The Structure of Properly Convex Manifolds

Sam Ballas

(joint with D. Long)

University of Pittsburgh

May 14, 2014

Some Questions

- What are convex projective manifolds?

Some Questions

- What are convex projective manifolds?
- How are they similar to hyperbolic manifolds? How are they different?

Some Questions

- What are convex projective manifolds?
- How are they similar to hyperbolic manifolds? How are they different?
- What sort of structure do convex projective manifolds have?

Some Questions

- What are convex projective manifolds?
  Generalizations of Hyperbolic manifolds
- How are they similar to hyperbolic manifolds? How are they different?

- What sort of structure do convex projective manifolds have?

Some Questions

- What are convex projective manifolds?
  Generalizations of Hyperbolic manifolds
- How are they similar to hyperbolic manifolds? How are they different?
  Strictly Convex $\Rightarrow$ *very similar. Properly convex* $\Rightarrow$ *less similar*
- What sort of structure do convex projective manifolds have?

Some Questions

- What are convex projective manifolds?
  Generalizations of Hyperbolic manifolds
- How are they similar to hyperbolic manifolds? How are they different?
  Strictly Convex $\Rightarrow$ *very similar. Properly convex* $\Rightarrow$ *less similar*
- What sort of structure do convex projective manifolds have?
  Deformations of finite volume strictly convex manifolds are structurally similar to complete finite volume hyperbolic manifolds

Projective Space

- $\mathbb{R}P^n$ is the space of lines through origin in $\mathbb{R}^{n+1}$.

Projective Space

- $\mathbb{R}P^n$ is the space of lines through origin in $\mathbb{R}^{n+1}$.
- Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}P^n$ be the obvious projection.

Projective Space

- $\mathbb{R}P^n$ is the space of lines through origin in $\mathbb{R}^{n+1}$.
- Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}P^n$ be the obvious projection.
- The automorphism group of $\mathbb{R}P^n$ is $\mathrm{PGL}_{n+1}(\mathbb{R}) := \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^\times$.

Projective Space

- $\mathbb{R}P^n$ is the space of lines through origin in $\mathbb{R}^{n+1}$.
- Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}P^n$ be the obvious projection.
- The automorphism group of $\mathbb{R}P^n$ is $\mathrm{PGL}_{n+1}(\mathbb{R}) := \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^\times$.
- A *codimension* k *projective plane* is the projectivization of a codimension k plane in $\mathbb{R}^{n+1}$

Projective Space

- $\mathbb{R}P^n$ is the space of lines through origin in $\mathbb{R}^{n+1}$.
- Let $P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ be the obvious projection.
- The automorphism group of $\mathbb{R}P^n$ is $\mathrm{PGL}_{n+1}(\mathbb{R}) := \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^{\times}$.
- A *codimension k projective plane* is the projectivization of a codimension k plane in $\mathbb{R}^{n+1}$
- A *projective line* is the projectivization of a 2-plane in $\mathbb{R}^{n+1}$

Projective Space

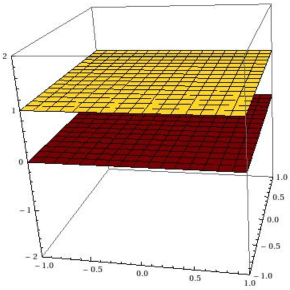
- $\mathbb{R}P^n$ is the space of lines through origin in $\mathbb{R}^{n+1}$.
- Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}P^n$ be the obvious projection.
- The automorphism group of $\mathbb{R}P^n$ is $\mathrm{PGL}_{n+1}(\mathbb{R}) := \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^\times$.
- A *codimension k projective plane* is the projectivization of a codimension k plane in $\mathbb{R}^{n+1}$
- A *projective line* is the projectivization of a 2-plane in $\mathbb{R}^{n+1}$
- A *projective hyperplane* is the projectivization of an n-plane in $\mathbb{R}^{n+1}$.

A Decomposition of $\mathbb{R}P^n$

- Let H be a hyperplane in $\mathbb{R}^{n+1}$.
- H gives rise to a Decomposition of $\mathbb{R}P^n = \mathbb{R}^n \sqcup \mathbb{R}P^{n-1}$ into an affine part and an ideal part.

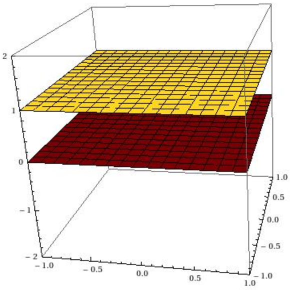
A Decomposition of $\mathbb{R}P^n$

- Let H be a hyperplane in $\mathbb{R}^{n+1}$.
- H gives rise to a Decomposition of $\mathbb{R}P^n = \mathbb{R}^n \sqcup \mathbb{R}P^{n-1}$ into an affine part and an ideal part.



A Decomposition of $\mathbb{R}P^n$

- Let H be a hyperplane in $\mathbb{R}^{n+1}$.
- H gives rise to a Decomposition of $\mathbb{R}P^n = \mathbb{R}^n \sqcup \mathbb{R}P^{n-1}$ into an affine part and an ideal part.

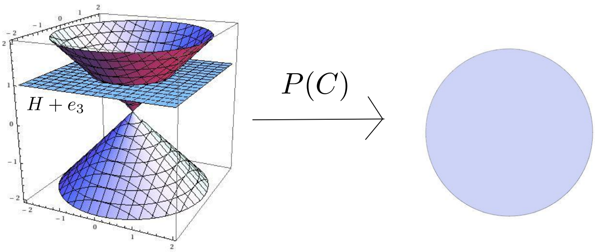


- $\mathbb{R}P^n \backslash P(H)$ is called an *affine patch*.

What is convex projective geometry?

Motivation from hyperbolic geometry

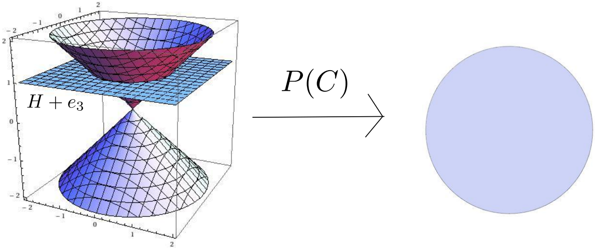
- Let $\langle x, y\rangle = x_1 y_1 + \ldots x_n y_n - x_{n+1} y_{n+1}$ be the standard bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$
- Let $C = \{x \in \mathbb{R}^{n+1} | \langle x, x\rangle < 0\}$



What is convex projective geometry?

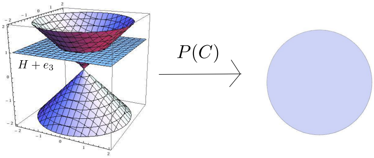
Motivation from hyperbolic geometry

- Let $\langle x, y\rangle = x_1y_1 + \ldots x_ny_n - x_{n+1}y_{n+1}$ be the standard bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$
- Let $C = \{x \in \mathbb{R}^{n+1} | \langle x, x\rangle < 0\}$
- $P(C)$ is the *Klein model* of hyperbolic space.
- $P(C)$ has isometry group $\mathrm{PSO}(n, 1) \leq \mathrm{PGL}_{n+1}(\mathbb{R})$



What is convex projective geometry?

Motivation from hyperbolic geometry

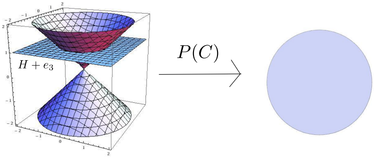


Nice Properties of Hyperbolic Space

- *Convex*: Intersection with projective lines is connected.

What is convex projective geometry?

Motivation from hyperbolic geometry

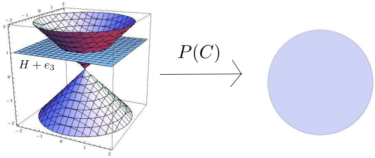


Nice Properties of Hyperbolic Space

- *Convex*: Intersection with projective lines is connected.
- *Properly Convex*: Convex and closure is contained in an affine patch $\iff$ Disjoint from some projective hyperplane.

What is convex projective geometry?

Motivation from hyperbolic geometry



Nice Properties of Hyperbolic Space

- *Convex*: Intersection with projective lines is connected.
- *Properly Convex*: Convex and closure is contained in an affine patch $\iff$ Disjoint from some projective hyperplane.
- *Strictly Convex*: Properly convex and boundary contains no non-trivial projective line segments.

What is convex projective geometry?

Motivation from hyperbolic geometry

Convex projective geometry focuses on the geometry of manifolds that are locally modeled on properly (strictly) convex domains.

What is convex projective geometry?

Motivation from hyperbolic geometry

Convex projective geometry focuses on the geometry of manifolds that are locally modeled on properly (strictly) convex domains.

Hyperbolic Geometry

$$\mathbb{H}^n/\Gamma$$

$$\Gamma \leq \mathrm{Isom}(\mathbb{H}^n)$$

Γ discrete + torsion free

What is convex projective geometry?

Motivation from hyperbolic geometry

Convex projective geometry focuses on the geometry of manifolds that are locally modeled on properly (strictly) convex domains.

Hyperbolic Geometry

$\mathbb{H}^n/\Gamma$

$\Gamma \leq \mathrm{Isom}(\mathbb{H}^n)$

Γ discrete + torsion free

Convex Projective Geometry

Ω/Γ

Ω properly (strictly) convex

$\Gamma \leq \mathrm{PGL}(\Omega)$

Γ discrete + torsion free

What is Convex Projective Geometry

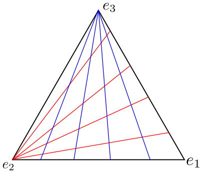
Examples

1. Hyperbolic manifolds

What is Convex Projective Geometry

Examples

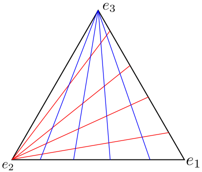
1. Hyperbolic manifolds
2. Let T be the interior of a triangle in $\mathbb{R}P^2$ and let $\Gamma \leq \text{Diag}^+$ be a suitable lattice inside the group of 3×3 diagonal matrices with determinant 1 and distinct positive eigenvalues. T/Γ is a properly convex torus.



What is Convex Projective Geometry

Examples

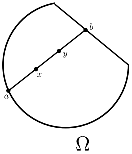
1. Hyperbolic manifolds
2. Let T be the interior of a triangle in $\mathbb{R}P^2$ and let $\Gamma \leq \text{Diag}^+$ be a suitable lattice inside the group of 3×3 diagonal matrices with determinant 1 and distinct positive eigenvalues. T/Γ is a properly convex torus.



These are extreme examples of properly convex manifolds. Generic examples interpolate between these extreme cases.

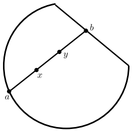
Hilbert Metric

Let Ω be a properly convex set and $\mathrm{PGL}(\Omega)$ be the projective automorphisms preserving Ω.



Hilbert Metric

Let Ω be a properly convex set and $\mathrm{PGL}(\Omega)$ be the projective automorphisms preserving Ω.

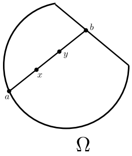


Every properly convex set Ω admits a Hilbert metric given by

$$d_\Omega(x, y) = \log[a, x; y, b] = \log\left(\frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}\right)$$

Hilbert Metric

Let Ω be a properly convex set and $\mathrm{PGL}(\Omega)$ be the projective automorphisms preserving Ω.



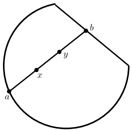
Every properly convex set Ω admits a Hilbert metric given by

$$d_\Omega(x, y) = \log[a, x; y, b] = \log\left(\frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}\right)$$

- When Ω is an ellipsoid d_Ω is twice the hyperbolic metric.

Hilbert Metric

Let Ω be a properly convex set and $\mathrm{PGL}(\Omega)$ be the projective automorphisms preserving Ω.



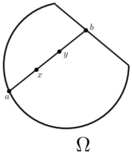
Every properly convex set Ω admits a Hilbert metric given by

$$d_\Omega(x, y) = \log[a, x; y, b] = \log\left(\frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}\right)$$

- When Ω is an ellipsoid d_Ω is twice the hyperbolic metric.
- $\mathrm{PGL}(\Omega) \leq \mathrm{Isom}(\Omega)$ and equal when Ω is strictly convex.

Hilbert Metric

Let Ω be a properly convex set and $\mathrm{PGL}(\Omega)$ be the projective automorphisms preserving Ω.



Every properly convex set Ω admits a Hilbert metric given by

$$d_\Omega(x,y) = \log[a,x;y,b] = \log\left(\frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}\right)$$

- When Ω is an ellipsoid d_Ω is twice the hyperbolic metric.
- $\mathrm{PGL}(\Omega) \leq \mathrm{Isom}(\Omega)$ and equal when Ω is strictly convex.
- Discrete subgroups of $\mathrm{PGL}(\Omega)$ act properly discontinuously on Ω.

Classification of Isometries

a la Cooper, Long, Tillmann

If Ω is open and properly convex then $\mathrm{PGL}(\Omega)$ embeds in $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ which allows us to talk about eigenvalues.

Classification of Isometries

a la Cooper, Long, Tillmann

If Ω is open and properly convex then $\mathrm{PGL}(\Omega)$ embeds in $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ which allows us to talk about eigenvalues.
If $\gamma \in \mathrm{PGL}(\Omega)$ then γ is

1. *elliptic* if γ fixes a point in Ω (zero translation length + realized),
2. *parabolic* if γ acts freely on Ω and has all eigenvalues of modulus 1 (zero translation length + not realized), and
3. *hyperbolic* otherwise (positive translation length)

Similarities to Hyperbolic Isometries

Strictly Convex Case

1. When Ω is an ellipsoid this classification is the same as the standard classification of hyperbolic isometries.

Similarities to Hyperbolic Isometries

Strictly Convex Case

1. When Ω is an ellipsoid this classification is the same as the standard classification of hyperbolic isometries.
2. When Ω is strictly convex, parabolic isometries have a unique fixed point on $\partial\Omega$.

Similarities to Hyperbolic Isometries

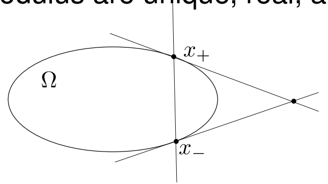
Strictly Convex Case

1. When Ω is an ellipsoid this classification is the same as the standard classification of hyperbolic isometries.
2. When Ω is strictly convex, parabolic isometries have a unique fixed point on $\partial\Omega$.
3. When Ω is strictly convex, hyperbolic isometries have 2 fixed points on $\partial\Omega$ and act by translation along the line connecting them.

Similarities to Hyperbolic Isometries

Strictly Convex Case

1. When Ω is an ellipsoid this classification is the same as the standard classification of hyperbolic isometries.
2. When Ω is strictly convex, parabolic isometries have a unique fixed point on $\partial\Omega$.
3. When Ω is strictly convex, hyperbolic isometries have 2 fixed points on $\partial\Omega$ and act by translation along the line connecting them.
4. In particular, when Ω is strictly convex, hyperbolic isometries are *positive proximal* (eigenvalues of minimum and maximum modulus are unique, real, and positive)



Similarities to Hyperbolic Isometries

The General Case

A properly convex domain is a compact convex subset of $\mathbb{R}^n$ and so if $\gamma \in \mathrm{PGL}(\Omega)$ then Brouwer fixed point theorem applies

Similarities to Hyperbolic Isometries

The General Case

A properly convex domain is a compact convex subset of $\mathbb{R}^n$ and so if $\gamma \in \mathrm{PGL}(\Omega)$ then Brouwer fixed point theorem applies

- Elliptic elements are all conjugate into O(n).

Similarities to Hyperbolic Isometries

The General Case

A properly convex domain is a compact convex subset of $\mathbb{R}^n$ and so if $\gamma \in \mathrm{PGL}(\Omega)$ then Brouwer fixed point theorem applies

- Elliptic elements are all conjugate into $O(n)$.
- Parabolic elements have a connected fixed set in $\partial\Omega$.

Similarities to Hyperbolic Isometries

The General Case

A properly convex domain is a compact convex subset of $\mathbb{R}^n$ and so if $\gamma \in \mathrm{PGL}(\Omega)$ then Brouwer fixed point theorem applies

- Elliptic elements are all conjugate into O(n).
- Parabolic elements have a connected fixed set in $\partial\Omega$.
- Hyperbolic elements have an attracting and repelling subspaces A_+ and A_- in $\partial\Omega$. The action on these sets is orthogonal and their dimension is determined by the number of "powerful" Jordan blocks of γ

Margulis Lemma

Let $\Omega \subset \mathbb{R}P^n$ is an open properly convex domain and let $\Gamma \leq \mathrm{PGL}(\Omega)$ be a discrete group. Then there exists a number μ_n (depending only on n) such that if $x \in \Omega$ then the group

$$\Gamma_x = \langle \gamma \in \Gamma | d_\Omega(x, \gamma x) < \mu_n \rangle$$

is virtually nilpotent.

Margulis Lemma

Let $\Omega \subset \mathbb{R}P^n$ is an open properly convex domain and let $\Gamma \leq \mathrm{PGL}(\Omega)$ be a discrete group. Then there exists a number μ_n (depending only on n) such that if $x \in \Omega$ then the group

$$\Gamma_x = \langle \gamma \in \Gamma | d_\Omega(x, \gamma x) < \mu_n \rangle$$

is virtually nilpotent.

- Γ_x can be thought of as the subgroup of Γ generated by loops in Ω/Γ of length at most μ_n passing through $[x]$.

Margulis Lemma

Let $\Omega \subset \mathbb{R}P^n$ is an open properly convex domain and let $\Gamma \leq \mathrm{PGL}(\Omega)$ be a discrete group. Then there exists a number μ_n (depending only on n) such that if $x \in \Omega$ then the group

$$\Gamma_x = \langle \gamma \in \Gamma | d_\Omega(x, \gamma x) < \mu_n \rangle$$

is virtually nilpotent.

- Γ_x can be thought of as the subgroup of Γ generated by loops in Ω/Γ of length at most μ_n passing through $[x]$.
- The Margulis lemma places restrictions on the topology and geometry of the “thin” part of Ω/Γ.

Margulis Lemma

Let $\Omega \subset \mathbb{R}P^n$ is an open properly convex domain and let $\Gamma \leq \mathrm{PGL}(\Omega)$ be a discrete group. Then there exists a number μ_n (depending only on n) such that if $x \in \Omega$ then the group

$$\Gamma_x = \langle \gamma \in \Gamma | d_\Omega(x, \gamma x) < \mu_n \rangle$$

is virtually nilpotent.

- Γ_x can be thought of as the subgroup of Γ generated by loops in Ω/Γ of length at most μ_n passing through $[x]$.
- The Margulis lemma places restrictions on the topology and geometry of the "thin" part of Ω/Γ.

Result due to Gromov-Margulis-Thurston for $\mathbb{H}^n$ and Cooper-Long-Tillmann in general.

Rigidity and Flexibility

When $n \geq 3$ Mostow-Prasad rigidity tells us that complete finite volume hyperbolic structures are very rigid

Theorem 1 (Mostow '70, Prasad '73)

Let $n \geq 3$ and suppose that $\mathbb{H}^n/\Gamma_1$ and $\mathbb{H}^n/\Gamma_2$ both have finite volume. If Γ_1 and Γ_2 are isomorphic then $\mathbb{H}^n/\Gamma_1$ and $\mathbb{H}^n/\Gamma_2$ are isometric.

Rigidity and Flexibility

When $n \geq 3$ Mostow-Prasad rigidity tells us that complete finite volume hyperbolic structures are very rigid

Theorem 1 (Mostow '70, Prasad '73)

Let $n \geq 3$ and suppose that $\mathbb{H}^n/\Gamma_1$ and $\mathbb{H}^n/\Gamma_2$ both have finite volume. If Γ_1 and Γ_2 are isomorphic then $\mathbb{H}^n/\Gamma_1$ and $\mathbb{H}^n/\Gamma_2$ are isometric.

There is no Mostow-Prasad rigidity for properly (strictly) convex domains.
There are examples of finite volume hyperbolic manifolds whose complete hyperbolic structure can be "deformed" to a non-hyperbolic convex projective structure.

Deformations

- Start with $M_0 = \Omega_0/\Gamma_0$ which is properly convex.

Deformations

- Start with $M_0 = \Omega_0/\Gamma_0$ which is properly convex.
- "Perturb" Γ_0 to $\Gamma_1 \leq \mathrm{PGL}(\Omega_1) \leq \mathrm{PGL}_{n+1}(\mathbb{R})$, where $\Gamma_0 \cong \Gamma_1$ and Ω_1 is properly convex.

Deformations

- Start with $M_0 = \Omega_0/\Gamma_0$ which is properly convex.
- "Perturb" Γ_0 to $\Gamma_1 \leq \mathrm{PGL}(\Omega_1) \leq \mathrm{PGL}_{n+1}(\mathbb{R})$, where $\Gamma_0 \cong \Gamma_1$ and Ω_1 is properly convex.
- We say that $M_1 = \Omega_1/\Gamma_1$ is a *deformation* of M_0

Deformations

- Start with $M_0 = \Omega_0/\Gamma_0$ which is properly convex.
- "Perturb" Γ_0 to $\Gamma_1 \leq \mathrm{PGL}(\Omega_1) \leq \mathrm{PGL}_{n+1}(\mathbb{R})$, where $\Gamma_0 \cong \Gamma_1$ and Ω_1 is properly convex.
- We say that $M_1 = \Omega_1/\Gamma_1$ is a *deformation* of M_0

Ex: Let $\Omega_0 \cong \mathbb{H}^n$, $\Gamma_0 \leq \mathrm{PSO}(n, 1)$, such that Ω_0/Γ_0 is finite volume and contains an embedded totally geodesic hypersurface Σ. Let Γ_1 be obtained by "bending" along Σ.

Structure of Hyperbolic Manifolds

The Closed Case

Let $\mathbb{H}^n/\Gamma$ be a closed hyperbolic manifold.

- Since Γ acts cocompactly by isometries on $\mathbb{H}^n$ we see that Γ is δ-hyperbolic group (Švarc-Milnor)

Structure of Hyperbolic Manifolds

The Closed Case

Let $\mathbb{H}^n/\Gamma$ be a closed hyperbolic manifold.

- Since Γ acts cocompactly by isometries on $\mathbb{H}^n$ we see that Γ is δ-hyperbolic group (Švarc-Milnor)
- By compactness, we see that if $1 \neq \gamma \in \Gamma$ then γ is hyperbolic

Structure of Hyperbolic Manifolds

The Closed Case

Let $\mathbb{H}^n/\Gamma$ be a closed hyperbolic manifold.

- Since Γ acts cocompactly by isometries on $\mathbb{H}^n$ we see that Γ is δ-hyperbolic group (Švarc-Milnor)
- By compactness, we see that if $1 \neq \gamma \in \Gamma$ then γ is hyperbolic
- In particular, if $1 \neq \gamma \in \Gamma$ then γ is positive proximal

Let $M = \Omega/\Gamma$ be a closed properly convex manifold that is a deformation of a closed strictly convex manifold $M_0 = \Omega_0/\Gamma_0$.

Structure of Convex Projective Manifolds

The Closed Case

Let $M = \Omega/\Gamma$ be a closed properly convex manifold that is a deformation of a closed strictly convex manifold $M_0 = \Omega_0/\Gamma_0$.

Theorem 2 (Benoist)

Suppose Ω/Γ is closed. Ω/Γ is strictly convex if and only if Γ is δ-hyperbolic.

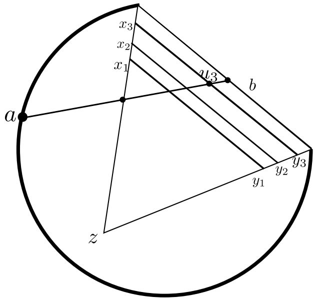
Let $M = \Omega/\Gamma$ be a closed properly convex manifold that is a deformation of a closed strictly convex manifold $M_0 = \Omega_0/\Gamma_0$.

Theorem 2 (Benoist)

Suppose Ω/Γ is closed. Ω/Γ is strictly convex if and only if Γ is δ-hyperbolic.

$\Longleftarrow$ Proof sketch.

If Ω is not strictly convex then it will contain arbitrarily fat triangles and is thus not δ-hyperbolic.



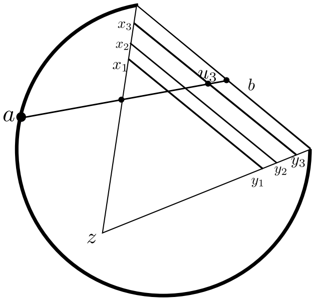
Let $M = \Omega/\Gamma$ be a closed properly convex manifold that is a deformation of a closed strictly convex manifold $M_0 = \Omega_0/\Gamma_0$.

Theorem 2 (Benoist)

Suppose Ω/Γ is closed. Ω/Γ is strictly convex if and only if Γ is δ-hyperbolic.

$\Longleftarrow$ Proof sketch.

If Ω is not strictly convex then it will contain arbitrarily fat triangles and is thus not δ-hyperbolic. Since Γ acts cocompactly by isometries on Ω, Švarc-Milnor tells us that Ω is q.i. to Γ and is thus δ-hyperbolic. □



Theorem 3 (Benoist)

Let $1 \neq \gamma \in \Gamma$ *then* γ *is positive proximal.*

Proof.

- Again by compactness we have that if $1 \neq \gamma \in \Gamma$ then γ is hyperbolic.

□

Theorem 3 (Benoist)

Let $1 \neq \gamma \in \Gamma$ *then* γ *is positive proximal.*

Proof.

- Again by compactness we have that if $1 \neq \gamma \in \Gamma$ then γ is hyperbolic.
- Since Ω is strictly convex and γ is hyperbolic we see that γ has exactly 2 fixed points in $\partial\Omega$ and acts as translation along the geodesic connecting them. γ is thus positive proximal.

□

Structure of Hyperbolic Manifolds

Finite Volume Case

Let $M = \mathbb{H}^n/\Gamma$ be a finite volume hyperbolic manifold. We can decompose M as

$$M = M_K \bigsqcup_i C_i,$$

where M_K is a compact and $\pi_1(M_K) = \Gamma$ and C_i are components of the thin part called *cusps*.

Structure of Hyperbolic Manifolds

Finite Volume Case

Let $M = \mathbb{H}^n/\Gamma$ be a finite volume hyperbolic manifold. We can decompose M as

$$M = M_K \bigsqcup_i C_i,$$

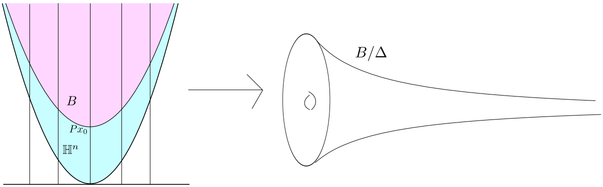
where M_K is a compact and $\pi_1(M_K) = \Gamma$ and C_i are components of the thin part called *cusps*.
As we will see, the Margulis lemma tells us that the C_i have relatively simple geometry.

Geometry of the Cusps

Let C be a cusp of a finite volume hyperbolic manifold and let

$$P = \left\{ \begin{pmatrix} 1 & v^T & |v|^2 \\ 0 & I_{n-1} & v \\ 0 & 0 & 1 \end{pmatrix} \middle| v \in \mathbb{R}^{n-1} \right\}$$

be the group of parabolic translations fixing ∞. Let $x_0 \in \mathbb{H}^n$, then $C \cong B/\Delta$ where B is horoball bounded by Px_0 and Δ is a finite extension of a lattice in P.



Structure of Hyperbolic Manifolds

The Finite Volume Case

- Γ no longer acts cocompactly on $\mathbb{H}^n$ and Γ is no longer δ-hyperbolic

- Γ no longer acts cocompactly on $\mathbb{H}^n$ and Γ is no longer δ-hyperbolic
- Instead Γ is δ-hyperbolic *relative to the cusps*

Structure of Hyperbolic Manifolds

The Finite Volume Case

- Γ no longer acts cocompactly on $\mathbb{H}^n$ and Γ is no longer δ-hyperbolic
- Instead Γ is δ-hyperbolic *relative to the cusps*
- If $1 \neq \gamma \in \Gamma$ is freely homotopic into a cusp then γ is parabolic, otherwise γ is hyperbolic (positive proximal)

Let Ω/Γ be a finite volume (Hausdorff measure of Hilbert metric) strictly convex manifold.

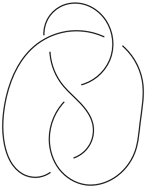
Theorem 4 (Cooper, Long, Tillmann '11)

Let $M = \Omega/\Gamma$ be as above then

- $M = M_K \bigsqcup_i C_i$, *where M_K is compact and C_i is projectively equivalent to the cusp of a finite volume hyperbolic manifold,*
- Γ *is δ-hyperbolic relative to its cusps, and*
- *If $1 \neq \gamma \in \Gamma$ is freely homotopic into a cusp then γ is parabolic. Otherwise γ is hyperbolic (positive proximal).*

Figure-8 Example

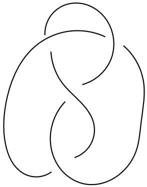
Consider the following example.



Let K be the figure-8 knot, let $M = S^3 \backslash K$, and let $G = \pi_1(M)$

Figure-8 Example

Consider the following example.



Let K be the figure-8 knot, let $M = S^3 \backslash K$, and let $G = \pi_1(M)$

Theorem 5 (B)

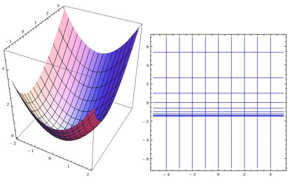
There exists $\varepsilon > 0$ such that for each $t \in (-\varepsilon, \varepsilon)$ there is a properly convex domain Ω_t and a discrete group $\Gamma_t \leq \mathrm{PGL}(\Omega_t)$ such that

- $\Omega_t/\Gamma_t \cong M$,
- Ω_0/Γ_0 *is the complete hyperbolic structure on M, and*
- *If $t \neq 0$ then Ω_t is not strictly convex.*

Figure-8 Example

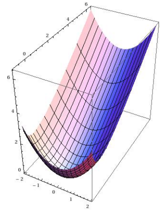
Theorem 6 (B)

For each $t \in (-\varepsilon, \varepsilon)$ we can decompose Ω_t/Γ_t as $M_K^t \bigsqcup C^t$, where M_K^t is compact and $C^t \cong T^2 \times [1, \infty)$.



- For each t, $C^t \cong B_t/\Delta_t$, where Δ_t is a lattice an Abelian group P_t of "translations," and B_t is a "horoball" bounded by an orbit of P_t.

Figure-8 Example



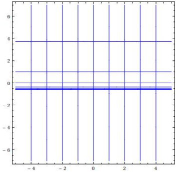


Figure-8 Example

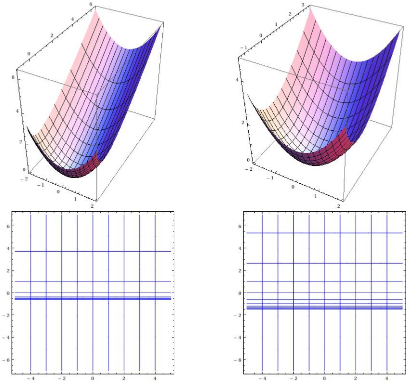


Figure-8 Example

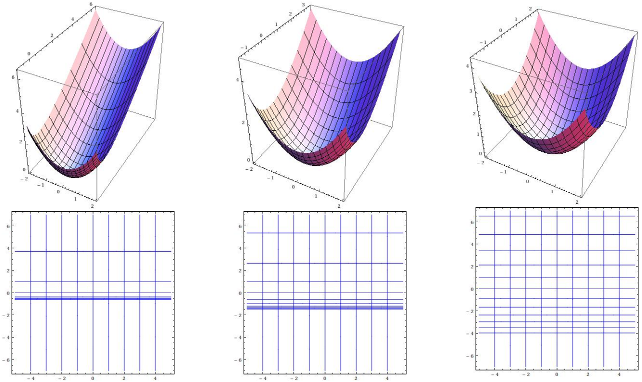


Figure-8 Example

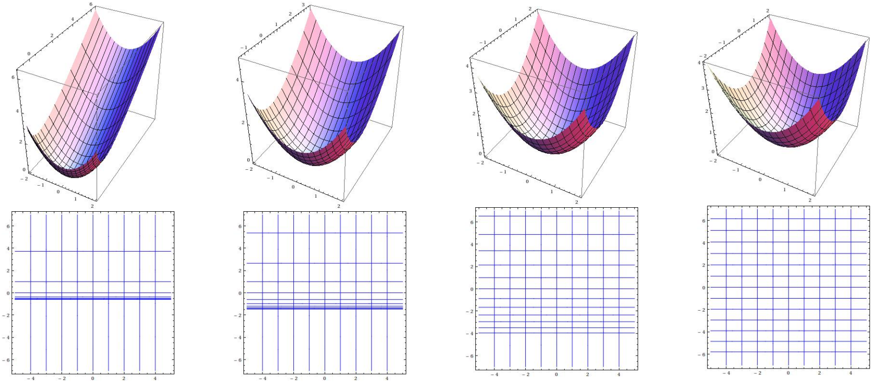
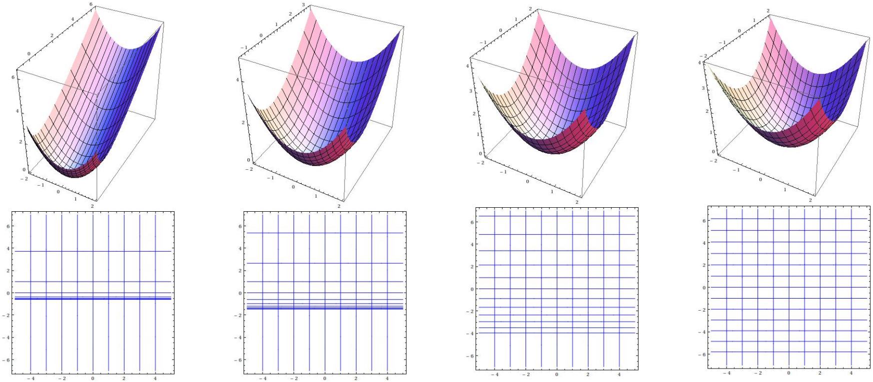
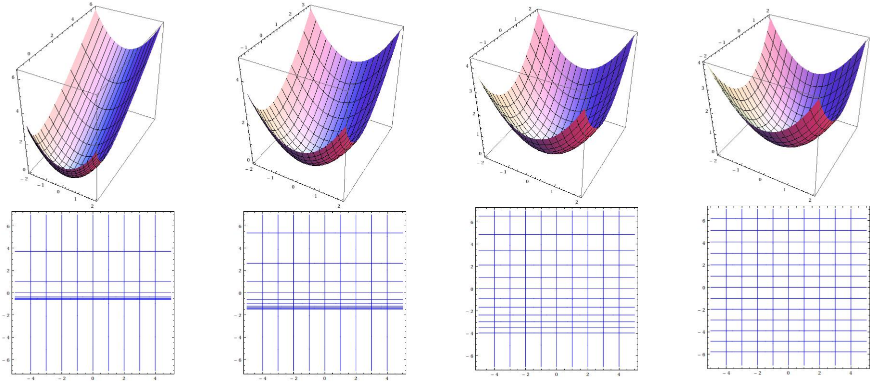


Figure-8 Example



- For each $t \neq 0$ there is $1 \neq \gamma_t \in \Gamma_t$ such that γ_t is hyperbolic, freely homotopic into C^t, but not positive proximal.

Figure-8 Example



- For each $t \neq 0$ there is $1 \neq \gamma_t \in \Gamma_t$ such that γ_t is hyperbolic, freely homotopic into C^t, but not positive proximal.
- Ω_t contains non-trivial line segments in $\partial\Omega_t$ that are preserved by conjugates of Δ_t. In particular, Ω_t is not δ-hyperbolic.

Figure-8 Example

Theorem 7 (B, Long)

$1 \neq \gamma \in \Gamma_t$ *is positive proximal if and only if it cannot be freely homotoped into* C^t.

Figure-8 Example

Theorem 7 (B, Long)

$1 \neq \gamma \in \Gamma_t$ *is positive proximal if and only if it cannot be freely homotoped into* C^t*.*

Proof.

$\Leftarrow$ Let $1 \neq \gamma \in \Gamma_t$. No elements of P_t are positive proximal, so if γ is freely homotopic to C^t then it is not positive proximal.

Figure-8 Example

Theorem 7 (B, Long)

$1 \neq \gamma \in \Gamma_t$ *is positive proximal if and only if it cannot be freely homotoped into* C^t*.*

Proof.

$\Leftarrow$ Let $1 \neq \gamma \in \Gamma_t$. No elements of P_t are positive proximal, so if γ is freely homotopic to C^t then it is not positive proximal.

$\Rightarrow$ If γ is not freely homotopic to C^t then γ has positive translation length and is thus hyperbolic. Furthermore, this translation length is realized by points on an axis.

Figure-8 Example

Proof (Continued).

Use Margulis lemma to construct a disjoint and Γ_t invariant collection $\mathcal{H}_t$ of horoballs in Ω_t.

Figure-8 Example

Proof (Continued).

Use Margulis lemma to construct a disjoint and Γ_t invariant collection $\mathcal{H}_t$ of horoballs in Ω_t.
let $\hat{\Omega}_t$ be the *electric space* obtained by collapsing the horospherical boundary components of $\Omega_t \backslash \mathcal{H}_t$.

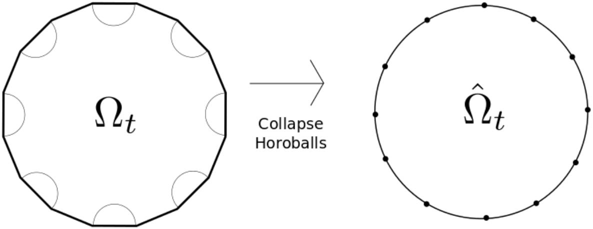
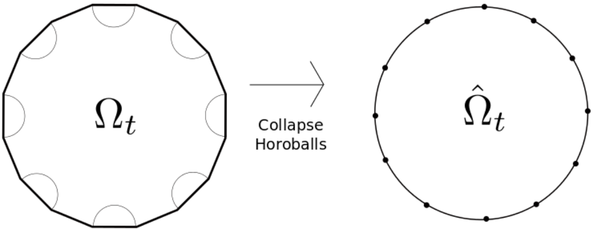


Figure-8 Example

Proof (Continued).

Use Margulis lemma to construct a disjoint and Γ_t invariant collection $\mathcal{H}_t$ of horoballs in Ω_t.
let $\hat{\Omega}_t$ be the *electric space* obtained by collapsing the horospherical boundary components of $\Omega_t \backslash \mathcal{H}_t$.



Lemma 8 (B, Long)

$\hat{\Omega}_t$ is δ-hyperbolic

Figure-8 Example

Proof (Continued).

- Since γ is hyperbolic and preserves Ω_t we know that γ has real eigenvalues of largest and smallest modulus and that these eigenvalues have the same sign.

Figure-8 Example

Proof (Continued).

- Since γ is hyperbolic and preserves Ω_t we know that γ has real eigenvalues of largest and smallest modulus and that these eigenvalues have the same sign.
- If γ is not positive proximal then there will be a γ-invariant set $T \subset \Omega_t$ disjoint from all the horoballs that contains a positive dimensional flat in its boundary

Figure-8 Example

Proof (Continued).

- Since γ is hyperbolic and preserves Ω_t we know that γ has real eigenvalues of largest and smallest modulus and that these eigenvalues have the same sign.
- If γ is not positive proximal then there will be a γ-invariant set $T \subset \Omega_t$ disjoint from all the horoballs that contains a positive dimensional flat in its boundary

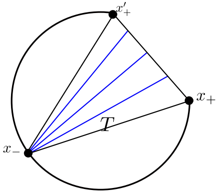
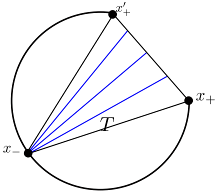


Figure-8 Example

Proof (Continued).

- Since γ is hyperbolic and preserves Ω_t we know that γ has real eigenvalues of largest and smallest modulus and that these eigenvalues have the same sign.
- If γ is not positive proximal then there will be a γ-invariant set $T \subset \Omega_t$ disjoint from all the horoballs that contains a positive dimensional flat in its boundary



- This gives rise to arbitrarily fat triangles in $\hat{\Omega}_t$

Summary and Questions

- The structure of a finite volume strictly convex manifold is well understood.

Summary and Questions

- The structure of a finite volume strictly convex manifold is well understood.
- As you deform the structure the “coarse” geometry of the compact part doesn’t change.

Summary and Questions

- The structure of a finite volume strictly convex manifold is well understood.
- As you deform the structure the “coarse” geometry of the compact part doesn’t change.
- The geometry of the cusps may change as we deform, but can be understood using the Margulis lemma.

Summary and Questions

- The structure of a finite volume strictly convex manifold is well understood.
- As you deform the structure the “coarse” geometry of the compact part doesn’t change.
- The geometry of the cusps may change as we deform, but can be understood using the Margulis lemma.
- Theorem 7 holds for all properly convex deformations of finite volume strictly convex manifolds in dimension 3

Summary and Questions

- The structure of a finite volume strictly convex manifold is well understood.
- As you deform the structure the “coarse” geometry of the compact part doesn’t change.
- The geometry of the cusps may change as we deform, but can be understood using the Margulis lemma.
- Theorem 7 holds for all properly convex deformations of finite volume strictly convex manifolds in dimension 3
- Theorem 7 should hold for higher dimensions.

Summary and Questions

- The structure of a finite volume strictly convex manifold is well understood.
- As you deform the structure the "coarse" geometry of the compact part doesn't change.
- The geometry of the cusps may change as we deform, but can be understood using the Margulis lemma.
- Theorem 7 holds for all properly convex deformations of finite volume strictly convex manifolds in dimension 3
- Theorem 7 should hold for higher dimensions.
- What can we say for deformations of deformations of infinite volume hyperbolic manifolds?