalize the theorem (3.8) to non-abelian context by not assuming that stacks and sheaves are necessarily Picard or abelian.

Theorem 3.8 has an immediate consequence.

Corollary 3.8.2. *The functor (3.8.1) induces an equivalence*

$$\mathrm{D}^{[-1,0]}(\mathsf{S}) \longrightarrow \mathrm{P{\small IC}}^{\flat}(\mathsf{S})$$

of categories where $\mathrm{D}^{[-1,0]}(\mathsf{S})$ *is the subcategory of the derived category of category of complexes of abelian sheaves* $A^{\bullet}$ *over a site* S *with* $H^{-i}(A^{\bullet}) \neq 0$ *only for* $i = 0, 1$ *and* $\mathrm{P{\small IC}}^{\flat}(\mathsf{S})$ *is the category of Picard stacks over* S *with 1-morphisms isomorphism classes of additive functors.*

Proof. The proof follows from the observation that isomorphism classes in $\mathrm{P{\small IC}}^{\flat}(\mathsf{S})$ correspond to isomorphism classes of flippable butterflies. A flippable butterfly from $A^{\bullet}$ to $B^{\bullet}$ corresponds to a zig-zag of complexes

$$A^{\bullet} \xleftarrow{\;p\;} M^{\bullet} \xrightarrow{\;q\;} B^{\bullet},$$

where both p and q are quasi-isomorphisms. □

CHAPTER 4

3-CATEGORY OF PICARD 2-STACKS

Itroduction paragraph

4.1 Picard 2-Categories

In this section, following [6] and [19], we define Picard 2-categories, additive 2-fuctors, natural 2-transformations, and modifications. Since our fundamental object of study is Picard 2-stack fibered in 2-groupoids, from now on, unless otherwise stated, we assume that all 2-categories are 2-groupoids (2.1.8). For compactness, in large diagrams we omit $\otimes$.

Definition 4.1.1. A 2-category $\mathbb{C}$ with the data

1. a 2-functor $\otimes : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$
2. a 2-natural transformation a,

$$\begin{array}{ccc} \mathbb{C}\times\mathbb{C}\times\mathbb{C} & \xrightarrow{\otimes\times 1} & \mathbb{C}\times\mathbb{C} \\ {\scriptstyle 1\times\otimes}\Big\downarrow & \swarrow_{\mathsf{a}} & \Big\downarrow{\scriptstyle\otimes} \\ \mathbb{C}\times\mathbb{C} & \xrightarrow[\otimes]{} & \mathbb{C} \end{array}$$

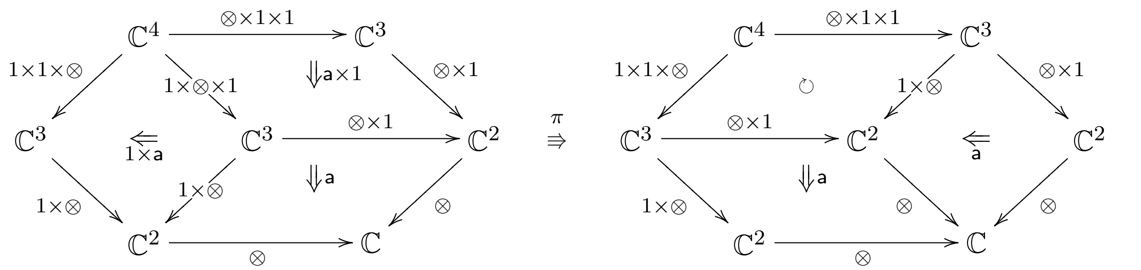
   expressing the associativity constraint.

3. a 2-natural transformation c,

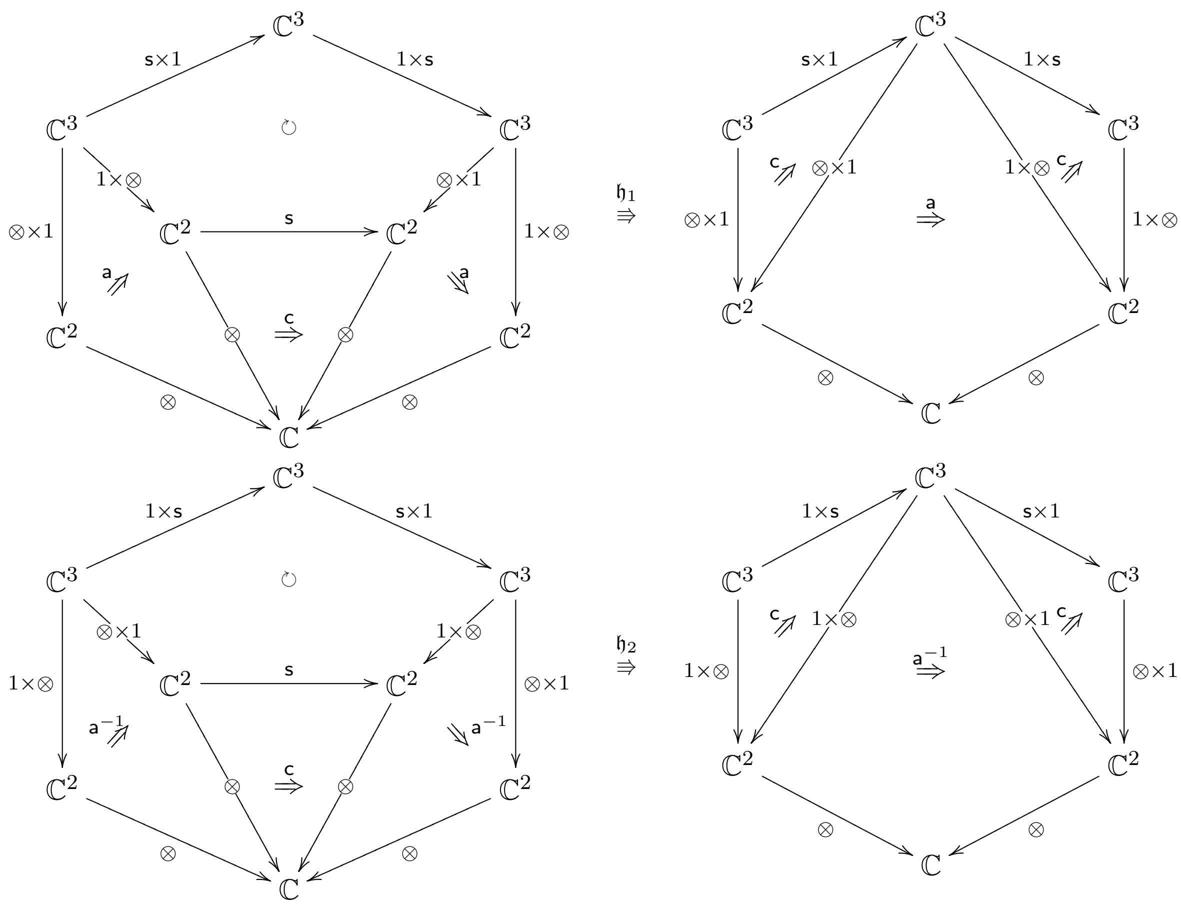
$$\begin{array}{ccc} \mathbb{C}\times\mathbb{C} & \xrightarrow{\mathsf{s}} & \mathbb{C}\times\mathbb{C} \\ {\scriptstyle\otimes}\searrow & \overset{\mathsf{c}}{\Rightarrow} & \swarrow{\scriptstyle\otimes} \\ & \mathbb{C} & \end{array}$$

   expressing the commutativity constraint.

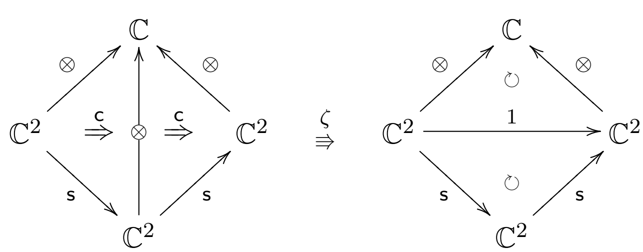
4. a modification π



5. two modifications $\mathfrak{h}_1$ and $\mathfrak{h}_2$

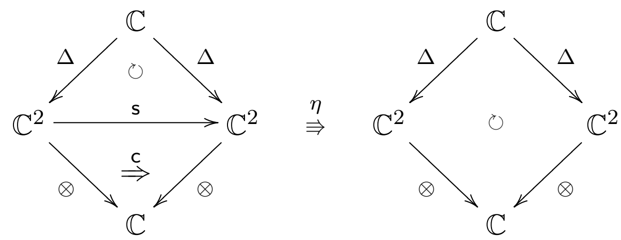


6. a modification ζ



(4.1.1)

7. a modification η



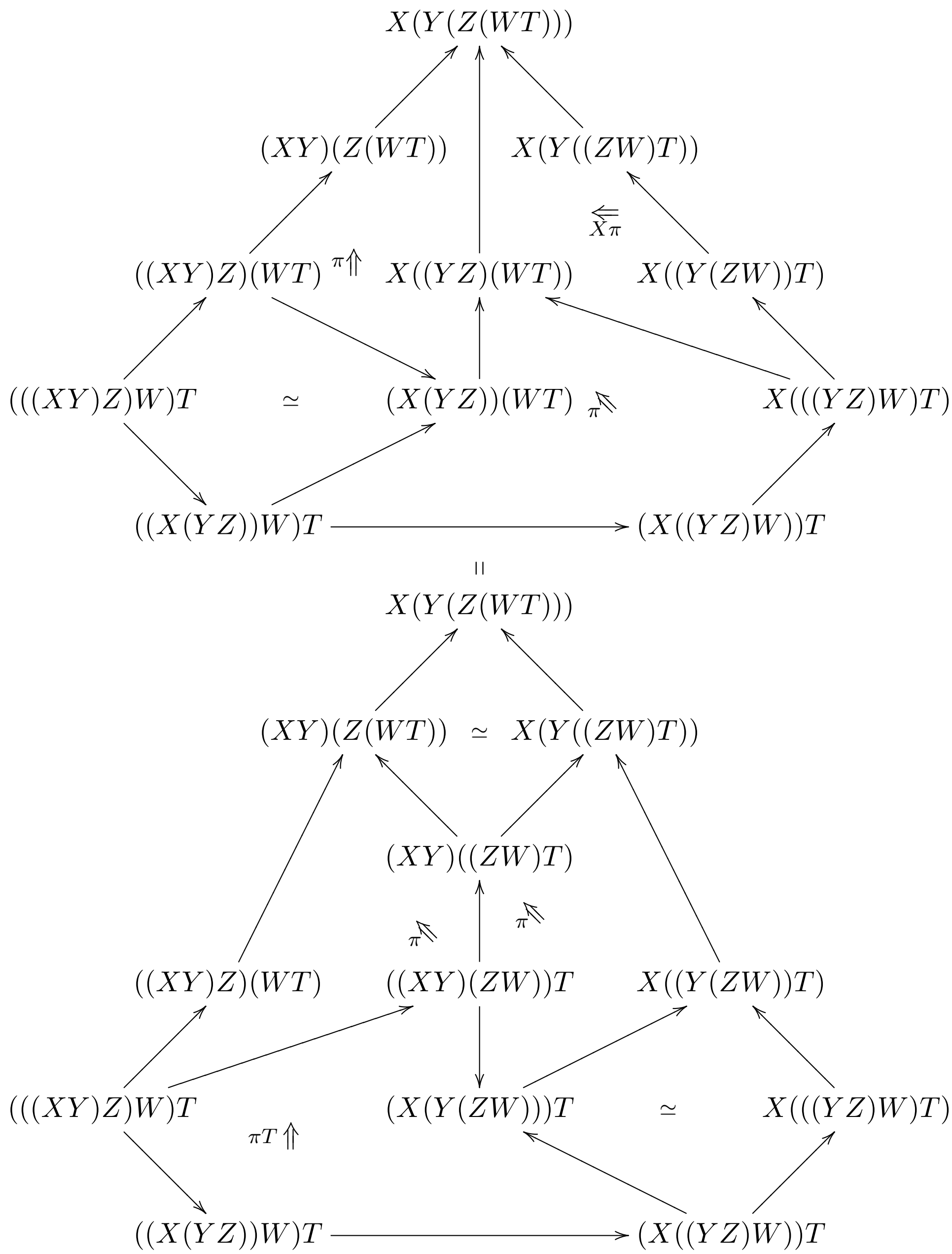
(4.1.2)

where Δ is the diagonal 2-functor.

These data must satisfy the conditions:

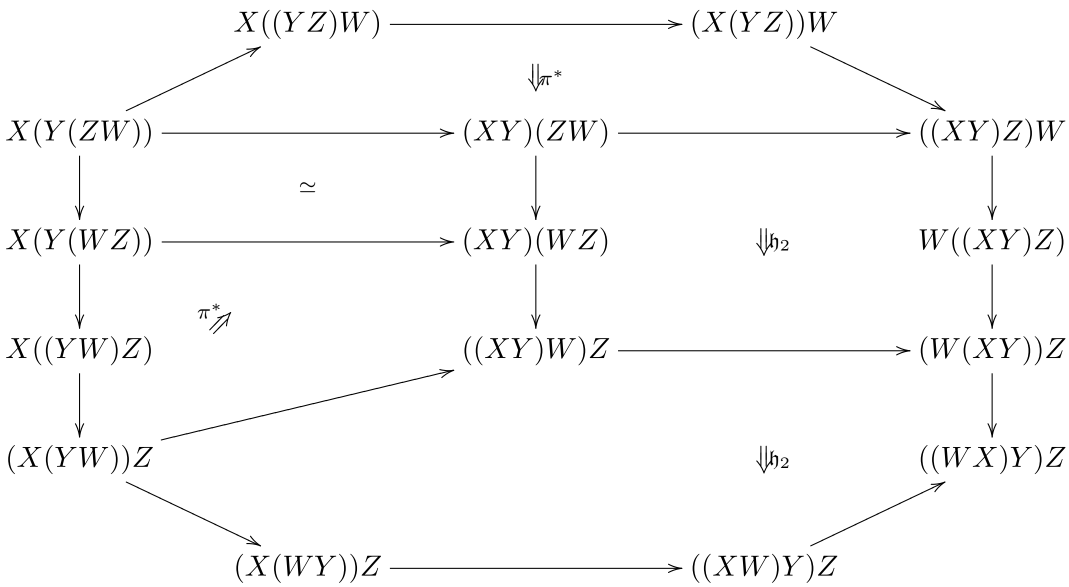
(i) for any object X in $\mathbb{C}$ the functor $X \otimes - : \mathbb{C} \longrightarrow \mathbb{C}$ is a biequivalence.

(ii) for all X, Y, Z, W, T objects in $\mathbb{C}$, the equation of 2-morphisms hold.

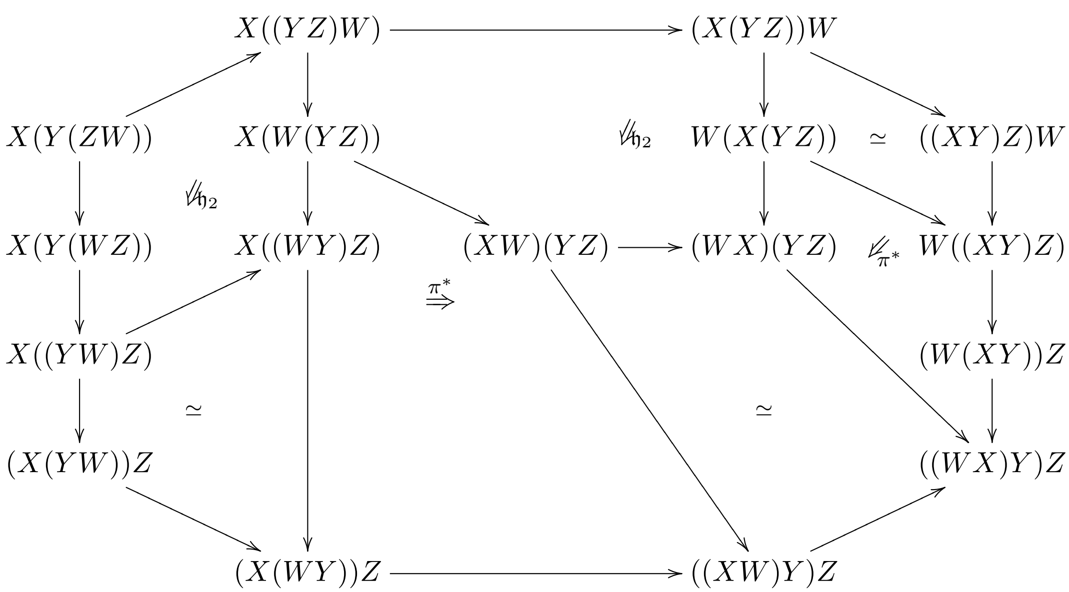


The natural 2-isomorphisms $\simeq$ are due to the functoriality of $\otimes$.

(iii) For any objects X, Y, Z, W, the equation of two 2-morphisms hold.

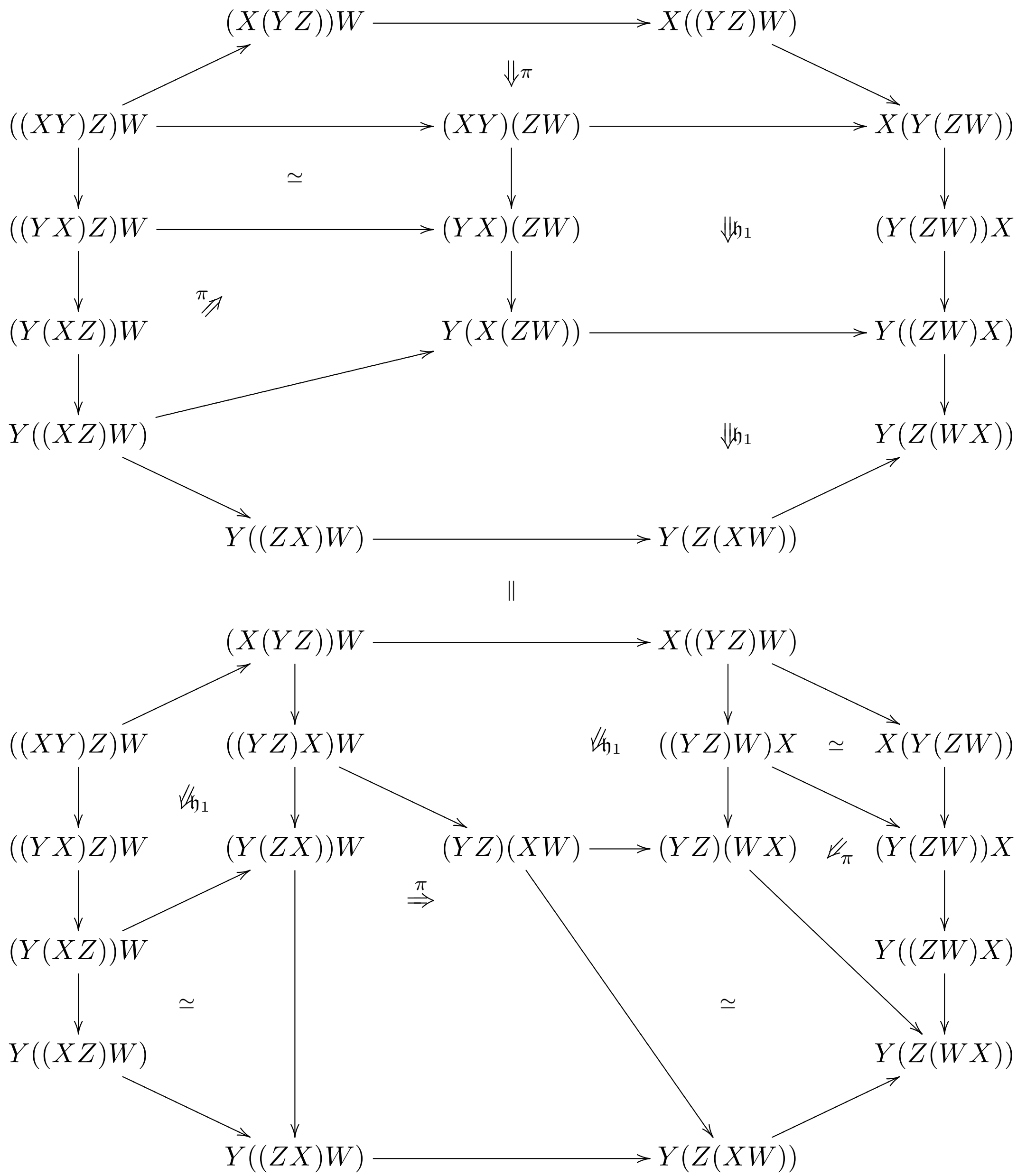


$$\|$$

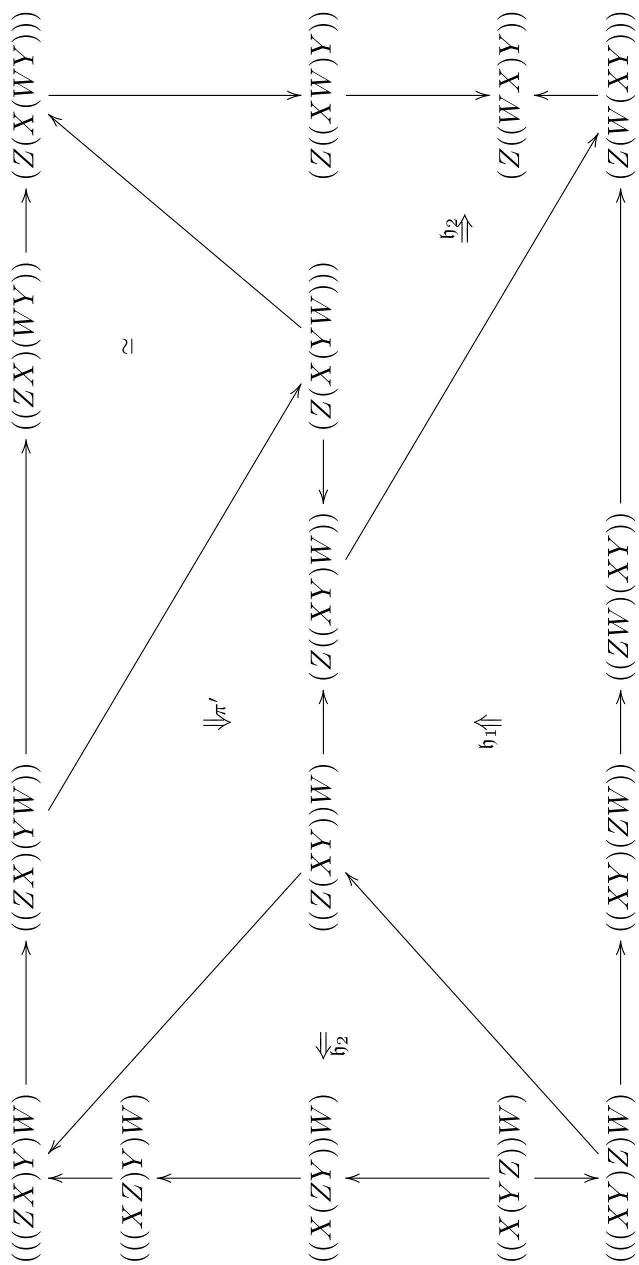


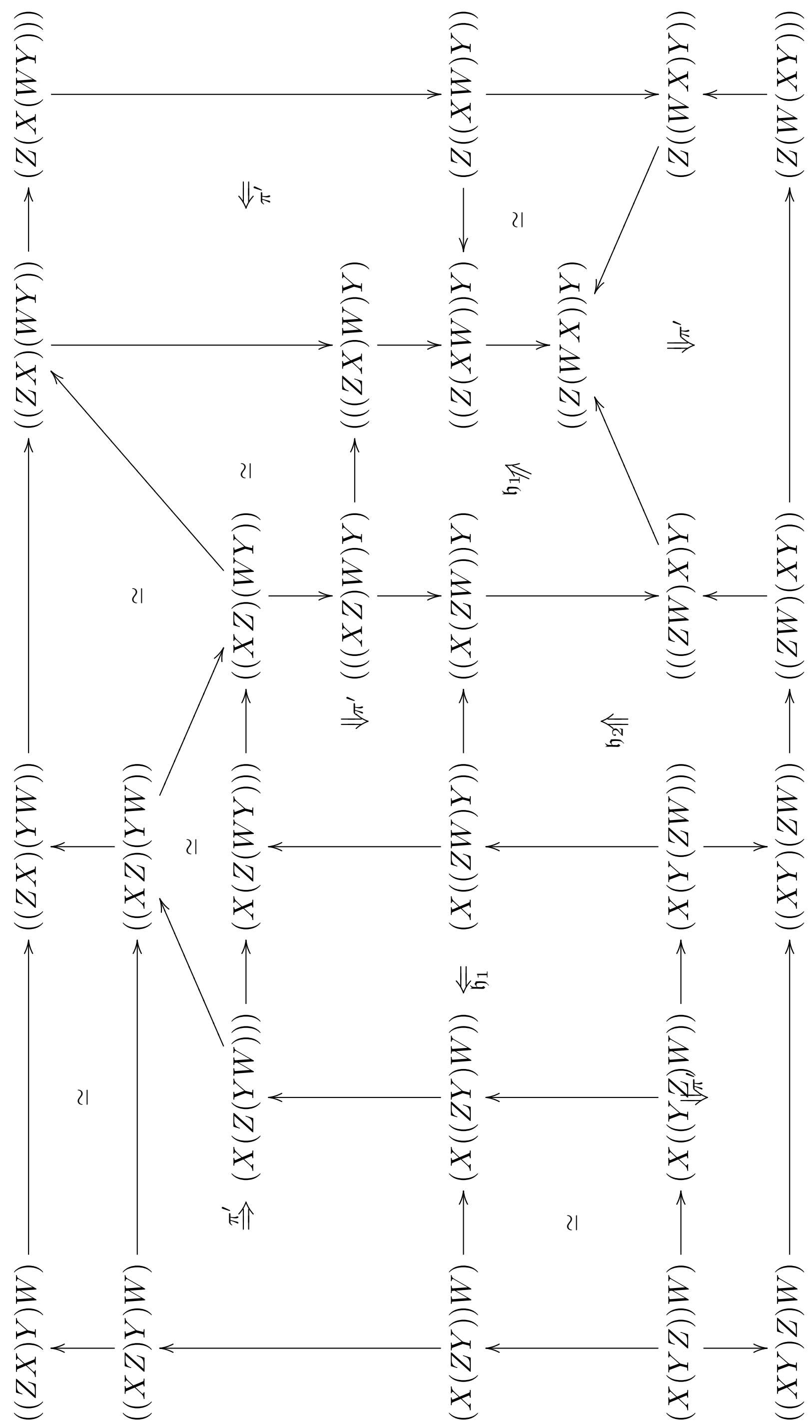
The modification π^* is defined in the same way as π using a^{-1} instead of a.

(iv) for any objects X, Y, Z, W, the equation of two 2-morphisms hold.



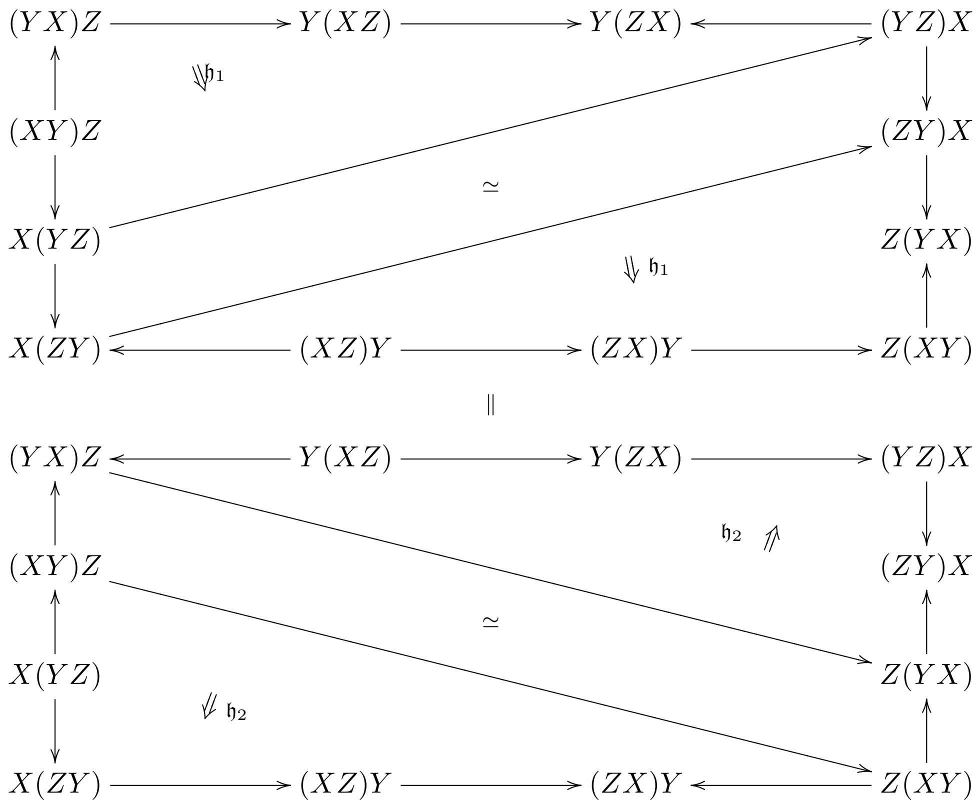
(v) for any objects X, Y, Z, W, the equation of two 2-morphisms hold.



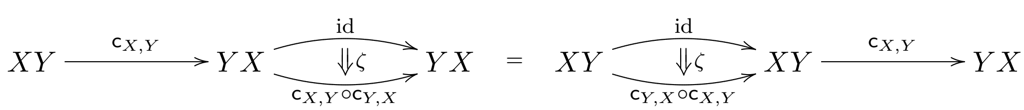


The modification π' is obtained from π by inverting one or more a's.

(vi) for all X, Y, Z objects in $\mathbb{C}$, the equation of two 2-morphisms hold.

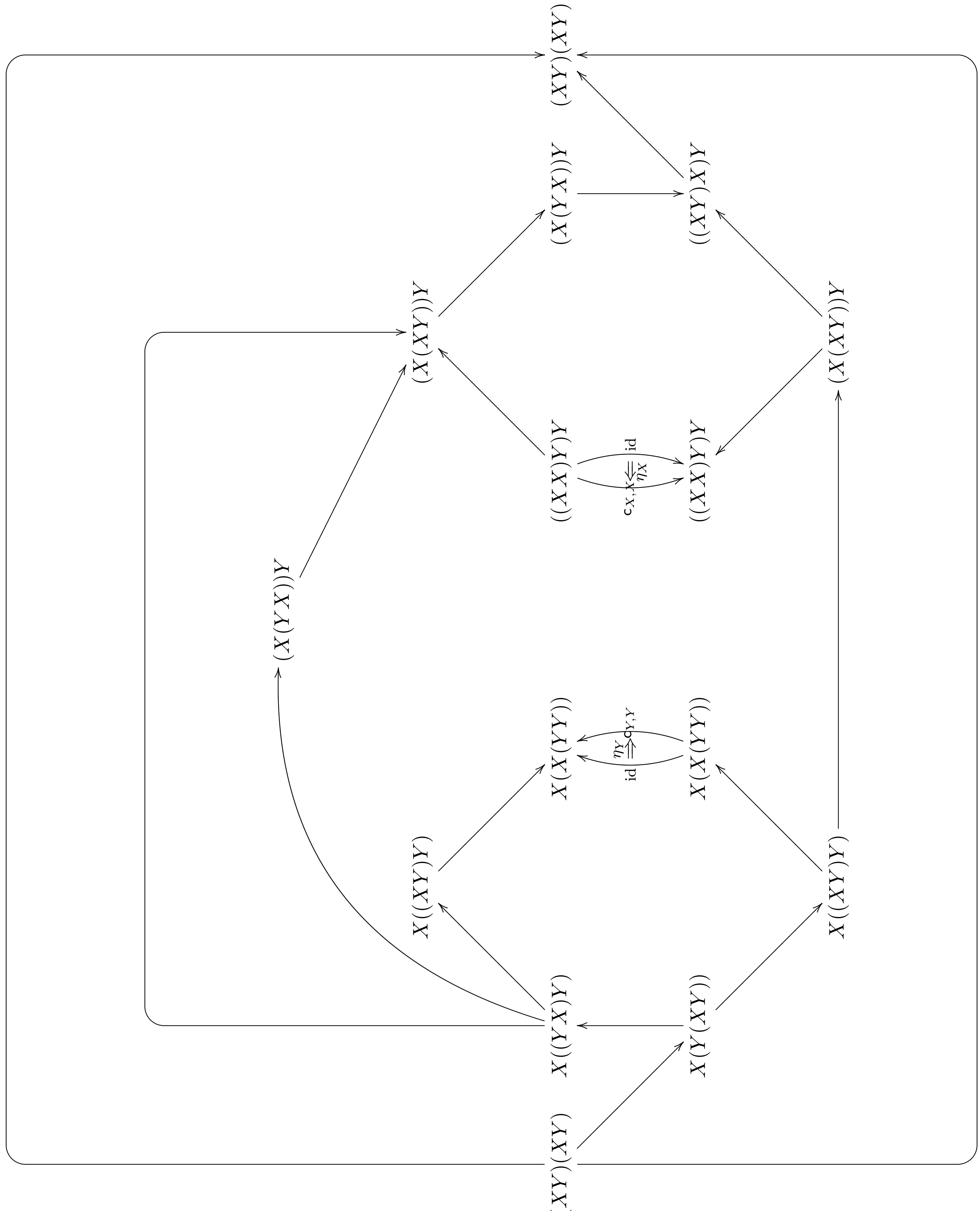


(vii) for all X, Y, Z objects in $\mathbb{C}$



(viii) $\eta * \eta = \zeta$.

(ix) for all X, Y objects in $\mathbb{C}$, there is an additive relation between η_X,η_Y and $\eta_{X \otimes Y}$. That is $\eta_{X \otimes Y}$ is equal to the pasting of the below diagram.


(XY)(XY)
X(Y(XY))
X((YX)Y)
X((XY)Y)
X(X(YY))
id
η_Y
c_{Y,Y}
(X(YX))Y
(X(XY))Y
((XX)Y)Y
c_{X,X}
η_X
((XY)X)Y
(XY)(XY)

Notation 4.1.2. We denote a Picard 2-category by $(\mathbb{C}, \otimes_{\mathbb{C}}, \mathsf{a}_{\mathbb{C}}, \mathsf{c}_{\mathbb{C}}, \pi_{\mathbb{C}}, \mathfrak{h}_{1\mathbb{C}}, \mathfrak{h}_{2\mathbb{C}}, \zeta_{\mathbb{C}}, \eta_{\mathbb{C}})$. In case there is no risk of confusion we are going to denote a Picard category by $(\mathbb{C}, \otimes)$.

Lemma 4.1.3. *Let $(\mathbb{C}, \otimes)$ be a Picard 2-category and $f : X \to Y$ be a morphism in $\mathbb{C}$. If f is weakly invertible then $f \otimes - : \mathrm{Hom}_{\mathbb{C}}(A, B) \to \mathrm{Hom}_{\mathbb{C}}(X \otimes A, Y \otimes B)$ is an equivalence of categories for all objects A, B in $\mathbb{C}$.*

Proof. Assume that f is weakly invertible. The functor $f \otimes -$ is defined as composition of the following two functors,

$$\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(X \otimes A, X \otimes B) \tag{4.1.3}$$
$$\mathrm{Hom}(X \otimes A, X \otimes B) \longrightarrow \mathrm{Hom}(X \otimes A, Y \otimes B) \tag{4.1.4}$$

Since the functor (4.1.3) is multiplication by X, it is an equivalence. The functor (4.1.4) is post composition by $f \otimes Y$. It is an equivalence since f is weakly invertible. Thus $f \otimes -$ is an equivalence. □

4.2 Units in Picard 2-Categories

We define unit element in Picard 2-categories. The only original result in this section is the Proposition 4.2.5 which says that every Picard 2-category possesses a unit element. All the definitions and other results can be found in the paper by Joyal and Kock [18]. The reason why we restate these results here is because originally they are given for strict monoidal 2-categories and in this thesis we interpret them for Picard 2-categories where the associativity is assumed to be non-strict.

Definition 4.2.1. Let $(\mathbb{C}, \otimes)$ be a Picard 2-category. A pair (e, φ) is called a unit element in $\mathbb{C}$ where e is an object in $\mathbb{C}$ and $\varphi : e \otimes e \to e$ is a weakly invertible 1-morphism.

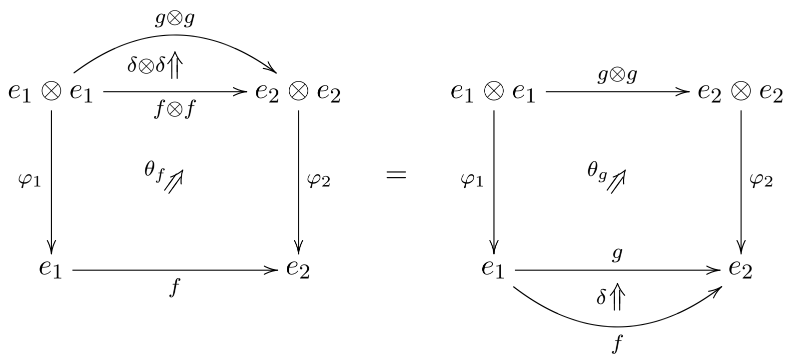
Definition 4.2.2. Let $(\mathbb{C}, \otimes)$ be a Picard 2-category and (e_1, φ_1) and (e_2, φ_2) be two unit elements. A 1-morphism $(e_1, \varphi_1) \to (e_2, \varphi_2)$ is given by a pair (f, θ_f) where $f : e_1 \to e_2$ is a weakly invertible 1-morphism and θ_f is the 2-isomorphism

$$\begin{array}{ccc} e_1 \otimes e_1 & \xrightarrow{f \otimes f} & e_2 \otimes e_2 \\ {\scriptstyle \varphi_1}\big\downarrow & \theta_f \nearrow & \big\downarrow{\scriptstyle \varphi_2} \\ e_1 & \xrightarrow[f]{} & e_2 \end{array} \tag{4.2.1}$$

We call such a pair (f, θ_f) a *unit morphism.*

Definition 4.2.3. Let $(\mathbb{C}, \otimes)$ be a Picard 2-category, and (f, θ_f) and (g, θ_g) be two unit morphisms from (e_1, φ_1) to (e_2, φ_2). A 2-morphism $(f, \theta_f) \Rightarrow (g, \theta_g)$ is given by a 2-isomorphism

$\delta : f \Rightarrow g$ in $\mathbb{C}$ such that

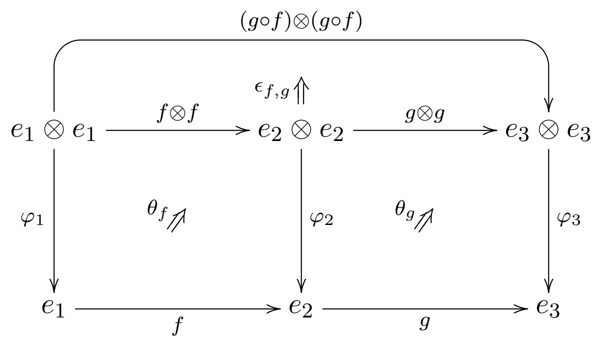


(4.2.2)

We call such a 2-isomorphism *unit 2-morphism.*

Unit elements, unit morphisms, and unit 2-morphisms of a Picard 2-category $(\mathbb{C}, \otimes)$ form a 2-category denoted by $\mathbb{U}(\mathbb{C})$ where

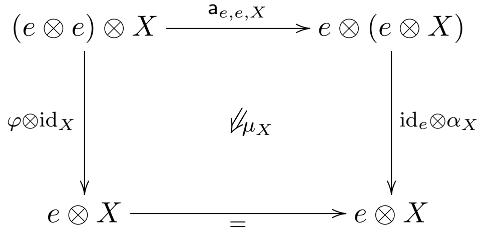
> the composition of 1-morphisms $(f, \theta_f) : (e_1, \varphi_1) \to (e_2, \varphi_2)$ and $(g, \theta_g) : (e_2, \varphi_2) \to (e_3, \varphi_3)$ is the pair $(h, \theta_h) : (e_1, \varphi_1) \to (e_3, \varphi_3)$ where $h = g \circ f$ and θ_h is the pasting of the 2-isomorphisms given in the diagram below.



> the vertical and the horizontal compositions are induced from the ones in $\mathbb{C}$.

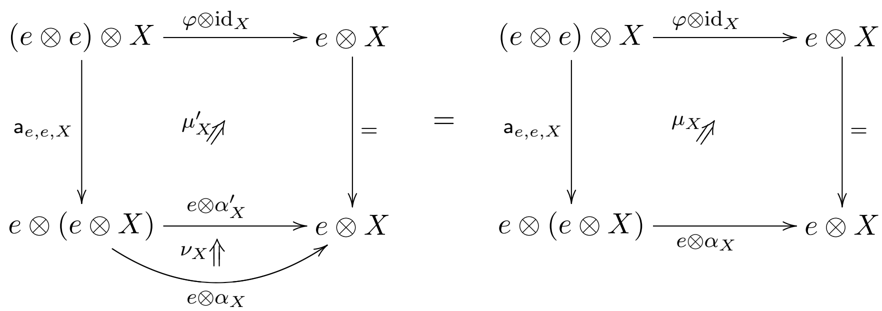
We remark that $\mathbb{U}(\mathbb{C})$ is in fact a 2-groupoid and call it *unit 2-groupoid.*

Proposition 4.2.4. *Let (e, φ) be a unit element in a Picard 2-category $(\mathbb{C}, \otimes)$. For every object X in $\mathbb{C}$, there exists a pair (α_X, μ_X) where $\alpha_X : e \otimes X \to X$ is a weakly invertible 1-morphism, μ_X is the 2-isomorphism,*



(4.2.3)

Also for any object X in $\mathbb{C}$, the pair (α_X, μ_X) is unique in the following sense. If there exists another pair (α'_X, μ'_X) then there exists a unique 2-isomorphism $\nu_X : \alpha_X \Rightarrow \alpha'_X$ such that the diagram



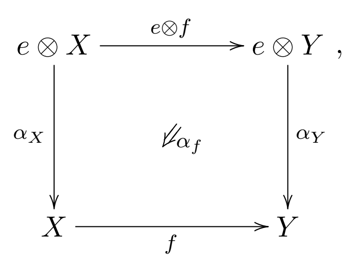
Proof. The main ideas of the proof are given in [18, §5]. However we want to give a detailed proof since we assume different from Joyal and Kock non-strict associativity.

Existence of (α_X, μ_X)*:* Let (e, φ) be a unit element and let X be an object in $\mathbb{C}$. The 1-morphism $\varphi \otimes \mathrm{id}_X$ is an object in the category $\mathrm{Hom}((e \otimes e) \otimes X, e \otimes X)$. This hom-category is equivalent to the hom-categories

$$\mathrm{Hom}((e \otimes e) \otimes X, e \otimes X) \simeq \mathrm{Hom}(e \otimes (e \otimes X), e \otimes X) \simeq \mathrm{Hom}(e \otimes X, X).$$

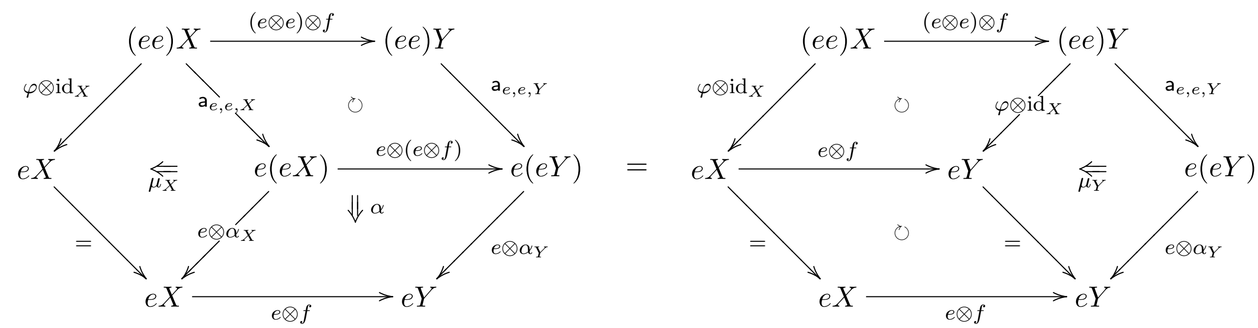
The first equivalence follows from the Lemma 4.1.3 since $\mathsf{a}_{e,e,X}$ is weakly invertible and the second follows from the biequivalence of the 2-functor $e \otimes -$. Under these equivalences, there exists $\alpha_X : e \otimes X \rightarrow X$ in $\mathrm{Hom}(e \otimes X, X)$ whose image in $\mathrm{Hom}((e \otimes e) \otimes X, e \otimes X)$ is $\mathrm{id}_e \otimes \alpha_X \circ \mathsf{a}_{e,e,X}$ and is 2-isomorphic to $\varphi \otimes \mathrm{id}_X$. That is there exists also a modification μ whose component at X is the 2-isomorphism μ_X as in diagram (4.2.3).

Naturality of (α_X, μ_X)*:* In order to show that α_X is functorial, we need to define for any 1-morphism $X \rightarrow Y$ in $\mathbb{C}$, a 2-isomorphism α_f



(4.2.4)

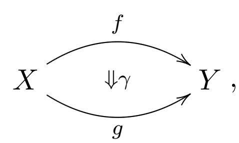
that is compatible with another choice of 1-morphism $g : X \rightarrow Y$. We let α_f be the inverse image of the 2-isomorphism α under the biequivalence $e \otimes -$. α is defined as the 2-morphism that makes the two different pastings of the 2-morphisms in the below diagram equal.



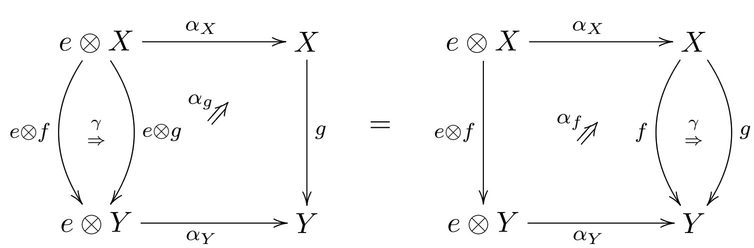
(4.2.5)

α is a uniquely defined 2-isomorphism since the other 2-morphisms in the diagram (4.2.5) are 2-isomorphisms.

The fact that the two different pastings in the diagram (4.2.5) are equal shows that μ_X is functorial. To verify that α_f is compatible with another choice of 1-morphism $g : X \to Y$, we have to show that for any 2-morphism

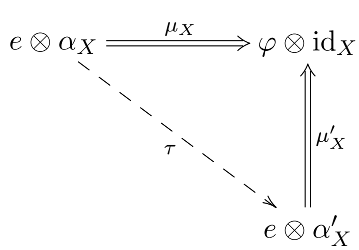


the two different pastings in the diagram



are equal. Since the 2-isomorphisms on the right half of the diagram (4.2.5) are independent of the morphism from X to Y, f and g satisfy the same diagram (4.2.5).

Uniqueness of (α_X, μ_X)*:* Let (α_X, μ_X) and (α'_X, μ'_X) be two pairs such that



(4.2.6)

Since μ_X and μ'_X are 2-isomorphisms, there exists a 2-isomorphism $\tau : e \otimes \alpha_X \Rightarrow e \otimes \alpha'_X$ that makes the diagram (4.2.6) commutative. Using the equivalence $\mathrm{Hom}(e \otimes (e \otimes X), e \otimes X) \simeq \mathrm{Hom}(e \otimes X, X)$, we deduce there exists a 2-isomorphism $\sigma : \alpha_X \Rightarrow \alpha'_X$ such that $e \otimes \sigma = \tau$. □

Proposition 4.2.5. *A Picard 2-category* $(\mathbb{C}, \otimes)$ *has a unit element.*

Proof. Let X be an object in $\mathbb{C}$. Since $X \otimes -$ is a biequivalence, for all $Y \in \mathbb{C}$ there exists $Z \in \mathbb{C}$ such that $X \otimes Z$ is equivalent to Z. That is there exists a weakly invertible 1-morphism $X \otimes Z \to Y$ in $\mathbb{C}$. In particular, we take $X = Y$ and there exists $f : e_X \in \mathbb{C}$ with a weakly invertible 1-morphism $X \otimes e_X \to X$. We claim that there exists $\varphi : e_X \otimes e_X \to e_X$ such that (e_X, φ) is a unit element in $\mathbb{C}$. The hom-categories

$$\mathrm{Hom}_{\mathbb{C}}((X \otimes e_X) \otimes e_X, X \otimes e_X) \simeq \mathrm{Hom}_{\mathbb{C}}(X \otimes (e_X \otimes e_X), X \otimes e_X) \simeq \mathrm{Hom}_{\mathbb{C}}(e_X \otimes e_X, e_X)$$

are equivalent. The first equivalence is defined by whiskering with the weakly invertible 1-morphism $\mathsf{a}_{X,e,e}$. The second equivalence follows from the fact that $X \otimes -$ is a biequivalence. Now $f \otimes e_X$ is a 1-morphism in the category $\mathrm{Hom}_{\mathbb{C}}((X \otimes e) \otimes e, X \otimes e)$. We define $\varphi : e_X \otimes e_X \to e_X$ as the image of $f \otimes e_X$ under these equivalences. φ is weakly invertible since f is. Hence, the pair (e_X, φ) is a unit element in $\mathbb{C}$. □

Proposition 4.2.6. *[18, Thereom C] Let* $(\mathbb{C}, \otimes)$ *be a Picard 2-category. The 2-groupoid of units* $\mathbb{U}(\mathbb{C})$ *is contractible. That is between any two units there exists a unit morphism and between any parallel two unit morphisms there exists a unique unit 2-morphism.*

Proof. The proof is given in [18, §5]. However one has to be careful since in [18] the associativity is assumed to be strict. □

The unit element in a Picard 2-category is first extracted from the definition of a tricategory. In [18], Joyal and Kock give another definition for the unit element 4.2.1. They show that

Theorem 4.2.7. *[18, Theorem E] The notion of unit element 4.2.1 is equivalent to the notion of the unit element extracted from the definition of tricategry.*

By the above theorem, the notion of unit element is not part of the Picard 2-category data, but it is already part of the Picard structure. This reduces significantly the number of compatibility conditions in the definition of Picard 2-category. As we are going to see in the next section, this simplifies the definition of the morphism of Picard 2-stacks. Before we want to point out some differences between the definitions and results given above and the same definitions and results enounced in [18].

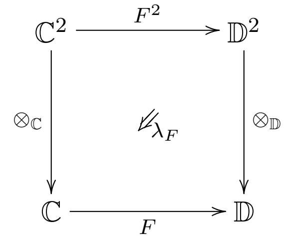
1. We assume that the 2-functors are weak(i.e. the composition is defined up to a 2-isomorphism) whereas in [18] a 2-functor means strong.

2. In the definition of unit 2-morphism, we assumed that the 2-morphism δ is an isomorphism, whereas Joyal and Kock only assumed that δ is a cancellable 2-morphism. That is a 2-morphism that induces bijection on the Hom sets of the Hom categories.

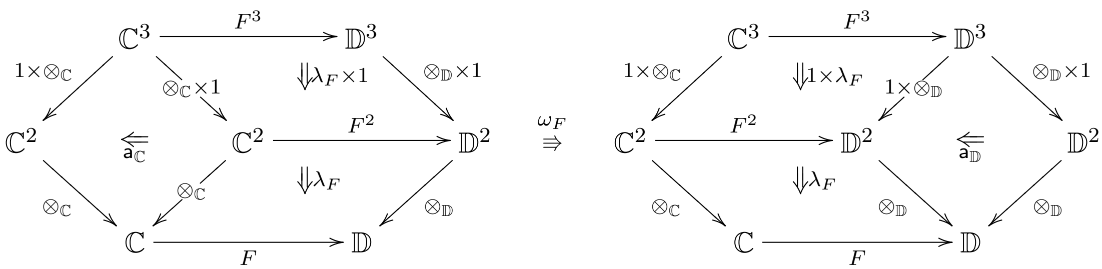
4.3 Morphisms of Picard 2-Categories

Definition 4.3.1. Let $(\mathbb{C}, \otimes_{\mathbb{C}}, \mathsf{a}_{\mathbb{C}}, \mathsf{c}_{\mathbb{C}}, \pi_{\mathbb{C}}, \mathfrak{h}_{1\mathbb{C}}, \mathfrak{h}_{2\mathbb{C}}, \zeta_{\mathbb{C}}, \eta_{\mathbb{C}})$ and $(\mathbb{D}, \otimes_{\mathbb{D}}, \mathsf{a}_{\mathbb{D}}, \mathsf{c}_{\mathbb{D}}, \pi_{\mathbb{D}}, \mathfrak{h}_{1\mathbb{D}}, \mathfrak{h}_{2\mathbb{D}}, \zeta_{\mathbb{D}}, \eta_{\mathbb{D}})$ be two Picard 2-categories. An additive 2-functor $(F, \lambda_F, \omega_F, \varepsilon_F) : \mathbb{C} \Rightarrow \mathbb{D}$ is given by the following data:

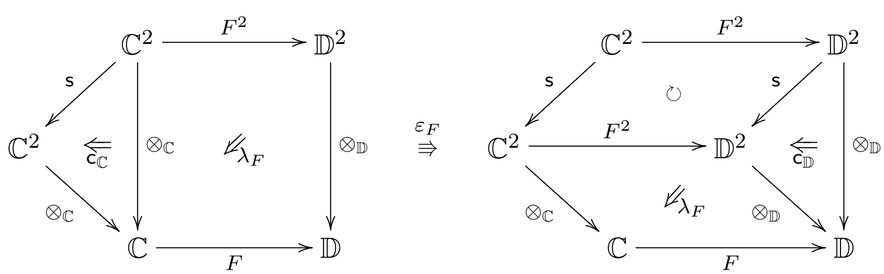
1. a 2-functor $F : \mathbb{C} \longrightarrow \mathbb{D}$,
2. a natural 2-transformation λ_F



3. a modification ω_F

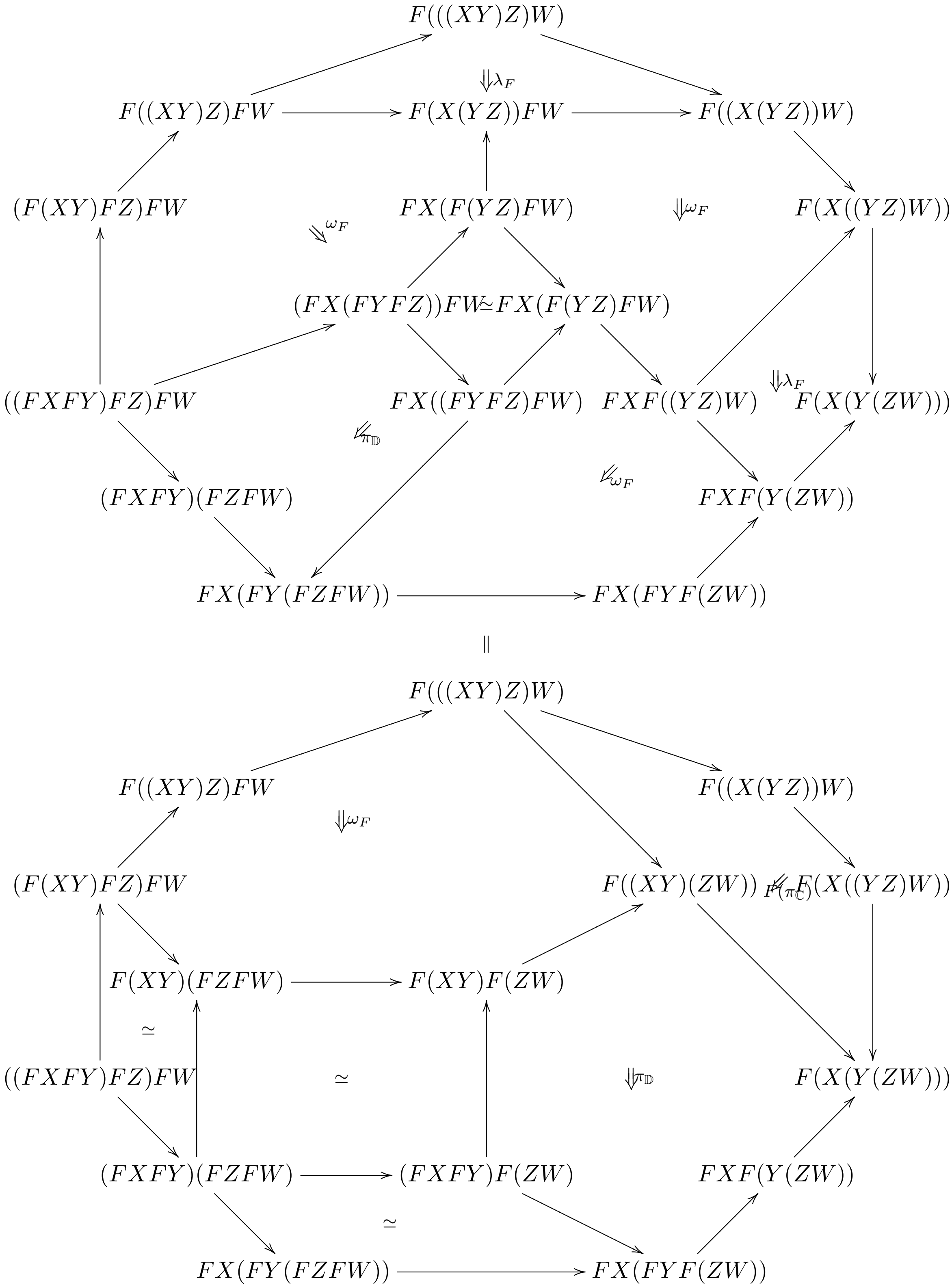


4. a modification ε_F

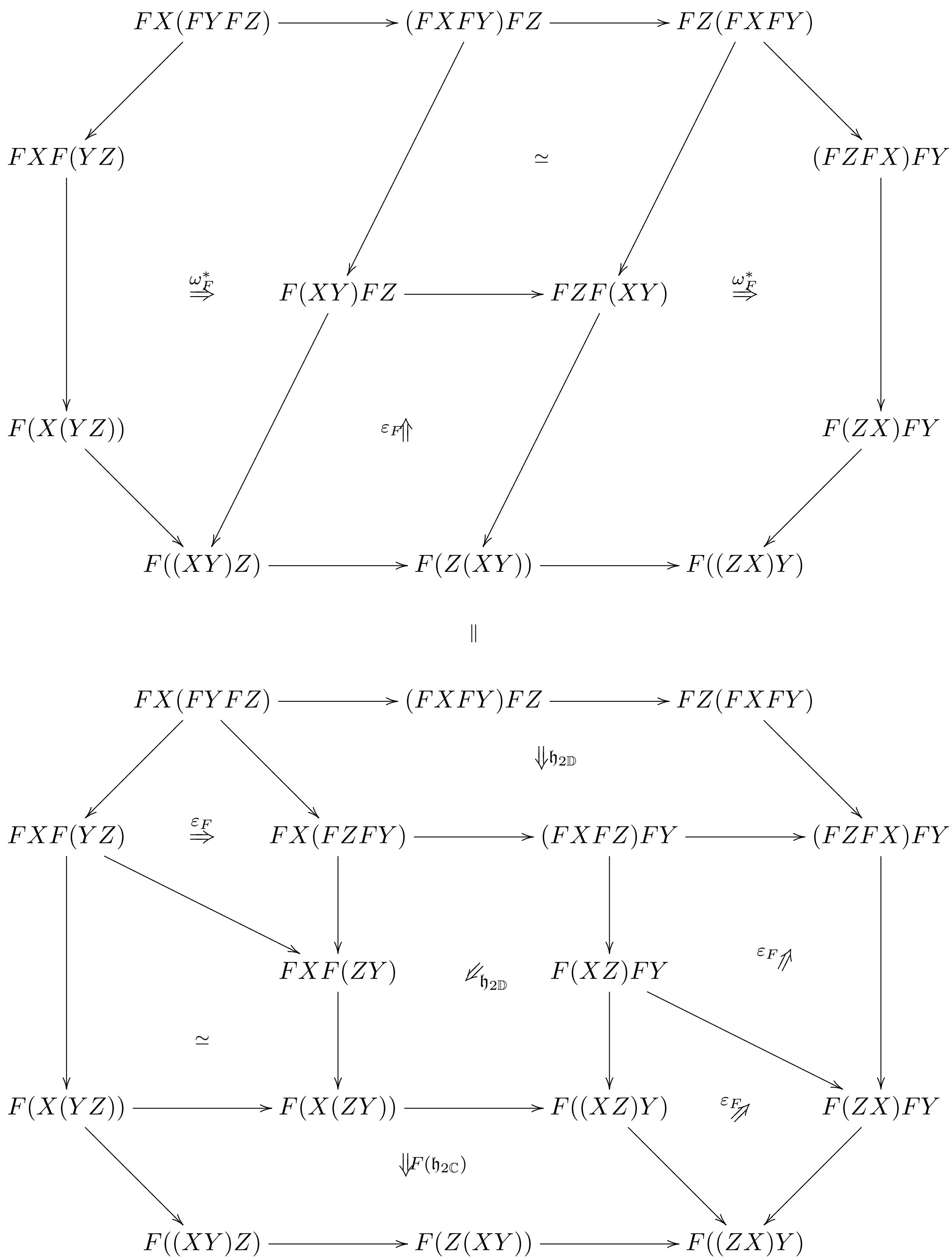


that satisfy the following conditions:

(i) For all X, Y, Z, W objects in $\mathbb{C}$, the equation of 2-morphisms holds in $\mathbb{D}$.

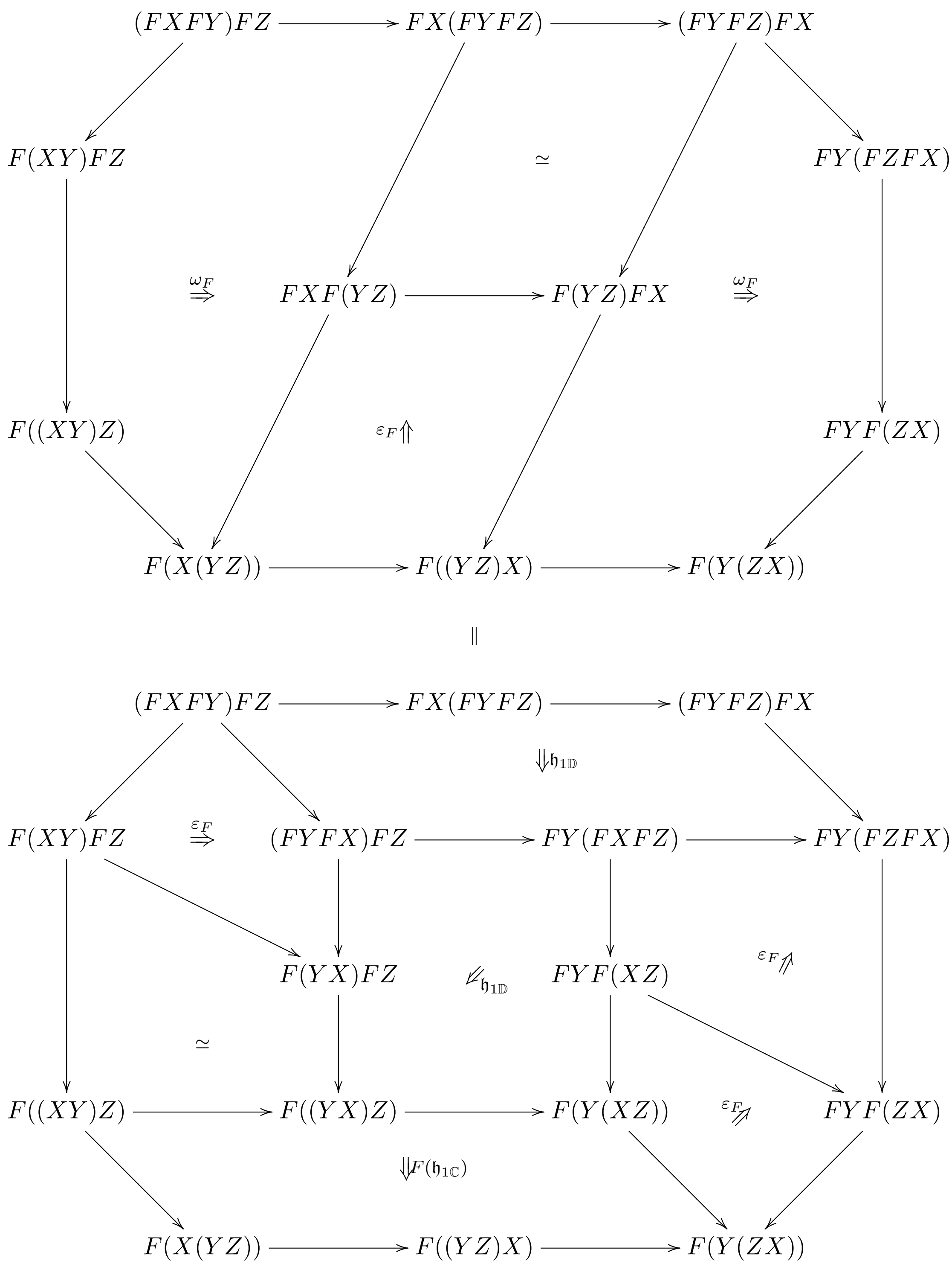

F(((XY)Z)W)
⇓λ_F
F((XY)Z)FW
F(X(YZ))FW
F((X(YZ))W)
(F(XY)FZ)FW
FX(F(YZ)FW)
⇓ω_F
F(X((YZ)W))
ω_F
(FX(FYFZ))FW
≃
FX(F(YZ)FW)
((FXFY)FZ)FW
FX((FYFZ)FW)
FXF((YZ)W)
⇓λ_F
F(X(Y(ZW)))
π_𝔻
ω_F
(FXFY)(FZFW)
FXF(Y(ZW))
FX(FY(FZFW))
FX(FYF(ZW))
‖
F(((XY)Z)W)
F((XY)Z)FW
F((X(YZ))W)
⇓ω_F
(F(XY)FZ)FW
F((XY)(ZW))
F(π_ℂ)
F(X((YZ)W))
F(XY)(FZFW)
F(XY)F(ZW)
≃
((FXFY)FZ)FW
≃
⇓π_𝔻
F(X(Y(ZW)))
(FXFY)(FZFW)
(FXFY)F(ZW)
FXF(Y(ZW))
≃
FX(FY(FZFW))
FX(FYF(ZW))

(ii) for all X, Y, Z objects in $\mathbb{C}$, the equation of 2-morphisms holds in $\mathbb{D}$.

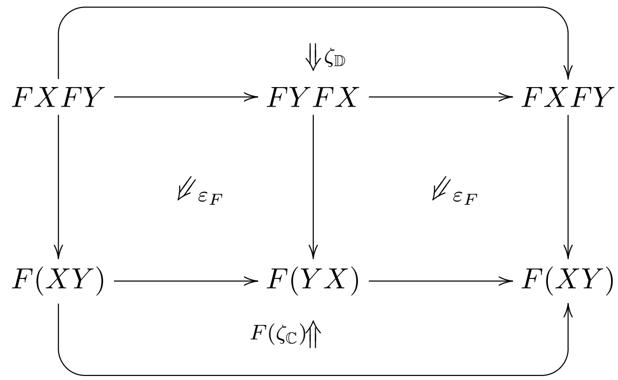


The modification ω_F^* is defined in the same way as ω_F using $\mathsf{a}_{\mathbb{C}}^{-1}$ and $\mathsf{a}_{\mathbb{D}}^{-1}$ except $\mathsf{a}_{\mathbb{C}}$ and $\mathsf{a}_{\mathbb{D}}$.

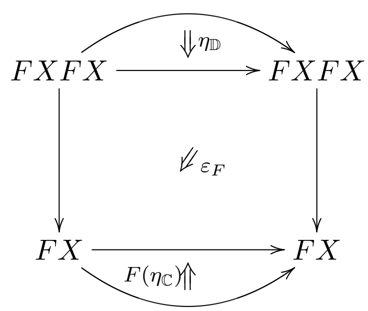
(iii) for all X, Y, Z objects in $\mathbb{C}$, the equation of 2-morphisms holds in $\mathbb{D}$.



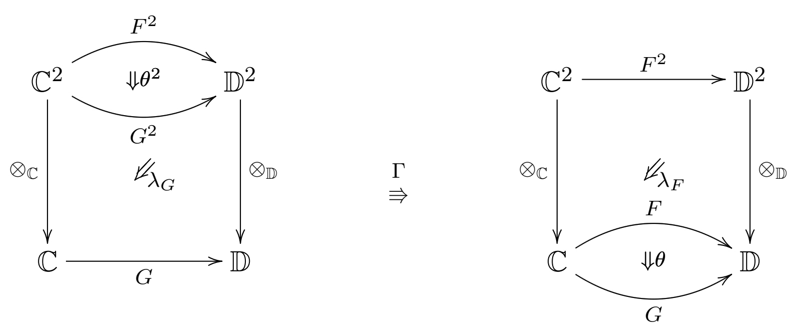
(iv) for all X, Y objects in $\mathbb{C}$, the pasting of the 2-morphisms in the below diagram is 1.



(v) for all X object in $\mathbb{C}$, the pasting of the 2-morphisms in the below diagram is 1.



Definition 4.3.2. A morphism of additive 2-functors $(F, \lambda_F, \omega_F, \varepsilon_F) \Rightarrow (G, \lambda_G, \omega_G, \varepsilon_G)$ is given by a pair (θ, Γ) where $\theta : F \Rightarrow G$ is a natural 2-transformation (2.1.6) and Γ is a modification (2.1.7)



where Γ satisfies two equations of modifications, one that involves ω_F and ω_G and another one that involves ε_F and ε_G.

Definition 4.3.3. A modification between two morphisms of additive 2-functors $(\theta_1, \Gamma_1) \Rrightarrow$

(θ_2, Γ_2) is given by a modification (2.1.7) $\Sigma : \theta_1 \Rrightarrow \theta_2$ such that the diagram of modifications

$$\begin{array}{ccc} \lambda_G * \theta_1^2 & \overset{\Gamma_1}{\Longrightarrow} & \theta_1 * \lambda_F \\ {\scriptstyle \lambda_G * \Sigma^2} \Downarrow & \circlearrowleft & \Downarrow {\scriptstyle \Sigma * \lambda_F} \\ \lambda_G * \theta_2^2 & \underset{\Gamma_2}{\Longrightarrow} & \theta_2 * \lambda_F \end{array}$$

commute.

4.4 Picard 2-Stacks

In this section, we are going to extend the discussion in Section 3.4 from categories to 2-categories. That is we investigate the naive notion of sheaf of 2-categories, namely 2-stacks over a site S. We discuss also fibered 2-categories over S which are analogues of presheaves of 2-categories. Lastly, we define Picard 2-stacks over S which are categorical analogues of abelian sheaves. Our main references for this section are [7], [8], and [16].

4.4.1 Fibered 2-Categories

In this Section, we study the 2-categories over a fixed site S, that is 2-categories $\mathbb{C}$ equipped with a strict 2-functor

$$p_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathsf{S}\,.$$

Definition 4.4.1. Let $\mathbb{C}$ be a 2-category over S and let U be an object of S. A *fiber* of $\mathbb{C}$ over U, denoted by $\mathbb{C}_U$ is a sub 2-category of $\mathbb{C}$ such that $p_{\mathbb{C}}$ maps its objects, 1-morphisms, and 2-morphisms to U, id_U, and $\mathrm{id}_{\mathrm{id}_U}$, respectively.

Let $\mathbb{C}$ be a 2-category over S and let $f : X \to Y$ be a 1-morphism in $\mathbb{C}$ such that

$$p_{\mathbb{C}}(X) = U \qquad p_{\mathbb{C}}(Y) = V \qquad p_{\mathbb{C}}(f) = i.$$

Post composing by f defines a natural 2-transformation $\tilde{f}$

$$\tilde{f} : \mathrm{Hom}_{\mathbb{C}_U}(-, X) \longrightarrow \mathrm{Hom}_i(-, Y)\,.$$

For any object A in $\mathbb{C}_U$, the component of $\tilde{f}$ at X is the functor

$$\tilde{f}_A : \mathrm{Hom}_{\mathbb{C}_U}(A, X) \longrightarrow \mathrm{Hom}_i(A, Y)\,, \tag{4.4.1}$$

defined by $\tilde{f}_A(g) = f \circ g$. $\mathrm{Hom}_i(A, Y)$ denotes the subcategory of $\mathrm{Hom}_{\mathbb{C}}(A, Y)$ whose objects are mapped to i by $p_{\mathbb{C}}$. For any morphism $\alpha : A \to B$, the component of $\tilde{f}$ at α is the natural transformation between the following composition of functors.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}_U}(B, X) & \longrightarrow & \mathrm{Hom}_i(B, Y) \\ \downarrow & \swArrow \tilde{f}_\alpha & \downarrow \\ \mathrm{Hom}_{\mathbb{C}_U}(A, X) & \longrightarrow & \mathrm{Hom}_i(A, Y) \end{array}$$

For any $g : B \to X$, $f_\alpha(g) : (f \circ g) \circ \alpha \Rightarrow f \circ (g \circ \alpha)$. However, in 2-categories composition of 1-morphisms is associative. Hence f_α is a trivial natural transformation.

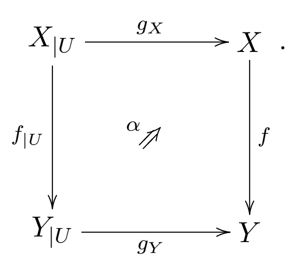
Definition 4.4.2. A morphism $f : X \to Y$ as above is called *cartesian* if $\tilde{f}$ is a weakly invertible 1-morphism in the 2-category of contravariant 2-functors from $\mathbb{C}_U$ to $\mathbb{C}\mathrm{at}$ the 2-category of categories. Equivalently by the above discussion, we say that $f : X \to Y$ is cartesian if for any object A in $\mathbb{C}$ the functor (4.4.1) is an equivalence.

Definition 4.4.3. Let $\mathbb{C}$ be a 2-category over S. We say that $\mathbb{C}$ is *fibered* over S if

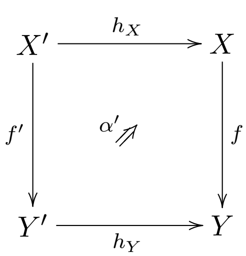
(i) for every $i : U \to V$ morphism in S and for every object Y in $\mathbb{C}_V$, there exists an object X in $\mathbb{C}_U$ and a cartesian morphism $f : X \to Y$ in $\mathbb{C}$ such that $p_{\mathbb{C}}(f) = i$.

(ii) composition of cartesian morphisms is cartesian.

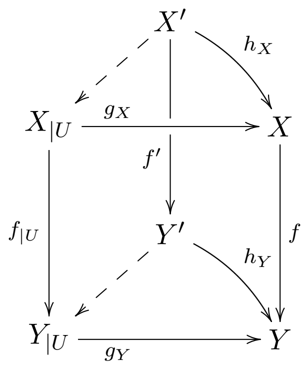
We observe that if $\mathbb{C}$ is a fibered 2-category over S, than for any object U in S $\mathrm{Hom}_{\mathbb{C}_U}(X, Y)$ is a fibered category for all X, Y objects in $\mathbb{C}_U$. In fact let $i : U \to V$ morphism in S and $f : X \to Y$ be a morphism in $\mathrm{Hom}_{\mathbb{C}_V}(X, Y)$. Since $\mathbb{C}$ is a fibered 2-category, we can pullback f to a morphism $f_{|U} : X_{|U} \to Y_{|U}$ where $X_{|U}$ and $Y_{|U}$ are pullbacks of X and Y into $\mathbb{C}_U$. This pullback $f_{|U}$ is defined up to 2-isomorphism α as shown in the diagram



where g_X and g_Y are cartesian morphisms. Let $f' : X' \to Y'$ be another morphism in $\mathbb{C}_U$ such that



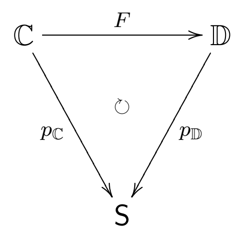
The unique 2-morphism β from f' to $f_{|U}$ is defined as the 2-morphism that makes the diagram



commute. The dotted arrows and the 2-isomorphisms on the top and bottom faces exist due to the fact that g_X and g_Y are cartesian arrows. Uniqueness of β follows from that fact that other 2-morphisms are isomorphisms.

Definition 4.4.4. Let $\mathbb{C}$ and $\mathbb{D}$ be two fibered 2-categories over S. A functor $F : \mathbb{C} \rightarrow \mathbb{D}$ is called a *morphism of fibered 2-categories* or a *cartesian 2-functor* if

(i) F preserves the base, that is if the diagram



commutes.

(ii) F maps cartesian morphisms to cartesian morphisms.

4.4.2 Sheaf Axiom for Fibered 2-Categories

Next, as we have done for fibered categories, we will talk about the analog of the sheaf axiom for fibered 2-categories. Since fibers are 2-categories, this axiom consists of conditions about objects, 1-morphisms, and 2-morphisms.

1. Axiom on 1-Morphisms and 2-Morphisms: for any two objects X, Y in $\mathbb{C}_U$, the fibered category $\mathrm{Hom}_{\mathbb{C}_U}(X, Y)$ is a stack over S/U.

2. Axiom on Objects: every decent 2-datum is effective.

A *decent 2-datum* is a collection $(V_\bullet \rightarrow U, X, \varphi, \alpha)$

1. $\ldots V_2 \overset{d_0}{\underset{d_2}{\overset{d_1}{\rightrightarrows}}} V_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} V_0 \xrightarrow{\delta} U$ is a hypercover over U,

2. X is an object in $\mathbb{C}_{V_0}$

3. $\varphi : d_0^* X \to d_1^* X$ is a 1-morphism in $\mathbb{C}_{V_1}$

4. $\alpha : d_1^* \varphi \Rightarrow d_2^* \varphi \circ d_0^* \varphi$ is a 2-morphism in $\mathbb{C}_{V_2}$

satisfying the 2-cocycle condition in $\mathbb{C}_{V_3}$

$$((d_2 d_3)^* \varphi * d_0^* \alpha) \circ d_2^* \alpha = (d_3^* \alpha * (d_0 d_1)^* \varphi) \circ d_1^* \alpha.$$

A decent 2-datum $(V_\bullet \to U, X, \varphi, \alpha)$ is *effective* if there exists an object $Y \in \mathbb{C}_U$ together with weakly invertible 1-morphisms $\psi : \delta^* Y \to X$ in $\mathbb{C}_{V_0}$ compatible with φ and α.

4.4.3 Picard 2-Stacks

The analog of a sheaf in 2-categorical context is a 2-stack. Hence, a *2-stack* over the site S is a fibered 2-category $\mathbb{C}$ that satisfies both axioms (4.4.2). If $\mathbb{C}$ satisfies only the first axiom, then $\mathbb{C}$ is called *pre 2-stack*. A (pre) 2-stack is *fibered in 2-groupoids* over S if for every $U \in \mathsf{S}$, $\mathbb{C}_U$ is a 2-groupoid (2.1.8). In this thesis, we assume that every (pre) 2-stack is fibered in 2-groupoid.

Definition 4.4.5. A *Picard 2-stack* $\mathbb{P}$ is a 2-stack equipped with a morphism of 2-stacks

$$\otimes : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P}$$

inducing a Picard structure (4.1.1) on $\mathbb{P}$. A *morphism of Picard 2-stacks* $F : \mathbb{P}_1 \to \mathbb{P}_2$ is an additive (4.3.1) and cartesian (4.4.4) 2-functor. By abuse of language, we call F additive 2-functor.

Picard 2-stacks over S form a 3-category, denoted by 2PIC(S), whose

- 1-morphisms are additive 2-functors,
- 2-morphisms are pairs (θ, Γ) of the form (4.3.2),
- 3-morphisms are modifications of the form (4.3.3).

Additive 2-functors between Picard 2-stacks $\mathbb{P}$ and $\mathbb{Q}$ form a 2-groupoid that we denote by $\mathsf{Hom}(\mathbb{P}, \mathbb{Q})$.

4.5 Associated Picard 2-Stack

An immediate example of a Picard 2-stack is the Picard 2-stack associated to a complex of abelian sheaves. It is already explained in [27] and in [3] how to associate a 2-groupoid to a length 3 complex. However, this 2-groupoid is not a 2-stack. It is not even a 2-prestack (i.e. 1-morphisms only form a prestack but not a stack and 2-descent data are not effective). Therefore to obtain a 2-stack one has to apply the stackification twice. Instead, we are going to use a torsor model for associated stacks. It is more geometric, intuitive, and can be found in [1] for the abelian case, and in [3] for the non-abelian case.

We start with a recall on torsors. Let $\mathscr{A}$ be a gr-stack, not necessarily Picard. A stack $\mathscr{L}$ is an (right) *$\mathscr{A}$-torsor* if there exists a morphism of stacks

$$\mathrm{m} : \mathscr{L} \times \mathscr{A} \longrightarrow \mathscr{L}$$

compatible with the group laws in $\mathscr{A}$, and the morphism

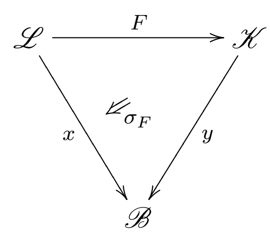
$$(\mathrm{pr}, \mathrm{m}) : \mathscr{L} \times \mathscr{A} \longrightarrow \mathscr{L} \times \mathscr{L}$$

is an equivalence, and for all $U \in \mathsf{S}$, $\mathscr{L}_U$ is not empty. [5, §6.1]

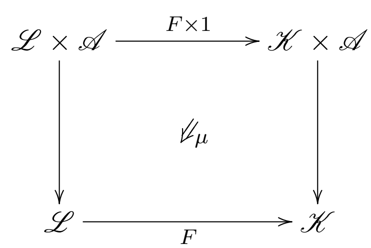
Let $\mathscr{A} \to \mathscr{B}$ be a morphism of gr-stacks. An *$(\mathscr{A}, \mathscr{B})$-torsor* is a pair $(\mathscr{L}, x)$, where $\mathscr{L}$ is an $\mathscr{A}$-torsor, and $x : \mathscr{L} \to \mathscr{B}$ is an $\mathscr{A}$-equivariant morphism of stacks [1, §6.1], [3, §6.3.4]. A *1-morphism of $(\mathscr{A}, \mathscr{B})$-torsors* is a pair

$$(F, \mu) : (\mathscr{L}, x) \longrightarrow (\mathscr{K}, y),$$

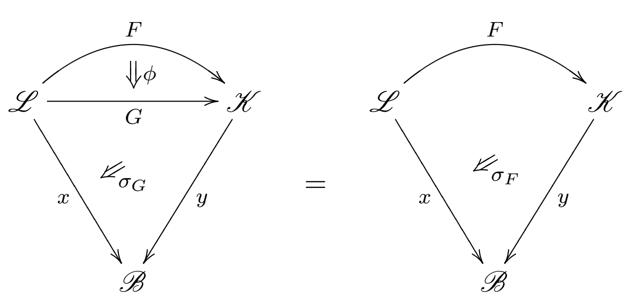
where $F : \mathscr{L} \to \mathscr{K}$ is a morphism of stacks such that



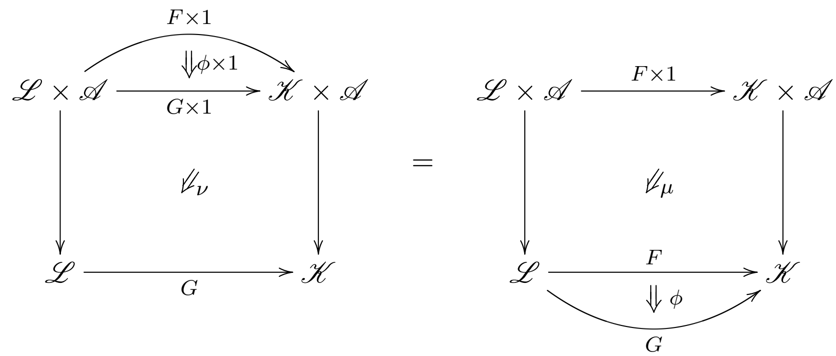
and μ is a natural transformation of stacks



expressing the compatibility of F with the torsor structure. Let $(F, \mu), (G, \nu) : (\mathscr{L}, x) \to (\mathscr{K}, y)$ be two parallel 1-morphisms of $(\mathscr{A}, \mathscr{B})$-torsors. *A 2-morphism of $(\mathscr{A}, \mathscr{B})$-torsors* $(F, \mu) \Rightarrow (G, \nu)$ is given by a natural transformation $\phi : F \Rightarrow G$ satisfying



and



$(\mathscr{A}, \mathscr{B})$-torsors with 1- and 2-morphisms as defined above form a 2-stack over the site S denoted by $\mathrm{TORS}(\mathscr{A}, \mathscr{B})$. Moreover, if $\mathscr{A}$ and $\mathscr{B}$ are Picard stacks, we can define a Picard group like structure on $\mathrm{TORS}(\mathscr{A}, \mathscr{B})$ as follows:

$$(\mathscr{L}, x) \otimes (\mathscr{K}, y) := (\mathscr{L} \wedge^{\mathscr{A}} \mathscr{K}, x \wedge y)$$

where $\mathscr{L} \wedge^{\mathscr{A}} \mathscr{K}$ is the contracted product and $x \wedge y$ is the $\mathscr{A}$-equivariant morphism from $\mathscr{L} \wedge^{\mathscr{A}} \mathscr{K}$ to $\mathscr{B}$ given by $x(l)y(k)$ where (l, k) is in $\mathscr{L} \wedge^{\mathscr{A}} \mathscr{K}$.

Now, consider $A^\bullet : [A^{-2} \to A^{-1} \to A^0]$ a complex of abelian sheaves. Let $\mathscr{A}$ be the Picard stack associated to $A^{-2} \to A^{-1}$, that is $\mathscr{A} := [A^{-2} \to A^{-1}]^\sim \simeq \mathrm{TORS}(A^{-2}, A^{-1})$ and let $\Lambda_A : \mathscr{A} \to A^0$ be an additive functor of Picard stacks, where A^0 is considered as a discrete stack (no non-trivial morphisms). It associates to an object (L, s) in $\mathrm{TORS}(A^{-2}, A^{-1})$ an element $\lambda_A(s)$ in A^0. It follows from the above discussion that $\mathrm{TORS}(\mathscr{A}, A^0)$ is a Picard 2-stack. Thus, we define

Definition 4.5.1. For any length 3 complex of abelian sheaves $A^\bullet : [A^{-2} \to A^{-1} \to A^0]$, the Picard 2-stack associated to $A^\bullet$ is $\mathrm{TORS}(\mathscr{A}, A^0)$. The hom-2-groupoid between two associated Picard 2-stacks $\mathrm{TORS}(\mathscr{A}, A^0)$ and $\mathrm{TORS}(\mathscr{B}, B^0)$ is denoted by $\mathsf{Hom}(A^\bullet, B^\bullet)$.

4.6 Homotopy Exact Sequence

Let $\mathrm{TORS}(\mathscr{A}, A^0)$ be the associated Picard 2-stack to $A^\bullet$, then there is a sequence of Picard 2-stacks

$$\mathscr{A} \xrightarrow{\Lambda_A} A^0 \xrightarrow{\pi_A} \mathrm{TORS}(\mathscr{A}, A^0), \tag{4.6.1}$$

where A^0 is considered as discrete Picard 2-stack (no non-trivial 1-morphisms and 2-morphisms). The morphism π_A assigns to an element a of $A^0(U)$ the pair $(\mathscr{A}, a)$, where a is identified with the morphism $\mathscr{A} \to A^0$ sending $1_{\mathscr{A}} = (A^{-2}, \delta_A)$ to a. (4.6.1) is homotopy exact in the sense that $\mathscr{A}$ satisfies the pullback diagram.

$$\begin{array}{ccc} \mathscr{A} & \longrightarrow & 0 \\ {\scriptstyle \Lambda_A}\big\downarrow & \nearrow\!\!\!\nearrow & \big\downarrow \\ A^0 & \xrightarrow[\pi_A]{} & \mathrm{T}\textsc{ors}(\mathscr{A}, A^0) \end{array} \tag{4.6.2}$$

Since $\mathscr{A}$ is the Picard stack associated to the morphism of abelian sheaves $\delta_A : A^{-2} \to A^{-1}$, it fits into the commutative pullback square of Picard stacks (see the proof of non-abelian version of Proposition 8.3.2 in [3]).

$$\begin{array}{ccc} A^{-2} & \longrightarrow & 0 \\ {\scriptstyle \delta_A}\big\downarrow & \nearrow\!\!\!\nearrow & \big\downarrow \\ A^{-1} & \xrightarrow[\pi_{\mathscr{A}}]{} & \mathscr{A} \end{array} \tag{4.6.3}$$

Then pasting the diagrams 4.6.2 and 4.6.3 at $\mathscr{A}$, we obtain

$$\begin{array}{ccccc} A^{-2} & \longrightarrow & 0 & & \\ {\scriptstyle \delta_A}\big\downarrow & \nearrow\!\!\!\nearrow & \big\downarrow & & \\ A^{-1} & \xrightarrow[\pi_{\mathscr{A}}]{} & \mathscr{A} & \longrightarrow & 0 \\ & {\scriptstyle \lambda_A}\searrow & {\scriptstyle \Lambda_A}\big\downarrow & \nearrow\!\!\!\nearrow & \big\downarrow \\ & & A^0 & \xrightarrow[\pi_A]{} & \mathrm{T}\textsc{ors}(\mathscr{A}, A^0) \end{array} \tag{4.6.4}$$

CHAPTER 5

TRICATEGORY OF LENGTH 3-COMPLEXES OF ABELIAN SHEAVES

In the previous Chapter 4, we have seen the relation between Picard 2-categories and length 3-complexes of abelian sheaves. Now, we look at the category of such complexes more in details. In this Chapter, we define $\mathrm{T}^{[-2,0]}(\mathsf{S})$ the tricategory of $A^\bullet : [A^{-1}{\to}A^{-1}{\to}A^0]$ length 3-complexes of abelian sheaves. In $\mathrm{T}^{[-2,0]}(\mathsf{S})$ a morphism between any two complexes $A^\bullet$ and $B^\bullet$ is called a weak morphism and we show that they form a bigroupoid denoted by $\mathsf{Frac}(A^\bullet, B^\bullet)$.

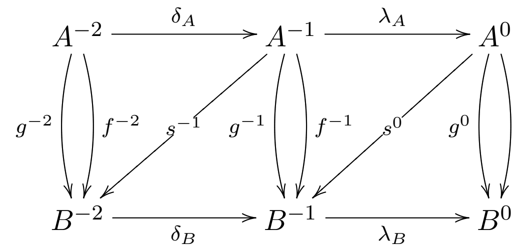
5.1 3-category of Complexes of Abelian Sheaves

The main purpose of this thesis is to construct a trifunctor (6.1.1) from the tricategory $\mathrm{T}^{[-2,0]}(\mathsf{S})$ to the 3-category 2Pic(S). This construction (see Lemma 6.1.1) is going to be first performed on $\mathrm{C}^{[-2,0]}(\mathsf{S})$ a strict sub 3-category of $\mathrm{T}^{[-2,0]}(\mathsf{S})$ and extended to $\mathrm{T}^{[-2,0]}(\mathsf{S})$. Although $\mathrm{C}^{[-2,0]}(\mathsf{S})$ is very well known, in order to setup our notation and terminology, we start this chapter with its explicit description. The objects of $\mathrm{C}^{[-2,0]}(\mathsf{S})$ are length 3 complexes of abelian sheaves placed in degrees $[-2, 0]$. For a pair of objects $A^\bullet$,$B^\bullet$, the hom-2-groupoid $\mathrm{Hom}_{\mathrm{C}^{[-2,0]}(\mathsf{S})(A^\bullet,B^\bullet)}$ is defined as follows:

- A 1-morphism $f^\bullet : A^\bullet{\to}B^\bullet$ is a degree 0 map given by strictly commutative squares.

$$\begin{array}{ccccc} A^{-2} & \xrightarrow{\delta_A} & A^{-1} & \xrightarrow{\lambda_A} & A^0 \\ \downarrow{\scriptstyle f^{-2}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} \\ B^{-2} & \xrightarrow[\delta_B]{} & B^{-1} & \xrightarrow[\lambda_B]{} & B^0 \end{array} \tag{5.1.1}$$

- A 2-morphism $s^\bullet : f^\bullet \Rightarrow g^\bullet$ is a homotopy map given by the diagram

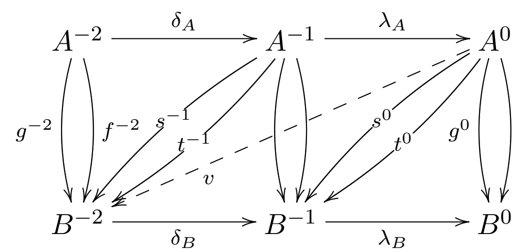


(5.1.2)

  satisfying the relations

$$g^0 - f^0 = \lambda_B \circ s^0, \quad g^{-1} - f^{-1} = \delta_B \circ s^{-1} + s^0 \circ \lambda_A, \quad g^{-2} - f^{-2} = s^{-1} \circ \delta_A.$$

- A 3-morphism $v^\bullet : s^\bullet \Rrightarrow t^\bullet$ is a homotopy map between homotopies $s^\bullet$ and $t^\bullet$ given by the diagram



(5.1.3)

  satisfying the relations

$$s^0 - t^0 = \delta_B \circ v, \quad s^{-1} - t^{-1} = -v \circ \lambda_A.$$

Remark 5.1.1. In fact, the hom-2-groupoid $\mathrm{Hom}_{\mathsf{C}^{[-2,0]}(\mathsf{S})(A^\bullet,B^\bullet)}$ is the 2-groupoid associated to $\tau^{\leq 0}(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet))$, the smooth truncation of the hom complex $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$, that is to the complex

$$\mathrm{Hom}^{-2}(A^\bullet, B^\bullet) \xrightarrow{\partial^{-2}} \mathrm{Hom}^{-1}(A^\bullet, B^\bullet) \xrightarrow{\partial^{-1}} Z^0(\mathrm{Hom}^0(A^\bullet, B^\bullet))$$

of abelian groups, where for $i = 1, 2$ the elements of $\mathrm{Hom}^{-i}(A^\bullet, B^\bullet)$ are morphisms of complexes from $A^\bullet$ to $B^\bullet$ of degree $-i$, and where $Z^0(\mathrm{Hom}^0(A^\bullet, B^\bullet))$ is the abelian group of cocycles. The differentials ∂^i are defined as

$$(\partial^{-i}(s^\bullet))^{-p} = \lambda_B^{-p-i} \circ s^{-p} + (-1)^{i+1} s^{-p+1} \circ \lambda_A^{-p}$$

for any $s^\bullet \in \mathrm{Hom}^{-i}(A^\bullet, B^\bullet)$ and $p = 0, 1, 2$.

5.2 Weak Morphisms of Complexes of Abelian Sheaves

We fix two complexes of abelian sheaves $A^\bullet$ and $B^\bullet$. We define $\mathsf{Frac}(A^\bullet, B^\bullet)$ a weakened analog of the hom-2-groupoid $\mathrm{Hom}_{\mathsf{C}^{[-2,0]}(\mathsf{S})}(A^\bullet, B^\bullet)$. We also prove that $\mathsf{Frac}(A^\bullet, B^\bullet)$ is a bigroupoid.

5.2.1 Definition of $\mathsf{Frac}(A^\bullet, B^\bullet)$

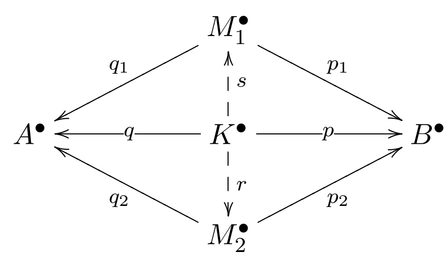
$\mathsf{Frac}(A^\bullet, B^\bullet)$ is a consists of objects, 1-morphisms, and 2-morphisms such that

- An object is an ordered triple $(q, M^\bullet, p)$, called fraction

$$(q, M^\bullet, p) : A^\bullet \xleftarrow{q} M^\bullet \xrightarrow{p} B^\bullet$$

  with $M^\bullet$ a complex of abelian sheaves, p a morphism of complexes, and q a quasi-isomorphism.

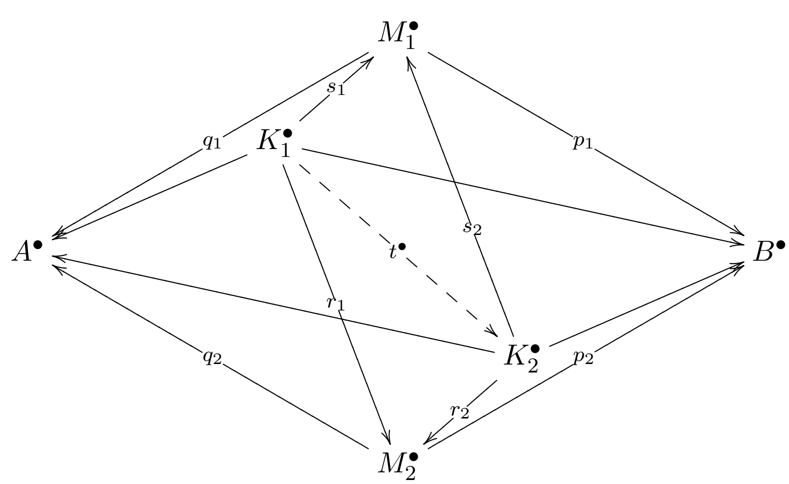
- A 1-morphism from the fraction $(q_1, M_1^\bullet, p_1)$ to the fraction $(q_2, M_2^\bullet, p_2)$ is an ordered triple $(r, K^\bullet, s)$ with $K^\bullet$ a complex of abelian sheaves, r and s quasi-isomorphisms making the diagram



(5.2.1)

  commutative.

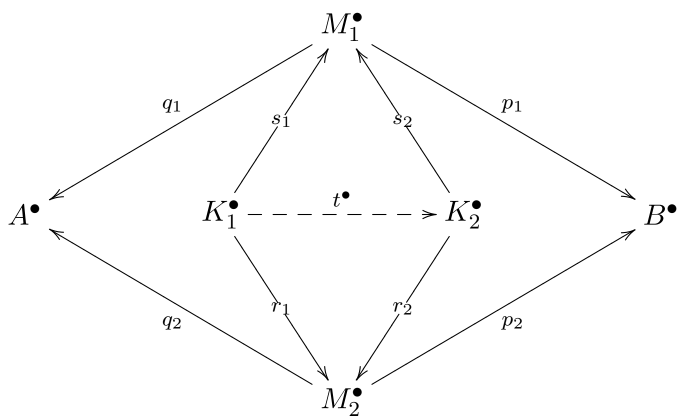
- A 2-morphism from the 1-morphism $(r_1, K_1^\bullet, s_1)$ to the 1-morphism $(r_2, K_2^\bullet, s_2)$ is an isomorphism $t^\bullet : K_1^\bullet \to K_2^\bullet$ of complexes of abelian sheaves such that the diagram that we will call "*diamond*"



(5.2.2)

  commutes.

Remark 5.2.1. For reasons of clarity, we will represent the above 2-morphism by the following planar commutative diagram



where we have ignored the maps from $K^\bullet$'s to $A^\bullet$ and $B^\bullet$.

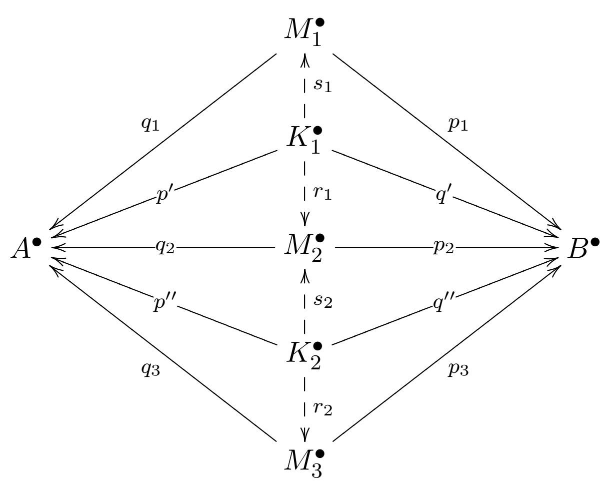
Remark 5.2.2. From the definition of 2-morphisms, it is immediate that all 2-morphisms are isomorphisms.

5.2.2 $\mathsf{Frac}(A^\bullet, B^\bullet)$ is a bigroupoid

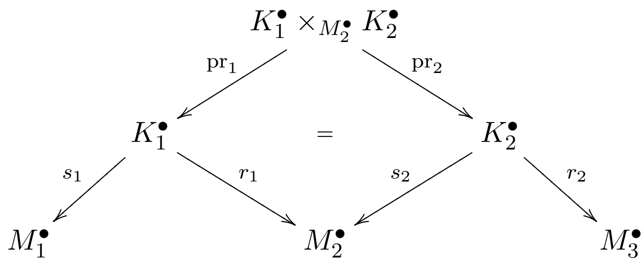
Proposition 5.2.3. *Let $A^\bullet$ and $B^\bullet$ be two complexes of abelian sheaves. Then $\mathsf{Frac}(A^\bullet, B^\bullet)$ is a bigroupoid.*

Proof. We will describe the necessary data to define the bigroupoid without verifying that they satisfy the required axioms.

- For any two composable morphisms $(r_1, K_1^\bullet, s_1) : (q_1, M_1^\bullet, p_1) \to (q_2, M_2^\bullet, p_2)$ and $(r_2, K_2^\bullet, s_2) : (q_2, M_2^\bullet, p_2) \to (q_3, M_3^\bullet, p_3)$ shown by the diagram

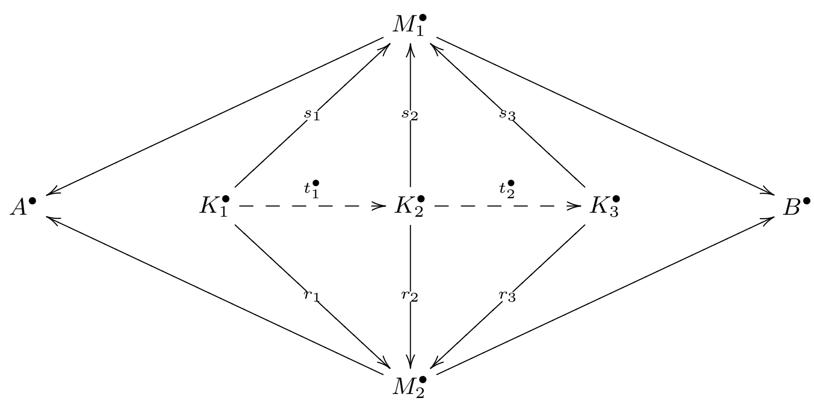


the composition is defined by the pullback diagram.



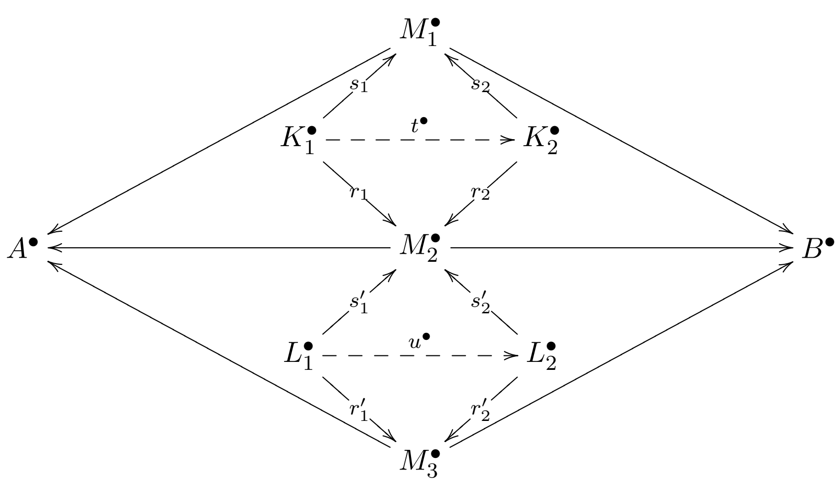
That is the composition is the triple $(r_2 \circ \mathrm{pr}_2, K_1^\bullet \times_{M_2^\bullet} K_2^\bullet, s_1 \circ \mathrm{pr}_1)$.

- For two 2-morphisms $t_1^\bullet : (r_1, K_1^\bullet, s_1) \Rightarrow (r_2, K_2^\bullet, s_2)$ and $t_2^\bullet : (r_2, K_2^\bullet, s_2) \Rightarrow (r_3, K_3^\bullet, s_3)$ shown by the diagram



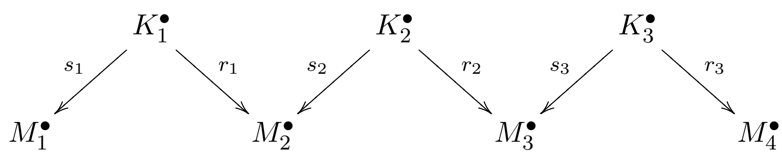
the vertical composition is defined by $t_2^\bullet \circ t_1^\bullet$.

- For two 2-morphisms $t^\bullet : (r_1, K_1^\bullet, s_1) \Rightarrow (r_2, K_2^\bullet, s_2)$ and $u^\bullet : (r_1', L_1^\bullet, s_1') \Rightarrow (r_2', L_2^\bullet, s_2')$ shown by the diagram



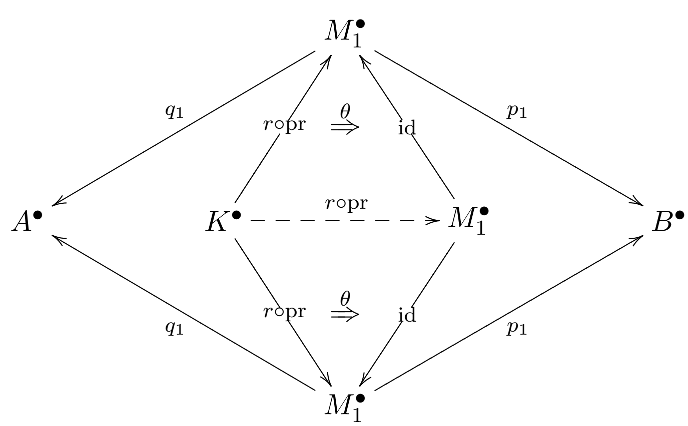
the horizontal composition is given by the natural morphism $K_1^\bullet \times_{M_2^\bullet} L_1^\bullet \to K_2^\bullet \times_{M_2^\bullet} L_2^\bullet$ between the pullbacks of pairs (r_1, s_1') and (r_2, s_2') over $M_2^\bullet$.

Any three composable 1-morphisms $(r_1, K_1^\bullet, s_1)$, $(r_2, K_2^\bullet, s_2)$, and $(r_3, K_3^\bullet, s_3)$ can be pictured as a sequence of three fractions



simply by ignoring the maps to $A^\bullet$ and $B^\bullet$. They can be composed in two different ways, either first by pulling back over $M_2^\bullet$ then over $M_3^\bullet$ or vice versa. The resulting fractions will be $(r, (K_1^\bullet \times_{M_2^\bullet} K_2^\bullet) \times_{M_3^\bullet} K_3^\bullet, s)$ and $(r', K_1^\bullet \times_{M_2^\bullet} (K_2^\bullet \times_{M_3^\bullet} K_3^\bullet), s')$, respectively, where r and r' (resp.s and s') are equal to r_3 (resp.s_1) composed with appropriate projection maps. The 2-isomorphism between these fractions is given by the natural isomorphism between the pullbacks. Thus, the associativity of composition of 1-morphisms is weak.

We also observe that 1-morphisms are weakly invertible. Let $(r, K^\bullet, s)$ be a 1-morphism from $(q_1, M_1^\bullet, p_1)$ to $(q_2, M_2^\bullet, p_2)$, then $(s, K^\bullet, r)$ is a weak inverse of $(r, K^\bullet, s)$ in the sense that the composition $(r \circ \mathrm{pr}, K^\bullet \times_{M_2^\bullet} K^\bullet, r \circ \mathrm{pr})$ is equivalent to the identity, that is there is a natural 2-transformation $\theta : r \circ \mathrm{pr} \Rightarrow \mathrm{id} \circ (r \circ \mathrm{pr})$ as shown in the below diagram.



Thus, $\mathsf{Frac}(A^\bullet, B^\bullet)$ is a bigroupoid. □

Remark 5.2.4. In the terminology of [2], what we have called fractions are called in the non-abelian context weak morphisms of 2-crossed modules or butterflies of gr-stacks or bats of sheaves.

5.3 Tricategory of Complexes of Abelian Sheaves

We define the tricategory $\mathrm{T}^{[-2,0]}(\mathsf{S})$ of length 3 complexes of abelian sheaves promised at the beginning of the section. To define a tricategory, one has to first define the data given in [15, Definition 3.3.1] and then verify that these data satisfy the axioms also given in [15, Definition 3.3.1]. Since this is a very long and dull procedure, we give here the some of the important data and leave the rest.

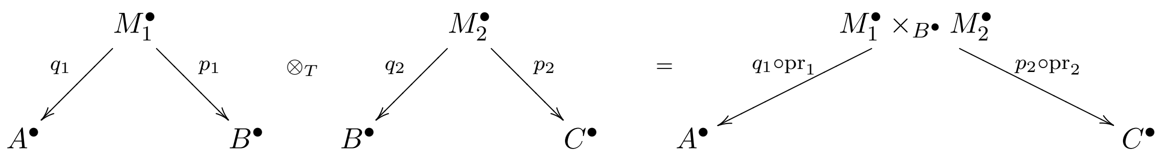
$\mathrm{T}^{[-2,0]}(\mathsf{S})$ is a tricategory equipped with the data

- Objects are length 3 complexes of abelian sheaves.
- For any two complexes of abelian sheaves $A^\bullet$ and $B^\bullet$, $\mathsf{Frac}(A^\bullet, B^\bullet)$ is the hom-bicategory.
- For any three complexes of abelian sheaves $A^\bullet$, $B^\bullet$, and $C^\bullet$, the composition is given by the weak functor

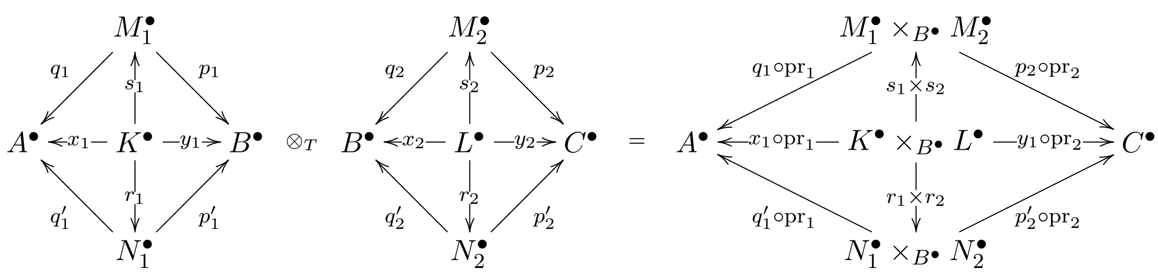
$$\otimes_T : \ \mathsf{Frac}(A^\bullet, B^\bullet) \times \mathsf{Frac}\,(B^\bullet, C^\bullet) \longrightarrow \mathsf{Frac}\,(A^\bullet, C^\bullet),$$

  which is defined on

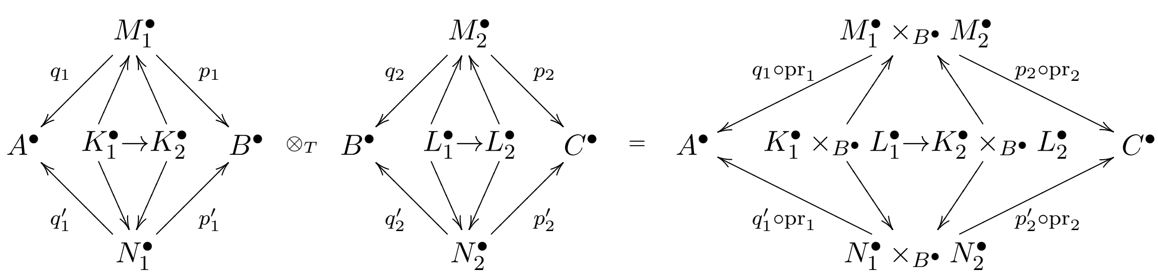
  1. objects, by



  2. 1-morphisms, by



  3. 2-morphisms, by



CHAPTER 6

CHARACTERIZATION THEOREM FOR PICARD 2-STACKS

This is the main chapter of the thesis where we prove the generalization of the theorem (3.8) to Picard 2-stacks. So far we have defined $2\mathrm{Pic}(\mathsf{S})$ the 3-category of Picard 2-stacks and $\mathrm{T}^{[-2,0]}(\mathsf{S})$ the tricategory of length 3 complexes of abelian sheaves.

6.1 Definition of the Trihomomorphism

We construct a trihomomorphism from $\mathrm{C}^{[-2,0]}(\mathsf{S})$ to $2\mathrm{Pic}(\mathsf{S})$.

Lemma 6.1.1. *There is a trihomomorphism*

$$2\wp : \mathrm{C}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{Pic}(\mathsf{S}) \tag{6.1.1}$$

between the 3-category $\mathrm{C}^{[-2,0]}(\mathsf{S})$ *of complexes of abelian sheaves and the 3-category* $2\mathrm{Pic}(\mathsf{S})$ *of Picard 2-stacks.*

Proof. We will give a step by step construction of the trihomomorphism and leave the verification of the axioms to the reader.

- Using the notations in section 4.5, given a complex $A^\bullet$, we define $2\wp(A^\bullet)$ as the associated Picard 2-stack, that is $2\wp(A^\bullet) := \mathrm{Tors}(\mathscr{A}, A^0)$.
- For any morphism $f^\bullet : A^\bullet \to B^\bullet$ of complexes (see diagram (5.1.1)), there exists a commutative square of Picard stacks

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\Lambda_A} & A^0 \\ {\scriptstyle F}\downarrow & = & \downarrow{\scriptstyle f^0} \\ \mathscr{B} & \xrightarrow[\Lambda_B]{} & B^0 \end{array} \tag{6.1.2}$$

  where F is induced by $f^{\bullet<0} : A^{\bullet<0} \to B^{\bullet<0}$. From the square (6.1.2), we construct a 1-morphism $2\wp(f^\bullet)$ in $2\mathrm{Pic}(\mathsf{S})$

$$2\wp(f^\bullet) : \textsc{Tors}(\mathscr{A}, A^0) \longrightarrow \textsc{Tors}(\mathscr{B}, B^0)$$

that sends an $(\mathscr{A}, A^0)$-torsor $(\mathscr{L}, x)$ to $(\mathscr{L} \wedge^{\mathscr{A}} \mathscr{B}, f^0 \circ x + \Lambda_B)$. For details, the reader can refer to [1, §6.1].

- For any 2-morphism $s^\bullet : f^\bullet \Rightarrow g^\bullet$ of complexes (see diagram 5.1.2), there exists a diagram of Picard stacks

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\Lambda_A} & A^0 \\ G \downarrow \downarrow F & \overset{\hat{s}^0}{\swarrow} & g^0 \downarrow \downarrow f^0 \\ \mathscr{B} & \xrightarrow[\Lambda_B]{} & B^0 \end{array} \tag{6.1.3}$$

such that for any (L, a) in $\mathscr{A}$, we have the relation

$$G(L, a) - F(L, a) = \hat{s}^0 \circ \Lambda_A(L, a) \quad \text{with} \quad \hat{s}^0(a) = (B^{-2}, s^0(a)).$$

From the relation, we construct a natural 2-transformation θ

$$\textsc{Tors}(\mathscr{A}, A^0) \underset{2\wp(g^\bullet)}{\overset{2\wp(f^\bullet)}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \theta}}}} \textsc{Tors}(\mathscr{B}, B^0)$$

in $2\textsc{Pic}(\mathsf{S})$ that assigns to any object $(\mathscr{L}, x)$ in $\textsc{Tors}(\mathscr{A}, A^0)$ a 1-morphism $\theta_{(\mathscr{L},x)}$

$$\theta_{(\mathscr{L},x)} : (\mathscr{L} \wedge^{\mathscr{A}}_{2\wp(f^\bullet)} \mathscr{B}, x_F) \longrightarrow (\mathscr{L} \wedge^{\mathscr{A}}_{2\wp(g^\bullet)} \mathscr{B}, x_G) \tag{6.1.4}$$

in $\textsc{Tors}(\mathscr{B}, B^0)$, where $x_F = f^0 \circ x + \Lambda_B$ and $x_G = g^0 \circ x + \Lambda_B$. The morphism (6.1.4) is defined by sending (l, b) to $(l, b - s^0 \circ x(l))$.

- For any 3-morphism $v^\bullet : s^\bullet \Rrightarrow t^\bullet$ of complexes (see diagram 5.1.3), there exists a modification Γ

$$\textsc{Tors}(\mathscr{A}, A^0) \underset{2\wp(f^\bullet)}{\overset{2\wp(f^\bullet)}{\underset{\longrightarrow}{\overset{\longrightarrow}{\theta\Downarrow \ \overset{\Rrightarrow}{\Gamma} \ \Downarrow\phi}}}} \textsc{Tors}(\mathscr{B}, B^0)$$

in $2\textsc{Pic}(\mathsf{S})$ that assigns to any $(\mathscr{L}, x)$ object of $\textsc{Tors}(\mathscr{A}, A^0)$ a natural 2-transformation $\Gamma_{(\mathscr{L},x)}$

$$\begin{array}{ccc} & \theta_{(\mathscr{L},x)} & \\ (\mathscr{L} \wedge^{\mathscr{A}}_{F} \mathscr{B}, x_F) & \Downarrow\Gamma_{(\mathscr{L},x)} & (\mathscr{L} \wedge^{\mathscr{A}}_{G} \mathscr{B}, x_G) \\ & \phi_{(\mathscr{L},x)} & \end{array}$$

in $\textsc{Tors}(\mathscr{B}, B^0)$, where $\theta_{(\mathscr{L},x)}$, $\phi_{(\mathscr{L},x)}$ are of the form (6.1.4). The natural 2-transformation $\Gamma_{(\mathscr{L},x)}$ is defined by assigning to any object (l,b) in $(\mathscr{L} \wedge^{\mathscr{A}}_{F} \mathscr{B}, x_F)$ a morphism

$$\Gamma_{(\mathscr{L},x)}(l,b) : (l, b - s^0 \circ x(l)) \longrightarrow (l, b - t^0 \circ x(l))$$

in $(\mathscr{L} \wedge^{\mathscr{A}}_{G} \mathscr{B}, x_G)$ given by the triple $(id_l, 1_{\mathscr{A}}, \beta)$ with β being the isomorphism

$$b - s^0 \circ x(l) \longrightarrow b - s^0 \circ x(l) + \delta_B \circ v \circ x(l),$$

and id_l the identity of l in $\mathscr{L}$, and $1_{\mathscr{A}}$ the unit element in $\mathscr{A}$.

□

6.2 Biequivalence of $\mathsf{Frac}(A^\bullet, B^\bullet)$ and $\mathsf{Hom}(A^\bullet, B^\bullet)$

We fix two length 3 complexes of abelian sheaves $A^\bullet$ and $B^\bullet$. In this Section, we prove that the bigroupoid $\mathsf{Frac}(A^\bullet, B^\bullet)$ of fractions defined in Section 5.2.1 is biequivalent to the 2-groupoid $\mathsf{Hom}(A^\bullet, B^\bullet)$ of additive 2-functors from $2\wp(A^\bullet)$ to $2\wp(A^\bullet)$ defined in Section 4.5(see Definition (4.5.1)).

6.2.1 Morphisms of Picard 2-Stacks as Fractions

Lemma 6.2.1. *Let $\mathbb{P}$ be a Picard 2-stack and A, B be two abelian sheaves with additive 2-functors $\phi : A \longrightarrow \mathbb{P}$ and $\psi : B \longrightarrow \mathbb{P}$. Then $A \times_{\mathbb{P}} B$ is a Picard stack.*

Proof. Proof of this technical lemma will be given in the Appendix (A.1.1). □

Lemma 6.2.2. *A morphism $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if*

$$2\wp(f) : 2\wp(A^\bullet) \longrightarrow 2\wp(B^\bullet)$$

is a biequivalence.

Proof. Given $f : A^\bullet \to B^\bullet$ a morphism of complexes, we know how to induce a morphism of Picard 2-stacks (see construction of trihomomorphism $2\wp(f)$). It is also known that a 2-stack (not necessarily Picard) can be seen as a 2-gerbe over its own π_0 bounded by the stack $\mathscr{A}ut(\mathrm{I})$ of automorphisms of identity [7, §8.1]. In particular, the Picard 2-stacks $\textsc{Tors}(\mathscr{A}, A^0)$ and $\textsc{Tors}(\mathscr{B}, B^0)$ are 2-gerbes over their own π_0 bounded by $\mathscr{A}ut(\mathrm{I}_{2\wp(A^\bullet)}) \simeq [A^{-2} \to \ker(\delta_A)]^\sim$ and $\mathscr{A}ut(\mathrm{I}_{2\wp(B^\bullet)}) \simeq [B^{-2} \to \ker(\delta_B)]^\sim$, respectively. Furthermore, if f is a quasi-isomorphism, then $H^{-i}(A^\bullet) \simeq H^{-i}(B^\bullet)$ for $i = 0, 1, 2$ and thus, $\pi_i(2\wp(A^\bullet)) \simeq \pi_i(2\wp(B^\bullet))$ for $i = 0, 1, 2$. So $\textsc{Tors}(\mathscr{A}, A^0)$ and $\textsc{Tors}(\mathscr{B}, B^0)$ are 2-gerbes with equivalent bands. Therefore they are equivalent. □

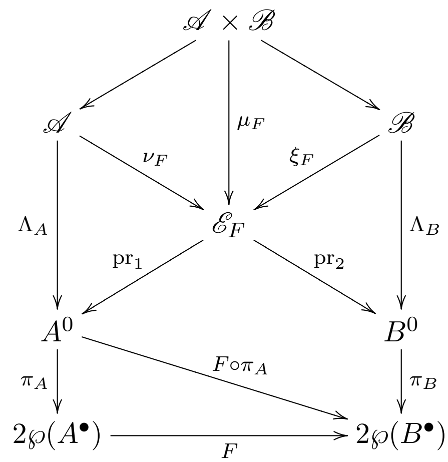
Given an additive 2-functor F in $\mathsf{Hom}(A^\bullet, B^\bullet)$, we will show in the next lemma that there is a corresponding object in $\mathsf{Frac}(A^\bullet, B^\bullet)$.

Lemma 6.2.3. *For any additive 2-functor* $F : 2\wp(A^\bullet) \to 2\wp(B^\bullet)$*, there exists a fraction* $(q, M^\bullet, p)$ *such that* $F \circ 2\wp(q) \simeq 2\wp(p)$.

Proof. From the sequences

$$\mathscr{A} \xrightarrow{\Lambda_A} A^0 \xrightarrow{\pi_A} 2\wp(A^\bullet) \quad \text{and} \quad \mathscr{B} \xrightarrow{\Lambda_B} B^0 \xrightarrow{\pi_B} 2\wp(B^\bullet)\ ,$$

we can construct the commutative diagram

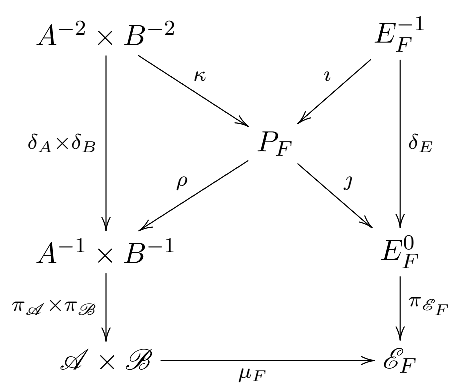


(6.2.1)

where $\mathscr{E}_F := A^0 \times_{F,B} B^0$. It follows from the commutativity of the above diagram that $\mu_F = (\Lambda_A, \Lambda_B)$. The sequence

$$\mathscr{B} \xrightarrow{\xi_F} \mathscr{E}_F \xrightarrow{\mathrm{pr}_1} A^0 \tag{6.2.2}$$

is homotopy exact since it is the pullback of the exact sequence $\mathscr{B} \to B^0 \to 2\wp(B^\bullet)$. From Lemma 6.2.1, it follows that $\mathscr{E}_F$ is a Picard stack. Therefore by [3, Proposition 8.3.2], there exists a length 2 complex $E^\bullet = [\delta_E : E_F^{-1} \to E_F^0]$ of abelian sheaves such that the associated Picard stack $\mathrm{T{\small ORS}}(E_F^{-1}, E_F^0)$ is equivalent to $\mathscr{E}_F$. Then by [3, Theorem 8.3.1], there exists a butterfly representing μ_F



(6.2.3)

with $P_F \simeq (A^{-1} \times B^{-1}) \times_{\mathscr{E}_F} E_F^0$. From a different perspective, this butterfly can be seen as

$$\begin{array}{ccccc} 0 & \longrightarrow & E_F^{-1} & \xrightarrow{\delta_E} & E_F^0 \\ \downarrow & & \downarrow \imath & & \downarrow \mathrm{id} \\ A^{-2} \times B^{-2} & \xrightarrow{\kappa} & P_F & \xrightarrow{\jmath} & E_F^0 \\ \mathrm{id} \downarrow & & \rho \downarrow & & \downarrow \\ A^{-2} \times B^{-2} & \xrightarrow[\delta_A \times \delta_B]{} & A^{-1} \times B^{-1} & \longrightarrow & 0 \end{array} \tag{6.2.4}$$

where each column is an exact sequence of abelian sheaves. The only non-trivial sequence is the second column and its exactness follows from the definition of a butterfly (3.7.1). So we have a short exact sequence of complexes of abelian sheaves

$$0 \longrightarrow E_F^\bullet \longrightarrow M_F^\bullet \longrightarrow A^{\bullet<0} \times B^{\bullet<0} \longrightarrow 0\ , \tag{6.2.5}$$

where

$$\begin{aligned} M_F^\bullet &:= A^{-2} \times B^{-2} \longrightarrow P_F \longrightarrow E_F^0, \\ E_F^\bullet &:= 0 \longrightarrow E_F^{-1} \longrightarrow E_F^0, \\ A^{\bullet<0} \times B^{\bullet<0} &:= A^{-2} \times B^{-2} \longrightarrow A^{-1} \times B^{-1} \longrightarrow 0. \end{aligned} \tag{6.2.6}$$

From the lower part of the diagram (6.2.4) and the definition of P_F, we deduce that there are morphisms of complexes

$$
\begin{array}{ccccc}
 & & A^{-2}\times B^{-2} & & \\
 & \swarrow^{\mathrm{pr}_1} & \downarrow^{\kappa} & \searrow^{\mathrm{pr}_2} & \\
A^{-2} & & & & B^{-2} \\
\downarrow^{\delta_A} & & P_F & & \downarrow^{\delta_B} \\
 & \swarrow^{\mathrm{pr}_2\circ\rho} & \downarrow^{\jmath} & \searrow^{\mathrm{pr}_1\circ\rho} & \\
A^{-1} & & & & B^{-1} \\
\downarrow^{\lambda_A} & & E_F^0 & & \downarrow^{\lambda_B} \\
 & \swarrow & & \searrow & \\
A^0 & & & & B^0
\end{array}
\tag{6.2.7}
$$

$$
\begin{array}{ccccc}
 & & M_F^\bullet & & \\
 & \swarrow^{q} & & \searrow^{p} & \\
A^\bullet & & & & B^\bullet
\end{array}
$$

We claim that q is a quasi-isomorphism, that is

$$H^{-2}(M_F^\bullet)\simeq \ker(\delta_A),\quad H^{-1}(M_F^\bullet)\simeq \ker(\lambda_A)/\operatorname{im}(\delta_A),\quad H^0(M_F^\bullet)\simeq \operatorname{coker}(\lambda_A).$$

Indeed, from the exact sequence (6.2.5), we obtain the long exact sequence of homology sheaves

$$
\begin{array}{l}
0\longrightarrow H^{-2}(M_F^\bullet)\longrightarrow H^{-2}(A^{\bullet<0})\times H^{-2}(B^{\bullet<0})\longrightarrow H^{-1}(E_F^\bullet)\xrightarrow{\ \partial\ } \\
H^{-1}(M_F^\bullet)\longrightarrow H^{-1}(A^{\bullet<0})\times H^{-1}(B^{\bullet<0})\longrightarrow H^0(E_F^\bullet)\longrightarrow H^0(M_F^\bullet)\longrightarrow 0
\end{array}
\tag{6.2.8}
$$

On the other hand, by [3, Proposition 6.2.6] applied to the exact sequence (6.2.2), we get a long exact sequence of homotopy groups

$$0\longrightarrow \pi_1(\mathscr{B})\longrightarrow \pi_1(\mathscr{E}_F)\longrightarrow \pi_1(A^0)\longrightarrow \pi_0(\mathscr{B})\longrightarrow \pi_0(\mathscr{E}_F)\longrightarrow \pi_0(A^0)\longrightarrow 0. \tag{6.2.9}$$

Since $\pi_1(A^0)=H^{-1}(A^0)=0$ and $\pi_0(A^0)=H^0(A^0)=A^0$, it follows from (6.2.9) that we have an isomorphism

$$H^{-2}(B^{\bullet<0}) \xrightarrow{\simeq} H^{-1}(E_F^{\bullet}) \tag{6.2.10}$$

and an exact sequence

$$0 \longrightarrow H^{-1}(B^{\bullet<0}) \longrightarrow H^0(E_F^{\bullet}) \longrightarrow A^0 \longrightarrow 0. \tag{6.2.11}$$

(6.2.10) implies that $\partial = 0$ in (6.2.8). Therefore from (6.2.8) again, we obtain a short exact sequence

$$0 \longrightarrow H^{-2}(M_F^{\bullet}) \longrightarrow H^{-2}(A^{\bullet<0}) \times H^{-2}(B^{\bullet<0}) \longrightarrow H^{-1}(E_F^{\bullet}) \longrightarrow 0$$

from which we deduce that $H^{-2}(M_F^{\bullet}) \simeq H^{-2}(A^{\bullet<0}) = \ker(\delta_A)$.

Now, apply the snake lemma to the short exact sequence (6.2.11) and to

$$0 \longrightarrow H^{-1}(B^{\bullet<0}) \longrightarrow H^{-1}(A^{\bullet<0}) \times H^{-1}(B^{\bullet<0}) \longrightarrow H^{-1}(A^{\bullet<0}) \longrightarrow 0$$

in order to get the dashed exact sequence

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & H^{-1}(M_F^{\bullet}) & \longrightarrow & \ker(\lambda_A)/\operatorname{im}(\delta_A) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{-1}(B^{\bullet<0}) & \longrightarrow & H^{-1}(A^{\bullet<0}) \times H^{-1}(B^{\bullet<0}) & \longrightarrow & H^{-1}(A^{\bullet<0}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{-1}(B^{\bullet<0}) & \longrightarrow & H^0(E_F^{\bullet}) & \longrightarrow & A_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \dashrightarrow & 0 & \longrightarrow & H^0(M_F^{\bullet}) & \longrightarrow & \operatorname{coker}(\lambda_A) & &
\end{array}$$

from which it follows $H^{-1}(M_F^{\bullet}) \simeq \ker(\lambda_A)/\operatorname{im}(\delta_A)$, and $H^0(M_F^{\bullet}) \simeq \operatorname{coker}(A^0)$ as wanted.

We end this proof by showing that $F \circ 2\wp(q) \simeq 2\wp(p)$. (6.2.7) induces a diagram of Picard 2-stacks

$$\begin{array}{ccc}
 & 2\wp(M_F^{\bullet}) & \\
{}^{2\wp(q)}\swarrow & & \searrow^{2\wp(p)} \\
2\wp(A^{\bullet}) & \xrightarrow[F]{} & 2\wp(B^{\bullet})
\end{array} \quad . \tag{6.2.12}$$

We claim that (6.2.12) commutes up to a natural 2-transformation. To show that, it is enough to look at $2\wp(M_F^\bullet)$ locally. Given $U \in \mathsf{S}$, $2\wp(M_F^\bullet)_U$ is the 2-groupoid associated to the complex of abelian groups (for the definition of the 2-groupoid associated to a complex see [3] or [27])

$$A^{-2}(U) \times B^{-2}(U) \xrightarrow{\delta} P_F(U) \xrightarrow{\lambda} E_F^0(U)$$

Then, an object of $2\wp(M_F^\bullet)_U$ is an element e of $E_F^0(U)$. Since $\mathscr{E}_F := A^0 \times_{F,B} B^0 \simeq \textsc{Tors}(E_F^{-1}, E_F^0)$, e can be taken as (a, f, b), where $a \in A^0(U)$, $b \in B^0(U)$, and $f : F(a) \to b$ is a 1-morphism in $2\wp(B^\bullet)_U$.

A 1-morphism of $2\wp(M_F^\bullet)_U$ from e_1 to e_2 is given by an element p of $P_F(U)$ such that $\lambda(p) + e_1 = e_2$ in $E_F^0(U)$. We can again take $\lambda(p)$, e_1, and e_2 as (a, f, b), (a_1, f_1, b_1), and (a_2, f_2, b_2), respectively. Therefore, the addition in $E_F^0(U)$ should be replaced by the monoidal operation on $\mathscr{E}_F$ between the triples, that is $(a, f, b) \otimes_{\mathscr{E}_F} (a_1, f_1, b_1) = (a_2, f_2, b_2)$. This monoidal operation is described in the proof of the technical Lemma 6.2.1. It creates a diagram commutative up to a 2-isomorphism in $2\textsc{Pic}(\mathsf{S})(B^\bullet)_U$ that defines f_2.

$$\begin{array}{ccc} F(a_2) & \xrightarrow{f_2} & b_2 \\ \simeq \downarrow & \theta \nearrow\!\!\!\nearrow & \downarrow \simeq \\ F(a) \otimes_B F(a_1) & \xrightarrow[f \otimes_B f_1]{} & b \otimes_B b_1 \end{array}$$

The collection (f, θ) gives the natural 2-transformation between $2\wp(q) \circ F$ and $2\wp(p)$.

Remark 6.2.4. Since q is a quasi-isomorphism in $\mathrm{C}^{[-2,0]}(\mathsf{S})$, the technical lemma 6.2.2 implies that $2\wp(q)$ is a biequivalence in $2\textsc{Pic}(\mathsf{S})(\mathsf{S})$. Therefore, by choosing an inverse of $2\wp(q)$ up to a natural 2-transformation we can write F as $F \simeq 2\wp(p) \circ 2\wp(q)^{-1}$.

□

6.2.2 Hom-categories of $\mathsf{Frac}(A^\bullet, B^\bullet)$ and $\mathsf{Hom}(A^\bullet, B^\bullet)$

In the next two lemmas, we are going to explore the relation between 1-morphisms (resp. 2-morphisms) of $\mathsf{Frac}(A^\bullet, B^\bullet)$ and natural 2-transformations (resp. modifications) of Picard 2-stacks.

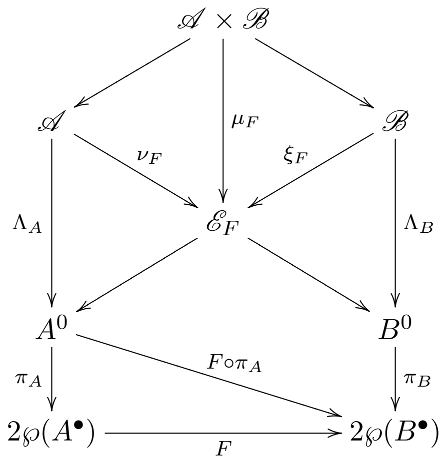
Suppose we have a natural 2-transformation θ

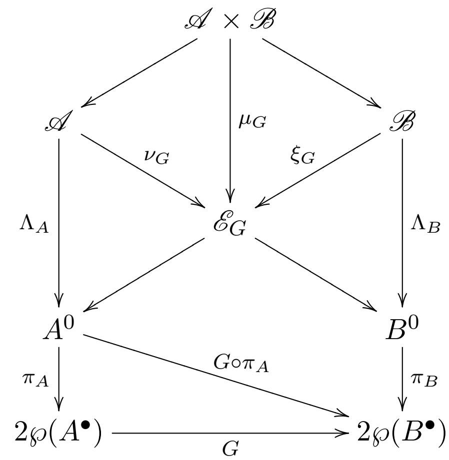
$$2\wp(A^\bullet) \underset{G}{\overset{F}{\rightrightarrows}} 2\wp(B^\bullet) \quad \Downarrow\theta \tag{6.2.13}$$

between the two additive 2-functors $F, G : 2\wp(A^\bullet) \to 2\wp(B^\bullet)$. By Lemma 6.2.3, we know that there are fractions $(q_F, M_F^\bullet, p_F)$ and $(q_G, M_G^\bullet, p_G)$ associated to F and G.

Lemma 6.2.5. *For any natural 2-transformation θ as in (6.2.13), there is a 1-morphism in $\mathsf{Frac}(A^\bullet, B^\bullet)$ between the fractions $(q_F, M_F^\bullet, p_F)$ and $(q_G, M_G^\bullet, p_G)$.*

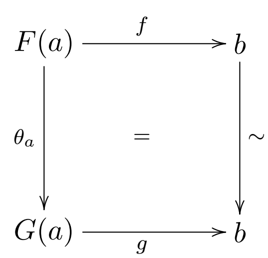
Proof. For F and G, we have the following diagrams similar to (6.2.1)



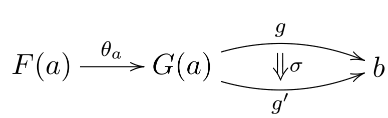


where $\mathscr{E}_F := A^0 \times_{F,B} B^0$ and $\mathscr{E}_G := A^0 \times_{G,B} B^0$ are Picard stacks by Lemma 6.2.1. Therefore by [3, Proposition 8.3.2], there exist $E_F^{-1} \to E_F^0$ and $E_G^{-1} \to E_G^0$ morphisms of abelian sheaves such that the Picard stack associated to them are respectively $\mathscr{E}_F$ and $\mathscr{E}_G$. The natural 2-transformation $\theta : F \Rightarrow G$ induces an equivalence $H : \mathscr{E}_G \to \mathscr{E}_F$ of Picard stacks defined as follows:

- For any (a, g, b) object of $(\mathscr{E}_G)_U$, $H((a, g, b)) := (a, f, b)$, where f fits into the commutative diagram

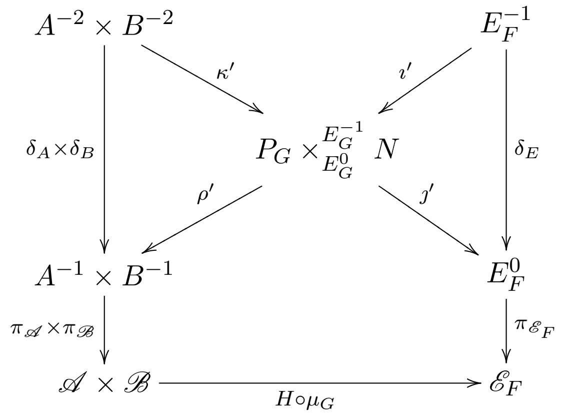


- For any (a, g, σ, g', b) morphism of $(\mathscr{E}_G)_U$, $H((a, g, \sigma, g', b)) := (a, f, \tau, f', b)$, where τ is defined by the following whiskering.



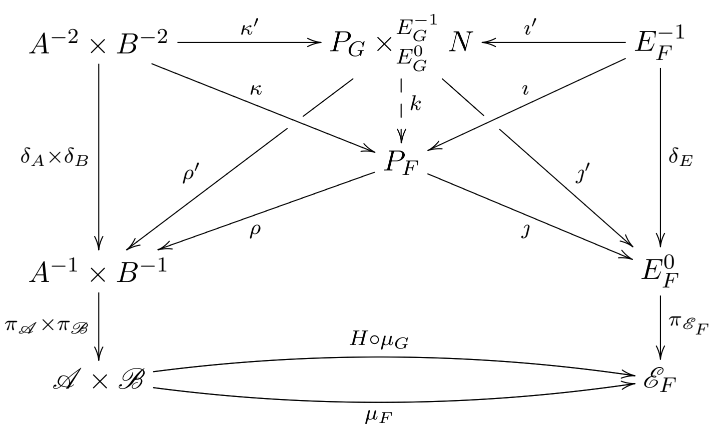
By [3, Theorem 8.3.1], H corresponds to a butterfly $[E_G^\bullet, N, E_F^\bullet]$. Since H is an equivalence, this butterfly is flippable.

We compose H and μ_G by composing their corresponding butterflies



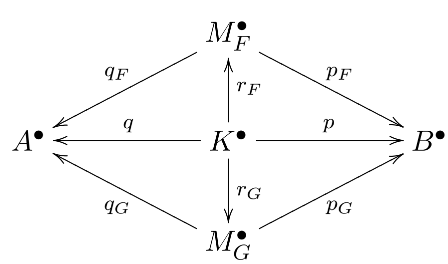
where $P_G \times_{E_G^0}^{E_G^{-1}} N$ is pull-out/pull-back construction as defined in [3, §5.1].

There is also a direct morphism μ_F from $\mathscr{A} \times \mathscr{B}$ to $\mathscr{E}_F$. μ_F is equivalent to $H \circ \mu_G$ since they both map an object of $\mathscr{A} \times \mathscr{B}$ to an object in $\mathscr{E}_F$ which is isomorphic to the unit object in $2\wp(B^\bullet)$. Then by [3, Theorem 8.3.1], there exists an isomorphism k between the corresponding butterflies of μ_F and $H \circ \mu_G$, that is the dotted arrow in the diagram below such that all regions commute.



(6.2.14)

Let $M_F^\bullet : A^{-2} \times B^{-2} \to P_F \to E_F^0$ and $M_G^\bullet : A^{-2} \times B^{-2} \to P_G \to E_G^0$. We claim that, there exists a complex $K^\bullet$ with quasi-isomorphisms r_F and r_G such that all regions in the diagram



(6.2.15)

commute.

Proof of the claim: Let $K^\bullet : A^{-2} \times B^{-2} \to P_G \times_{E_G^0} N \to N$ and define r_F by the composition

$$
\begin{array}{ccccccc}
K^\bullet & & A^{-2}\times B^{-2} & \longrightarrow & P_G \times_{E^0_G} N & \longrightarrow & N \\
\Big\downarrow r_F & & \Big\| & & \Big\downarrow \text{quotient} & & \Big\downarrow \text{quotient} \\
 & & A^{-2}\times B^{-2} & \longrightarrow & P_G \times^{E^{-1}_G}_{E^0_G} N & \longrightarrow & N/E^{-1}_G \\
 & & \Big\| & & \Big\downarrow & & \Big\downarrow \\
M^\bullet_F & & A^{-2}\times B^{-2} & \longrightarrow & P_F & \longrightarrow & E^0_F
\end{array}
\tag{6.2.16}
$$

and r_G by the diagram

$$
\begin{array}{ccccccc}
K^\bullet & & A^{-2}\times B^{-2} & \longrightarrow & P_G \times_{E^0_G} N & \longrightarrow & N \\
\Big\downarrow r_G & & \Big\| & & \Big\downarrow & & \Big\downarrow \\
M^\bullet_G & & A^{-2}\times B^{-2} & \longrightarrow & P_G & \longrightarrow & E^0_G
\end{array}
\tag{6.2.17}
$$

The commutativity of the diagram (6.2.16) follows from composition of butterflies. Since $P_G \times^{E^{-1}_G}_{E^0_G} N \simeq P_F$ and the butterfly $[E^\bullet_G, N, E^\bullet_F]$ is flippable, r_F is a quasi-isomorphism. The diagram (6.2.17) commutes because its left square is a pullback. This implies that r_G is a quasi-isomorphism.

It remains to show that $q_F \circ r_F = q_G \circ r_G$, that is in the diagram below each column closes to a commutative square.

$$
\begin{array}{ccccccc}
A^\bullet & & A^{-2} & \longrightarrow & A^{-1} & \longrightarrow & A^0 \\
\Big\uparrow q_F & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow \\
M^\bullet_F & & A^{-2}\times B^{-2} & \longrightarrow & P_F & \longrightarrow & E^0_F \\
\Big\uparrow r_F & & \Big\| & & \Big\uparrow & & \Big\uparrow \\
K^\bullet & & A^{-2}\times B^{-2} & \longrightarrow & P_G \times_{E^0_G} N & \longrightarrow & N \\
\Big\downarrow r_G & & \Big\| & & \Big\downarrow & & \Big\downarrow \\
M^\bullet_G & & A^{-2}\times B^{-2} & \longrightarrow & P_G & \longrightarrow & E^0_G \\
\Big\downarrow q_G & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
A^\bullet & & A^{-2} & \longrightarrow & A^{-1} & \longrightarrow & A^0
\end{array}
$$

It is obvious for the first column. The commutativity of the triangles

$$\begin{array}{ccc} P_G \times_{E_G^0}^{E_G^{-1}} N & \xrightarrow{k} & P_F \\ & \searrow^{\rho'} & \downarrow \rho \\ & & A^{-1} \times B^{-1} \end{array} \qquad \begin{array}{ccc} \mathscr{E}_G & \xrightarrow{H} & \mathscr{E}_F \\ & \searrow_{\mathrm{pr}_1} \quad \swarrow_{\mathrm{pr}_2} & \\ & A^0 & \end{array}$$

imply that the middle and last columns close to a commutative square, respectively (the first triangle is extracted from diagram (6.2.14)).

In the same way, we also show that $p_F \circ r_F = p_G \circ r_G$. □

Now, suppose we have a modification Γ

$$2\wp(A^\bullet) \underset{G}{\overset{F}{\rightrightarrows}} 2\wp(B^\bullet), \quad \theta\Downarrow \overset{\Rrightarrow}{\Gamma} \Downarrow\phi \tag{6.2.18}$$

between two natural 2-transformations $\theta, \phi : F \Rightarrow G$. We have proved in Lemmas 6.2.3 and 6.2.5 that both θ and ϕ correspond to a 1-morphism in $\mathsf{Frac}(A^\bullet, B^\bullet)$.

Lemma 6.2.6. *Given a modification Γ as in (6.2.18), there exists a 2-morphism between the two 1-morphisms corresponding to θ and ϕ.*

Proof. Using the same notations as in Lemma 6.2.5, we construct a diagram of Picard stacks

$$\mathscr{E}_G \underset{H_\phi}{\overset{H_\theta}{\rightrightarrows}} \mathscr{E}_F, \quad \Downarrow T$$

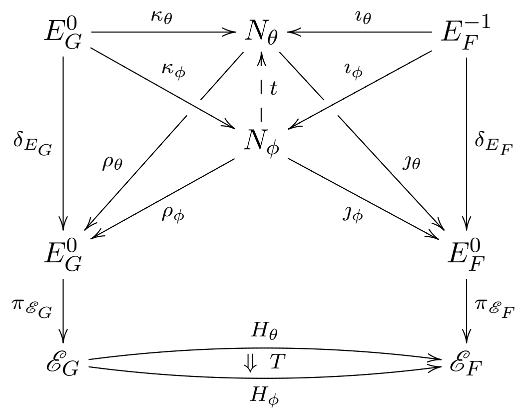
where T is a natural transformation. For any object (a, g, b) in $\mathscr{E}_G$, $T_{(a,g,b)}$ is a morphism in $\mathscr{E}_F$ defined by

$$F(a) \underset{f_\phi}{\overset{f_\theta}{\rightrightarrows}} b, \quad \Downarrow 1_g * \Gamma_a$$

where

$$F(a) \underset{\phi_a}{\overset{\theta_a}{\rightrightarrows}} G(a), \quad \Downarrow \Gamma_a$$

and $H_\theta(a, g, b) = (a, f_\theta, b)$, $H_\phi(a, g, b) = (a, f_\phi, b)$. By [3, Theorem 5.3.6], the natural transformation T corresponds to an isomorphism t between the centers of the butterflies associated to H_θ and H_ϕ.



(6.2.19)

t induces an isomorphism of complexes $t^\bullet$.

$$\begin{array}{ccccccc} K^\bullet_\phi & & A^{-2}\times B^{-2} & \longrightarrow & P_G \times_{E^0_G} N_\phi & \longrightarrow & N_\phi \\ {\scriptstyle t^\bullet}\big\downarrow & & \big\| & & \big\downarrow{\scriptstyle \mathrm{id}\times t} & & \big\downarrow{\scriptstyle t} \\ K^\bullet_\theta & & A^{-2}\times B^{-2} & \longrightarrow & P_G \times_{E^0_G} N_\theta & \longrightarrow & N_\theta \end{array}$$

The proof finishes by showing that all the regions in the diagram (5.2.2) commute. The only regions, whose commutativity are non-trivial, are the triangles in the middle sharing an edge marked by the isomorphism $t^\bullet$. They commute as well since in the diagram below

$$\begin{array}{ccccccc} M^\bullet_G & & A^{-2}\times B^{-2} & \longrightarrow & P_G & \longrightarrow & E^0_G \\ {\scriptstyle r_{G,\phi}}\big\uparrow & & \big\| & & \big\uparrow{\scriptstyle \mathrm{pr}_1} & & \big\uparrow{\scriptstyle \rho_\phi} \\ K^\bullet_\phi & & A^{-2}\times B^{-2} & \longrightarrow & P_G \times_{E^0_G} N_\phi & \longrightarrow & N_\phi \\ {\scriptstyle t^\bullet}\big\downarrow & & \big\| & & \big\downarrow{\scriptstyle \mathrm{id}\times t} & & \big\downarrow{\scriptstyle t} \\ K^\bullet_\theta & & A^{-2}\times B^{-2} & \longrightarrow & P_G \times_{E^0_G} N_\theta & \longrightarrow & N_\theta \\ {\scriptstyle r_{G,\theta}}\big\downarrow & & \big\| & & \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle \rho_\theta} \\ M^\bullet_G & & A^{-2}\times B^{-2} & \longrightarrow & P_G & \longrightarrow & E^0_G \end{array}$$

each column closes to a commutative triangle. This is immediate for the first two columns. The triangle formed by the last column commutes as well, since it is a piece of the commutative diagram (6.2.19). $\square$

For any two complexes of abelian sheaves $A^\bullet$ and $B^\bullet$, the proofs of Lemmas 6.2.3 and 6.2.5 define us a 2-functor

$$2\wp_{(A^\bullet,B^\bullet)} : \ \mathsf{Frac}(A^\bullet, B^\bullet) \longrightarrow \mathsf{Hom}(A^\bullet, B^\bullet) \qquad (6.2.20)$$

between the bigroupoid $\mathsf{Frac}(A^\bullet, B^\bullet)$ and the 2-groupoid $\mathsf{Hom}(A^\bullet, B^\bullet)$ of additive 2-functors between $2\wp(A^\bullet)$ and $2\wp(B^\bullet)$ considered as a bigroupoid. In fact, we have proved:

Theorem 6.2.7. *For any two complexes of abelian sheaves $A^\bullet$ and $B^\bullet$, $2\wp_{(A^\bullet,B^\bullet)}$ is a biequivalence of bigroupoids.*

The trihomomorphism (6.1.1) extends to a trihomomorphism

$$2\wp : \mathrm{T}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{P{\small IC}}(\mathsf{S}) \tag{6.2.21}$$

on the tricategory $\mathrm{T}^{[-2,0]}(\mathsf{S})$ as follows[1]: It sends any length 3 complex of abelian sheaves to the associated Picard 2-stack. The biequivalence (6.2.20) defines it on 1-,2-,3-morphisms.

6.3 From Picard 2-Stacks to Complexes of Abelian Sheaves

In this Section, we show that for any Picard 2-stack $\mathbb{P}$, there exists a length 3 complex of abelian sheaves whose associated Picard 2-stack (see Section 4.5) is equivalent to $\mathbb{P}$. Said differently, we prove (Lemma 6.3.2) that the trifunctor (6.2.21) is essential surjectivity. This proof depends on the following technical result, which is similar to Lemme 1.4.3 in [9]. We give its proof in the Appendix (A.2.1).

Proposition 6.3.1. *For any set E, denote by $\mathbb{Z}(E)$ the free abelian group generated by E. Let $\mathbb{C}$ be a Picard 2-category and $F_0 : E \to \mathbb{C}$ be a set map. Then F_0 extends to an additive 2-functor $F : \mathbb{Z}(E) \to \mathbb{C}$ where $\mathbb{Z}(E)$ is considered as a 2-category (trivially Picard).*

Lemma 6.3.2. *Let $\mathbb{P}$ be a Picard 2-stack, then there exists a complex of abelian sheaves $A^\bullet$ such that $2\wp(A^\bullet)$ is biequivalent to $\mathbb{P}$.*

Proof. There is a construction analogous to the skeleton of categories. For any 2-category $\mathbb{P}$, we construct $2\mathrm{sk}(\mathbb{P})$ a 2-category that has one object per equivalence class in $\mathbb{P}$. We observe that $2\mathrm{sk}(\mathbb{P})$ is a full sub 2-category of $\mathbb{P}$, that is the inclusion $2\mathrm{sk}(\mathbb{P}) \to \mathbb{P}$ is a biequivalence. Let $\mathbb{P}$ be a Picard 2-stack. We note that $\mathrm{Ob}\, 2\mathrm{sk}(\mathbb{P}) : U \to \mathrm{Ob}(2\mathrm{sk}(\mathbb{P}_U))$ is a presheaf of sets. We consider A^0 the abelian sheaf over S associated to the presheaf $\{U \to \mathbb{Z}(\mathrm{Ob}(2\mathrm{sk}(\mathbb{P}_U)))\}$ where $\mathbb{Z}(\mathrm{Ob}(2\mathrm{sk}(\mathbb{P}_U)))$ is the free abelian group associated to $\mathrm{Ob}(2\mathrm{sk}(\mathbb{P}_U))$. By Proposition 6.3.1, the inclusion $i : \mathrm{Ob}\, 2\mathrm{sk}(\mathbb{P}) \to \mathbb{P}$ extends to

$$\pi_{\mathbb{P}} : A^0 \longrightarrow \mathbb{P}$$

an essentially surjective additive 2-functor on A^0.

Define $\mathscr{A}$ by the pullback diagram

$$\begin{array}{ccc} \mathscr{A} & \longrightarrow & 0 \\ {\scriptstyle \Lambda_A}\big\downarrow & \nearrow & \big\downarrow \\ A^0 & \xrightarrow[\pi_{\mathbb{P}}]{} & \mathbb{P} \end{array} \tag{6.3.1}$$

[1]We commit an abuse of notation by calling both functors (6.1.1) and (6.2.21) by $2\wp$.

of morphisms of Picard 2-stacks, which is similar to (4.6.2). Then, the sequence of Picard 2-stacks

$$\mathscr{A} \longrightarrow A^0 \longrightarrow \mathbb{P}$$

is exact sequence in the sense of Section 4.6.

On the other hand, from Lemma 6.2.1, it follows that $\mathscr{A}$ is a Picard stack. Therefore by [3, Proposition 8.3.2], there exists a morphism of abelian sheaves $\delta_A : A^{-2} \to A^{-1}$, where A^{-2} is defined by the pullback diagram

$$\begin{array}{ccc} A^{-2} & \longrightarrow & 0 \\ {\scriptstyle \delta_A}\Big\downarrow & \nearrow & \Big\downarrow \\ A^{-1} & \xrightarrow[\pi_{\mathscr{A}}]{} & \mathscr{A} \end{array} \tag{6.3.2}$$

and $\mathscr{A} := \textsc{Tors}(A^{-2}, A^{-1})$.

Now putting the diagrams (6.3.1) and (6.3.2) together,

$$\begin{array}{ccccc} A^{-2} & \longrightarrow & 0 & & \\ {\scriptstyle \delta_A}\Big\downarrow & \nearrow & \Big\downarrow & & \\ A^{-1} & \xrightarrow[\pi_{\mathscr{A}}]{} & \mathscr{A} & \longrightarrow & 0 \\ & {\scriptstyle \lambda_A}\searrow & {\scriptstyle \Lambda_A}\Big\downarrow & \nearrow & \Big\downarrow \\ & & A^0 & \xrightarrow[\pi_{\mathbb{P}}]{} & \mathbb{P} \end{array} \tag{6.3.3}$$

we have a diagram of Picard 2-stacks. It implies that $A^\bullet : A^{-2} \xrightarrow{\delta_A} A^{-1} \xrightarrow{\lambda_A} A^0$ is a complex.

The Picard 2-stack associated to $A^\bullet$, that is $2\wp(A^\bullet) := \textsc{Tors}(\mathscr{A}, A^0)$, verifies by definition the above diagram (see 4.6.4).

The biequivalence $2\wp(A^\bullet) \simeq \mathbb{P}$ is almost immediate. Essential surjectivity follows from the definition of $\pi_{\mathbb{P}}$ and equivalence of hom-categories from the fact that A^0 and 0 pull back to $\mathscr{A}$ over $2\wp(A^\bullet)$ and over $\mathbb{P}$. □

6.4 The Main Theorem

Considering $2\textsc{Pic}(\mathsf{S})$ as a tricategory, our main result follows from Theorem 6.2.7 and Lemma 6.3.2.

Theorem 6.4.1. *The trihomomorphism (6.2.21) is a triequivalence.*

An immediate consequence of the Theorem 6.4.1, which was also the motivation for this work, is the following.

Let $2\mathrm{PIC}(\mathsf{S})^{\flat\flat}$ denote the category of Picard 2-stacks obtained from $2\mathrm{PIC}(\mathsf{S})$ by ignoring the modifications and taking as morphisms the equivalence classes of additive 2-functors. Let $\mathrm{D}^{[-2,0]}(\mathsf{S})$ be the subcategory of the derived category of category of complexes of abelian sheaves $A^\bullet$ over S with $H^{-i}(A^\bullet) \neq 0$ for $i = 0, 1, 2$. We deduce from Theorem 6.4.1 the following, which generalizes Deligne's result [9, Proposition 1.4.15] from Picard stacks to Picard 2-stacks.

Corollary 6.4.2. *The functor (6.2.21) induces an equivalence*

$$2\wp^{\flat\flat} : \mathrm{D}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{PIC}(\mathsf{S})^{\flat\flat} \tag{6.4.1}$$

of categories.

Proof. It is enough to observe from the calculations in Section 5.2.2 that $\pi_0(\mathsf{Frac}(A^\bullet, B^\bullet)) \simeq \mathrm{Hom}_{\mathrm{D}^{[-2,0]}(\mathsf{S})(A^\bullet, B^\bullet)}$. Since the objects of $\mathrm{D}^{[-2,0]}(\mathsf{S})$ are same as the objects of $\mathrm{T}^{[-2,0]}(\mathsf{S})$, the essential surjectivity follows from the Lemma 6.3.2. $\square$

CHAPTER 7

CONCLUSION

The purpose of this thesis is to generalize Deligne's characterization theorem for Picard stacks which is recalled in Chapter 3 to Picard 2-stacks. For that, we closely follow Deligne's proof in [9].

In Chapter 4, we first define the 3-category $2\mathrm{P\scriptsize{IC}}(\mathsf{S})$ of Picard 2-stacks. Our definition of Picard 2-stacks (4.4.5) differs from the usual definition (see [7, Chapter 8]). The difference is that, we assume the multiplication by an object in a Picard 2-category is a biequivalence. This implies existence of a unit object in the sense of Joyal-Kock [18] and an inverse of an object. The advantage of this definition is that we do not need to add any data about unit objects or inverses to the definition of Picard 2-stack which therefore reduces significantly the number of coherence conditions. This definition of Picard 2-stacks also makes easier to define the morphism of Picard 2-stacks (4.4.5). In Chapter 5, we introduce length 3 complexes of abelian sheaves and their tricategory $\mathrm{T}^{[-2,0]}(\mathsf{S})$. In Chapter 6, we show that the tricategory $\mathrm{T}^{[-2,0]}(\mathsf{S})$ is triequivalent to the 3-category $2\mathrm{P\scriptsize{IC}}(\mathsf{S})$ (Theorem 6.4.1).

We want to conclude this thesis with an informal discussion of stack versions of some of our results. We will assume that all structures are strict unless otherwise stated. Throughout the thesis, we dealt with 2- and 3-categories and their weakened versions bi- and tricategories. They can be stackified.

2-stacks over a site are well known [7]. The collection of 2-stacks over S, denoted by $2\mathrm{S\scriptsize{TACK}}(\mathsf{S})$, comprise a 3-category structure. We can consider the fibered 3-category $2\mathfrak{S}\mathrm{\scriptsize{TACK}}(\mathsf{S})$, whose fiber over U is the 3-category $2\mathrm{S\scriptsize{TACK}}(\mathsf{S}/U)$ of 2-stacks over S/U. In [7, Remark 1.12], Breen claims that $2\mathfrak{S}\mathrm{\scriptsize{TACK}}(\mathsf{S})$ is a 3-stack. Hirschowitz and Simpson in [17], generalize this result to weak n-stacks.

Theorem. *[17, Théorème 20.5] The weak $(n+1)$-prestack of weak n-stacks $nW\mathfrak{S}\mathrm{\scriptsize{TACK}}(\mathsf{S})$ is a weak $(n+1)$-stack over S.*

We can use the above facts to deduce that the 3-prestack of Picard 2-stacks $2\mathfrak{P}\mathrm{\scriptsize{IC}}(\mathsf{S})$ with fibers $2\mathrm{P\scriptsize{IC}}(\mathsf{S})(\mathsf{S}/U)$ over U is a 3-stack.

Claim. *$\mathbb{H}\mathrm{om}(A^\bullet, B^\bullet)$ fibered over S in 2-groupoids is a 2-stack where for any $U \in \mathsf{S}$, the 2-groupoid $\mathsf{Hom}(A^\bullet_{|U}, B^\bullet_{|U})$ of additive 2-functors from $2\wp(A^\bullet)_{|U}$ to $2\wp(B^\bullet)_{|U}$ defines the fiber over U.*

We have also fibered analogs for each hom-bicategory $\mathsf{Frac}(A^\bullet, B^\bullet)$ and for $\mathrm{T}^{[-2,0]}(\mathsf{S})$. It follows from the above claim and Theorem 6.2.7 that the bi-prestack $\mathbb{F}\mathsf{rac}(A^\bullet, B^\bullet)$ of fractions from $A^\bullet$ to $B^\bullet$ with fibers defined by $\mathsf{Frac}(A^\bullet_{|U}, B^\bullet_{|U})$ is a bistack. Then, once an appropriate notion of 3-descent has been specified and all descent data are shown to be effective, we conclude by the characterization proposition [17, Proposition 10.2] for n-stacks that the tri-prestack of complexes $\mathfrak{T}^{[-2,0]}(\mathsf{S})$ with fibers $\mathrm{T}^{[-2,0]}(\mathsf{S})(\mathsf{S}/U)$ is a tristack. The characterization proposition cited above briefly says that $\mathfrak{P}$ is an n-stack over S if and only if all descent data are effective and for any X, Y objects of $\mathfrak{P}_U$, $\mathrm{Hom}_{\mathfrak{P}_U}(X, Y)$ is an $n-1$ stack over S/U.

Remark 7.0.3. The characterization proposition in [17, Proposition 10.2] is originally enounced for Segal n-categories, n-prestacks, and n-stacks. But again in the same paper, it has been remarked that the proposition holds for non Segal structures [17, §20] where in this case, the weak structure is assumed to be the one defined by Tamsamani. Its definition can be found in [31] and [32]. However, we are being very informal and not discussing here the connection of the weak structure of our categories, pre-stacks and, stacks with the ones mentioned above.

Finally, we define the trihomomorphism of tristacks by localizing the triequivalence (6.2.21).

$$\mathfrak{T}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathfrak{P}\mathrm{IC}(\mathsf{S}), \tag{7.0.1}$$

where $2\mathfrak{P}\mathrm{IC}(\mathsf{S})$ is considered naturally as a tristack. We deduce then its stack analog

Theorem 7.0.4. *(7.0.1) is a triequivalence of tristacks.*

APPENDIX A

TECHNICAL LEMMAS

A.1 Lemma 1

In the appendix we give the proof of two technical lemmas.

Lemma A.1.1. *Let $\mathbb{P}$ be a Picard 2-stack and A, B be two abelian sheaves with additive 2-functors $\phi : A \longrightarrow \mathbb{P}$ and $\psi : B \longrightarrow \mathbb{P}$. Then $A \times_{\mathbb{P}} B$ is a Picard stack.*

Proof. The fibered category $A \times_{\mathbb{P}} B$ with fibers $(A \times_{\mathbb{P}} B)_{|U}$ consisting of

- objects (a, f, b), where $a \in A(U)$, $b \in B(U)$, and $f : \phi(a) \to \psi(b)$ is a 1-morphism in $\mathbb{P}_U$;

- morphisms (a, f, α, g, b), where $\phi(a) \underset{g}{\overset{f}{\Downarrow\alpha}} \psi(b)$ is a 2-morphism in $\mathbb{P}_U$;

is a prestack since for any $U \in \mathsf{S}$, 1-morphisms of $\mathbb{P}$ form a stack over S/U. It is in fact a stack.

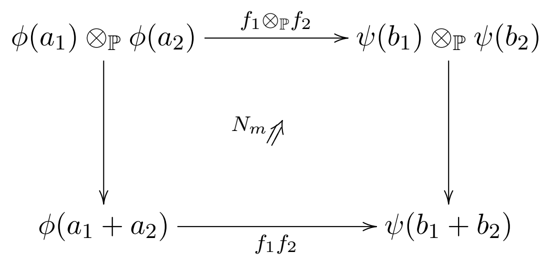
Let $((U_i \to U), (a_i, f_i, b_i), \alpha_{i,j})_{i,j\in I}$ be a descent datum with $(U_i \to U)_{i\in I}$ a covering of U, (a_i, f_i, b_i) an object in $(A \times_{\mathbb{P}} B)_{U_i}$ and $\alpha_{i,j}$ a 1-morphism in $(A \times_{\mathbb{P}} B)_{U_{ij}}$ between $(a_j, f_j, b_j)_{|U_{ij}}$ and $(a_i, f_i, b_i)_{|U_{ij}}$. Since $a_{i|U_{ij}} = a_{j|U_{ij}}$, $b_{i|U_{ij}} = b_{j|U_{ij}}$ and A and B are sheaves, there exist $a \in A(U)$ and $b \in B(U)$ such that $a_{|U_i} = a_i$ and $b_{|U_i} = b_i$. Then the collection $((U_i \to U), f_i, \alpha_{i,j})_{i,j\in I}$ satisfies the descent in $\mathrm{Hom}(\phi(a), \psi(b))$, which is effective since $\mathbb{P}$ is a Picard 2-stack. That is, there exists $f \in \mathrm{Hom}(\phi(a), \psi(b))$ and $\beta_i : f_{|U_i} \Rightarrow f_i$ compatible with $\alpha_{i,j}$ such that for all $i \in I$, $(a_i, f_{|U_i}, \beta_i, f_i, b_i)$ is a morphism from $(a, f, b)_{|U_i}$ to (a_i, f_i, b_i). Thus, the descent $((U_i \to U), (a_i, f_i, b_i), \alpha_{i,j})_{i,j\in I}$ is effective.

Next, we show that $A \times_{\mathbb{P}} B$ is Picard. Let $\mathfrak{a}$ and $\mathfrak{c}$ represent the associativity and commutativity constraints in $\mathbb{P}$. According to the definition (3.1.1), the Picard structure is given by

1. a 2-functor $\otimes : A \times_{\mathbb{P}} B \times A \times_{\mathbb{P}} B \longrightarrow A \times_{\mathbb{P}} B$ defined as

$$(a_1, f_1, b_1) \otimes (a_2, f_2, b_2) := (a_1 + a_2, f_1 f_2, b_1 + b_2),$$

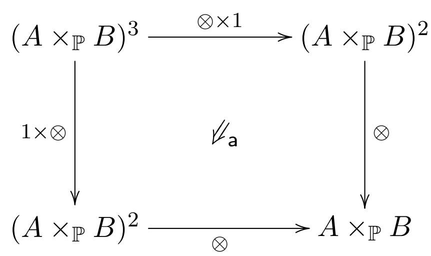
where $f_1 f_2$ is the morphism that makes the diagram



(A.1.1)

commute up to a 2-isomorphism N_m.

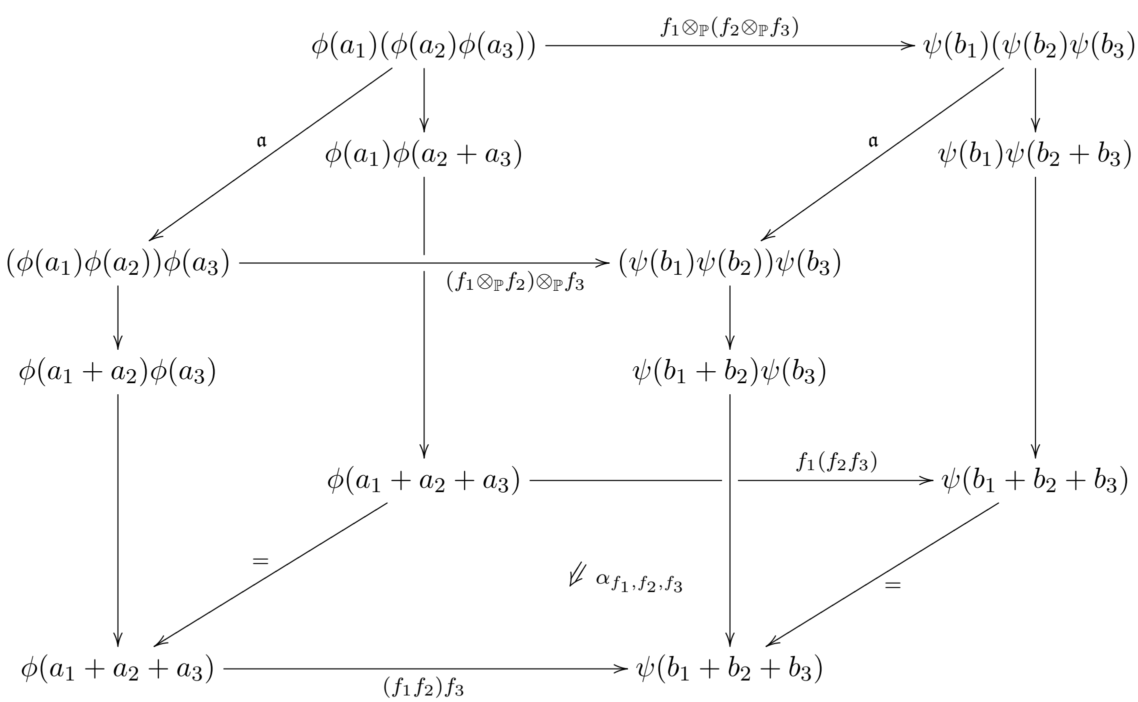
2. a functorial isomorphism a



such that for any three objects $(a_i, f_i, b_i)_{\{}1, 2, 3\}$, $\mathsf{a}_{1,2,3}$ is the associator morphism

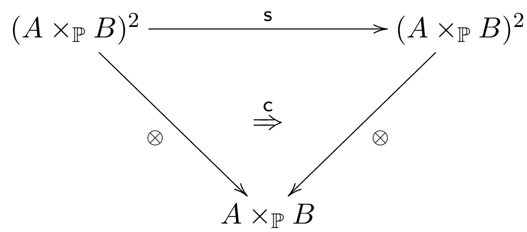
$$\mathsf{a}_{1,2,3} := (a_1 + a_2 + a_3, f_1(f_2 f_3), \alpha_{f_1, f_2, f_3}, (f_1 f_2) f_3, b_1 + b_2 + b_3),$$

where α_{f_1, f_2, f_3} is defined as the 2-isomorphism of the bottom face that makes the following cube commutative (we ignored $\otimes_{\mathbb{P}}$ for compactness).



The other 2-isomorphisms of the cube are, the left and right 2-isomorphisms represent the compatibility of the associator of $\mathbb{P}$ respectively with the strict associators of A and B(i.e. they are the 2-morphisms defined by the modifications ω_ϕ and ω_ψ which are similar to ω_F in the definition 4.3.1), the back and front ones are of the form N_m, the top one is of the form $\mathfrak{a}_{f_1,f_2,f_3} : f_1 \otimes_{\mathbb{P}} (f_2 \otimes_{\mathbb{P}} f_3) \Rightarrow (f_1 \otimes_{\mathbb{P}} f_2) \otimes_{\mathbb{P}} f_3$.

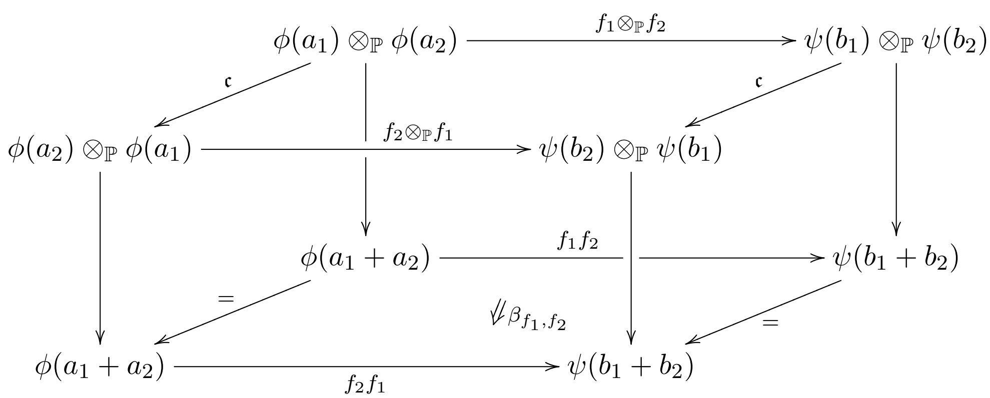
3. a functorial isomorphism c



such that for any two objects (a_1, f_1, b_1) and (a_2, f_2, b_2), $\mathsf{c}_{1,2}$ is the morphism from $(a_1, f_1, b_1) \otimes (a_2, f_2, b_2)$ to $(a_2, f_2, b_2) \otimes (a_1, f_1, b_1)$ defined by

$$\mathsf{c}_{1,2} = (a_1 + a_2, f_1 f_2, \beta_{f_1,f_2}, f_2 f_1, b_1 + b_2),$$

where β_{f_1,f_2} is the 2-isomorphism of the bottom face of the commutative cube.



(A.1.2)

The other 2-isomorphisms of the cube are, the left and right 2-isomorphisms represent the compatibility of the braiding of $\mathbb{P}$ respectively with the strict braidings of A and B(i.e. they are the 2-morphisms defined by the modifications ε_ϕ and ε_ψ which are similar to ε_F in the definition 4.3.1), the front and back ones are of the form N_m, and the top one is of the form $\mathfrak{c}_{f_1,f_2} : f_1 \otimes_{\mathbb{P}} f_2 \Rightarrow f_2 \otimes_{\mathbb{P}} f_1$.

We need to verify that these data satisfy the condition given in definition (3.1.1).

(i) *equivalence*:Let (a, f, b) be an object in $A \times_{\mathbb{P}} B$. We claim that

$$(a, f, b) \otimes - : A \times_{\mathbb{P}} B \longrightarrow A \times_{\mathbb{P}} B$$

is an equivalence. To show essential surjectivity, we need to find for any (x, g, y) object in $A \times_{\mathbb{P}} B$, an object (a', f', b') such that $(a, f, b) \otimes (a', f', b')$ is isomorphic to (x, g, y). Define $a' = x - a$ and $b' = b - a$. Since f is a 1-morphism in $\mathbb{P}$, by Lemma (4.1.3)

$$f \otimes_{\mathbb{P}} - : \mathrm{Hom}(\phi(a'), \psi(b')) \longrightarrow \mathrm{Hom}(\phi(a) \otimes_{\mathbb{P}} \phi(a'), \psi(b) \otimes_{\mathbb{P}} \psi(b')) \qquad \text{(A.1.3)}$$

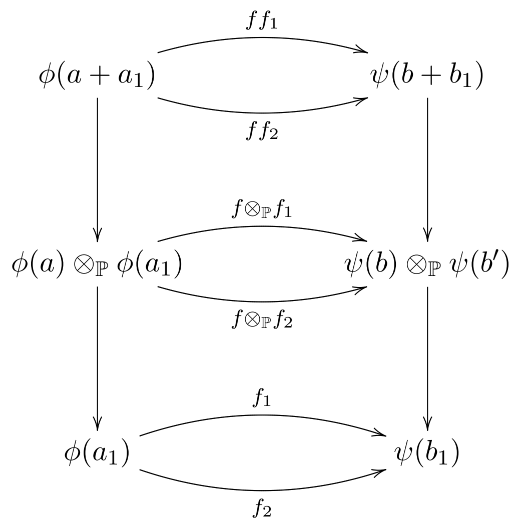
is an equivalence. We also have the equivalence

$$\mathrm{Hom}(\phi(a) \otimes_{\mathbb{P}} \phi(a'), \psi(b) \otimes_{\mathbb{P}} \psi(b')) \longrightarrow \mathrm{Hom}(\phi(x), \psi(y)) \qquad \text{(A.1.4)}$$

So we let f' be the inverse image of $g \in \mathrm{Hom}(\phi(x), \psi(y))$ under the composition (A.1.3) and (A.1.4). To show $(a, f, b) \otimes -$ is fully-faithful, we need to show for any two objects (a_1, f_1, b_1) and (a_2, f_2, b_2), the map

$$\mathrm{Hom}((a_1, f_1, b_1), (a_2, f_2, b_2)) \longrightarrow \mathrm{Hom}((a + a_1, ff_1, b + b_1), (a + a_2, ff_2, b + b_2))$$

is a bijection.



(A.1.5)

In the above 2-commutative cylinder, the 2-morphisms of the lateral faces of the top half are of the form (A.1.1) and the 2-morphisms of the lateral faces of the bottom half are uniquely defined by the fact that $f \otimes -$ is an equivalence. Therefore the 2-morphisms of the top and bottom faces are in 1-1 correspondence which are elements in the sets $\mathrm{Hom}((a_1, f_1, b_1), (a_2, f_2, b_2))$ and $\mathrm{Hom}((a+a_1, ff_1, b+b_1), (a+a_2, ff_2, b+b_2))$, respectively.

(ii) Verifying the commutativity of the pentagon and the two hexagons is trivial.

(iii) To verify symmetry we need to show that the 2-morphism of the bottom face of the diagram obtained by concatenation of the appropriate two cubes of the form (A.1.2) is identity(the back face of one of the cubes overlaps with the front face of the other cube). This follows from the fact that, 2-morphism of the top face of the concatenated cube pastes to identity with the help of the 2-morphisms defined by the modification of the form (4.1.1).

(iv) The morphism from $(a, f, b) \otimes (a, f, b)$ to itself is identity because the 2-morphism of the top face of the diagram (A.1.2) pastes to identity with the help f the 2-morphism defined by the modification of the form (4.1.2).

□

A.2 Lemma 2

Proposition A.2.1. *For any set E, denote by $\mathbb{Z}(E)$ the free abelian group generated by E. Let $\mathbb{C}$ be a Picard 2-category and $F_0 : E \to \mathbb{C}$ be a set map. Then F_0 extends to an additive 2-functor $F : \mathbb{Z}(E) \to \mathbb{C}$ where $\mathbb{Z}(E)$ is considered as a 2-category (trivially Picard).*

Proof. We assume that the set E is well-ordered and denote the order on E by $\preceq$. In what follows, we define

1. a 2-functor $F : \mathbb{Z}(E) \longrightarrow \mathbb{C}$,
2. for any two words w_1 and w_2 in $\mathbf{Z}(E)$, a functorial 1-morphism λ_{w_1,w_2}

$$\lambda_{w_1,w_2} : F(w_1) \otimes F(w_2) \longrightarrow F(w_1 + w_2),$$

3. for any three words w_1, w_2, and w_3 in $\mathbf{Z}(E)$, a 2-morphism ψ_{w_1,w_2,w_3} (A.2.8),
4. for any two words w_1 and w_2 in $\mathbf{Z}(E)$, a 2-morphism ϕ_{w_1,w_2} (A.2.10).

A.2.1 Definition of F

We construct the 2-functor $F : \mathbb{Z}(E) \to \mathbb{C}$ as follows:

- For any generator $a \in E$, $Fa := F_0 a$,
- For any generator $a \in E$, $F(-a) := (Fa)^*$, where $(Fa)^*$ is inverse of Fa in $\mathbb{C}$,
- $F(0)$ is the unit element in $\mathbb{C}$, where 0 denotes the unit element in $\mathbb{Z}(E)$.
- For any word w in $\mathbb{Z}(E)$, we
  - simplify w so that there are no cancelations and denote the simplified word by w_c,
  - order the letters of w_c from least to greatest and denote the simplified and ordered word by $w_{c,o}$

  $F(w)$ is defined by multiplying the letters of $w_{c,o}$ from left to right.

For instance let $w = 2a + b - c - a - 2b$. After cancelations and ordering the letters $w_{c,o} = a - b - c$ and

$$F(w) = F(w_{c,o}) = ((Fa \otimes (Fb)^*) \otimes Fc)$$

.

The order on the set E is needed since without the order two words that differ by the position of letters would map to different objects in $\mathbb{C}$ although they are the same word in $\mathbb{Z}(E)$. For the reasons of compactness, we use juxtaposition for the group operation $\otimes$ on the 2-category $\mathbb{C}$.

A.2.2 Monoidal Case

The items (2), (3), and (4) describes the additive structure of the 2-functor F. We first define them on the words that do not have negative coefficients. That is, they are constructed first on the free abelian monoid $\mathbb{N}(E)$. In the next section A.2.3, we extend their definition to the free abelian group $\mathbb{Z}(E)$.

Definition of λ_{w_1,w_2}. Let $w_1 = a_1 + \ldots + a_m$ and $w_2 = b_1 + \ldots + b_n$ be two words in $\mathbb{N}(E)$. The word $w_1 + w_2$ is defined by concatenation of w_1 and w_2 and then by an (m,n)-shuffle so that the letters of w_1 and w_2 are ordered from least to greatest. We denote $w_1 + w_2$ by $c_1 + \ldots + c_{m+n}$. From the definition of F,

$$F(w_1) \otimes F(w_2) = (\ldots((Fa_1Fa_2)Fa_3)\ldots Fa_m) \otimes (\ldots((Fb_1Fb_2)Fb_3)\ldots Fb_n) \quad \text{(A.2.1)}$$

$$F(w_1 + w_2) = (\ldots((Fc_1Fc_2)Fc_3)\ldots Fc_{m+n}) \quad \text{(A.2.2)}$$

We define the functorial morphism $\lambda_{w_1+w_2} : F(w_1) \otimes F(w_2) \rightarrow F(w_1 + w_2)$ in two steps as follows:

Step 1: Correct Bracketing

In this step, we define the morphism

$$\begin{aligned}(\ldots((Fa_1Fa_2)Fa_3)\ldots Fa_m) \otimes (\ldots((Fb_1Fb_2)Fb_3)\ldots Fb_n) \rightarrow \\ ((((\ldots((Fa_1Fa_2)Fa_3)\ldots Fa_m)Fb_1)Fb_2)\ldots Fb_n),\end{aligned} \quad \text{(A.2.3)}$$

which moves the pairs of parenthesis of $F(w_2)$ one by one to the left from the outer most to the inner most without changing the place of parenthesis of $F(w_1)$. (A.2.3) is composition of $n-1$ many morphisms of the form

$$(\ldots((F(w_1)(F(w_2')Fb_i))Fb_{i+1})\ldots Fb_n) \rightarrow (\ldots(((F(w_1)F(w_2'))Fb_i)Fb_{i+1})\ldots Fb_n), \quad \text{(A.2.4)}$$

where w_2' is a subword of w_2.

Step 2: Ordering Letters

Once (A.2.3) is applied, the letters of w_1 and w_2 are parenthesized from left. Next, we define the morphism

$$((((\ldots((Fa_1Fa_2)Fa_3)\ldots Fa_m)Fb_1)Fb_2)\ldots Fb_n) \rightarrow (\ldots((Fc_1Fc_2)Fc_3)\ldots Fc_{m+n}), \quad \text{(A.2.5)}$$

that shuffles the letters of w_1 and w_2 to order them from least to greatest, that is $c_1 \preceq c_2 \preceq \ldots \preceq c_{m+n}$.

The rule is as follows,

1. find the smallest letter of w_2 in $w_1 + w_2$ such that it has a letter of w_1 on its left that is greater,
2. change their places. Depending on the position of the letters, there are two cases. Either the letters are in the same parenthesis , then (A.2.5) simply permutes them

$$(\ldots((Fc_1Fc_2)Fc_3)\ldots Fc_{m+n}) \rightarrow (\ldots((Fc_2Fc_1)Fc_3)\ldots Fc_{m+n}), \quad \text{(A.2.6)}$$

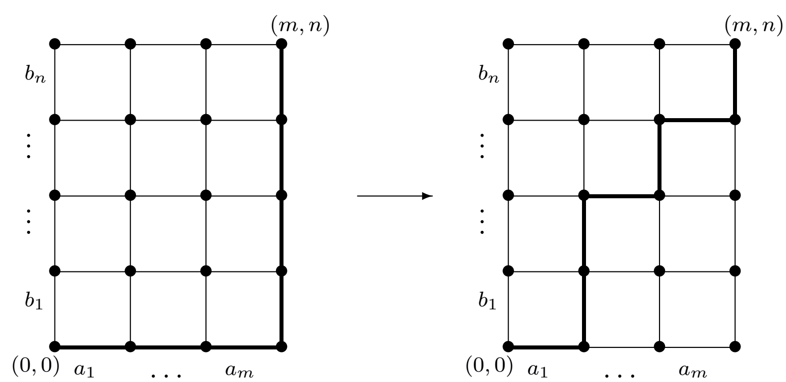
or they are in different pairs of parenthesis and (A.2.5) first groups them together by moving the appropriate pair of parenthesis to the right, then permutes the letters, and moves the pair of parenthesis moved to the right to the left, that is

$$\begin{aligned}
&((\ldots((((\ldots(Fc_1Fc_2)\ldots)Fc_{k-1})Fc_{k+1})Fc_k)\ldots)Fc_{m+n})\to\\
&((\ldots(((\ldots(Fc_1Fc_2)\ldots)Fc_{k-1})(Fc_{k+1}Fc_k))\ldots)Fc_{m+n})\to\\
&((\ldots(((\ldots(Fc_1Fc_2)\ldots)Fc_{k-1})(Fc_kFc_{k+1}))\ldots)Fc_{m+n})\to\\
&((\ldots((((\ldots(Fc_1Fc_2)\ldots)Fc_{k-1})Fc_k)Fc_{k+1})\ldots)Fc_{m+n})
\end{aligned} \tag{A.2.7}$$

where c_k is a letter of w_2 in w_1+w_2 with $1<k<m+n$ and c_{k-1} is a letter of w_1 such that $c_k \prec c_{k-1}$.

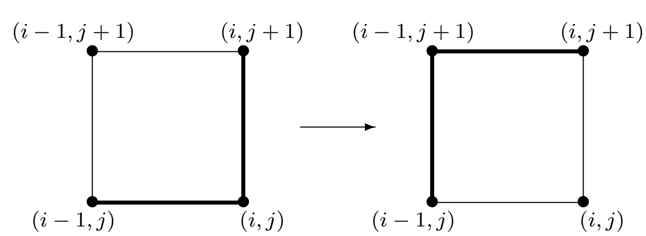
We repeat the above process to every letter of w_2 in w_1+w_2. We define the morphism (A.2.5) as composition of the morphisms of the form (A.2.6) or (A.2.7).

We can illustrate the map (A.2.5) by the lattice paths [12, Chapter 7.3D]. It is clear that there is a 1-1 correspondence between the lattice paths from $(0,0)$ to (m,n) and the (m,n)-shuffles. (A.2.2) can be seen as the lattice path corresponding to the (m,n)-shuffle of the words w_1, w_2 that defines w_1+w_2 and (A.2.1) as the lattice path corresponding to the concatenation of the words w_1 and w_2 (i.e. the empty (m,n)-shuffle). We denote these paths by $L_{w_1+w_2}$ and L_{w_1,w_2}, respectively. From this perspective, the map (A.2.5) can be thought as applying an (m,n)-shuffle to the concatenation of the words w_1 and w_2.

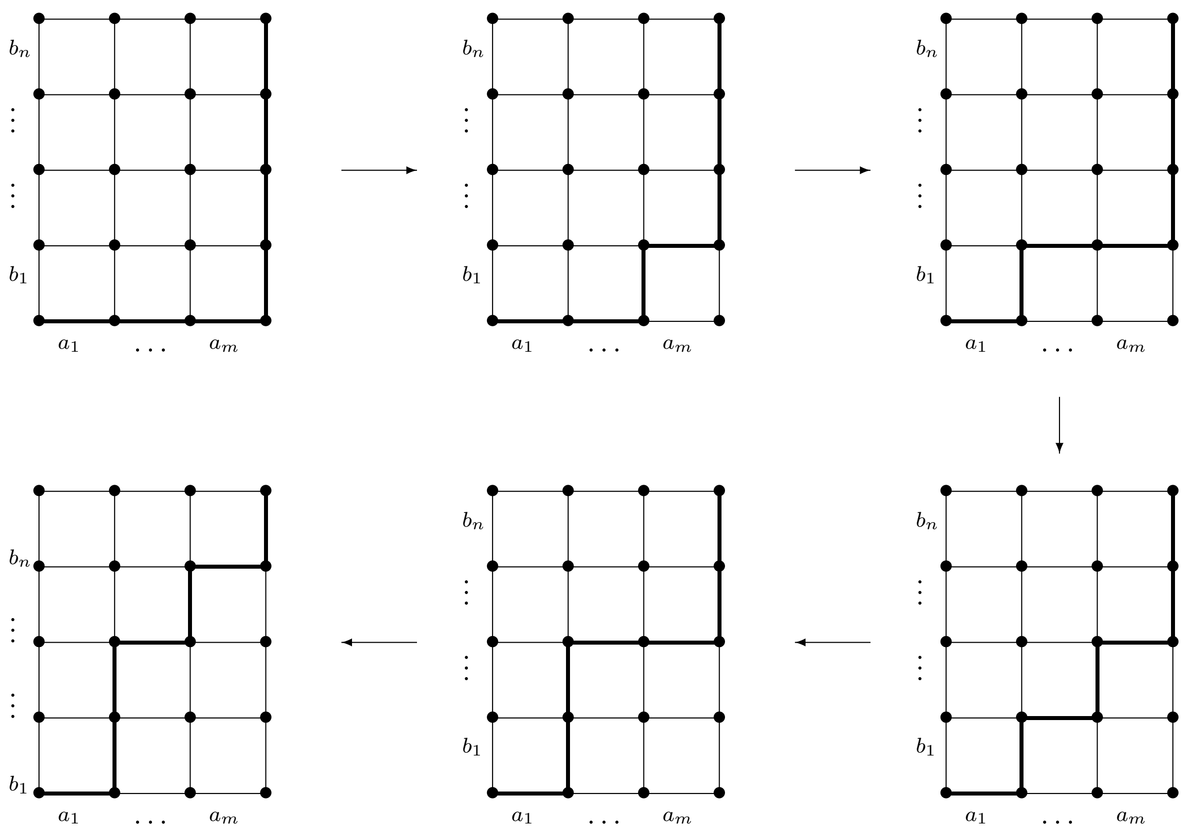


Lattice Path L_{w_1,w_2} Lattice Path $L_{w_1+w_2}$

The morphisms (A.2.6) and (A.2.7) describe the basic movement. They substitute the point (i,j) on the lattice path with the point $(i-1,j+1)$ as shown in the picture below.



The overall movement is described by the morphism (A.2.5) where each step is a basic movement. We define the following special point on the lattice path in order to explain the mechanism of the movements. We call the point (i, j) on the lattice path the *corner point* if the points $(i-1, j)$ and $(i, j+1)$ are on the lattice path, as well. The morphism (A.2.5) picks at every step the corner point (i, j) with the least y-coordinate that is not on the lattice path $L_{w_1+w_2}$ and substitutes it with $(i-1, j+1)$. We show in the picture below the transformation of the lattice path L_{w_1,w_2} to the lattice path $L_{w_1+w_2}$.



The morphism (A.2.3) obtained in the first step followed by the morphism (A.2.5) constructed in the second step defines λ_{w_1,w_2}.

We remark that if all the letters of w_1 are less than all the letters of w_2, then $w_1 + w_2$ is obtained by concatenating the words w_1 and w_2 without the shuffle. That is $L_{w_1+w_2}$ coincides with L_{w_1,w_2}. In this case λ_{w_1,w_2} is of the form (A.2.3).

We also observe that the morphism λ_{w_1,w_2} is a path in the 1-skeleton of permuto-associahedron $K\Pi_{m+n-1}$ where m and n are lengths of the words w_1 and w_2, respectively. $K\Pi_{m+n-1}$ is a polytope whose vertices are all possible orderings and groupings of strings of length $m + n$ and whose edges are all possible adjacent permutations and all possible parenthesis movements. For more details about permuto-associahedron, we refer to [20] and

[34].

Definition of ψ_{w_1,w_2,w_3}. For any three words w_1, w_2, w_3 in $\mathbb{N}(E)$, we define the 2-morphism ψ_{w_1,w_2,w_3}

$$
\begin{array}{ccccc}
((F(w_1)F(w_2))F(w_3)) & \xrightarrow{\lambda_{w_1,w_2}} & F(w_1+w_2)F(w_3) & \xrightarrow{\lambda_{w_1+w_2,w_3}} & F(w_1+w_2+w_3) \\
\Big\downarrow{\scriptstyle \mathsf{a}} & & \Swarrow \psi_{w_1,w_2,w_3} & & \Big\| \\
(F(w_1)(F(w_2)F(w_3))) & \xrightarrow[\lambda_{w_2,w_3}]{} & F(w_1)F(w_2+w_3) & \xrightarrow[\lambda_{w_1,w_2+w_3}]{} & F(w_1+w_2+w_3)
\end{array}
\tag{A.2.8}
$$

between the 1-morphisms $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a}$ and $\lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$ from $((F(w_1)F(w_2))F(w_3))$ to $F(w_1+w_2+w_3)$[1]. These 1-morphisms are paths in the 1-skeleton of $K\Pi_{m+n+p-1}$ where n,m, and p are the lengths of the words w_1, w_2, and w_3, respectively. This follows from the fact that every map in the diagram (A.2.8) is in the 1-skeleton of $K\Pi_{m+n+p-1}$.

In order to better understand these paths, we interpret them in terms of 3-dimensional lattice paths. Assume that the letters of the words w_1, w_2, and w_3 represent respectively the unit intervals on the x, y, and z-axis. $F(w_1 + w_2 + w_3)$ can be represented by the 3-dimensional lattice path corresponding to the (m, n, p)-shuffle of the words w_1, w_2, w_3 that defines $w_1 + w_2 + w_3$ and $((F(w_1)(F(w_2))F(w_3))$ by the 3-dimensional lattice path corresponding to the empty shuffle of the words w_1, w_2, w_3. Therefore, the paths $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a}$ and $\lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$ can be thought as two different ways of shuffling w_1, w_2, w_3 to obtain $w_1+w_2+w_3$. The path $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a}$ first does the (n, p)-shuffle then the (m, n)-shuffle. On the other hand the path $\lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$ does the (m, n)-shuffle first, then the (n, p)-shuffle. In this sense the 2-morphism ψ_{w_1,w_2,w_3} can be seen as the connection between the two different ways of doing the (m, n, p)-shuffle.

To define the 2-morphism ψ_{w_1,w_2,w_3}, we need the following Lemmas .

Lemma A.2.2. *Let w_1 and w_2 be two elements of $\mathbb{N}(E)$. $\lambda_{w_2,w_3} = \mathsf{c}$ and $\lambda_{w_1,w_2} = \mathrm{id}$ if and only if $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a} = \lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$*

Proof. We first remark that $\lambda_{w_2,w_3} = \mathsf{c}$ and $\lambda_{w_1,w_2} = \mathrm{id}$ is equivalent to assuming w_2 and w_3 are letters such that w_2 is greater than w_3 and w_2 is greater than or equal to all letters of w_1. These facts imply that the map $\lambda_{w_1+w_2,w_3}$ first permutes $F(w_2)$ and $F(w_3)$ then shuffles $F(w_1)$ and $F(w_3)$ without changing the position of $F(w_2)$. Thus $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a} = \lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$.

In the other direction, we observe that the morphism a can be only part of the morphism λ_{w_1,w_2+w_3} which means $\lambda_{w_1,w_2} = \mathrm{id}$. This requires w_2 to be a letter greater than or equal to all letters of w_1 and $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a} = \lambda_{w_1+w_2,w_3}$. We also observe that a parenthesis movement caused by λ_{w_2,w_3} effects only the places of the parenthesis around the letters of w_2 and w_3 and such a movement can not be caused by $\lambda_{w_1+w_2,w_3}$. This means λ_{w_2,w_3} does not cause any parenthesis movements. Hence, we deduce that w_3 is also a letter. If $w_2 \preceq w_3$ then λ_{w_2,w_3} and $\lambda_{w_1+w_2,w_3}$ become identity morphisms and we obtain $\lambda_{w_1,w_2+w_3} \circ \mathsf{a} = \mathrm{id}$ which is not possible. Therefore λ_{w_2,w_3} should consist of a single permutation. $\square$

[1] We commit an abuse of notation in diagram (A.2.8). By λ_{w_1,w_2} and λ_{w_2,w_3} we mean $\lambda_{w_1,w_2} \otimes \mathrm{id}_{w_3}$ and $\mathrm{id}_{w_1} \otimes \lambda_{w_2,w_3}$, respectively.

Lemma A.2.3. *Let w_1, w_2, and w_3 be three elements of $\mathbb{N}(E)$. Then the followings are equivalent.*

1. *The path $\lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$ is strictly included in $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a}$. That is $V_{(w_1,w_2|w_3)}$ the vertex set of the path $\lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$ is strictly included in $V_{(w_1|w_2,w_3)}$ the vertex set of the path $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a}$.*

2. $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} = \lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2} \circ \mathsf{a}^{-1}$.

3. $\lambda_{w_2,w_3} = \mathrm{id}$.

Proof. It is clear that (2) implies (1).

(3) $\Rightarrow$ (2): $\lambda_{w_2,w_3} = \mathrm{id}$ is equivalent to assuming that both w_2 and w_3 are letters and $w_2 \prec w_3$. This requires $F(w_1)F(w_2 + w_3)$ to be of the form $F(w_1)(F(w_2)F(w_3))$. Since all the morphisms λ's start with moving parenthesis to the left, λ_{w_1,w_2+w_3} starts exactly with a^{-1}. Therefore $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} = \lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2} \circ \mathsf{a}^{-1}$.

(1) $\Rightarrow$ (3): In all the vertices that λ_{w_2,w_3} pass through, $F(w_1)$ is grouped separately from $F(w_2)$ and $F(w_3)$. Therefore any parenthesis movement or permutation that is part of λ_{w_2,w_3} does not change the parenthesis around $F(w_1)$. However, on the path $\lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$ the same movements that describe λ_{w_2,w_3} are part of the morphism $\lambda_{w_1+w_2,w_3}$. Since this path passes through the vertices that group $F(w_1)$ and $F(w_2)$, the parenthesis movements and permutations change the parenthesis around $F(w_1)$. This contradicts to the fact that $\lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$ is included in $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a}$. ☐

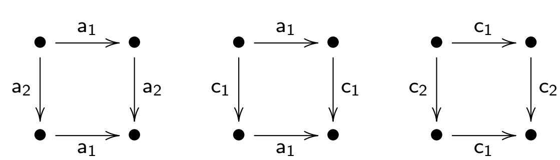
We remark that the Lemma (A.2.3) can be also expressed as $\lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$ is strictly included in $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a}$ if and only if $V_{(w_1|w_2,w_3)} = V_{(w_1,w_2|w_3)} \cup \{(F(w_1)(F(w_2)F(w_3)))\}$.

We can return to the definition of the 2-morphism ψ_{w_1,w_2,w_3}. By the Lemmas (A.2.2) and (A.2.3), the paths $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a}$ and $\lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$ are going to satisfy one of the following three cases.

1. The paths may be the same. In this case, the 2-morphism ψ_{w_1,w_2,w_3} is identity.

2. The path $\lambda_{w_1+w_2,w_3} \circ \lambda_{w_1,w_2}$ is strictly included in $\lambda_{w_1,w_2+w_3} \circ \lambda_{w_2,w_3} \circ \mathsf{a}$. In this case, by Lemma (A.2.3), the 2-morphism ψ_{w_1,w_2,w_3} is $\mathsf{a}\mathsf{a}^{-1} \Rightarrow \mathrm{id}$.

3. The paths may enclose a 2-cell. This 2-cell is a tiling of pentagonal and rectangular 2-cells. The pentagonal 2-cells are either MacLane Pentagones or their derivatives obtained by inverting the direction of an edge. The rectangular 2-cells are of the form



(A.2.9)

where a_1, a_2 are either leftward or rightward parenthesis movements and c_1, c_2 permute adjacent objects. Rectangular 2-cells can be also derived from (A.2.9) by inverting the direction of an edge. These 2-cells commute up to structural 2-morphisms defined by the Picard structure of the 2-category $\mathbb{C}$. The Theorem 3.3 in [29] implies that these 2-morphisms compose in a unique way. We let ψ_{w_1,w_2,w_3} be this composition.

Definition of ϕ_{w_1,w_2}. The last piece of the additive structure of F is the 2-morphism ϕ_{w_1,w_2}

$$\begin{array}{ccc} F(w_1)F(w_2) & \xrightarrow{\lambda_{w_1,w_2}} & F(w_1+w_2) \\ {\scriptstyle \mathsf{c}}\downarrow & \Downarrow\phi_{w_1,w_2} & \| \\ F(w_2)F(w_1) & \xrightarrow[\lambda_{w_2,w_1}]{} & F(w_2+w_1) \end{array} \tag{A.2.10}$$

between the 1-morphisms $\lambda_{w_2,w_1} \circ \mathsf{c}$ and λ_{w_1,w_2} from $F(w_1)F(w_2)$ to $F(w_1+w_2)$ where w_1 and w_2 are any two words in $\mathbb{N}(E)$. We notice that the path $\lambda_{w_2,w_1} \circ \mathsf{c}$ is not necessarily in the 1-skeleton of $K\Pi_{m+n-1}$. The reason is that the braiding c is not an adjacent permutation unless w_1 and w_2 are letters.

In the case where the words w_1 and w_2 are letters, ϕ_{w_1,w_2} is defined by the table

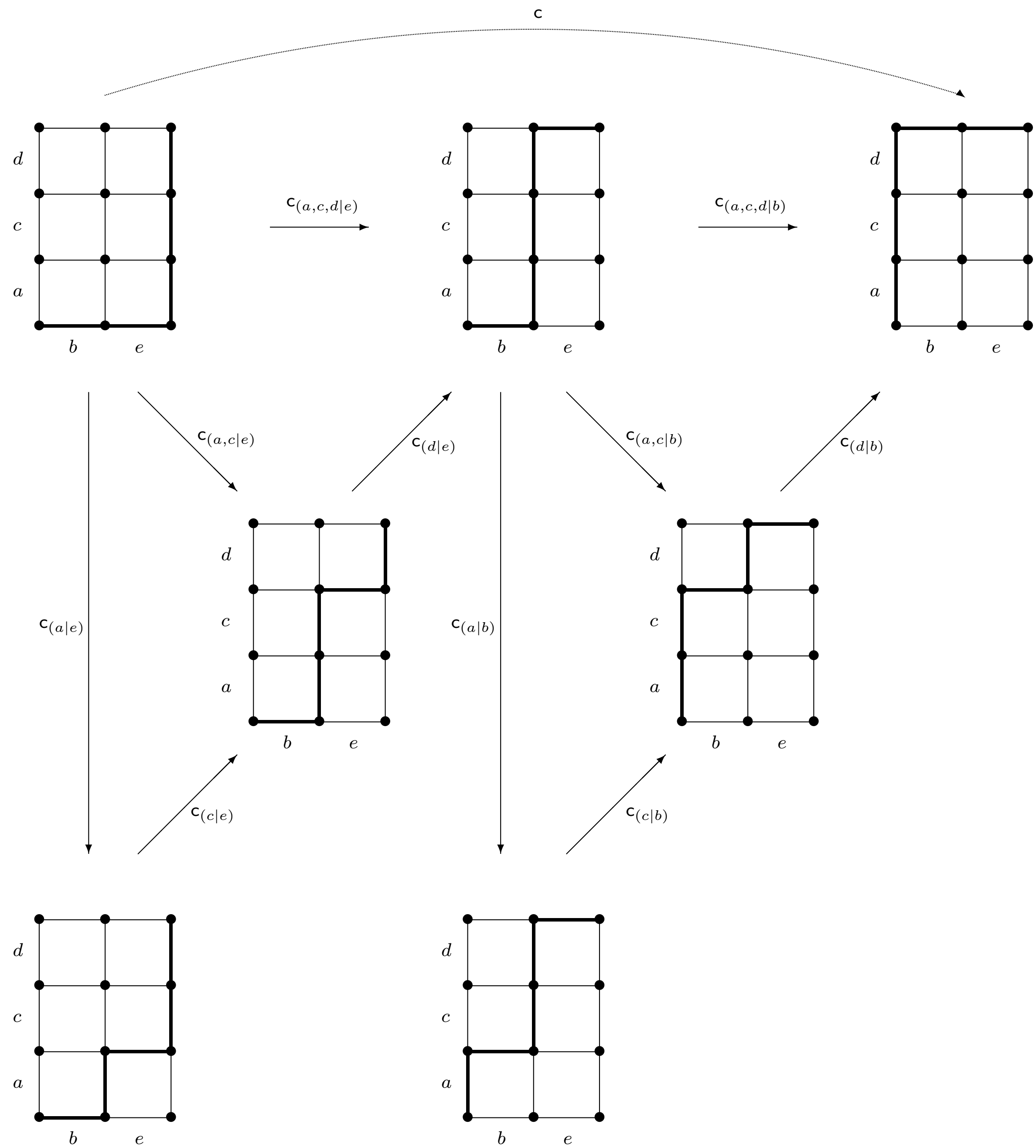
$\mathbf{w_1}$	$\mathbf{w_2}$	ϕ_{w_1,w_2}
a	a	id
a	b	id$\Rightarrow\mathsf{c}^2$
b	a	id

where id$\Rightarrow\mathsf{c}^2$ is given by the Picard structure of the 2-category $\mathbb{C}$.

Now, we assume that w_1 and w_2 are two words such that their sum of lengths is $m+n \geq 3$. The 2-morphism ϕ_{w_1,w_2} is defined in the following way. We first transform the path $\lambda_{w_2,w_1} \circ \mathsf{c}$ to a path in the 1-skeleton of $K\Pi_{m+n-1}$. Second we apply the process that defines ψ_{w_1,w_2,w_3} to the new path and the path λ_{w_1,w_2}. ϕ_{w_1,w_2} is then defined as the appropriate composition of the 2-morphisms obtained at the first and the second step. Therefore to define ϕ_{w_1,w_2}, it suffices to describe how we transform the path $\lambda_{w_2,w_1} \circ \mathsf{c}$ into a path in the 1-skeleton of $K\Pi_{m+n-1}$.

The main idea is to substitute the edge c that is not in the 1-skeleton by a sequence of five other edges. This sequence is an alternating collection of three leftward or rightward parenthesis movements and two braidings. The parenthesis movements are certainly in the 1-skeleton; however the braidings may not be. If they are not, then we substitute each of those braidings by a sequence of five other edges as above. We keep substituting until all the braidings become permutations of adjoint objects, therefore part of the 1-skeleton. We know that the substitution process is going to terminate because after each substitution braidings permute parenthesized objects with shorter length.

We describe this process on the sample $w_1 = b + e$ and $w_2 = a + c + d$. The braiding c permutes $F(w_1)$ and $F(w_2)$. First, we substitute c by the braidings $\mathsf{c}_{(a,c,d|e)}$ and $\mathsf{c}_{(a,c,d|b)}$. $\mathsf{c}_{(a,c,d|e)}$ permutes the parenthesized object $((FaFc)Fd)$ with Fe and $\mathsf{c}_{(a,c,d|b)}$ permutes $((FaFc)Fd)$ with Fb. They are going to be substituted by $\mathsf{c}_{(d|e)}$ and $\mathsf{c}_{(a,c|e)}$ and by $\mathsf{c}_{(a,c|b)}$ and $\mathsf{c}_{(d|b)}$, respectively. Since $\mathsf{c}_{(d|e)}$ permutes Fd and Fe and $\mathsf{c}_{(d|b)}$ permutes Fd and Fb, they are edges in the 1-skeleton and therefore can not be substituted. In the diagram below, we illustrate the complete process of substituting c by adjacent permutations $\mathsf{c}_{(a|b)}$, $\mathsf{c}_{(c|b)}$, $\mathsf{c}_{(d|b)}$, $\mathsf{c}_{(a|e)}$, $\mathsf{c}_{(c|e)}$, and $\mathsf{c}_{(d|e)}$ using lattice paths.



This process defines a 2-morphism as follows. Substituting a braiding by an alternating sequence of three leftward or rightward parenthesis movements and two braidings means substituting an edge in a hexagonal 2-cell by the other five edges. Such hexagonal 2-cells commute up to a 2-morphism given by the Picard structure of the 2-category $\mathbb{C}$. The appropriate composition of these 2-morphisms defines the 2-morphism of the first step.

A.2.3 Extending the Additive Structure to Free Abelian Gorup

Here we extend the additive structure of the 2-functor F to the free abelian group $\mathbb{Z}(E)$ generated by the set E.

Extending λ_{w_1,w_2}. The extension of λ_{w_1,w_2}, denoted by $\widetilde{\lambda}_{w_1,w_2}$, to the words in $\mathbb{Z}(E)$ should take into consideration the cancelations that might occur in $w_1 + w_2$. If w_2 does not have a letter that appears with an opposite sign in w_1 then there aren't any cancelations in $w_1 + w_2$ and $\widetilde{\lambda}_{w_1,w_2} = \lambda_{w_1,w_2}$. Otherwise, $\widetilde{\lambda}_{w_1,w_2}$ orders the letters of w_1 and w_2 from least to greatest, left parenthesizes, and does the cancelations starting with the image of the least letter. That is $\widetilde{\lambda}_{w_1,w_2}$ is equal to post composition of λ_{w_1,w_2} with the morphisms of the form

$$\begin{aligned} &(\dots(((F(w)Fc_i)(Fc_i)^*)Fc_{i+1})\dots Fc_{n+m}) \longrightarrow (\dots((F(w)(Fc_i(Fc_i)^*))Fc_{i+1})\dots Fc_{n+m}) \\ &\xrightarrow{\ \mathrm{inv}_{Fc_i}\ } (\dots((F(w)I)Fc_{i+1})\dots Fc_{n+m}) \xrightarrow[\ \mathrm{r}_{F(w)}\ ]{} (\dots(F(w)Fc_{i+1})\dots Fc_{n+m}) \end{aligned} \tag{A.2.11}$$

for every cancelation. In (A.2.11) w is a subword of w_1+w_2, I is a unit element in the Picard 2-category and inv_{Fc_i} and $\mathrm{r}_{F(w)}$ are structural morphisms due to the Picard structure of the 2-category. By the Picard structure, we can also assume for simplicity that when $\widetilde{\lambda}_{w_1,w_2}$ orders letters from least to greatest the inverse of an object is always adjacent to the object and it is on its left. We note that using λ_{w_1,w_2} for the morphism that orders the letters of w_1 and w_2 from least to greatest and left parenthesizes them is an abuse of notation. Here λ_{w_1,w_2} does not map to the object $F(w_1 + w_2)$ but to an object that we denote $F(w_{1,2})$. $F(w_{1,2})$ is product of the images of all letters in w_1 and w_2 parenthesized from the left, ordered from least to greatest, and if there exists inverse of an object is placed on its left. For instance, if $w_1 = b + c$ and $w_2 = a - b$, then

$$\lambda_{w_1,w_2} : (FbFc)(Fa(Fb)^*) \longrightarrow (((FaFb)Fb)^*)Fc),$$

where $F(w_{1,2}) = ((FaFb)(Fb)^*)Fc)$. Thus $\widetilde{\lambda}_{w_1,w_2}$ can be expressed as composition of

$$F(w_1)F(w_2) \xrightarrow{\lambda_{w_1,w_2}} F(w_{1,2}) \xrightarrow{\tau_{w_1,w_2}} F(w_1 + w_2), \tag{A.2.12}$$

where τ_{w_1,w_2} is composition of morphisms of the form (A.2.11) for every cancelation. We remark that λ_{w_1,w_2} as in the monoidal case defines a path in the 1-skeleton of the permuto-associahedron $K\Pi_{m+n-1}$. However if there are cancelations, $\widetilde{\lambda}_{w_1,w_2}$ is not a path in the 1-skeleton of $K\Pi_{m+n-1}$.

Extending ψ_{w_1,w_2,w_3}. The extension of ψ_{w_1,w_2,w_3}, denoted by $\widetilde{\psi}_{w_1,w_2,w_3}$, to the words w_1, w_2, w_3 in $\mathbb{Z}(E)$ is a 2-morphism

$$
\begin{array}{ccccc}
((F(w_1)F(w_2))F(w_3)) & \xrightarrow{\widetilde{\lambda}_{w_1,w_2}} & F(w_1+w_2)F(w_3) & \xrightarrow{\widetilde{\lambda}_{w_1+w_2,w_3}} & F(w_1+w_2+w_3) \\
\Big\downarrow{\scriptstyle \mathsf{a}} & & \Downarrow \widetilde{\psi}_{w_1,w_2,w_3} & & \Big\| \\
(F(w_1)(F(w_2)F(w_3))) & \xrightarrow[\widetilde{\lambda}_{w_2,w_3}]{} & F(w_1)F(w_2+w_3) & \xrightarrow[\widetilde{\lambda}_{w_1,w_2+w_3}]{} & F(w_1+w_2+w_3)
\end{array}
\tag{A.2.13}
$$

between the 1-morphisms $\widetilde{\lambda}_{w_1,w_2+w_3} \circ \widetilde{\lambda}_{w_2,w_3} \circ \mathsf{a}$ and $\widetilde{\lambda}_{w_1+w_2,w_3} \circ \widetilde{\lambda}_{w_1,w_2}$. As noticed, these paths may not be in the 1-skeleton of $K\Pi_{m+n+p-1}$. However, there exists a vertex V_0 of the permuto-associahedron $K\Pi_{m+n+p-1}$ that both paths $\widetilde{\lambda}_{w_1+w_2,w_3} \circ \widetilde{\lambda}_{w_1,w_2}$ and $\widetilde{\lambda}_{w_1,w_2+w_3} \circ \widetilde{\lambda}_{w_2,w_3}$ pass through it. Therefore the diagram (A.2.13) can be rewritten as:

$$
\begin{array}{ccccccc}
((F(w_1)F(w_2))F(w_3)) & \longrightarrow & V_0 & \longrightarrow & F(w_1+w_2)F(w_3) & \xrightarrow{\widetilde{\lambda}_{w_1+w_2,w_3}} & F(w_1+w_2+w_3) \\
\Big\downarrow{\scriptstyle \mathsf{a}} & \Downarrow \psi'_{w_1,w_2,w_3} & \Big\| & & & \Downarrow \rho_{w_1,w_2,w_3} & \Big\| \\
(F(w_1)(F(w_2)F(w_3))) & \longrightarrow & V_0 & \longrightarrow & F(w_1)F(w_2+w_3) & \xrightarrow[\widetilde{\lambda}_{w_1,w_2+w_3}]{} & F(w_1+w_2+w_3)
\end{array}
\tag{A.2.14}
$$

where both vertical morphisms to V_0 are paths on $K\Pi_{m+n+p-1}$. So we compute ψ'_{w_1,w_2,w_3} in the same way as ψ of the monoidal case. After the vertex V_0, the morphisms on the diagram (A.2.14) are not any more in the 1-skeleton of $K\Pi_{m+n+p-1}$ because of the cancelations. The region between the two paths from V_0 to $F(w_1+w_2+w_3)$ can be filled with the structural 2-morphisms of the Picard structure in particular involving the ones related with inverse and unit objects. The 2-morphism ρ_{w_1,w_2,w_3} is then the unique pasting of those structural 2-morphisms. Hence, we define ψ_{w_1,w_2,w_3} as pasting of ψ'_{w_1,w_2,w_3} and ρ_{w_1,w_2,w_3}.

Extending ϕ_{w_1,w_2}. The extension of ϕ_{w_1,w_2}, denoted by $\widetilde{\phi}_{w_1,w_2}$ is a 2-morphism

$$
\begin{array}{ccc}
F(w_1)F(w_2) & \xrightarrow{\widetilde{\lambda}_{w_1,w_2}} & F(w_1+w_2) \\
\Big\downarrow{\scriptstyle \mathsf{c}} & \Downarrow \widetilde{\phi}_{w_1,w_2} & \Big\| \\
F(w_2)F(w_1) & \xrightarrow[\widetilde{\lambda}_{w_2,w_1}]{} & F(w_2+w_1)
\end{array}
\tag{A.2.15}
$$

between the 1-morphisms $\widetilde{\lambda}_{w_2,w_1} \circ \mathsf{c}$ and $\widetilde{\lambda}_{w_1,w_2}$ from $F(w_1)F(w_2)$ to $F(w_1+w_2)$ where w_1 and w_2 are any two words in $\mathbb{Z}(E)$. We rewrite the diagram (A.2.15) by expressing $\widetilde{\lambda}_{w_1,w_2}$ and $\widetilde{\lambda}_{w_2,w_1}$ as compositions using (A.2.12).

$$
\begin{array}{ccccc}
F(w_1)F(w_2) & \xrightarrow{\lambda_{w_1,w_2}} & F(w_{1,2}) & \xrightarrow{\tau_{w_1,w_2}} & F(w_1+w_2) \\
\Big\downarrow{\scriptstyle \mathsf{c}} & \Downarrow \phi'_{w_1,w_2} & \Big\| & & \Big\| \\
F(w_2)F(w_1) & \xrightarrow[\lambda_{w_2,w_1}]{} & F(w_{2,1}) & \xrightarrow[\tau_{w_2,w_1}]{} & F(w_2+w_1)
\end{array}
\tag{A.2.16}
$$

The square on the left commutes up to the 2-morphism ϕ'_{w_1,w_2} obtained inthe same way as ϕ of the monoidal case. The square on the right commutes since $F(w_{1,2}) = F(w_{2,1})$ and therefore $\tau_{w_1,w_2} = \tau_{w_2,w_1}$. Hence, $\widetilde{\phi}_{w_1,w_2}$ is the whiskering $\phi'_{w_1,w_2} * \tau_{w_1,w_2}$. □

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BIOGRAPHICAL SKETCH

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