The Structure of Properly Convex Manifolds

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(joint with D. Long)

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- What sort of structure do convex projective manifolds have?
  Deformations of finite volume strictly convex manifolds are structurally similar to complete finite volume hyperbolic manifolds

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- A *projective hyperplane* is the projectivization of an n-plane in $\mathbb{R}^{n+1}$.

A Decomposition of $\mathbb{R}P^n$

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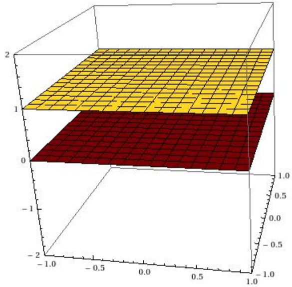

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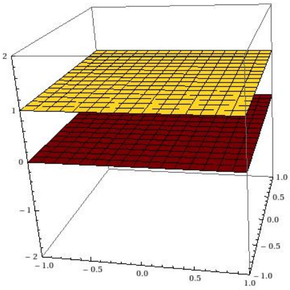
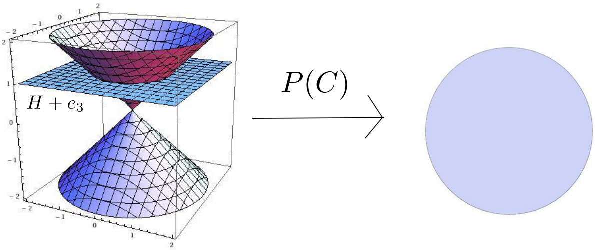


- $\mathbb{R}P^n \backslash P(H)$ is called an *affine patch*.

What is convex projective geometry?

Motivation from hyperbolic geometry

- Let $\langle x, y\rangle = x_1 y_1 + \ldots x_n y_n - x_{n+1} y_{n+1}$ be the standard bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$
- Let $C = \{x \in \mathbb{R}^{n+1} | \langle x, x\rangle < 0\}$



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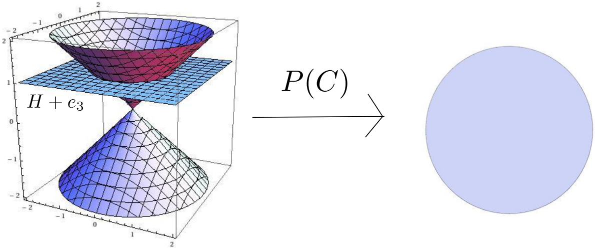
- Let $C = \{x \in \mathbb{R}^{n+1} | \langle x, x\rangle < 0\}$
- $P(C)$ is the *Klein model* of hyperbolic space.
- $P(C)$ has isometry group $\mathrm{PSO}(n, 1) \leq \mathrm{PGL}_{n+1}(\mathbb{R})$



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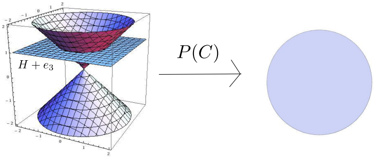

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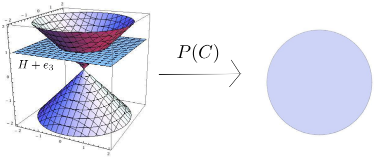

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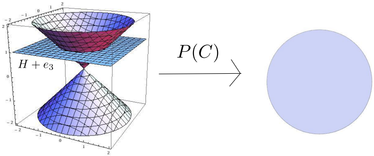

Nice Properties of Hyperbolic Space

- *Convex*: Intersection with projective lines is connected.
- *Properly Convex*: Convex and closure is contained in an affine patch $\iff$ Disjoint from some projective hyperplane.
- *Strictly Convex*: Properly convex and boundary contains no non-trivial projective line segments.

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Motivation from hyperbolic geometry

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Convex Projective Geometry

Ω/Γ

Ω properly (strictly) convex

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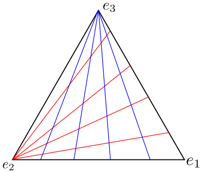
2. Let T be the interior of a triangle in $\mathbb{R}P^2$ and let $\Gamma \leq \text{Diag}^+$ be a suitable lattice inside the group of 3×3 diagonal matrices with determinant 1 and distinct positive eigenvalues. T/Γ is a properly convex torus.



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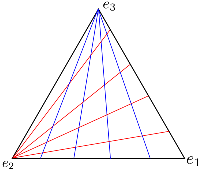

These are extreme examples of properly convex manifolds. Generic examples interpolate between these extreme cases.

Hilbert Metric

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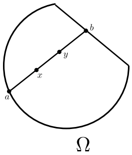

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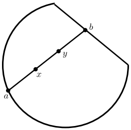

Ω

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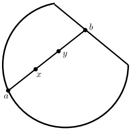

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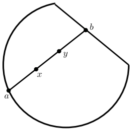

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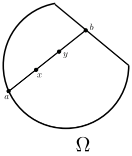

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- Discrete subgroups of $\mathrm{PGL}(\Omega)$ act properly discontinuously on Ω.

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If $\gamma \in \mathrm{PGL}(\Omega)$ then γ is

1. *elliptic* if γ fixes a point in Ω (zero translation length + realized),
2. *parabolic* if γ acts freely on Ω and has all eigenvalues of modulus 1 (zero translation length + not realized), and
3. *hyperbolic* otherwise (positive translation length)

Similarities to Hyperbolic Isometries

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4. When Ω is strictly convex, parabolic and hyperbolic elements in a common discrete subgroup do not share fixed points.
5. When Ω is strictly convex, a discrete torsion-free subgroup of elements fixing a geodesic is infinite cyclic.

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The General Case

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- Parabolic elements have a connected fixed set in $\partial\Omega$.
- Hyperbolic elements have an attracting and repelling subspaces A_+ and A_- in $\partial\Omega$. The action on these sets is orthogonal and their dimension is determined by the number of “powerful” Jordan blocks of γ

Margulis Lemma

Let $\Omega \subset \mathbb{R}P^n$ is an open properly convex domain and let $\Gamma \leq \mathrm{PGL}(\Omega)$ be a discrete group. Then there exists a number μ_n (depending only on n) such that if $x \in \Omega$ then the group

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Result due to Gromov-Margulis-Thurston for $\mathbb{H}^n$ and Cooper-Long-Tillmann in general.

Rigidity and Flexibility

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Theorem 1 (Mostow '70, Prasad '73)

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There is no Mostow-Prasad rigidity for properly (strictly) convex domains.
There are examples of finite volume hyperbolic manifolds whose complete hyperbolic structure can be "deformed" to a non-hyperbolic convex projective structure.

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Ex: Let $\Omega_0 \cong \mathbb{H}^n$, $\Gamma_0 \leq \mathrm{PSO}(n, 1)$, such that Ω_0/Γ_0 is finite volume and contains an embedded totally geodesic hypersurface Σ. Let Γ_1 be obtained by "bending" along Σ.

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The Closed Case

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- By compactness, we see that if $1 \neq \gamma \in \Gamma$ then γ is hyperbolic
- In particular, if $1 \neq \gamma \in \Gamma$ then γ is *positive proximal* (eigenvalues of minimum and maximum modulus are unique, real, and positive)

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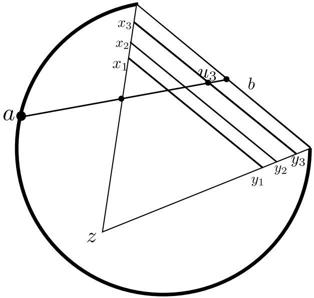
If Ω is not strictly convex then it will contain arbitrarily fat triangles and is thus not δ-hyperbolic.



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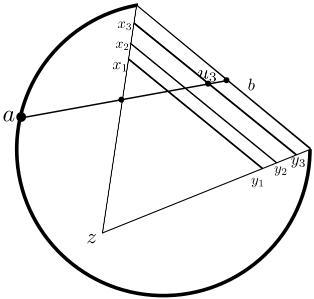
If Ω is not strictly convex then it will contain arbitrarily fat triangles and is thus not δ-hyperbolic. Since Γ acts cocompactly by isometries on Ω, Švarc-Milnor tells us that Ω is q.i. to Γ and is thus δ-hyperbolic. □



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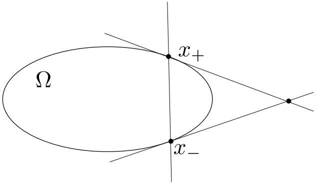
- Since Ω is strictly convex and γ is hyperbolic we see that γ has exactly 2 fixed points in $\partial\Omega$ and acts as translation along the geodesic connecting them. γ is thus positive proximal.



Structure of Hyperbolic Manifolds

Finite Volume Case

Let $M = \mathbb{H}^n/\Gamma$ be a finite volume hyperbolic manifold. We can decompose M as

$$M = M_K \bigsqcup_i C_i,$$

where M_K is a compact and $\pi_1(M_K) = \Gamma$ and C_i are components of the thin part called *cusps*.

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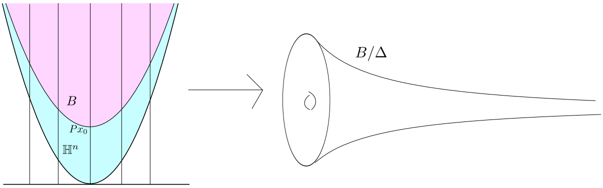
As we will see, the Margulis lemma tells us that the C_i have relatively simple geometry.

Geometry of the Cusps

Let C be a cusp of a finite volume hyperbolic manifold and let

$$P = \left\{ \begin{pmatrix} 1 & v^T & |v|^2 \\ 0 & I_{n-1} & v \\ 0 & 0 & 1 \end{pmatrix} \middle| v \in \mathbb{R}^{n-1} \right\}$$

be the group of parabolic translations fixing ∞. Let $x_0 \in \mathbb{H}^n$, then $C \cong B/\Delta$ where B is horoball bounded by Px_0 and Δ is a finite extension of a lattice in P.



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- Instead Γ is δ-hyperbolic *relative to the cusps*
- If $1 \neq \gamma \in \Gamma$ is freely homotopic into a cusp then γ is parabolic, otherwise γ is hyperbolic (positive proximal)

Let Ω/Γ be a finite volume (Hausdorff measure of Hilbert metric) strictly convex manifold.

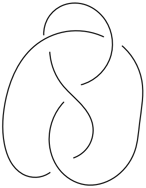
Theorem 4 (Cooper, Long, Tillmann '11)

Let $M = \Omega/\Gamma$ be as above then

- $M = M_K \bigsqcup_i C_i$, *where M_K is compact and C_i is projectively equivalent to the cusp of a finite volume hyperbolic manifold,*
- Γ *is δ-hyperbolic relative to its cusps, and*
- *If $1 \neq \gamma \in \Gamma$ is freely homotopic into a cusp then γ is parabolic. Otherwise γ is hyperbolic (positive proximal).*

Figure-8 Example

Consider the following example.



Let K be the figure-8 knot, let $M = S^3 \backslash K$, and let $G = \pi_1(M)$

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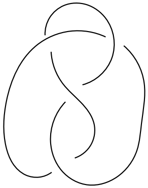


Let K be the figure-8 knot, let $M = S^3 \backslash K$, and let $G = \pi_1(M)$

Theorem 5 (B)

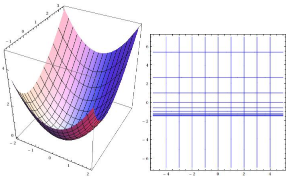
There exists $\varepsilon > 0$ such that for each $t \in (-\varepsilon, \varepsilon)$ there is a properly convex domain Ω_t and a discrete group $\Gamma_t \leq \mathrm{PGL}(\Omega_t)$ such that

- $\Omega_t / \Gamma_t \cong M$,
- Ω_0 / Γ_0 *is the complete hyperbolic structure on M, and*
- *If $t \neq 0$ then Ω_t is not strictly convex.*

Figure-8 Example

Theorem 6 (B)

For each $t \in (-\varepsilon, \varepsilon)$ we can decompose Ω_t/Γ_t as $M_K^t \bigsqcup C^t$, where M_K^t is compact and $C^t \cong T^2 \times [1, \infty)$.



- For each t, $C^t \cong B_t/\Delta_t$, where Δ_t is a lattice an Abelian group P_t of "translations," and B_t is a "horoball" bounded by an orbit of P_t.

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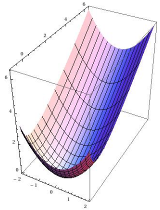

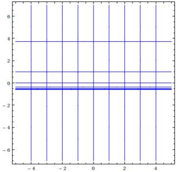

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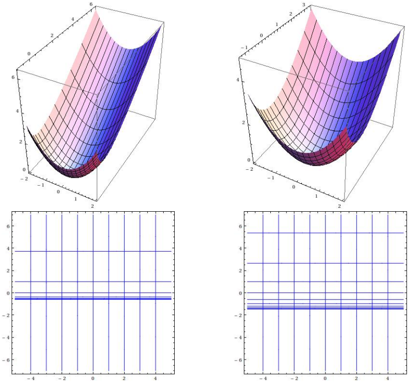

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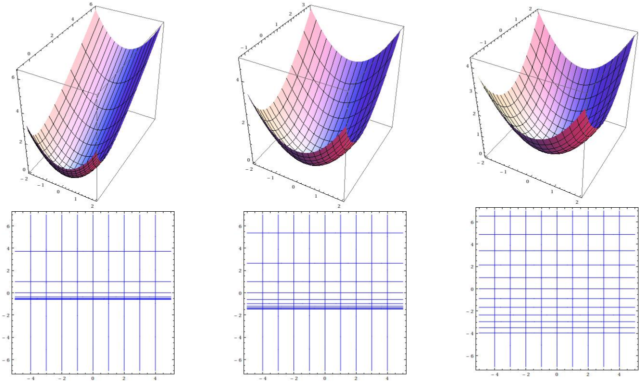

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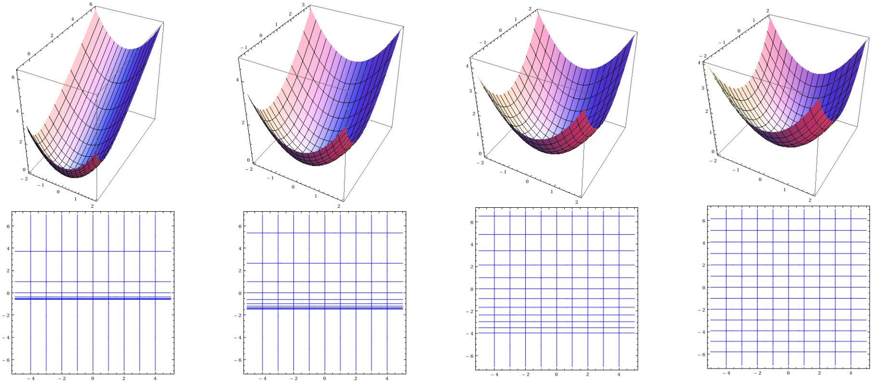

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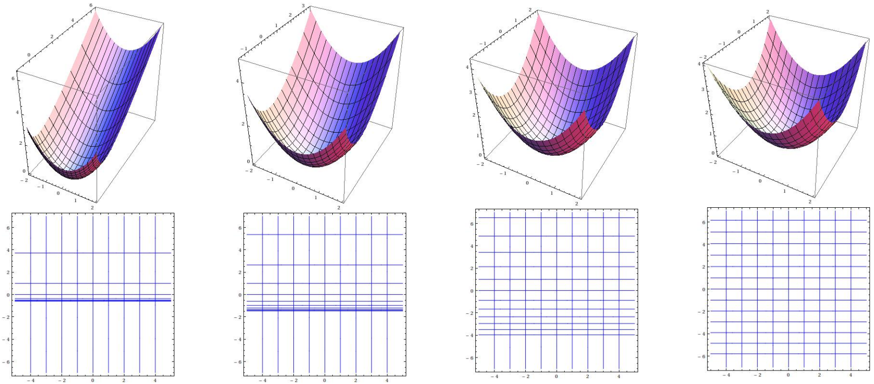

- For each $t \neq 0$ there is $1 \neq \gamma_t \in \Gamma_t$ such that γ_t is hyperbolic, freely homotopic into C^t, but not positive proximal.

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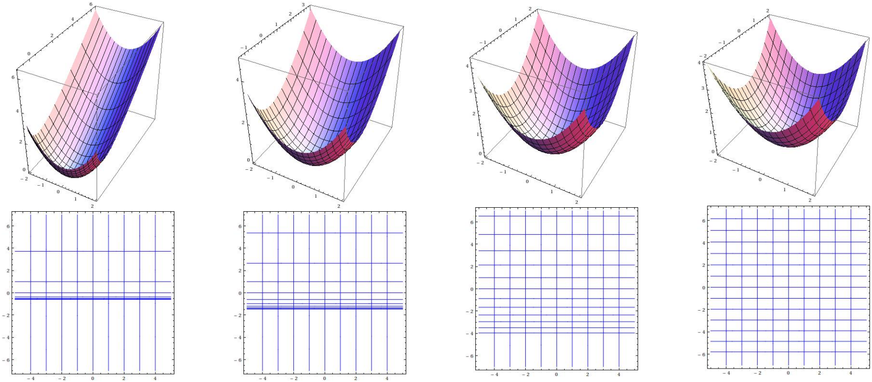

- For each $t \neq 0$ there is $1 \neq \gamma_t \in \Gamma_t$ such that γ_t is hyperbolic, freely homotopic into C^t, but not positive proximal.
- Ω_t contains non-trivial line segments in $\partial\Omega_t$ that are preserved by conjugates of Δ_t. In particular, Ω_t is not δ-hyperbolic.

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$\Leftarrow$ Let $1 \neq \gamma \in \Gamma_t$. No elements of P_t are positive proximal, so if γ is freely homotopic to C^t then it is not positive proximal.

$\Rightarrow$ If γ is not freely homotopic to C^t then γ has positive translation length and is thus hyperbolic. Furthermore, this translation length is realized by points on an axis.

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Proof (Continued).

Use Margulis lemma to construct a disjoint and Γ_t invariant collection $\mathcal{H}_t$ of horoballs in Ω_t.

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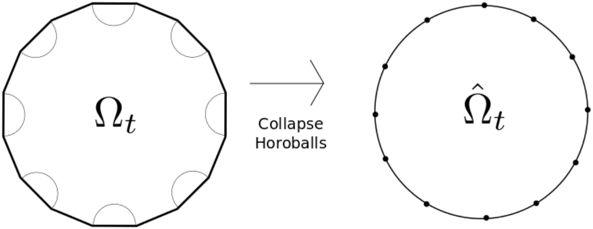

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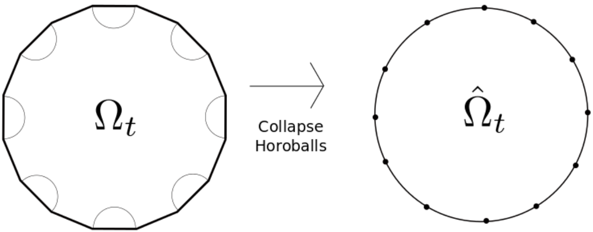

Lemma 8 (B, Long)

$\hat{\Omega}_t$ is δ-hyperbolic

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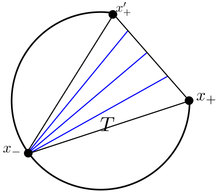

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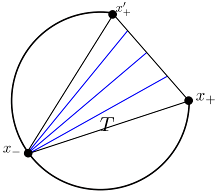

- This gives rise to arbitrarily fat triangles in $\hat{\Omega}_t$

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- What can we say for deformations of deformations of infinite volume hyperbolic manifolds?