Triangulations in Low Dimensional Geometry & Topology

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Florida State University

Cal Poly San Luis Obispo
Colloquium
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Geometric Topology

A biased and oversimplified viewpoint

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For this talk we typically assume $n = 2$ or 3.

From topology to algebra and geometry

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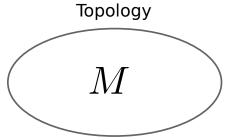

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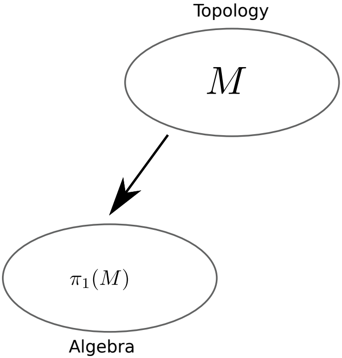

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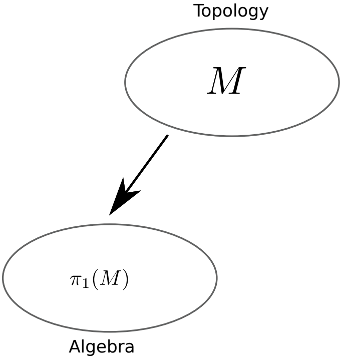

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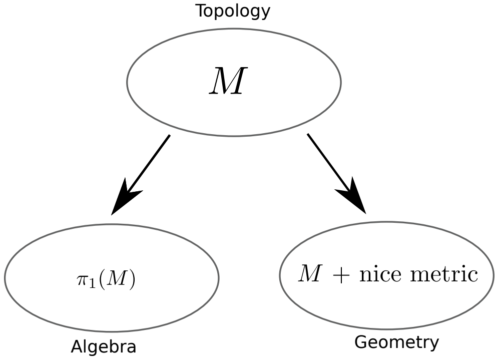

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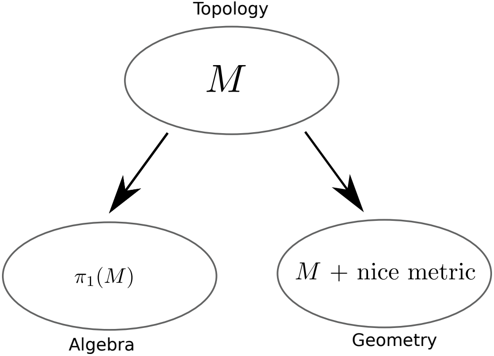

- Forgets structure
- Adds structure

Quantitative questions

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- What is the rank of $H_1(M)$?
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Even better, answering these questions is algorithmic
A computer can do it for you!!

Simplices

An *n-simplex* is given by

$$\Delta^n = \left\{ (c_1, \ldots, c_{n+1}) \in \mathbb{R}^{n+1} \mid c_i \geqslant 0, \quad \sum_i c_i = 1 \right\}$$

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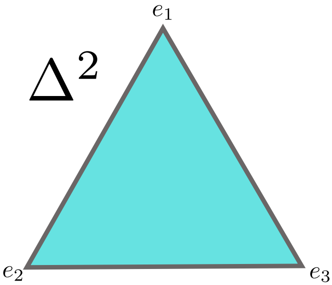

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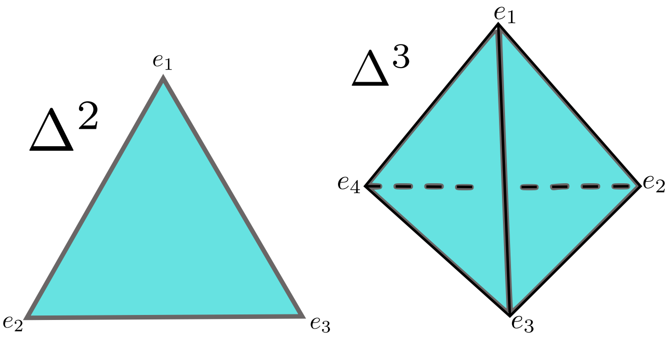
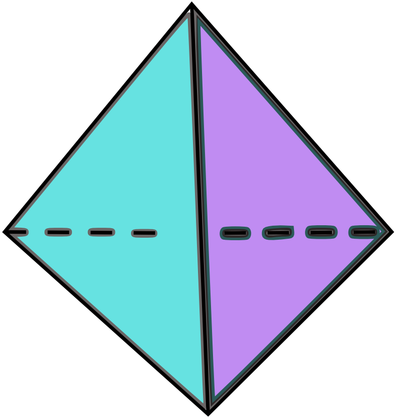


Faces

A *face* of an n-simplex is obtained by restricting a coordinate to zero

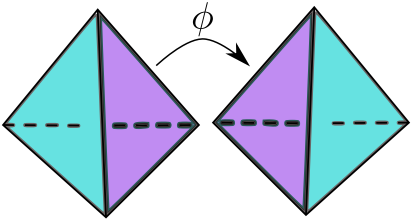


Face pairings

Let $\hat{\Delta} = \{\Delta_1^n, \ldots, \Delta_k^n\}$ (Disjoint union of n-simplices)

A collection Φ of orientation reversing affine maps between faces of simplices in $\hat{\Delta}$ is a *face pairing* if

- $\phi \in \Phi$ iff $\phi^{-1} \in \Phi$
- every face of every simplex in $\hat{\Delta}$ is the domain of a unique $\phi \in \Phi$.



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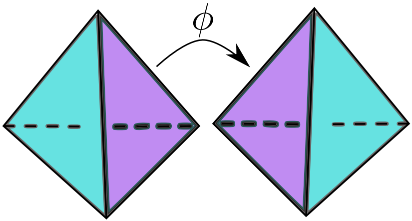
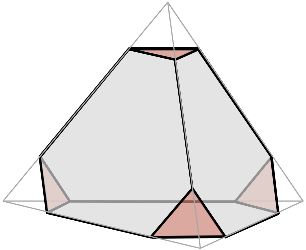
Let $\hat{M} := \hat{\Delta}/\Phi$ (*a triangulated pseudo-manifold*)



Pseudo-manifolds

$\hat{M}$ is almost, but not quite, a manifold.

$\hat{M}$ may contain a "small" subset of non-manifold points (they live in the $(n-3)$-skeleton)



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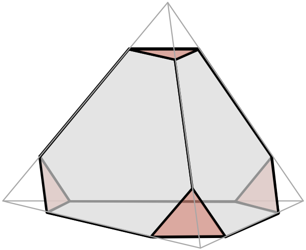

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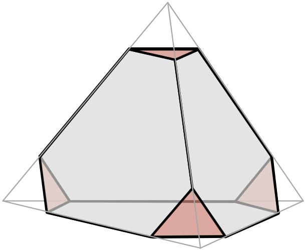
If $n = 2$ then $M = \hat{M}$ and if $n = 3$ then $M = \hat{M} \setminus \{\text{vertices}\}$



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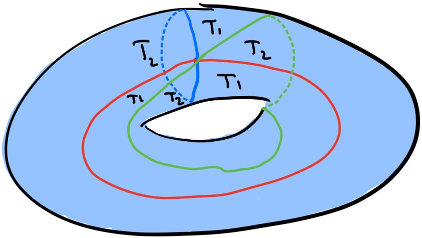
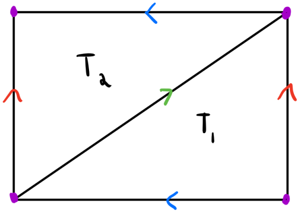
Examples

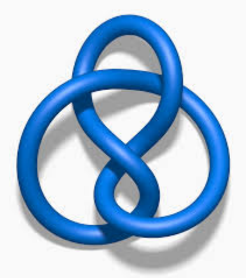
Torus

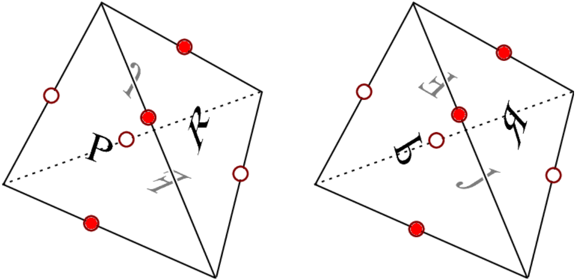




Examples

Figure-eight complement





Motivation

Triangulations

Calculating $\pi_1(M)$

Building hyperbolic metrics

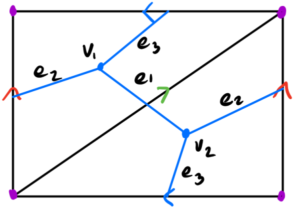
Recent work

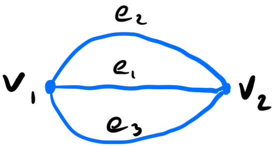
The dual graph

We can build an embedded (multi)-graph Γ with

- a vertex for each simplex of M
- and edge if two simplices are glued along a face.

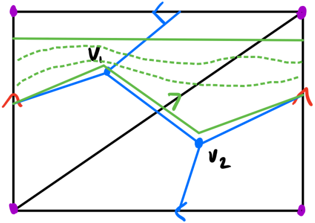
Γ is called the *dual graph* of M.

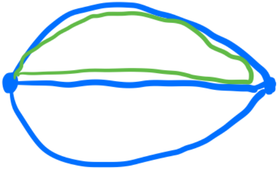




Generators

Every curve in M can be homotoped onto Γ





Inclusion $\iota : \Gamma \to M$ gives $\iota_* : \pi_1(\Gamma) \twoheadrightarrow \pi_1(M)$.

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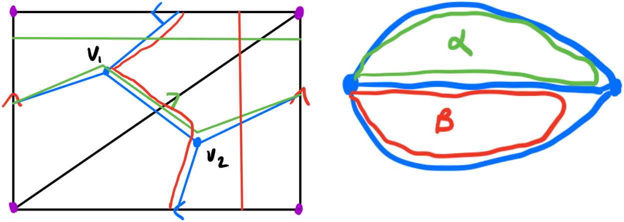

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Generators for $\pi_1(\Gamma)$ give generators for $\pi_1(M)$

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(There are some "obvious" elements in the kernel)

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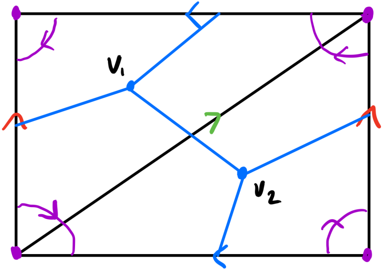

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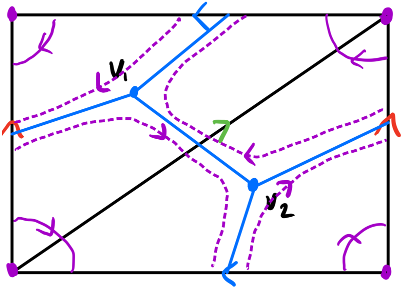

These are all the relations, so

$$\pi_1(M) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1} \rangle$$

Summary

In general

- Dual graph gives generators for $\pi_1(M)$
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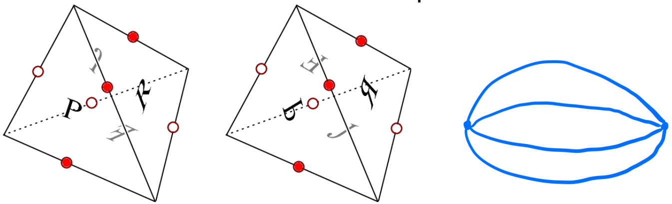

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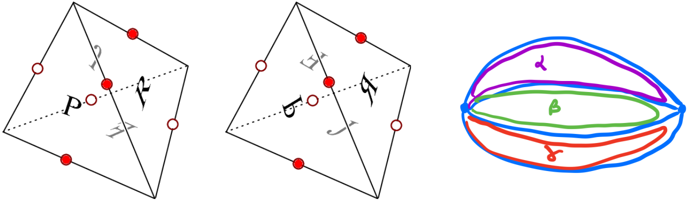

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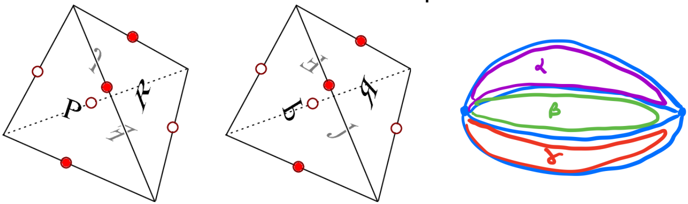

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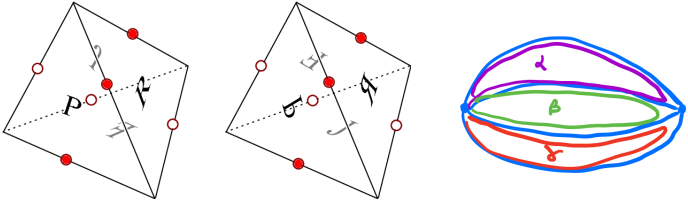

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so $H_1(M) = \mathbb{Z}$

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Let Σ_g be a surface of genus g. We want to build a nice metric on Σ_g

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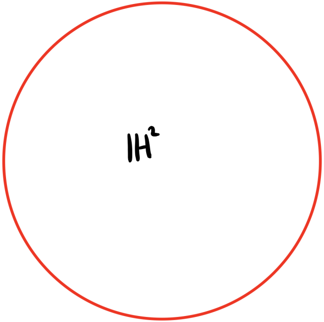
- $g = 0$: $\Sigma_g \cong S^2$ (*spherical metric*)
- $g = 1$: $\Sigma_g \cong T^2$ (*Euclidean metric*)
- $g \geqslant 2$: Σ_g admits a hyperbolic metric (Lots of them!)

Hyperbolic 2-space

A crash course

- $\mathbb{H}^2 \cong B^2$



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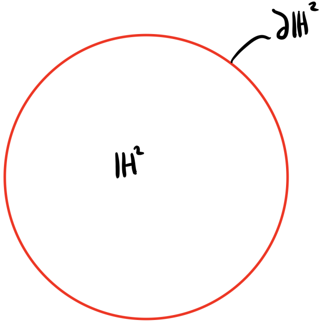

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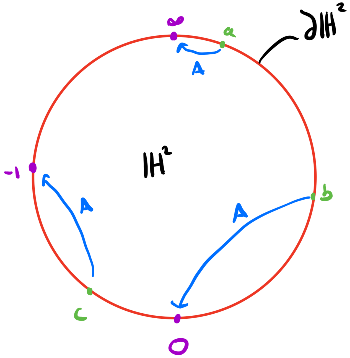
- $\partial\mathbb{H}^2 \cong S^1 \cong \mathbb{R} \cup \{\infty\}$
- $G = \mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$
- G acts on $\partial\mathbb{H}^2$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}$$

- G acts simply transitively on triples of distinct points in $\partial\mathbb{H}^2$



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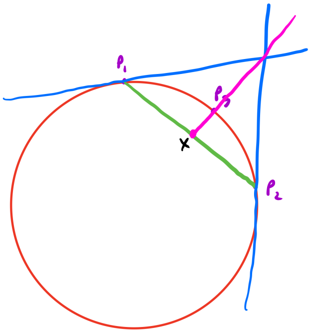

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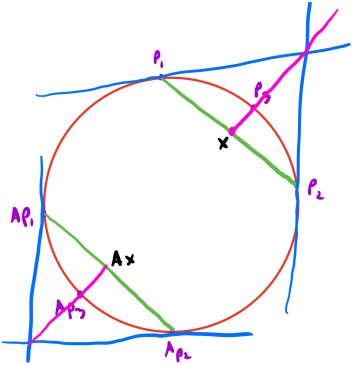

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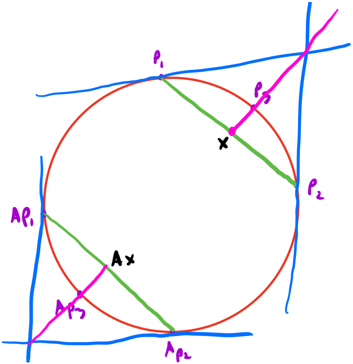

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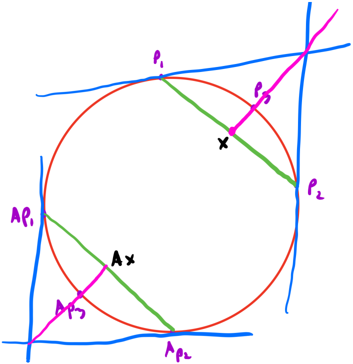

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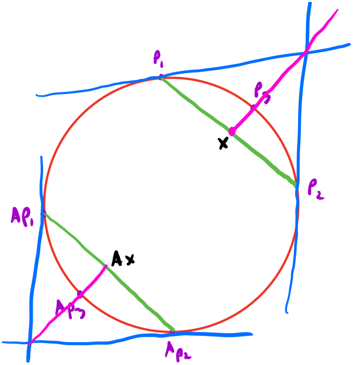
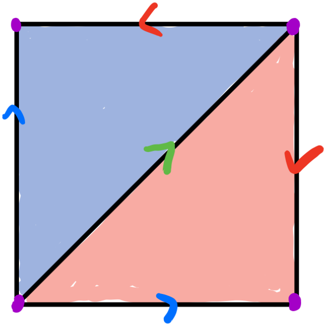
- Geodesics in this metric are straight lines

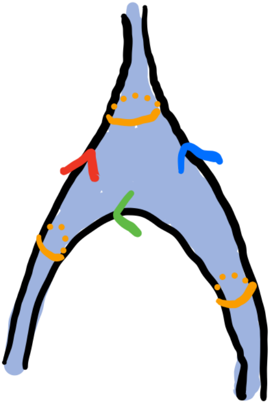


Pair of pants

A toy example

Triangulate a pair of pants, P, using two ideal (*no vertices*) triangles





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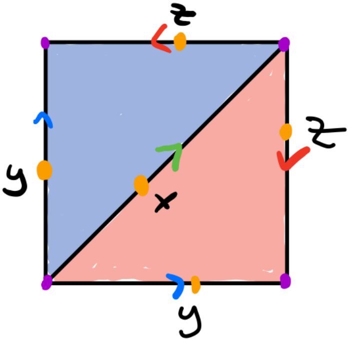
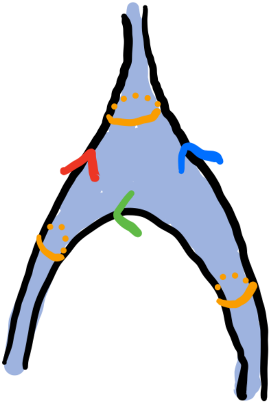
Decorate the edges of P with positive real numbers





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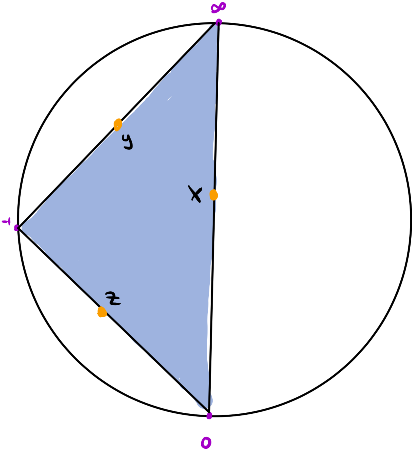
Get a tiling in $\mathbb{H}^2$.



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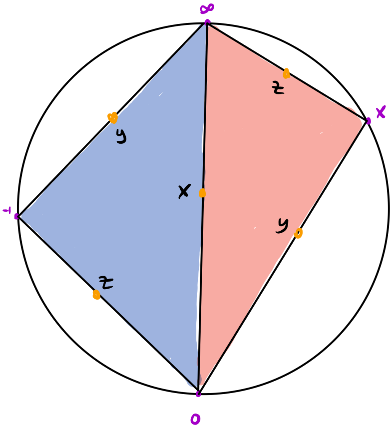

Triangles disjoint $\Leftrightarrow x > 0$

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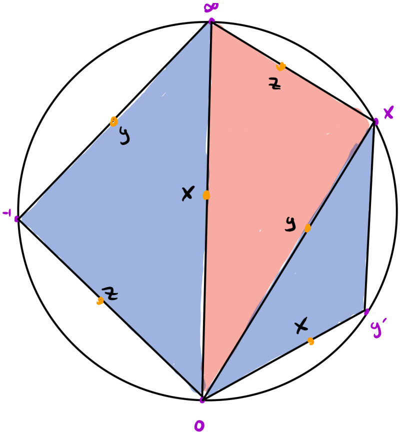

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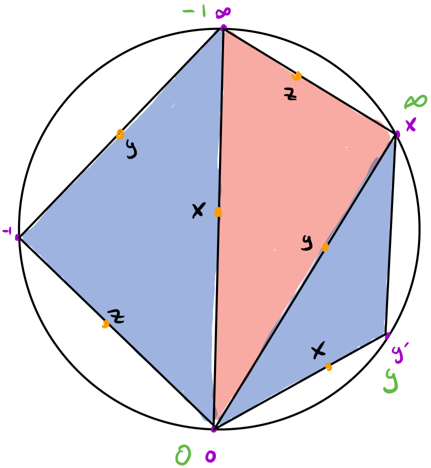

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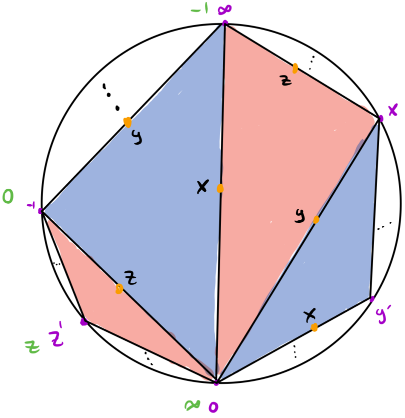
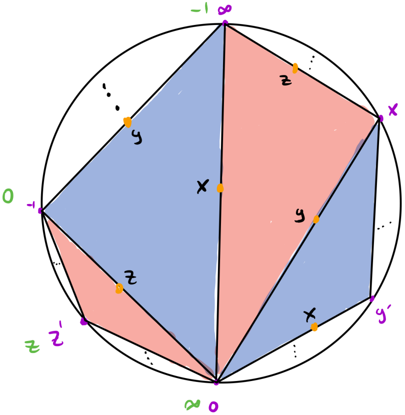


Pair of pants

Get a tiling in $\mathbb{H}^2$.
Metric on $\mathbb{H}^2$ pulls back to a metric on P!



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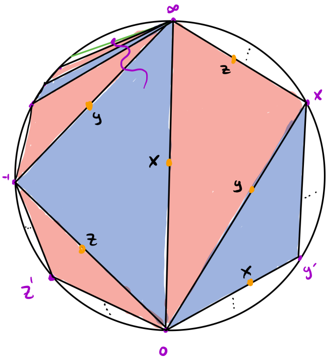
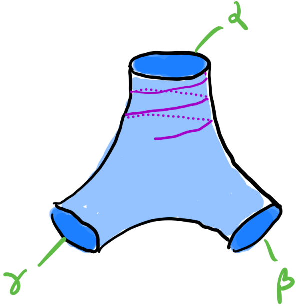
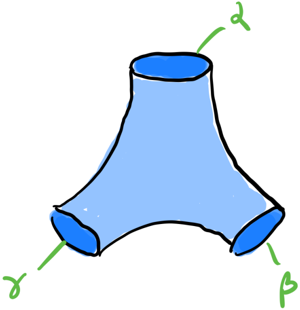




Pair of pants

This metric is typically not complete
Metric completion is closed pair of pants with geodesic boundary

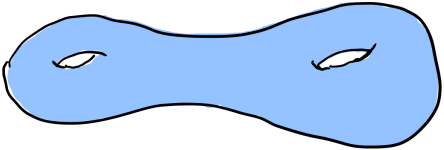


$$\{(x, y, z) \in \mathbb{R}^3_{>0}\}$$
$$\text{“} \cong \text{“}$$
$$\{\text{Pants with boundary lengths } \alpha, \beta, \gamma > 0\}$$

(Thurston's shear coordinates)

Other surfaces

Let S be a closed surface of genus $g \geqslant 2$.



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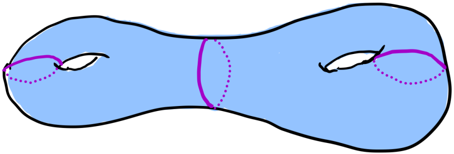
Decompose S into pants by cutting along $3g - 3$ curves



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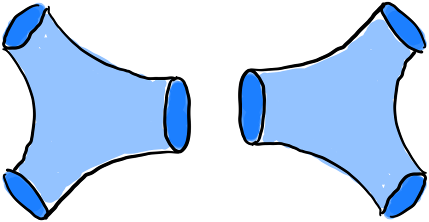

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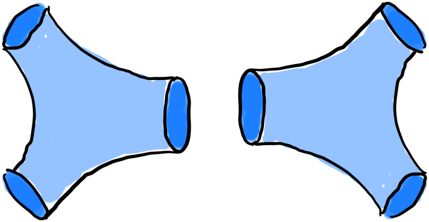

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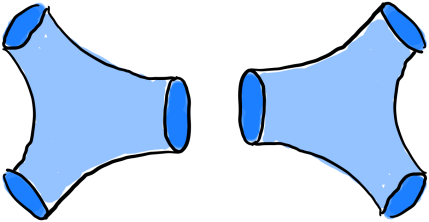

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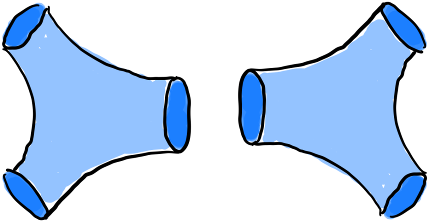

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$\mathcal{T}(S)$ " $=$ " $\{\text{hyperbolic metrics on } S\}/\text{isometries} \cong \mathbb{R}^{6g-6}$
(Teichmüller space)

Metrics on 3-manifolds

Let M be a closed 3-manifold.

Fact: “Most” closed 3-manifolds admit hyperbolic metrics

We want to construct a hyperbolic metric on M.

Dehn Filling

Let M' be a manifold with torus boundary and let D be a solid torus.

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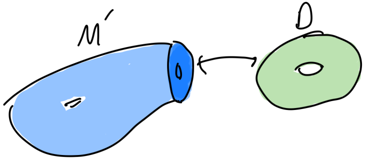
We can build a closed manifold M by gluing M' and D along their boundaries (*Dehn filling*)



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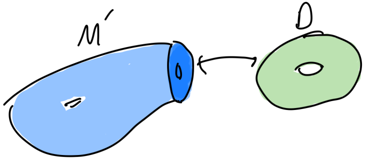

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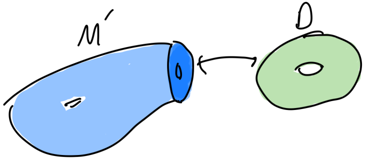


(Lickorish-Wallace, 60's): All closed 3-manifolds are obtained via Dehn filling
Idea: Start by constructing metric on M'

Hyperbolic 3-space

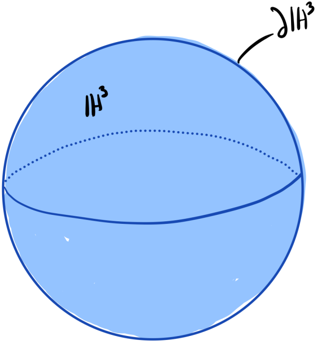
A crash course

Story is similar to dimension 2

- $\mathbb{H}^3 \cong B^3$
- $\partial\mathbb{H}^3 \cong S^2 \cong \mathbb{C} \cup \{\infty\}$
- $G = \mathrm{PSL}_2(\mathbb{C}) := \mathrm{SL}_2(\mathbb{C})/\{\pm I\}$
- G acts on $\partial\mathbb{H}^3$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

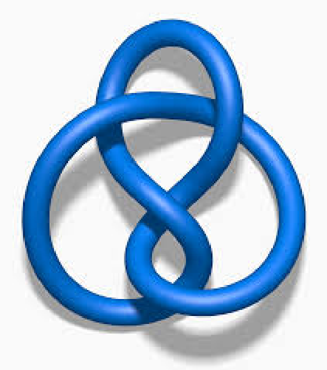
- G acts simply transitively on triples of distinct points in $\partial\mathbb{H}^3$
- $G \curvearrowright \partial\mathbb{H}^3$ induces $G \curvearrowright \mathbb{H}^3$



Metrics for 3-manifolds

Let $\overline{M}$ be a 3-manifold with torus boundary components

Let M be its interior



Metrics for 3-manifolds

Let $\overline{M}$ be a 3-manifold with torus boundary components

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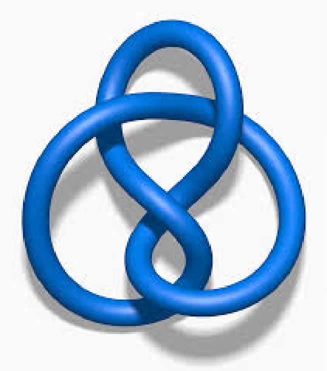
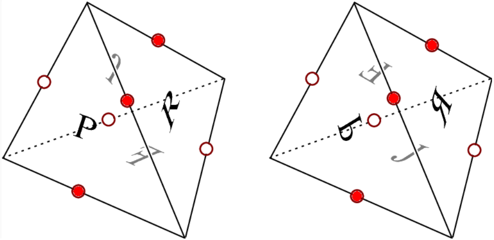
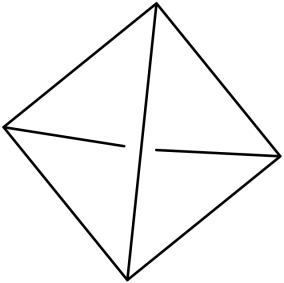
Take an ideal triangulation of $\mathcal{T}$ of M.





Coordinates for tetrahedra

Take an ideal (no vertices) tetrahedron T



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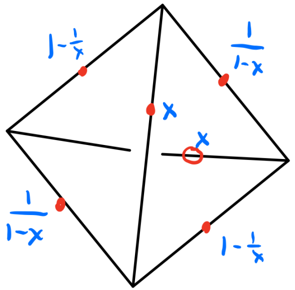
Take an ideal (no vertices) tetrahedron T

Label the edges of T with complex numbers



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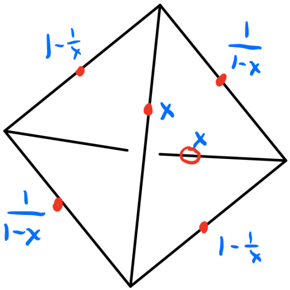
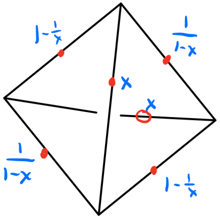
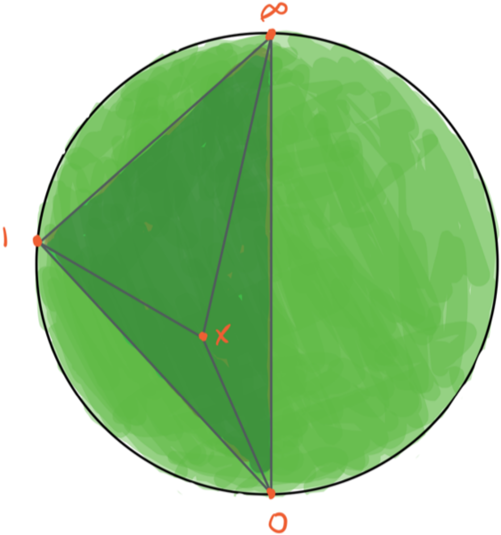


Labelling tells us how to build T in $\mathbb{H}^3$

Tetrahedra in $\mathbb{H}^3$





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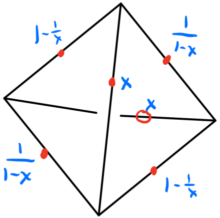

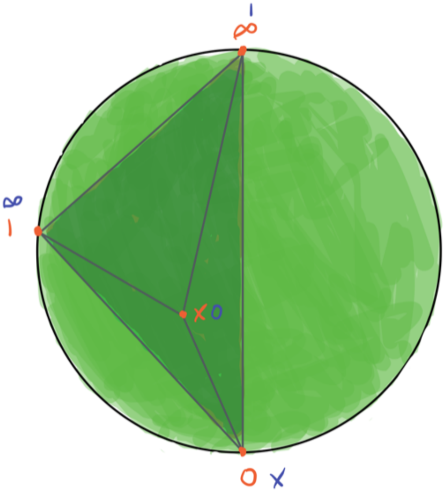

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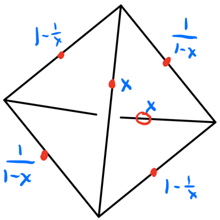
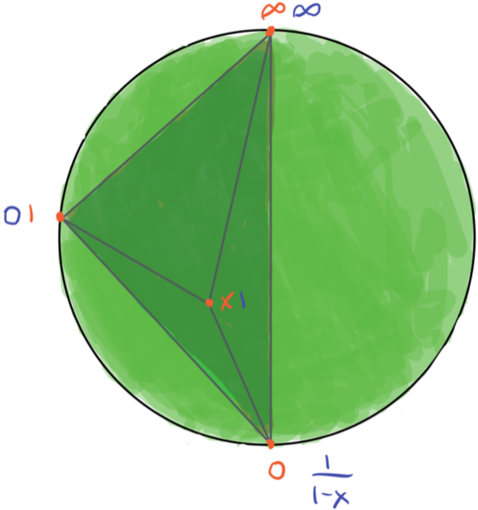
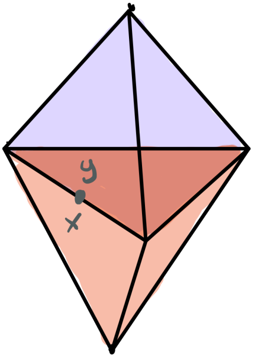
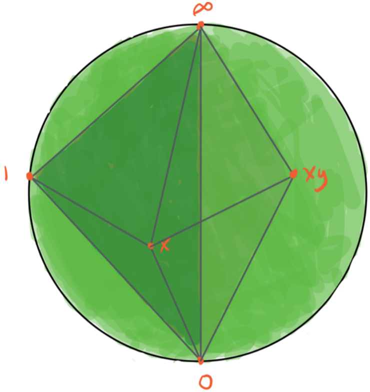




Gluing Tetrahedra

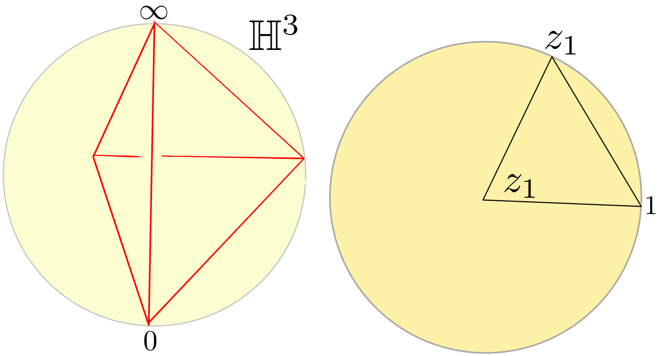
Tetrahedra can be glued along faces





Thurston's gluing equations

Given a collection of ideal tetrahedra, we can glue them together around an edge



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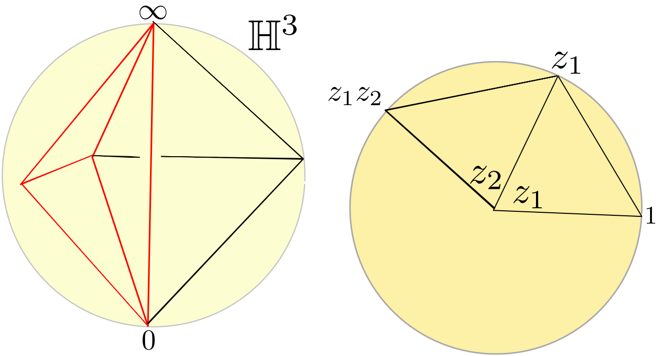

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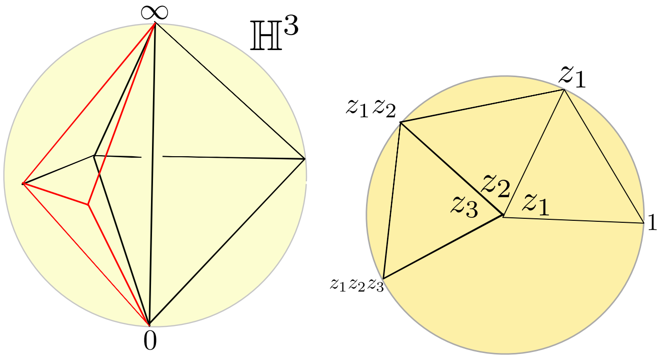

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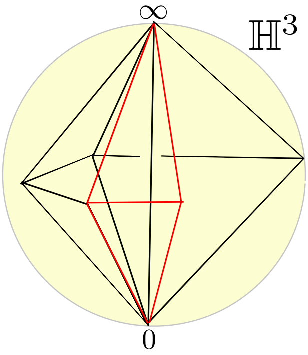

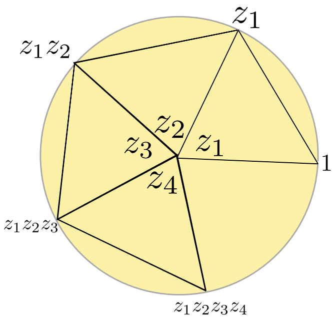

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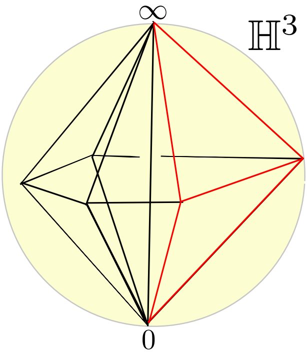

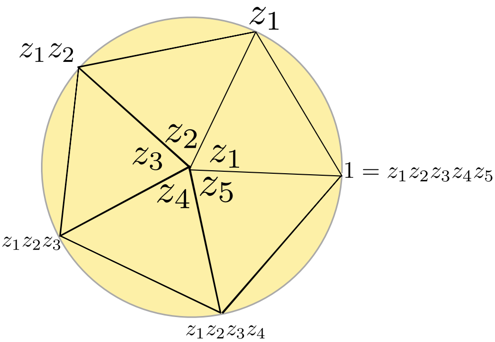

In order for the cycle to close up we need to impose an equation

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A solution to these equations is *geometric* if each component has positive imaginary part (No inside out tetrahedra)

Building the metric

Start with geometric solution to gluing equations

1. Build tetrahedra comprising M in $\mathbb{H}^3$
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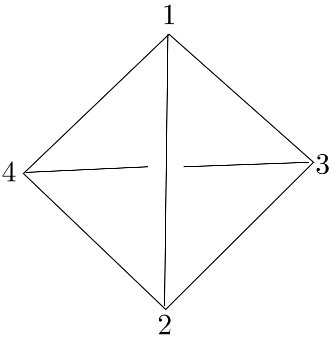
Coordinates for projective strucures

Previous approach is constrained to build tetrahedra inscribed in $\partial\mathbb{H}^3$.
In recent work with A. Casella we extend these techniques to build arbitrary straight tetrahedra in $\mathbb{R}^3$ (really $\mathbb{RP}^3$)



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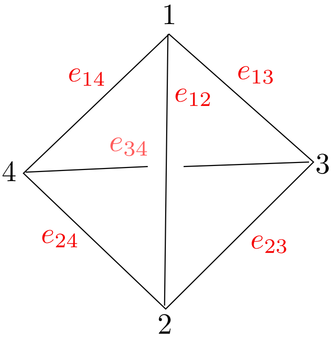

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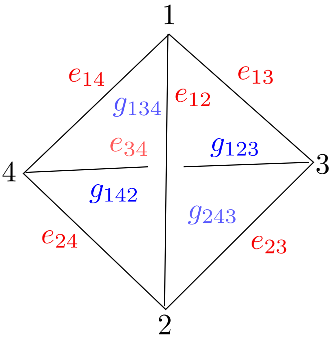
- 4 Gluing coordinates: 1 per face: Describe how this tetrahedron will be glued to adjacent tetrahedra.



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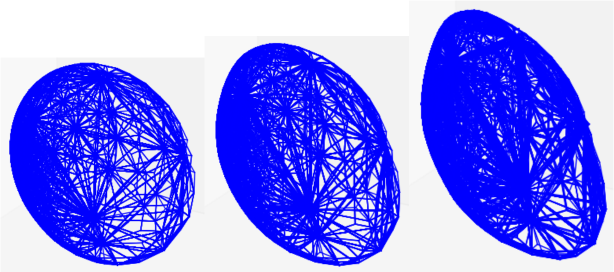
Some pictures

Families of solutions give rise to tilings of families of convex regions in $\mathbb{R}^3$



Thank you