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Flexibility and Rigidity of Three-Dimensional Convex Projective Structures

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Flexibility and Rigidity of Three-Dimensional Convex Projective Structures

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Dedicated to my wife Nicole.

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Flexibility and Rigidity of Three-Dimensional Convex Projective Structures

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This thesis investigates various rigidity and flexibility phenomena of convex projective structures on hyperbolic manifolds, particularly in dimension 3. Let M^n be a finite volume hyperbolic n-manifold where $n \geq 3$ and Γ be its fundamental group. Mostow rigidity tells us that there is a unique conjugacy class of discrete faithful representation of Γ into $\mathrm{PSO}(n, 1)$. In light of this fact we examine when this representations can be non-trivially deformed into the larger Lie group of $\mathrm{PGL}_{\mathrm{n+1}}(\mathbb{R})$ as well as the relationship between these deformations and convex projective structures on M. Specifically, we show that various two-bridge knots do not admit such deformations into $\mathrm{PGL}_4(\mathbb{R})$ satisfying certain boundary conditions. We subsequently use this result to show that certain orbifold surgeries on amphicheiral knot complements do admit deformations.

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Chapter 1

Introduction

Mostow-Prasad rigidity for hyperbolic manifolds is a crucial tool for understanding the deformation theory of lattices in $\text{Isom}(\mathbb{H}^n)$. Specifically, it tells us that the fundamental groups of hyperbolic manifolds of dimension $n \geq 3$ admit a unique conjugacy class of discrete, faithful representations of their fundamental group into $\text{Isom}(\mathbb{H}^n)$.

Using the Klein model we can view hyperbolic structures on manifolds as specific instances of strictly convex projective structures. Recent work of [5–7, 17] has revealed several parallels between the geometry of hyperbolic n-space and the geometry of arbitrary strictly convex domains in $\mathbb{RP}^n$. For example, the classification and interaction of isometries of strictly convex domains is analogous to the situation in hyperbolic geometry. Additionally, if the isometry group of the domain is sufficiently large then the strictly convex domain is known to be δ-hyperbolic. Despite the many parallels between these two types of geometry, there is no analogue of Mostow-Prasad rigidity for arbitrary strictly convex domains. This observation prompts the following question: is it possible to deform the hyperbolic structure on a finite volume manifold to a non-hyperbolic strictly convex structure on the same manifold?

Currently, the answer is known only in certain special cases. For example, when the manifold contains a totally geodesic, hypersurface there exist non-trivial deformations at the level of representations coming from the bending construction of Johnson and Millson [30]. In the closed case, work of Koszul [33] shows that these new projective structures arising from bending remain properly convex. Further work of Benoist [6] shows that these structures are actually strictly convex. In the non-compact case recent work of Marquis [37] has shown that the projective structures arising from bending remain properly convex in this setting as well.

In contrast to the previous results, there are examples of closed 3-manifolds for which no such deformations exist (see [16]). Additionally, there exist 3-manifolds that contain no totally geodesic surfaces, yet still admit deformations (see [15]). Henceforth, we will refer to these deformations that do not arise from bending as *flexing deformations*. Prompted by these results a natural question to ask is whether or not there exist flexing deformations for non-compact finite volume hyperbolic manifolds.

Subtleties arising from the presence of peripheral subgroups complicate the non-compact situation making it more difficult to analyze. For example, while Mostow-Prasad rigidity guarantees the uniqueness of *complete* structures on finite volume 3-manifolds, work of Thurston [47] shows that if we remove the completeness hypothesis then there is an interesting deformation theory of representations into $\mathrm{PSL}_2(\mathbb{C})$ for cusped hyperbolic 3-manifolds. In order to obtain a complete hyperbolic structure we must insist that the holonomy of the peripheral subgroup be parabolic. As we shall see, there is a similar boundary condition for representations that must be satisfied in order for the deformation to correspond to complete strictly convex structures, and we use this fact to analyze strictly convex structures on certain knot and link complements.

Two bridge knots and links provide a good place to begin our analysis of strictly convex deformations because they have particularly simple presentations for their fundamental groups making them amenable to the normal forms techniques. Additionally, work of [25] has shown that they contain no closed, totally geodesic, embedded surfaces and thus there are no bending deformations. Using the normal form techniques developed in Chapter 4 we are able to prove that several two bridge knot and link complements enjoy a certain rigidity property.

Theorem 4.2.1. *The two bridge knots and links with rational number* $\frac{5}{3}$ *(figure-eight),* $\frac{7}{3}$*,* $\frac{9}{5}$*, and* $\frac{8}{3}$ *(Whitehead link) do not admit strictly convex deformations near their complete hyperbolic structures.*

In [28] it is shown that there is a strong relationship between deformations of a cusped hyperbolic 3-manifold and deformations of surgeries on that manifold. In particular they are able to use the fact that the figure-eight knot is infinitesimally projectively rigid relative to the boundary to deduce that there are strictly convex deformations of certain orbifold surgeries of the figure-eight knot. We are able to extend this result to all amphicheiral knot complements that enjoy a certain rigidity property in the following theorem.

Theorem 5.3.1. *Let M be the complement of a hyperbolic, amphicheiral knot, and suppose that M is infinitesimally projectively rigid relative to the boundary and the longitude is a rigid slope. Then for sufficiently large n, $M(n/0)$ has a one dimensional space of strictly convex projective deformations near the complete hyperbolic structure.*

Here $M(n/0)$ is the orbifold obtained by surgering a solid torus with cone singularities along its longitude of cone angle $2\pi/n$ along the meridian of M.

The organization of the thesis is as follows. Chapter 2 provides some background material on geometric structures, hyperbolic geometry, and representation spaces. Chapter 3 discusses projective geometry, convex structures, and convex deformations. Section 4 is dedicated to the setup and proof of Theorem 4.2.1. In Chapter 5 set up and prove Theorem 5.3.1. Finally, in Chapter 6 we discuss future directions of research.

Chapter 2

Background

Throughout this chapter, unless otherwise stated, M will be a connected, compact, orientable manifold (possibly with boundary).

2.1 (G, X)-structures

Let X be a manifold on which a Lie group G acts analytically by diffeomorphisms, where analyticity means that the action of an element of G is determined by its restriction to an open subset of X. If U is an open subset of X, then a function $f : U \to X$ is *locally* (G, X) if for each connected component U_i of U there exists a (necessarily unique) $g_i \in G$ such that $g_i \mid_{U_i} = f$. A (G, X)-*atlas* on M is a collection $\{U_\alpha, \phi_\alpha\}$ such that

1. $\{U_\alpha\}$ is a covering of M by open sets,
2. $\{\phi_\alpha : U_\alpha \to X\}$ is a collection of maps such that if $U_\alpha \cap U_\beta \neq \emptyset$ then $\phi_\alpha \circ \phi_\beta^{-1}$ restricted to $\phi_\beta(U_\alpha \cap U_\beta)$ is locally (G, X), and
3. $\phi_\alpha(\partial M \cap U_\alpha)$ is a smooth embedding into X.

A (G, X)-*structure* on M is a maximal (G, X)-atlas on M. It is worth noting that this definition implies that M and X have the same dimension. If M and

N are two (G, X)-manifolds, then a map ξ between them is a (G, X)-*map* if for each chart (U, ϕ) of M and (V, ψ) of N such that U and $\xi^{-1}(V)$ overlap the function

$$\psi\xi\phi^{-1} : \phi(U \cap \xi^{-1}(V)) \to \psi(\xi(U) \cap V)$$

agrees with an element of G in a neighborhood of each point of its domain. Typically we think of X as being endowed with some sort of geometry and G as being the group of transformations that preserve the geometry.

A simple way to get new (G, X)-manifolds from existing ones it to pass to covers. If $p : M' \to M$ is a covering and M has a (G, X)-structure then we get a (G, X)-structure on M' as follows. Pick an atlas for M consisting of simply connected, evenly covered (with respect to p) sets $\{(U_\alpha, \phi_\alpha)\}$. For each α let $\{(U^j_\alpha, \phi^j_\alpha)\}$ be pairs such that U^j_α evenly covers U_α and ϕ^j_α is a lift of ϕ_α restricted to U^j_α. The union of these sets over α forms a (G, X)-atlas of M' for which p is a (G, X)-map. In fact this is the unique (G, X)-structure on M' such that p is a (G, X)-map.

The local nature of our definition of (G, X)-structures makes it hard to gain insight into global properties of (G, X)-manifolds. Fortunately, there is a tool for globalizing the data of a (G, X)-structure on a manifold M, which takes the shape of a local diffeomorphism $D : \tilde{M} \to X$ where $\tilde{M}$ is the universal cover of M. The map D is called the *developing map* and is constructed via analytic continuation as follows (see Figure 2.1). Pick a chart (U_0, ϕ_0) and a base point $q \in U_0$ and recall that $\tilde{M}$ can be identified with the space of homotopy classes relative to endpoints of paths $\gamma : [0, 1] \to M$ such that $\gamma(0) = q$. Suppose

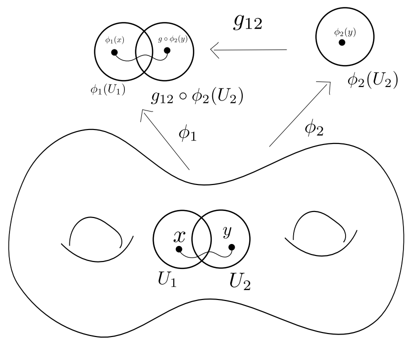

Figure 2.1: Developing via analytic continuation

that $[\gamma(t)]$ is a homotopy class with representative $\gamma(t)$. If $\gamma(t)$ is contained entirely in U_0 then $D(\gamma(t)) = \phi_0(\gamma(1))$. If $\gamma(t)$ is not contained in U_0 then pick charts $\{(U_i, \phi_i)\}_{i=1}^k$ that cover $\gamma(t)$ such that consecutive charts overlap. By the definition of the (G, X)-structure we know that there exists transition functions $g_{ii+1} \in G$ that agree with $\phi_i \circ \phi_{i+1}^{-1}$. Therefore, if $\gamma(1) \in U_i$ then we define

$$D(\gamma(t)) = g_{01} g_{12} \dots g_{i-1i} \phi_i(\gamma(1)).$$

It is a simple exercise to show that this map is well defined independent of the covering and the choice of representative $\gamma(t)$. In fact, the only choice that our construction depended on was the initial chart U_0 and the base point q. Additionally, D is a (G, X)-map if we equip $\tilde{M}$ with the (G, X)-structure induced by M. Next we examine the ambiguity of developing maps arising from choices of initial chart and base point. Before proceeding we prove the following proposition

Proposition 2.1.1. *Let M be a simply connected (G, X)-manifold and let $f_1, f_2 : M \to X$ be (G, X)-maps, then there is a unique element $g \in G$ such that $f_2 = gf_1$*

Proof. Pick a chart $\phi : U \to X$ such that ϕf_i^{-1} is a chart of X for $i = 1, 2$. Using the same analytic continuation technique we used to construct the developing map we can extend ϕf_i^{-1} to a (G, X)-map on all of X. However, a self (G, X)-map of X is just an element $g_i \in G$. Therefore we see that $\phi = g_i f_i$ for $i = 1, 2$ and since $g_1 f_1 = g_2 f_2$ on an open set, analyticity tells us that they must be equal and so we can choose $g = g_2^{-1} g_1$. Finally, analyticity again tells us that g is unique. □

Let D_1 and D_2 be developing maps coming from different choices of initial charts, then Proposition 2.1.1 tells us that $D_2 = gD_1$ for a unique $g \in G$, which we record in the following corollary.

Corollary 2.1.2. *Let D_1 and D_2 be developing maps for a (G, X)-structure on M then there exists a unique $g \in G$ such that $D_2 = gD_1$.*

Another consequence of Proposition 2.1.1 is that it allows us to attach an algebraic object to a (G, X)-structure. Let M be a (G, X)-manifold, let $\Gamma = \pi_1(M)$, and let D be a developing map for this structure. If $\gamma \in \Gamma$ (which we think of as being the group of deck transformations of $\tilde{M}$) then Proposition 2.1.1 tells us that there is a unique element $\rho(\gamma) \in G$ such that

$$D\gamma = \rho(\gamma)D. \tag{2.1}$$

Equation (2.1) tells us that the map D is *equivariant* with respect to ρ. It is a simple exercise to show that $\rho : \Gamma \to G$ defines a homomorphism which we call a *holonomy representation*. Finally, Corollary 2.1.2 also tells us that if we choose a different developing map $D' = gD$ then the holonomy ρ' is given by

$$\rho'(\gamma) = g\rho(\gamma)g^{-1} \tag{2.2}$$

2.1.1 Representation and Character Varieties

The relationship between (G, X)-structures and representations requires us to understand the set of representations from a fixed finitely presented group Γ into a fixed Lie group G. When the group G is algebraic this set can be endowed with extra algebraic and geometric structure that can simplify their study. In this section we review some basics of representation varieties for algebraic groups. Let G be an algebraic group, let $\Gamma = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_m \rangle$ be a finitely presented group, and let $\mathcal{R}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)$, which we now refer to as the *G-representation variety* of Γ into G or just the *representation variety* when no confusion concerning Γ or G will arise. This presentation allows us to identify $\mathcal{R}(\Gamma, G)$ with the subset in G^n of points where each of the words r_i is equal to the identity in G. Since G is algebraic, multiplication and inversion are polynomial maps and this identifies $\mathcal{R}(\Gamma, G)$ with an algebraic variety. If we select a different presentation $\Gamma = \langle g'_1, \ldots, g'_k \mid r_1, \ldots, r_l \rangle$, then we will get a different realization of $\mathcal{R}(\Gamma, G)$ inside G^k, but the two will be isomorphic as varieties (see [46] for details). One of the utilities of this construction is that it provides a way to concretely think about points of $\mathcal{R}(\Gamma, G)$

as solutions to a finite set of polynomial equations.

There is a natural action of G on $\mathcal{R}(\Gamma, G)$ by conjugation, and we define the *G-character variety* to be the set of conjugacy classes of representations of Γ into G, which we denote $\mathfrak{X}(\Gamma, G)$. When no confusion can occur we will often refer to this set as the *character variety*. Due to pathologies of this action, $\mathfrak{X}(\Gamma, G)$ may not have the global structure of a variety (see [40] for details). However, in all cases of interest to us the action of G will be nice enough to guarantee that $\mathfrak{X}(\Gamma, G)$ has the local structure of a variety, and in many instances $\mathfrak{X}(\Gamma, G)$ can be identified with the set of characters of representations of Γ into G. For more details about the variety structure of $\mathfrak{X}(\Gamma, G)$ and the relationship to characters see [22, §3] and [18].

2.1.2 Deformation Spaces

The discussion of the previous few paragraphs can be packaged nicely as follows. Let $S(M; G, X)$ be the set of all (G, X)-structures on M. $S(M; G, X)$ can be realized as the quotient of G acting on the space of developing maps of M by post-composition, and so the compact C^∞ topology on the space of developing maps induces a topology on $S(M; G, X)$. If we topologize $\mathfrak{X}(\Gamma, G)$ using the compact-open topology[1] on $\mathcal{R}(\Gamma, G)$ then there is a continuous map

$$hol : S(M; G, X) \to \mathfrak{X}(\Gamma, G)$$

[1]Since Γ is finitely generated this is the same as the topology of pointwise convergence on a generating set.

that associates to a (G, X)-structure the conjugacy class of the holonomy of one of its developing maps. By Corollary 2.1.2 and the discussion immediately thereafter we see that this map is well defined.

In general, the fibers of this map can be quite complicated, but they have nice local structure. In particular, they form equivalence classes of a natural equivalence relation on $S(M; G, X)$ which we now describe. A *marked* (G, X)*-structure* on M is a pair (N, f) where N is a (G, X)-manifold and $f : M \to N$ is a diffeomorphism, which we think of a way of identifying the topological manifold M with the geometric manifold N. Two marked (G, X)-structures (N_i, f_i) $i = 1, 2$ are *isotopic* if there exists a (G, X)-diffeomorphism ϕ *defined on all but a collar neighborhood of* ∂N_1 *and onto all but a collar neighborhood of* ∂N_2 (see Remark 2.1.1) such that ϕ is isotopic to $f_2 f_1^{-1}$. We call the set of isotopy classes of (G, X)-structures on M the *deformation space* of (G, X)-structures on M and denote it $D(M; G, X)$. We can topologize $D(M; G, X)$ using the quotient topology. It is a simple exercise to see that isotopic (G, X)-structures induce the same conjugacy class of representations and so *hol* descends to a map which we continue to call *hol* from $D(M; G, X)$ to $\mathfrak{X}(\Gamma, G)$. The following theorem concerning *hol* is originally due to Ehresmann and Thurston (see [13, 22] for details).

Theorem 2.1.3. *The map hol descends to* $D(M; G, X)$ *on which it is a local homeomorphism.*

We think of Theorem 2.1.3 as saying that, up to isotopy, (G, X) structures on M are locally parameterized by conjugacy class of representation from

Γ to G.

Remark 2.1.1. In [13] it is shown that when M has boundary, any (G, X)-structure on M is induced by a (G, X)-structure on a manifold without boundary, of the same dimension, containing M as a a submanifold. The manifold without boundary is called a thickening and is denoted M_T. Topologically, M_T is obtained from M by adding an open collar, and therefore has the same fundamental group as M. Theorem 2.1.3 is proven by defining structures on M_T and restricting to M. If two (G, X)-structures on M have the same holonomy they are induced by two different embeddings M into M_T. For this reason we will often blur the distinction between (G, X) structures on M and (G, X)-structures on the interior of M when M has boundary.

As an example consider a hyperbolic surface S_1 with totally geodesic boundary of length L. This can be embedded into a hyperbolic surface S_∞ with an infinite volume funnel as an end. If we truncate the end of S_∞ so that it is totally geodesic of length $2L$ we will get another hyperbolic surface S_2 that is isotopic to S_1 and thus has the same holonomy (see Figure 2.2).

We close this section by discussing how properties of the developing map translate into properties of (G, X)-structures. We say a representation ρ is *discrete* if it has discrete image and that a group Γ acts *properly discontinuously* on X if for each compact set $K \subset X$ the set $\{\gamma \in \Gamma | \gamma K \cap K \neq \emptyset\}$ is finite. When M has no boundary we say that a (G, X)-structure on M is *complete* if the developing map D is a covering map onto its image. In the case where X is simply connected the developing map is a homeomorphism onto its image and

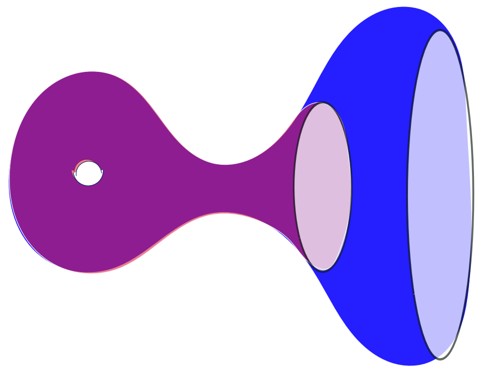

Figure 2.2: Two isotopic hyperbolic surfaces S_1 (purple) and S_2 (blue) are superimposed on top of one another.

we can identify $D(\tilde{M}) \subset X$ with $\tilde{M}$. In this case the holonomy representation ρ is discrete, faithful, and has a properly discontinuous action that allows us to identify M with $D(\tilde{M})/\rho(\Gamma)$.

2.2 Local and Infinitesimal Deformations

We have previously seen that $\mathfrak{X}(\Gamma, G)$ locally parameterizes isotopy classes of (G, X)-structures on M, and so to understand small deformations of geometric structures it suffices to understand the local structure of $\mathfrak{X}(\Gamma, G)$. Let $\rho_0 : \Gamma \to G$ be a representation and $[\rho_0]$ be the class it represents in $\mathfrak{X}(\Gamma, G)$. A *deformation* of ρ_0 is a smooth[2] map $\sigma(t) : (-\varepsilon, \varepsilon) \to \mathfrak{R}(\Gamma, G)$ such that $\sigma(0) = \rho_0$. To simplify notation we will denote $\sigma(t)$ by ρ_t. A deformation is *non-trivial* if $t \mapsto [\rho_t]$ is not the constant path in $\mathfrak{X}(\Gamma, G)$. If $[\rho_0]$ is an isolated point of $\mathfrak{X}(\Gamma, G)$ then we say that $[\rho_0]$ is *locally G-rigid* at ρ_0 or just

[2]By smooth we mean that for each $\gamma \in \Gamma$, the map $t \mapsto \sigma(t)(\gamma)$ is smooth near $t = 0$

locally rigid when the group G and the representation ρ_0 are clear from the context. From the correspondence arising from Theorem 2.1.3 we see that if $[\rho_0]$ is locally rigid then the isotopy class of any geometric structure on M with holonomy ρ_0 is an isolated point in $D(M; G, X)$ and its geometry cannot be deformed.

2.2.1 Infinitesimal Deformations

In practice it is often very difficult to understand $\mathfrak{X}(\Gamma, G)$ locally near a particular representation ρ_0. In order to simplify the problem we will study $\mathfrak{X}(\Gamma, G)$ infinitesimally, which allows us to linearize the problem.

Let $\mathfrak{g}$ be the Lie algebra of G (which we think of as being the tangent space to G at the identity). The group G admits an action on $\mathfrak{g}$ as follows. For each $g \in G$ define $\psi_g : G \to G$ by $\psi_g(h) = ghg^{-1}$. Since ψ_g fixes the identity, the derivative of ψ_g at the identity gives an automorphism of $\mathfrak{g}$. Thus we get a map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ by mapping g to the derivative of ψ_g at the identity. This gives rise to an action that we call the *adjoint action* and we denote the adjoint action of an element $g \in G$ on an element $h \in \mathfrak{g}$ by $\mathrm{Ad}_g \cdot h$. When $G = \mathrm{GL}_n(\mathbb{R})$ then $\mathfrak{g} = \mathrm{End}(\mathbb{R}^n)$ and the adjoint action is given by matrix conjugation.

Let $\rho_t : \Gamma \to G$ be a deformation. Then we can define a map $z : \Gamma \to \mathfrak{g}$ by $\gamma \mapsto \rho'(\gamma)\rho_0(\gamma)^{-1}$, where $\rho'(\gamma) = \left(\frac{d}{dt}\big|_{t=0}\rho_t(\gamma)\right)$. From the homomorphism condition for ρ_t and the Leibniz rule for derivatives of products, we see that if

$\gamma, \gamma' \in \Gamma$ then

$$\begin{aligned} z(\gamma\gamma') &= \rho'(\gamma\gamma')\rho_0(\gamma\gamma')^{-1} = (\rho(\gamma)\rho(\gamma'))'\,\rho_0(\gamma\gamma')^{-1} \\ &= (\rho'(\gamma)\rho_0(\gamma') + \rho_0(\gamma)\rho'(\gamma'))\,\rho_0(\gamma\gamma')^{-1} \\ &= \rho'(\gamma)\rho_0(\gamma)^{-1} + \rho_0(\gamma)\rho'(\gamma')\rho_0(\gamma')\rho_0(\gamma) \\ &= \rho'(\gamma)\rho_0(\gamma)^{-1} + \mathrm{Ad}_{\rho_0(\gamma)} \cdot \rho'(\gamma')\rho_0(\gamma')^{-1} \\ &= z(\gamma) + \mathrm{Ad}_{\rho_0(\gamma)} \cdot z(\gamma'). \end{aligned} \tag{2.3}$$

Equation (2.3) is known as the *cocycle condition* and we denote the space of functions satisfying the cocycle condition by $Z^1(\Gamma, \mathfrak{g}_{\rho_0})$ and refer to its elements as *group cocycles.* Next, suppose that ρ_t is a trivial deformation, that is $\rho_t = g_t\rho_0 g_t^{-1}$, where g_t is a smooth path in G such that g_0 is the identity. A simple computation similar to Equation (2.3) shows that the cocycle of this path is given by

$$z(\gamma) = g' - \mathrm{Ad}_{\rho_0(\gamma)} \cdot g'. \tag{2.4}$$

Cocycles that satisfy Equation (2.4) for some element $g' \in \mathfrak{g}$ are called *group coboundaries* and we denote the space of coboundaries by $B^1(\Gamma, \mathfrak{g}_{\rho_0})$. The quotient $H^1(\Gamma, \mathfrak{g}_{\rho_0}) := Z^1(\Gamma, \mathfrak{g}_{\rho_0})/B^1(\Gamma, \mathfrak{g}_{\rho_0})$ is the first group cohomology of Γ with coefficients in $\mathfrak{g}$ (see [11] for more details about group cohomology).

There is also a notion of cohomology with twisted coefficients for manifolds which we denote $H^*(M, \mathfrak{g}_{\rho_0})$, where ρ_0 is a representation from $\pi_1(M)$ to G (see [26, 28] for details). In most cases of interest to us the manifold M will be aspherical, in which case there is a natural isomorphism between

$H^*(\pi_1(M), \mathfrak{g}_{\rho_0})$ and $H^*(M, \mathfrak{g}_{\rho_0})$ [51]. For this reason we will frequently not distinguish between these two cohomology theories.

In this context we think of elements of $Z^1(\Gamma, \mathfrak{g}_{\rho_0})$ as parameterizing *infinitesimal deformations* near ρ_0 and that if ρ_0 is a smooth point of $\mathcal{R}(\Gamma, G)$ then $Z^1(\Gamma, \mathfrak{g}_{\rho_0})$ can be identified with the tangent space to $\mathcal{R}(\Gamma, G)$ at ρ_0 (when ρ_0 is not a smooth point $Z^1(\Gamma, \mathfrak{g}_{\rho_0})$ can still be identified with the Zariski tangent space at ρ_0 [35, §2]). The coboundaries infinitesimally parametrize trivial deformations, and the group $H^1(\Gamma, \mathfrak{g}_{\rho_0})$ infinitesimally parameterizes $\mathfrak{X}(\Gamma, G)$ near ρ_0.

When $H^1(\Gamma, \mathfrak{g}_{\rho_0}) = 0$ we say that Γ is *infinitesimally G-rigid* at ρ_0 or just *infinitesimally rigid* if no confusion about G or ρ_0 will arise. The following theorem of Weil [50], whose proof is essentially contained in the previous paragraphs, determines the relationship between infinitesimal and local rigidity.

Theorem 2.2.1. *If Γ is infinitesimally G rigid at ρ_0 then Γ is locally G rigid at ρ_0.*

Remark 2.2.1. More generally, the dimension of $H^1(\Gamma, \mathfrak{g}_{\rho_0})$ is an upper bound for the dimension of $\mathfrak{X}(\Gamma, G)$ at $[\rho_0]$ (see [35]), however if $[\rho_0]$ is not a smooth point of $\mathfrak{X}(\Gamma, G)$ then this bound need not be sharp and so the converse to Theorem 2.2.1 is false in general.

2.3 Projective Geometry

In this section we review some preliminaries of projective geometry and its relationship to other geometries

2.3.1 Projective Space

Let $\mathbb{R}^{n+1}$ be the real vector space of dimensions $n+1$. We define an equivalence relationship $\sim$ on $\mathbb{R}^{n+1}\backslash\{0\}$ by $x \sim \lambda x$, where $\lambda \in \mathbb{R}^{\times}$ is a non-zero real number. Equivalence classes $[v]$ of $\sim$ are called *lines* and the quotient of $\mathbb{R}^{n+1}$ by $\sim$ is known as *real projective n-space* and is denoted $\mathbb{RP}^n$. $\mathbb{RP}^n$ can also be realized as the quotient of the n-dimensional sphere $\mathbb{S}^n$ by the antipodal map, and is thus easily seen to be a manifold of dimension n. A *projective line* is the image of a 2-dimensional subspace of $\mathbb{R}^{n+1}$ in $\mathbb{RP}^n$ and a *projective hyperplane* is the image of an n-dimensional subspace of $\mathbb{R}^{n+1}$ in $\mathbb{RP}^n$.

An element $A \in \mathrm{GL_{n+1}}(\mathbb{R})$ sends lines to lines and thus descends to a self map of $\mathbb{RP}^n$. Conversely, every automorphism of $\mathbb{RP}^n$ arises in this way. Additionally, an element acts as the identity on $\mathbb{RP}^n$ if and only if $A = \lambda I_{n+1}$, where I_{n+1} is the $n \times n$ identity matrix and $\lambda \in \mathbb{R}^{\times}$, thus we identify the automorphism group of $\mathbb{RP}^n$ with $\mathrm{GL_{n+1}}(\mathbb{R})/\{\lambda I_{n+1}\}$ and denote it $\mathrm{PGL_{n+1}}(\mathbb{R})$.

The structure of $\mathbb{RP}^n$ and $\mathrm{PGL_{n+1}}(\mathbb{R})$ depends on whether n is even or odd. When n is even the antipodal map is orientation reversing and $\mathbb{RP}^n$ is non-orientable. In this case, $\mathrm{PGL_{n+1}}(\mathbb{R})$ is connected and isomorphic to

$\mathrm{SL}_{n+1}(\mathbb{R})$. When n is odd the antipodal map is orientation preserving and $\mathbb{RP}^n$ is orientable. In this case, $\mathrm{PGL_{n+1}}(\mathbb{R})$ has two connected components and the identity component is isomorphic to $\mathrm{PSL}_{n+1}(\mathbb{R})$ (see [23, §2] for details)

Fixed points of elements in $\mathrm{PGL_{n+1}}(\mathbb{R})$ correspond to eigenvectors of elements of $\mathrm{GL_{n+1}}(\mathbb{R})$ with *real* eigenvalues. As a consequence we see that when n is even every element of $\mathrm{PGL_{n+1}}(\mathbb{R})$ fixes a point in $\mathbb{RP}^n$, whereas when n is odd there exist elements of $\mathrm{PGL_{n+1}}(\mathbb{R})$ that have no fixed points.

We call (G, X)-structures where $X = \mathbb{RP}^n$ and $G = \mathrm{PGL_{n+1}}(\mathbb{R})$ *projective structures*. In most examples of interest the developing map of a projective structure is not a surjection. We will discuss several such examples in the subsequent section. One benefit of studying projective geometry is that it contains many other geometries and thus provides a unified setting to examine the interaction between different types of geometries.

2.3.2 Affine/Euclidean Geometry

Let $\mathbb{A}^n$ denote the affine plane in $\mathbb{R}^{n+1}$ given by the equation $x_{n+1} = 1$. No two points in $\mathbb{A}^n$ belong to the same line and so we get an embedding of $\mathbb{A}^n$ into $\mathbb{RP}^n$ which we also refer to as $\mathbb{A}^n$. Additionally, $\mathbb{A}^n$ contains a representative of every line not contained in the plane $x_{n+1} = 0$ and we see that $\mathbb{RP}^n$ can be decomposed as $\mathbb{A}^n \sqcup \mathbb{RP}^{n-1}$. If $A \in \mathrm{PGL_{n+1}}(\mathbb{R})$ preserves $\mathbb{A}^n$ then A must also preserve the copy of $\mathbb{RP}^{n-1}$ coming from the plane $x_{n+1} = 0$

and therefore we can find a representative of A in $\mathrm{GL_{n+1}}(\mathbb{R})$ of the form

$$\begin{pmatrix} B & c \\ 0 & 1 \end{pmatrix},$$

where $B \in \mathrm{GL_n}(\mathbb{R})$ and $c \in \mathbb{R}^n$ is a vector. Under the map

$$\begin{pmatrix} B & c \\ 0 & 1 \end{pmatrix} \mapsto Bx + c, \tag{2.5}$$

we get an identification between the subgroup of $\mathrm{PGL_{n+1}}(\mathbb{R})$ preserving $\mathbb{A}^n$ and the affine group $\mathrm{Aff}(\mathbb{R}^n) \cong \mathbb{R}^n \rtimes \mathrm{GL_n}(\mathbb{R})$. If we place the standard Euclidean metric on $\mathbb{A}^n$ and we restrict to the case where $B \in \mathrm{O}(n)$ then (2.5) gives an identification with $\mathrm{Euc}(\mathbb{R}^n) \cong \mathbb{R}^n \rtimes \mathrm{O}(n)$ (Euclidean isometries of $\mathbb{R}^n$). Notable examples of (G, X)-structures arising in this context are $(\mathrm{Aff}(\mathbb{R}^n), \mathbb{A}^n)$ structures (affine structures) and $(\mathrm{Euc}(\mathbb{R}^n), \mathbb{A}^n)$ structures (Euclidean structures), which can be thought of in this way as specific instances of projective structures.

2.3.3 Hyperbolic Geometry

We define a bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$ by

$$\langle x, y \rangle = x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}. \tag{2.6}$$

Using (2.6) we define a cone (see Figure 2.3) in $\mathbb{R}^{n+1}$ by

$$C^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle < 0\} \tag{2.7}$$

The image $\mathbb{D}^n$ of C^n in $\mathbb{RP}^n$ is commonly referred to as the *Klein model* of hyperbolic space (see [41, §6.1] for more details). There is a convenient

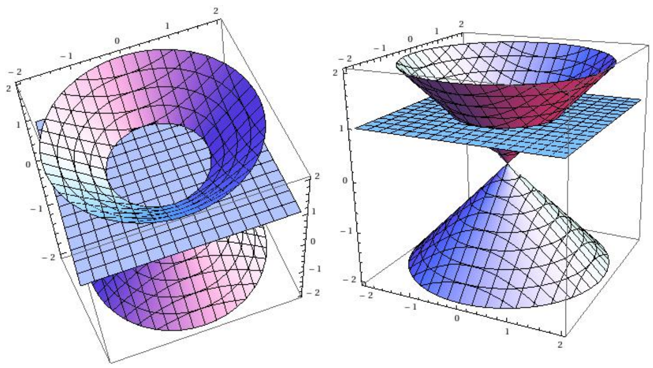

Figure 2.3: Multiple views of C^n and $\mathbb{A}^n$ when $n = 2$

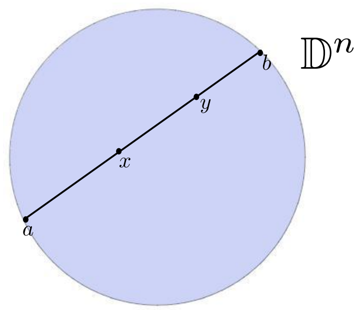

Figure 2.4: The Klein model $\mathbb{D}^n$ sitting inside $\mathbb{A}^n$

way to visualize $\mathbb{D}^n$ inside of $\mathbb{A}^n \subset \mathbb{RP}^n$ as follows: observe that the plane $x_n = 1$ in $\mathbb{R}^{n+1}$ used in Section 2.3.2 to define $\mathbb{A}^n$ intersects C in the unit disk (see Figure 2.4). Given two points $x, y \in \mathbb{D}^n$ the affine line segment between x and y intersects $\partial\mathbb{D}^n$ in two points a and b (see Figure 2.4). Let $[a, x; y, b] := \frac{|y-a||x-b|}{|y-b||x-a|}$ be the *projective cross ratio* of a, x, y, and b. Then we can define the hyperbolic metric on $\mathbb{D}^n$ by

$$d_H(x, y) := \frac{|\log([a, x; y, b])|}{2}. \tag{2.8}$$

With the metric d_H, $\mathbb{D}^n$ is a complete Riemannian metric space with constant curvature -1 whose geodesics are intersections of straight lines in $\mathbb{A}^n$ with $\mathbb{D}^n$. Let $\mathrm{PGL}(\mathbb{D}^n)$ be the subgroup of $\mathrm{PGL}_{n+1}(\mathbb{R})$ that preserves $\mathbb{D}^n$. A simple exercise shows that $\mathrm{PGL}(\mathbb{D}^n) = \mathrm{PO}(n, 1)$ (the projective orthogonal group of the form (2.6)). Additionally, it is well known that the projective cross ratio of 4 collinear points is preserved by any projective transformation, and so we see that $\mathrm{PGL}(\mathbb{D}^n) \subseteq \mathrm{Isom}(\mathbb{D}^n)$, where $\mathrm{Isom}(\mathbb{D}^n)$ is the set of isometries of $\mathbb{D}^n$ with the metric d_H. Furthermore, it is shown in [41, §6.1] that the previous inclusion is actually an equality.

Isometries of $\mathbb{D}^n$ can be classified by their fixed points as follows [41]. Let $A \in \mathrm{PGL}(\mathbb{D}^n)$, then A is

1. *elliptic* if it fixes a point in $\mathbb{D}^n$
2. *parabolic* if it acts freely on $\mathbb{D}^n$ and fixes a unique point on $\partial\mathbb{D}^n$, or
3. *hyperbolic* if it acts freely on $\mathbb{D}^n$ and fixes exactly two points on $\partial\mathbb{D}^n$.

2.3.3.1 Hyperbolic Manifolds

Any Riemannian manifold M of constant sectional curvature -1 is called a *hyperbolic manifold* and can be realized as a (G, X)-manifold where $X = \mathbb{D}^n$ and $G = \mathrm{Isom}(\mathbb{D}^n)$. These types of (G, X)-structures are called *hyperbolic structures* and serve as another example of projective structures. In this setting completeness of the metric is equivalent to completeness of the (G, X)-structure [47]. Therefore, when M is complete it can always be realized as

$\mathbb{D}^n/\Gamma$, where $\Gamma = \pi_1(M)$ and the action is given by the holonomy representation.

If $M = \mathbb{D}^n/\Gamma$ is a complete hyperbolic n-manifold, then M inherits a volume form dV from the Riemannian metric on $\mathbb{D}^n$ and we say that M is *finite volume* if $\int_M dV < \infty$. We close this section by introducing a dichotomy concerning hyperbolic structures on finite volume hyperbolic manifolds based on dimension. When $n = 2$ there is a rich deformation theory of complete finite volume structures on M and these structures are parameterized by the Teichmüller space, which is a topologically a ball of dimension $6g - 6 + 2n$ where g is the genus of the surface and n is the number of cusps (see [20] for details). On the other hand, when $n \geq 3$ and M has finite volume there is a strong uniqueness theorem about complete hyperbolic structures on M.

Theorem 2.3.1 (Mostow-Prasad Rigidity [39])**.** *Let $M = \mathbb{D}^n/\Gamma_1$ and $N = \mathbb{D}^n/\Gamma_2$ be complete, finite volume hyperbolic n-manifolds with $n \geq 3$. Suppose we have a homotopy equivalence $F : M \to N$, then there exists $g \in \mathrm{Isom}(\mathbb{D}^n)$ homotopic to F. Furthermore, Γ_1 and Γ_2 are conjugate subgroups of* $\mathrm{Isom}(\mathbb{D}^n)$.

Mostow-Prasad rigidity implies that if Γ is the fundamental group of a finite volume hyperbolic manifold of dimension at least 3 then there is a unique conjugacy class of discrete, faithful representations from Γ to $\mathrm{Isom}(\mathbb{D}^n)$. We refer to a representative of this conjugacy class as the *geometric representation* of M (or Γ) and denote it by ρ_{geo}. We now observe a few consequences of Mostow-Prasad rigidity. First, in dimension at least 3 there are no deformations of complete hyperbolic structures on M *through complete structures.*

As we will see later, when M is non-compact and dimension 3 then there is an interesting deformation theory of incomplete structures on M. Additionally, when M is closed then it is automatically metrically complete and thus complete as a (G, X) manifold. In this case we see that M is locally $\text{Isom}(\mathbb{D}^n)$-rigid.

Chapter 3

Convex Projective Structures on Manifolds

In this chapter we will introduce convex projective geometry, convex projective structures on manifolds, and deformations in this setting.

3.1 Convex Projective Geometry

Let $\Omega \subset \mathbb{A}^n \subset \mathbb{RP}^n$ be an open set that is convex and has compact closure. Such sets are called *properly convex* and are the object of study in convex projective geometry. If Ω is properly convex and $\partial\Omega$ contains no non-trivial affine line segments then Ω is *strictly convex*. It is easily seen that $\mathbb{D}^n$ is both properly and strictly convex. A standard example of a properly, but not strictly convex set is a simplex in $\mathbb{A}^n$.

Similar to the case of $\mathbb{D}^n$, we can put a metric on a properly convex set Ω. Let $x, y \in \Omega$, then by proper convexity, the affine line segment between x and y intersects $\partial\Omega$ in two points a and b (where a is closer to x and b is closer to y see Figure 3.1) and we define a metric as

$$d_\Omega(x, y) = \left|\log([a, x; y, b])\right|. \tag{3.1}$$

The function d_Ω defines a complete Finsler metric on Ω. When Ω is strictly convex this makes Ω a geodesic metric space, where geodesics are

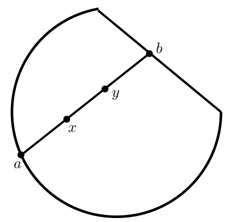

Figure 3.1: A properly, but not strictly, convex domain

intersections of affine lines with Ω. However, if Ω is not strictly convex then geodesics between points are not (even locally) unique. We now use Figure 3.2 to explain this fact. From the definition of the cross ratio, we see that $[a, x; y, b] = [a, x; z', b][a, z'; y, b]$. Additionally, by projective invariance of the cross ratio we see that $[a, x; z', b] = [a', x; z, b']$ and $[a, z'; y, b] = [a'', z; y, b'']$ and so we see that

$$d_\Omega(x, y) = |\log[a, x; y, b]| = |\log[a, x; z', b]| + |\log[a, z'; y, b]|$$

$$= |\log[a', x; z, b']| + |\log[a'', z; y, b'']| = d_\Omega(x, z) + d_\Omega(z, y).$$

Thus we see that the segments $[x, y]$ and $[x, z] \cup [z, y]$ are both geodesics connecting x and y.

Let $\mathrm{PGL}(\Omega)$ be the elements of $\mathrm{PGL_{n+1}}(\mathbb{R})$ that preserve Ω and $\mathrm{Isom}(\Omega)$ be the set of isometries of Ω with respect to the metric d_Ω. Again, since the projective cross ratio is invariant under projective transformations we see that $\mathrm{PGL}(\Omega) \subseteq \mathrm{Isom}(\Omega)$. If Ω is strictly convex then this inclusion is an equality, however, if Ω is not strictly convex then the inclusion may be proper. For example, when Δ is a 2-simplex in $\mathbb{A}^2$ then $\mathrm{PGL}(\Delta)$ is an index 2 subgroup of $\mathrm{Isom}(\Delta)$ (see [19] for details).

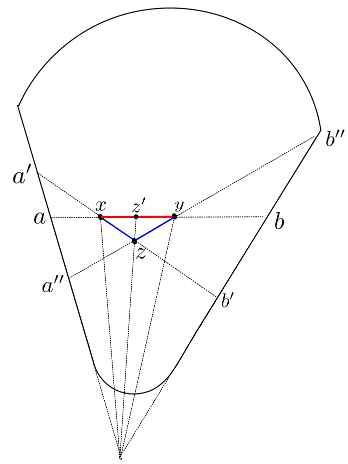

Figure 3.2: An example of non-unique geodesics in a properly convex domains

As observed by Benoist [5], there is a strong relationship between strict convexity and δ-hyperbolicity. Recall that in a geodesic metric space a geodesic triangle T is *δ-thin* if each side of T is contained in a δ-neighborhood of the union of the other two sides and a metric space is *δ-hyperbolic* if all geodesic triangles are δ-thin. We can now state the following result

Theorem 3.1.1 (Benoist [5])**.** *Let Ω be a properly convex domain. If Ω is δ-hyperbolic then Ω is strictly convex. Additionally, when $\mathrm{PGL}(\Omega)$ contains a discrete subgroup such that Ω/Γ is compact, then the converse is true.*

Figure 3.3 demonstrates how to construct fat triangles when Ω is not strictly convex. The converse of Theorem 3.1.1 is also true in cases where Ω admits certain non-compact quotients [17].

One benefit of a $\mathrm{PGL}(\Omega)$ invariant metric is that it allows us to prove that discrete subgroups always act properly discontinuously [41, §5.3]. As

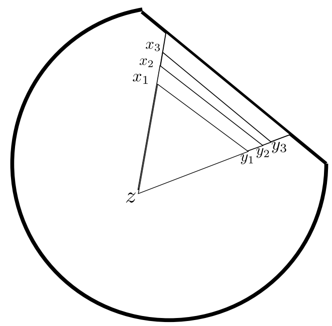

Figure 3.3: A sequence of fat triangles in a non-strictly convex domain

a consequence we see that if we take the quotient of Ω by Γ, where Γ is a discrete, torsion-free subgroup of $\mathrm{PGL}(\Omega)$ then Ω/Γ will be a complete (with respect to metric induced by d_Ω) manifold. In this way we can often reduce questions of geometry on a manifold to more manageable questions about discrete subgroups.

3.1.1 Convex Projective Isometries

In general, elements of $\mathrm{PGL_{n+1}}(\mathbb{R})$ are equivalence classes of matrices and so subgroups of $\mathrm{PGL_{n+1}}(\mathbb{R})$ do not have well defined lifts to $\mathrm{GL}_{n+1}(\mathbb{R})$. However, when Ω is properly convex then there is a canonical lift of $\mathrm{PGL}(\Omega)$ into $\mathrm{GL}_{n+1}(\mathbb{R})$. Let $\mathbb{S}^n = \tilde{\mathbb{RP}}^n$ be the universal cover of $\mathbb{RP}^n$, which we identify with $\mathbb{R}^{n+1}\backslash\{0\}/\mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive scalars. The automorphism group of S^n is the group of matrices with determinant ± 1 which we denote $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$. Since Ω is properly convex, its preimage in $\mathbb{S}^n$ has two disjoint components. Each element of $\mathrm{PGL}(\Omega)$ has two lifts to $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$, one that preserves the components and another that interchanges them. By selecting the lift that preserves the components we get an isomorphism with the subgroup of $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ that preserves the components of Ω which we denote $\mathrm{SL}^{\pm}(\Omega)$. In light of this identification we will freely identify $\mathrm{PGL}(\Omega)$ with $\mathrm{SL}^{\pm}(\Omega)$ and will regard elements of $\mathrm{PGL}(\Omega)$ as matrices when convenient.

Following [17] we can separate isometries in $\mathrm{PGL}(\Omega)$ into 3 types. Since Ω is properly convex, its closure can be realized as a compact, convex subset of $\mathbb{A}^n$ and thus by the Brouwer fixed point theorem every element of $\mathrm{PGL}(\Omega)$ fixed a point in the closure $\overline{\Omega}$ of Ω. Thus we see that generic elements of $\mathrm{PGL}_{\mathrm{n+1}}(\mathbb{R})$ cannot preserve a properly convex domain. With this in mind we say that $A \in \mathrm{PGL}(\Omega)$ is:

1. *elliptic* if it fixes a point in Ω
2. *parabolic* if it acts freely on Ω and all of its eigenvalues have modulus one, or
3. *hyperbolic* otherwise.

In the case where $\Omega = \mathbb{D}^n$ this classification agrees with the standard classification of isometries of hyperbolic space (see [41]). Additionally, there are strong similarities between isometries of strictly convex domains and their hyperbolic counterparts as illustrated by the next few results. The first of which is contained in [17].

Theorem 3.1.2. *Let Ω be a strictly convex domain and $A \in \mathrm{SL}^{\pm}(\Omega)$ an isometry. If A is parabolic then A fixes precisely one point in $\partial\Omega$. If A is* *hyperbolic then it has precisely two fixed points on $\partial\Omega$ and acts as a translation on the geodesic in Ω connecting the fixed points.*

The main idea behind the proof of Theorem 3.1.2 is that fixed points of projective isometries correspond to the projectivization of eigenspaces with real eigenvalues. If A is parabolic or hyperbolic it will act freely on Ω and so these eigenspaces can only intersect $\overline{\Omega}$ in $\partial\Omega$. However, strict convexity then implies that this intersection is a single point for each eigenspace. When Ω is properly, but not strictly, convex then the intersection with these eigenspaces with the boundary can be larger and give rise to isometries that fix higher dimensional portions of $\partial\Omega$.

The next result, which is also contained in [17], shows that parabolic elements must satisfy certain linear algebraic constraints.

Theorem 3.1.3. *Suppose that Ω is a properly convex domain and that $A \in \mathrm{SL}^{\pm}(\Omega)$ is parabolic. Then one of largest Jordan blocks of A has eigenvalue 1. Additionally, the size of this Jordan block is odd and at least 3. If Ω is strictly convex then this is the only Jordan block of this size.*

Later we will make use of the following corollary of this result which tells us that in dimension 3 there is a unique conjugacy class of parabolic elements preserving any properly convex domain.

Corollary 3.1.4. *Let Ω be a properly convex domain of dimension 3 and that*

$A \in \mathrm{SL}^{\pm}(\Omega)$ *is parabolic. Then* A *is conjugate in* $\mathrm{SL}_4(\mathbb{R})$ *to*

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The next result can be viewed as a properness property of the action of $\mathrm{PGL}(\Omega)$ on Ω.

Lemma 3.1.5. *Let* Ω *be a properly convex domain and let* x *be a point in the interior of* Ω*, then the set* $\Omega_x^K = \{T \in \mathrm{PGL}(\Omega) \mid \mathrm{d}_\Omega(\mathrm{x}, \mathrm{Tx}) \leq \mathrm{K}\}$ *is compact.*

Proof. Let $\mathcal{B} = \{x_0, \ldots, x_n\}$ be a projective basis of $\mathbb{RP}^n$ (that is a set of $n+1$ points no n of which live in a common hyperplane) that are contained in Ω and such that $x_0 = x$. There is a homeomorphism between $\mathrm{PGL_{n+1}}(\mathbb{R})$ and an open set of $(\mathbb{RP}^n)^{n+1}$ that arises in much the same way that the action of an element of $\mathrm{GL}_{n+1}(\mathbb{R})$ on a basis gives rise a homeomorphism to an open set in $(\mathbb{R}^n)^n$. Next, let γ_i be a sequence of elements of Ω_x^K. The elements $\gamma_i x_0$ all live in the compact ball of radius K centered at x and so by passing to a subsequence we can assume that $\gamma_i x_0 \to x_0^\infty \in \Omega$. Next, we claim that by passing to subsequences that $\gamma_i x_j \to x_j^\infty \in \Omega$ for $1 \leq j \leq n$. To see this observe that

$$d_\Omega(x_0^\infty, \gamma_i x_j) \leq d_\Omega(x_0^\infty, \gamma_i x_0) + d_\Omega(\gamma_i x_0, \gamma_i x_j) = d_\Omega(x_0^\infty, \gamma_i x_0) + d_\Omega(x_0, x_j),$$

and so all of the $\gamma_i x_j$ live in a compact ball centered at x_0^∞, and so we can find the desired subsequence. Therefore the proof will be complete if we can show that the set $\{x_0^\infty \dots, x_n^\infty\}$ is a projective basis. Suppose that this set is not a basis, then without loss of generality we can assume that the set $\{v_0^\infty, \dots, v_{n-1}^\infty\}$ is linearly dependent, where v_i^∞ is a vector in the class of x_i^∞. Thus $v_0^\infty = c_1 v_1^\infty + \dots + c_{n-1} v_{n-1}^\infty$ is a non-trivial linear combination, and we find that $\gamma_i[c_1 v_1 + \dots + c_{n-1} v_{n-1}] \to [v_0^\infty]$. However,

$$d_\Omega([v_0], [c_1 v_1 + \dots + c_{n-1} v_{n-1}]) = d_\Omega(\gamma_i[v_0], \gamma_i[c_1 v_1 + \dots + c_{n-1} v_{n-1}])$$

and

$$d_\Omega(\gamma_i[v_0], \gamma_i[c_1 v_1 + \dots + c_{n-1} v_{n-1}]) \to d_\Omega([v_0^\infty], [c_1 v_1^\infty + \dots + c_{n-1} v_{n-1}^\infty]) = 0,$$

which contradicts the fact that $\mathcal{B}$ is a basis. □

Lemma 3.1.5 helps us prove the following proposition which will be useful in our analysis of strictly convex geometry on manifolds.

Proposition 3.1.6. *Let Ω be a strictly convex domain and $\phi, \psi \in \mathrm{PGL}(\Omega)$ with ϕ hyperbolic. If ϕ and ψ have exactly one fixed point in common, then the subgroup generated by ϕ and ψ is not discrete.*

Similar proofs of this proposition can be found in [3, 17], and both proofs use an adaptation of a well known argument in hyperbolic geometry (see [41, Thm 5.5.4]). A simple corollary of Proposition 3.1.6 is the following

Corollary 3.1.7. *Let Ω be a strictly convex domain and let $\Gamma \leq \mathrm{PGL}(\Omega)$ be a discrete subgroup. If $A, B \in \Gamma$ are parabolic and hyperbolic, respectively, then A and B do not commute.*

We close this section with a lemma about the behavior of elements that preserve a common geodesic.

Lemma 3.1.8. *Let Ω be strictly convex and let $\Gamma \leq \mathrm{PGL}(\Omega)$ be a discrete torsion free subgroup of elements that all fix a common geodesic in Ω, then Γ is an infinite cyclic group generated by a hyperbolic element.*

Proof. Since the elements of Γ all preserve a geodesic γ there is a homomorphism from Γ to $\mathbb{R}$ that assigns to each element its translation length (measured in the Hilbert metric) along γ. Since Γ is torsion free, it acts freely on Ω and we see that this map has trivial kernel. Therefore Γ is isomorphic to a discrete subgroup of $\mathbb{R}$ and is thus infinite cyclic. The last statement follows from fact that parabolic elements have unique fixed points and thus do not preserve geodesics. $\square$

3.2 Convex Projective Manifolds

Let M be a manifold and let $\Xi \in D(M; \mathrm{PGL}_{\mathrm{n+1}}(\mathbb{R}), \mathbb{RP}^{\mathrm{n}})$ be an isotopy class of projective structures on M. We say that Ξ is a *convex projective structure* if any (hence all) developing map D is a homeomorphism onto a properly convex set and we denote the set of such isotopy classes by $\mathcal{CP}(M)$. From the definition, we see that convex projective structures are always complete. If in addition the image of D is a strictly convex set then we say that Ξ is a *strictly convex projective structure* and we denote the set of such isotopy classes by $\mathcal{SCP}(M)$. The simplest examples of (strictly) convex projective structures on manifolds come from complete hyperbolic structures. A *convex projective deformation* (resp. *strictly convex projective deformation*) of Ξ_0 is a smooth path in $\mathcal{CP}(M)$ (resp. in $\mathcal{SCP}(M)$) through Ξ_0. With this in mind we ask the following question.

Question 1. *Let M be a complete finite volume hyperbolic n-manifold. Does M admit non-trivial, strictly convex projective deformations near the*[1] *complete hyperbolic structure?*

This question naturally breaks into two cases according to whether M is closed or not and we discuss these two cases separately.

3.2.1 The Closed Case

Throughout this section let M be a closed manifold and let $\Gamma = \pi_1(M)$, then via the holonomy map we see that a strictly convex projective structure gives rise to a discrete, faithful representation $\rho : \Gamma \to \mathrm{PGL}_{n+1}(\mathbb{R})$ that preserves a strictly convex open subset of $\mathbb{RP}^n$. When M is hyperbolic we see that strictly convex projective deformations of the complete hyperbolic structure give rise to curves in $\mathfrak{X}(\Gamma, \mathrm{PGL}_{n+1}(\mathbb{R}))$ passing through $[\rho_{\mathrm{geo}}]$. On the other hand Theorem 2.1.3 tells us that if we have a curve in $\mathfrak{X}(\Gamma, \mathrm{PGL}_{n+1}(\mathbb{R}))$ passing through $[\rho_{\mathrm{geo}}]$ and consisting of discrete, faithful representations then we will get a curve of projective structures on M passing through the complete hyperbolic structure, but a priori we have no guarantee that these projective

[1] when $n = 2$ there is not a unique complete hyperbolic structure and we look at deformations near a given hyperbolic structure.

structures are strictly or even properly convex. However in the closed case we have the following theorem of Koszul [33]

Theorem 3.2.1. *Let M be a closed manifold, then $\mathcal{CP}(M)$ is an open subset of $D(M; \mathrm{PGL_{n+1}}(\mathbb{R}), \mathbb{RP}^{\mathrm{n}})$.*

When M is hyperbolic we get the following corollary by combining the previous result and Theorem 3.1.1.

Corollary 3.2.2 (Benoist [5])**.** *If M is hyperbolic, then $\mathcal{CP}(M) = \mathcal{SCP}(M)$*

Proof. The inclusion $\mathcal{SCP}(M) \subseteq \mathcal{CP}(M)$ follows directly from the definition. Conversely, suppose that we are given a convex projective structure with holonomy ρ. Observe that Γ acts cocompactly by isometries on $\mathbb{D}^n$ and so by the Švarc-Milnor lemma (see [10, Ch 8] for details) Γ is quasi-isometrically equivalent to $\mathbb{D}^n$ and hence δ-hyperbolic (with respect to the word metric on Γ). As ρ is the holonomy of a convex projective structure $\rho(\Gamma)$ preserves a properly convex set Ω and acts cocompactly on Ω by isometries. Therefore by applying the Švarc-Milnor lemma again we see that Ω is quasi-isometrically equivalent to Γ and hence δ-hyperbolic. Finally, since Ω is δ-hyperbolic Theorem 3.1.1 implies that Ω is strictly convex. □

Combining these results with Theorem 2.1.3 we see that if M is hyperbolic and $[\rho]$ is close enough to $[\rho_{\mathrm{geo}}]$ in $\mathfrak{X}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$ then ρ is the holonomy of a strictly convex projective structure on M, in particular ρ is discrete and faithful. One benefit of Theorem 3.2.1 is that it tells us that if we are able to deform ρ_{geo} through a family of non-conjugate representations then this corresponds to a strictly convex deformation of M.

In dimension 2 the convex deformation theory is well understood thanks to work of Goldman [24] and Choi and Goldman [14]. Here the situation is similar to that of hyperbolic structures in that there is a well understood space of convex projective structures on M that is topologically a ball, this time of dimension $16g-16$, where g is the genus of M. By Corollary 3.2.2 we see that these structures are all strictly convex.

3.2.1.1 Bending

In this section we will discuss a technique for finding strictly convex projective deformations of closed hyperbolic manifolds in arbitrary dimensions. The rough idea is that totally geodesic hypersurfaces give rise to non-trivial curves of representations in $\mathfrak{X}(\Gamma, \mathrm{PGL}_{n+1}(\mathbb{R}))$. This idea goes back to Apanasov [1] and Thurston [47] in the case of quasi-Fuchsian deformations. Later the construction was generalized by Johnson and Millson [30], and it is their approach that we will take moving forward.

Let M be a closed hyperbolic manifold and let S be an orientable totally geodesic hypersurface and let $\Delta = \pi_1(S)$. Recall that the Lie algebra of $\mathrm{PGL}_{n+1}(\mathbb{R})$ is $\mathfrak{sl}(n+1)$ and Γ acts on $\mathfrak{sl}(n+1)$ via

$$\gamma \cdot x = \mathrm{Ad}(\rho_{\text{geo}}(\gamma)) \cdot x.$$

The following lemma tells us that in this situation we can find a unique 1-

dimensional subspace of $\mathfrak{sl}(n+1)$ that is invariant under the adjoint action restricted to Δ.

Remark 3.2.1. An element of $\mathfrak{sl}(n+1)$ is invariant under the adjoint action of γ precisely when $\rho_{\text{geo}}(\gamma)$ and x are commuting matrices.

Lemma 3.2.3. *Let M be a closed hyperbolic manifold and S a totally geodesic hypersurface. Then there exists a unique 1-dimensional subspace generated by a vector $x_S \in \mathfrak{sl}(n+1)$ that is invariant under the action of Δ. Furthermore this subspace is generated by a conjugate in* $\mathrm{PGL_{n+1}}(\mathbb{R})$ *of*

$$\begin{pmatrix} -n & 0 \\ 0 & I \end{pmatrix},$$

where I is the $n \times n$ identity matrix.

Proof. Γ is a subgroup of $PO(n,1)$ (the projective orthogonal group of the form $x_1^2 + x_2^2 + \ldots x_n^2 - x_{n+1}^2$). Since S is totally geodesic we can assume after conjugation that Δ preserves both the hyperplane where $x_1 = 0$ and its orthogonal complement which is generated by $(1, 0, \ldots, 0)$. Hence if $A \in \Delta$ then

$$A = \begin{pmatrix} 1 & 0^T \\ 0 & \tilde{A} \end{pmatrix},$$

where $\tilde{A} \in PO(n-1,1)$ (the projective orthogonal group of the form $x_2^2 + x_3^2 + \ldots + x_n^2 - x_{n+1}^2$) and $0 \in \mathbb{R}^n$. If $x \in \mathfrak{sl}(n+1)$ is invariant under Δ then we know that $B(t) = \exp(tx)$ commutes with every $A \in \Delta$. If we write $B(t) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, where $b_{11} \in \mathbb{R}$, $b_{12}^T, b_{21} \in \mathbb{R}^n$, and $b_{22} \in \mathrm{SL_n}(\mathbb{R})$, then

$$\begin{pmatrix} b_{11} & b_{12} \\ \tilde{A}b_{21} & \tilde{A}b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\tilde{A} \\ b_{21} & b_{22}\tilde{A} \end{pmatrix}.$$

From this computation we learn that b_{12} and b_{21} are invariants of Δ and that b_{22} is in the centralizer of Δ in $SL_n(\mathbb{R})$. However, the representation of Δ into $PO(n-1,1)$ is irreducible, and so the only matrices that commute with every element of Δ are scalar matrices and the only invariant vector of Δ is 0, and so

$$B = \begin{pmatrix} e^{-n\lambda t} & 0 \\ 0 & e^{\lambda t} I \end{pmatrix},$$

where I is the identity matrix. Differentiating $B(t)$ at $t = 0$ we find that

$$x = \begin{pmatrix} -n\lambda & 0 \\ 0 & \lambda I \end{pmatrix},$$

and the result follows. □

The vector x_S from Lemma 3.2.3 will be called a *bending cocycle*. We can now define a family of deformations of ρ_{geo}. The construction breaks into two cases depending on whether or not S is separating.

If S is separating then Γ splits as the following amalgamated free product:

$$\Gamma \cong \Gamma_1 *_\Delta \Gamma_2,$$

where Γ_i are the fundamental groups of the components of the complement of S in M, and we can define a family of representations ρ_t as follows. Since ρ_{geo} is an irreducible representation we know that x_S is not invariant under all of Γ and so we can assume without loss of generality that it is not invariant under Γ_2. So let $\rho_t|_{\Gamma_1} = \rho_{\text{geo}}$ and $\rho_t|_{\Gamma_2} = \text{Adj}(\exp(tx_S)) \cdot \rho_{\text{geo}}$. By the construction of x_S these two maps agree on Δ and so they give a well defined family of homomorphisms of Γ, such that $\rho_0 = \rho_{\text{geo}}$.

If S is nonseparating, then Γ is realized as the following HNN extension:

$$\Gamma \cong \Gamma' *_{\Delta},$$

where Γ' is the fundamental group of $M \backslash S$. If we let α be a curve dual to S then we can define a family of homomorphisms through ρ_{geo} by $\rho_t|_{\Gamma'} = \rho_{\text{geo}}$ and $\rho_t(\alpha) = \exp(tx_S)\rho_{\text{geo}}(\alpha)$. Since x_S is invariant under Δ the values of $\rho_t(\iota_1(\Delta))$ do not depend on t, where ι_1 is the inclusion of the positive boundary component of a regular neighborhood of S into $M \backslash S$, and so we have well defined homomorphisms of the HNN extension.

In both cases the fact that x_S is not invariant under the action of Γ implies that ρ_t gives rise to a non-trivial curve of representations. The family ρ_t of representations is called a *bending deformation of M along S*. Furthermore, by examining the cohomology class coming from ρ_t, Johnson and Millson [30] showed that $[\rho_t]$ actually defines a non-trivial path in $\mathfrak{X}(\Gamma, \mathrm{PGL}_{n+1}(\mathbb{R}))$. Finally, by applying Theorem 3.2.1 and Corollary 3.2.2 we see that these deformed representations give rise to strictly convex projective deformations of M.

3.2.1.2 Flexing

By work of Cooper, Long, and Thistlethwaite [15, 16] there are examples of closed hyperbolic 3-manifolds that contain no totally geodesic hypersurfaces, but nevertheless admit strictly convex projective deformations. A deformation that does not result from the bending procedure is known as a *flexing*. The class of manifolds that they examined were closed census 3-manifolds with 2-generator fundamental groups. In this context, 2-generator 3-manifolds are a natural class in which to look for flexing deformations as their fundamental groups admit fairly simple presentations and by work of Mensaso and Reid [38] they do not contain totally geodesic closed hypersurfaces.

In [16] a procedure is outlined to compute dimensions of cohomology groups of 2-generator groups, and using these techniques the authors are able to examine 4500 such manifolds, and they subsequently discovered that most 2-generator census 3-manifolds were infinitesimally projectively rigid at ρ_{geo}. In particular they found that only 61 of the manifolds they examined admitted infinitesimal deformations at ρ_{geo}. Using the numerical techniques developed in [15] they were able to explicitly construct families of deformations of ρ_{geo} for 25 of these 61 manifolds, and found strong numerical evidence that 27 others admit actual deformations. Finally, Theorem 3.2.1 and Corollary 3.2.2 imply that the deformations of ρ_{geo} described above correspond to strictly convex projective deformations.

3.2.2 The Non-compact Case

In this section M will be a non-compact finite volume n-hyperbolic manifold and $\Gamma = \pi_1(M)$. The primary difference between the closed and non-compact cases is the fact that Theorem 3.2.1 is no longer true if M

is non-compact. The failure of Theorem 3.2.1 in the non-compact case can be better understood by examining non-compact finite volume hyperbolic 3-manifolds. By work of Thurston [47, §5] there are always deformations of ρ_{geo} and these deformations correspond to incomplete hyperbolic structures on M with holonomies that are always either indiscrete or non-faithful. Furthermore, the developing map has non-convex image near portions of $\tilde{M}$ that correspond to ends of M. As a result we see that there are hyperbolic (and hence projective) structures near the complete structure of M that are not properly convex.

Next, we discuss some restrictions that are imposed on the holonomies of strictly convex structures on finite volume hyperbolic manifolds. A standard result in hyperbolic manifolds (see [41, Thm 12.7.4] for example) states that M admits a decomposition

$$M = M_K \sqcup \left(\sqcup_{i=1}^{k} C_i\right), \tag{3.2}$$

where M_K is is a compact submanifold and the C_i are called *cusps* and are diffeomorphic to $E_i \times [0, \infty)$, where E_i is a closed Euclidean manifold of dimension $n-1$. We call $\pi_1(C_i)$ a *peripheral subgroup* of M. Moreover, the universal cover of C_i is $\mathbb{A}^{n-1} \times [0, \infty)$ and the action of ρ_{geo} on the universal cover respects this product structure, acting discontinuously by Euclidean isometries on the first factor and trivially on the second factor. Using this structure of the cusps we can prove the following proposition.

Proposition 3.2.4. *Let M have dimension at least 3 and let ρ be the holonomy* *of a strictly convex projective structure on M and let $\{C_i\}_{i=1}^k$ be the set of cusps of M. If $1 \neq \gamma \in \pi_1(C_i)$ then $\rho(\gamma)$ is parabolic.*

Proof. Let Ω be a strictly convex domain preserved by $\rho(\Gamma)$ and let $\Delta_i = \pi_1(C_i)$. Since Γ is torsion free and ρ is a discrete faithful representation we see that $\rho(\gamma)$ is either hyperbolic or parabolic, and so it suffices to show that $\rho(\gamma)$ is not hyperbolic.

We will start by showing that $\rho(\Delta_i)$ has a global fixed point in $\partial\Omega$. It follows from the previous paragraph that Δ_i can be realized as a discrete group of $\mathrm{Euc}(\mathbb{R}^n)$. By work of Bieberbach [8], it follows that Δ_i is virtually abelian, and in particular Δ_i contains a finite index abelian subgroup Δ_i' of rank $n-1$. Let $\gamma' \in \Delta'$, then γ' has exactly one or two fixed points and we pick a fixed point x of γ'. Since every element of Δ_i' commutes with γ' we see that the elements of Δ_i' permute the fixed points of γ' and thus there is a subgroup Δ_i^0 of index at most 2 in Δ_i' (and hence of finite index in Δ_i) that fixes x. Observe that if $\delta \in \Delta_i$ is non-trivial then there exists some m such that $\delta^m \in \Delta_i^0$ and thus fixes x. However, the fixed point set of a parabolic or a hyperbolic is unchanged by taking powers and so δ fixes x, thus proving the claim.

Next, suppose for contradiction that $\rho(\gamma)$ is hyperbolic. If this is the case then $\rho(\Delta_i)$ cannot contain any parabolic elements because by the previous claim they would share a single fixed point with $\rho(\gamma)$ which contradicts Proposition 3.1.6. Therefore every element of $\rho(\Delta_i)$ is hyperbolic and Proposition 3.1.6 tells us that each of these elements has the same fixed point set, and thus they preserve a common geodesic. Finally, Lemma 3.1.8 tells us that $\rho(\Delta_i)$ is infinite cyclic which contradicts the fact that ρ is faithful. □

Let ρ_t be a family of holonomies of a strictly convex deformation of a finite volume hyperbolic n-manifold. One consequence of Proposition 3.2.4 is that if $n \geq 3$ then for each t, the holonomy of each peripheral subgroup consists entirely of parabolic isometries. With this in mind we define the following refinement of the character variety. A matrix $A \in \mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ is a *strictly convex parabolic* if all of its eigenvalues have modulus 1 and there is a unique largest Jordan block that has odd dimension and eigenvalue 1. An element of of $\mathrm{PGL_{n+1}}(\mathbb{R})$ is *strictly convex parabolic* if one of its lifts to $\mathrm{SL}^{\pm}_{n+1}\mathbb{R}$ is strictly convex parabolic. Next, let $\mathrm{Hom}_{scp}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$ be the set of representations from Γ into $\mathrm{PGL_{n+1}}(\mathbb{R})$ such that all non-identity elements of each peripheral subgroup are mapped to strictly convex parabolics. We now define the *relative character variety* which we denote by $\mathfrak{X}_{\mathrm{scp}}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$ to be the quotient of $\mathrm{Hom}_{scp}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$ by $\mathrm{PGL_{n+1}}(\mathbb{R})$ acting by conjugation. If $[\rho_0]$ is an isolated point of $\mathfrak{X}_{\mathrm{scp}}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$ then we say that M (or Γ) is *locally projectively rigid relative to the boundary* at ρ_0. A simple corollary of Theorem 3.1.3 and the preceding paragraph is the following

Corollary 3.2.5. *Let $n \geq 3$ and ρ_t be a family of holonomies corresponding to a strictly convex deformation of M^n. Then $[\rho_t] \in \mathfrak{X}_{\mathrm{scp}}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$. Furthermore, if M is locally projectively rigid relative to the boundary at ρ_{geo}*

then there are no non-trivial strictly convex deformations near the complete hyperbolic structure on M.

3.2.2.1 More Bending

We now return our attention to the bending construction. In the construction of the bending deformations we did not use the fact that M was closed, and in fact the construction continues to work if M is non-compact and S is finite volume. Let ρ_t be a family of representations obtained by bending M along a finite volume surface S. A priori we have no guarantee that these deformed representations that we have constructed correspond to properly convex deformations of M. Fortunately, by work of Marquis [37, Thm 3.7] these representations do correspond to properly convex deformations.

The next question that we attempt to answer is when are these properly convex deformations strictly convex. In light of Corollary 3.2.5 we see that a necessary condition for the structures to be strictly convex is that the $[\rho_t] \in \mathfrak{X}_{\mathrm{scp}}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$. We begin by analyzing the simpler case where S is closed.

Proposition 3.2.6. *Let S be a closed totally geodesic hypersurface of M and let ρ_t be the family of representations obtained by bending along S. Then $[\rho_t]$ is contained in $\mathfrak{X}_{\mathrm{scp}}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$.*

Proof. In the case that S is closed and separating a peripheral element, $\gamma \in \Gamma$, is contained in either Γ_1 or Γ_2 since it is disjoint from S. In this case $\rho_t(\gamma)$ is either $\rho(\gamma)$ or some conjugate of $\rho(\gamma)$. In either case we have not changed the conjugacy class of any peripheral elements and so $[\rho_t]$ is a curve of representations in $\mathfrak{X}_{\mathrm{scp}}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$. Similarly if S is closed and non-separating then if $\gamma \in \Gamma$ is peripheral, then $\gamma \in \pi_1(M \backslash S)$, and so its conjugacy class does not depend on t. $\square$

The case where S has finite volume, but is non-compact is more subtle. However, when all of the cusps have torus cross sections then the situation is greatly simplified. Let $C = T \times [1, \infty)$, where T is an $n-1$ dimensional Euclidean torus, be a cusp of M. Since S is non-compact and properly embedded we see that $S \cap C = \sqcup_{i=1}^k (t_i \times [1, \infty))$, where t_i is an $n-2$ dimensional Euclidean torus. Let $\Delta = \pi_1(T)$ and $\delta_i = \pi_1(t_i)$ and we think of the situation in the universal cover of M, which we view as the upper half space model, see Figure 3.4. The universal cover of C is a horoball B with boundary $\widetilde{T}$. Since S is properly embedded and totally geodesic in M we see after possibly shrinking B that the lifts $\widetilde{S}$ of S intersect B in a collection of parallel vertical half spaces. The universal covers of the t_i are realized as the intersection of $\widetilde{S}$ and $\widetilde{T}$.

Next, we examine how δ_i sits inside of Δ. Observe that $\Delta = \mathbb{Z}^{n-1}$ and $\delta_i = \mathbb{Z}^{n-2}$, and since t_i is embedded in T we see that under the inclusion $\delta_i \hookrightarrow \Delta$ that primitive elements are taken to primitive elements. Therefore $\Delta/\delta_i \cong \mathbb{Z}$ (this fact is easily seen by examining the Smith normal form of a presentation matrix). Let α denote a choice of lift of the generator of Δ/δ_i to Δ, and we say that α is *dual* to t_i. It should be noted that since the $\tilde{t}_i$ are all parallel that the same α is dual to each t_i. The element α can be realized as

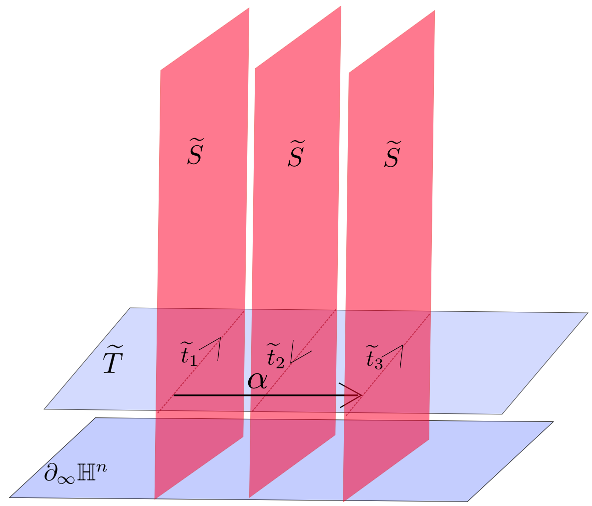

Figure 3.4: Bending viewed in upper half space with signed intersection +1

a curve in ∂M that is transverse to the intersection of S and ∂M and we can compute the signed intersection $\iota(\alpha, S \cap T)$ of $S \cap T$ and α In this setting we can prove the following proposition.

Proposition 3.2.7. *Let S be a non-compact, finite volume, totally geodesic hypersurface of M. Suppose that each cusp C_i of M is diffeomorphic to $T_i \times [1, \infty)$, where T_i is a torus. If ρ_t is the family of representations obtained by bending along S then $[\rho_t]$ is contained in $\mathfrak{X}_{\mathrm{scp}}(\Gamma, \mathrm{PGL}_{\mathrm{n+1}}(\mathbb{R}))$ if and only if $\iota(\alpha, S \cap T_i) = 0$ for each i.*

Proof. To simplify the exposition we assume that the manifold has a single cusp $T \times [1, \infty)$ as the general case follows if and only if it is true for each individual cusp. The result of bending along S on the cusp can be realized by bending T along the totally geodesic hypersurface surface $S \cap T$. If this surface is disconnected then we are simultaneously bending along each of the connected components. Each component of $S \cap T$ inherits an orientation from S and this orientation determines the direction (i.e. a choice of x_S or $-x_S$ as the bending cocylce) of the bending and as a result it is possible for bendings to cancel one another. More precisely, bending along oppositely oriented components of $S \cap T$ cancel. Thus we see that if $\iota(\alpha, S \cap T) = 0$ then all the bendings will cancel and ρ_t will be constant when restricted to ∂M. Since $\rho_0 = \rho_{\mathrm{geo}}$ we see that $[\rho_t] \in \mathfrak{X}_{\mathrm{scp}}(\Gamma, \mathrm{PGL_{n+1}}(\mathbb{R}))$ for all t.

If $\iota(\alpha, S \cap T) \neq 0$, then by the previous argument some net bending is occurring in the cusp and we will show that such a bending gives rise to hyperbolic elements. We now switch to working in the hyperboloid model of hyperbolic space. By conjugating we can assume that the fixed point of $\rho_0(\alpha)$ is $v_0 = (0, 0, \ldots, 0, -1, 1)$. Since $\rho_0(\alpha)$ is parabolic we see that v_0 is an eigenvector for $\rho_0(\alpha)$ of eigenvalue 1 and so

$$\rho_t(\alpha) = \begin{pmatrix} e^{-nt} & 0 \\ 0 & e^t I \end{pmatrix} \rho_0(\alpha) \cdot v_0 = \begin{pmatrix} e^{-nt} & 0 \\ 0 & e^t I \end{pmatrix} \cdot v_0 = e^t v_0.$$

Thus we see that for $t \neq 0$ that $\rho_t(\alpha)$ has e^t as an eigenvalue and is thus hyperbolic. □

It should be noted that for orientable, finite volume hyperbolic 3-manifolds the cusps are always of the form $T^2 \times [1, \infty)$ and so Proposition 3.2.7 and Proposition 3.2.7 allow us to analyze all orientable finite volume hyperbolic 3-manifolds.

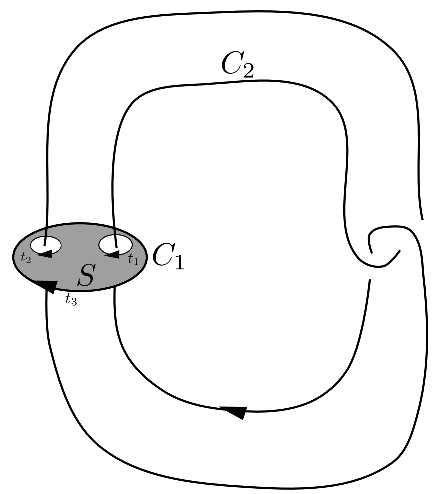

Figure 3.5: The Whitehead link and its totally geodesic surface S

We conclude this chapter by discussing an example of bending along a non-compact surface. Consider the Whitehead link (see Figure 3.5) along with the totally geodesic thrice punctured sphere S, its boundary components t_1, t_2 and t_3, and the cusps C_1 and C_2. The cusp C_1 intersects S in t_3 and experiences non-trivial bending in this cusp. The cusp C_2 intersects S in the parallel 1-tori t_1 and t_2. These tori are oppositely oriented and cancel and so this cusp experiences no bending. The result is a representation that is constant on the fundamental group of C_2 but non-trivial on the fundamental group of C_1. From a symmetry of the Whitehead link that exchanges C_1 and C_2 we see that there is another totally geodesic thrice punctured sphere which bends C_2 trivially and C_1 non-trivially.

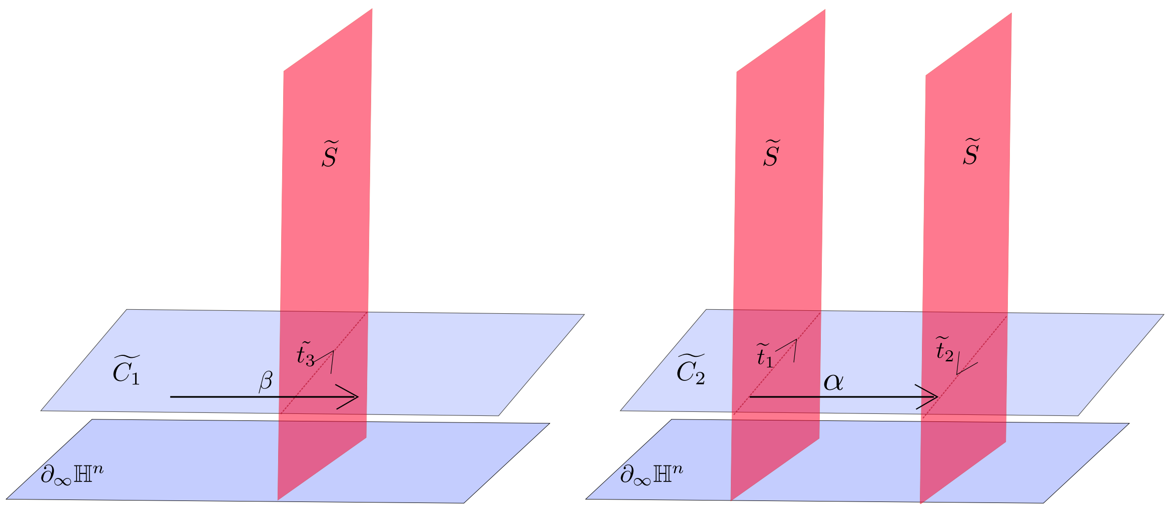

Figure 3.6: Bending of the cusps of the Whitehead link viewed from the universal cover

Chapter 4

Rigid Two-Bridge Knots and Links

In this chapter we will focus our attention on strictly convex deformations in dimension 3 and develop the necessary tools to demonstrate that several two-bridge knots and links do not admit strictly convex deformations near their complete hyperbolic structure. The rough idea is to prove that there are no conjugacy classes of representations near $[\rho_{\mathrm{geo}}]$ where the peripheral elements are all mapped to strictly convex parabolics. We begin by describing some useful normal forms for matrices.

4.1 Some Normal Forms

In this section we will examine various normal forms into which we can put two non-commuting strictly convex parabolic elements of $\mathrm{PGL}_4(\mathbb{R})$. One of the difficulties in working with $\mathfrak{X}(\Gamma, \mathrm{PGL}_4(\mathbb{R}))$ is that you are dealing with conjugacy classes instead of actual representations. One way to deal with this difficulty is to use normal forms which contain exactly one representation from each conjugacy class. In this way we can reduce proofs to examining actual representations which are nicer algebraic objects. The normal forms that we have chosen are very much in the spirit of those introduced by Riley [42], and we now recall his construction. Let a and b be two non-commuting parabolic (here parabolic is being used in the sense of hyperbolic geometry) elements of $\mathrm{SL}_2(\mathbb{C})$, then a and b can be simultaneously conjugated into the following form:

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}, \tag{4.1}$$

where ω is a non-zero complex number. By Corollary 3.1.4 we know that any strictly convex parabolic element in $\mathrm{PGL}_4(\mathbb{R})$ must be conjugate to

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.2}$$

With this fact in mind we would like to show that if A_0 and B_0 are non commuting parabolic elements of $\mathrm{SO}(3,1)$ then any two strictly convex parabolic elements A and B that are sufficiently close to A_0 and B_0 of $\mathrm{PGL}_4(\mathbb{R})$ can be simultaneously conjugated into a normal form similar to (4.1). Before we proceed we set some notation. Let $F_2 =< \alpha, \beta >$ be the free group on two letters and let $\mathrm{Hom}_{scp}(F_2, \mathrm{PGL}_4(\mathbb{R}))$ be the set of homomorphism of F_2 that send α and β to strictly convex parabolics. The remainder of this section will be dedicated to proving the following proposition.

Proposition 4.1.1. *Let $f_0 \in \mathrm{Hom}_{scp}(F_2, \mathrm{PGL}_4(\mathbb{R}))$ satisfy the following conditions*

1. *$\langle f_0(\alpha), f_0(\beta)\rangle$ is irreducible and conjugate into* $\mathrm{SO}(3,1)$.

2. *$\langle f_0(\alpha), f_0(\beta)\rangle$ is not conjugate into* $\mathrm{SO}(2,1)$.

Then for $f \in \mathrm{Hom}_{scp}(F_2, \mathrm{PGL}_4(\mathbb{R}))$ sufficiently close to f_0 there exists a unique (up to $\pm I$) element $G \in \mathrm{SL}_4(\mathbb{R})$ such that

$$G^{-1}f(\alpha)G = \begin{pmatrix} 1 & 0 & 1 & a_{14} \\ 0 & 1 & 1 & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad G^{-1}f(\beta)G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_{21} & 1 & 0 & 0 \\ b_{31} & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

Additionally, the map taking f to its normal form is continuous.

In order to do this we will first show that A and B can be conjugated into the normal form

$$A = \begin{pmatrix} 1 & 0 & 2 & 1+a_{14} \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_{21} & 1 & 0 & 0 \\ b_{31}+b_{21}b_{32} & 2b_{32} & 1 & 0 \\ b_{21}+b_{41} & 2 & 0 & 1 \end{pmatrix}. \tag{4.4}$$

At first, (4.4) may appear an odd normal form, however it provides a nice symmetry between A and B and their inverses. For example,

$$A^{-1} = \begin{pmatrix} 1 & 0 & -2 & 1-a14 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and there is a similar relation for B^{-1}. To begin we need to show that if A_0 and B_0 are conjugate into a fixed copy of $\mathrm{SO}(3,1)$ then they can be put into our normal form. In particular, we would like to build a homomorphism from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{PGL}_4(\mathbb{R})$ that send elements in the normal form (4.1) to elements in the normal form (4.4).

The standard way to build homomorphisms from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{PGL}_4(\mathbb{R})$ is by using quadratic forms of signature $(3,1)$. The standard way to realize the group $\mathrm{SO}(3,1)$ is the isometry group of the quadratic form coming from the symmetric matrix

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

There is a map, ψ, from $\mathbb{R}^4$ into 2×2 hermitian matrices given by

$$\psi((x,y,z,t)) = \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix}.$$

The negative determinant gives a quadratic form on the set of hermitian matrices and it is a simple computation to see that ψ respects these quadratic forms. Since hyperbolic space can be realized as vectors $v \in \mathbb{R}^4$ such that $v^T J v = -1$, we see that this map takes hyperbolic space onto the set of hermitian 2×2 matrices of determinant 1. Next, let M be a 2×2 matrix and $N \in \mathrm{SL}_2(\mathbb{C})$. Then we can define an action on via $N \cdot M = NMN^*$, where $*$ denotes the conjugate transpose operator. This action is clearly linear and preserves determinants and thus gives us a representation ψ from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{SO}(3,1)$. The problem with ψ is that it does not take elements in the normal form (4.1) to elements in the normal form (4.4). To fix this problem we need to choose our quadratic form more judiciously.

Let

$$X = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ x_{12} & x_{22} & x_{23} & x_{24} \\ x_{13} & x_{23} & x_{33} & x_{34} \\ x_{14} & x_{24} & x_{34} & x_{44} \end{pmatrix}$$

be a symmetric matrix and observe that X has signature $(3,1)$ (up to $\pm I$) if and only if $\det X < 0$. We now examine the restriction placed on the coefficients of A, B, and X by knowing that the quadratic form determined by X is preserved by matrices of the type (4.4). The restrictions that $A^T X A = X$ and $B^T X B = X$ tell us that

$$a_{14} = -2b_{32},\ b_{31} = 0,\ b_{41} = -2,\ x_{12} = -1,\ x_{13} = 2b_{32},\ x_{14} = -b_{21}/2, \quad (4.5)$$

$$x_{22} = 1,\ x_{23} = -2b_{32},\ x_{24} = 2b_{32}^2,\ x_{33} = b_{21},\ x_{34} = -b_{21}b_{32},\ x_{44} = b_{21}b_{32}^2.$$

With these restrictions, we see that our matrix X looks like

$$\begin{pmatrix} 1 & -1 & 2b_{32} & -b_{21}/2 \\ -1 & 1 & -2b_{32} & 2b_{32}^2 \\ 2b_{32} & -2b_{32} & b_{21} & -b_{21}b_{32} \\ -b_{21}/2 & 2b_{32}^2 & -b_{21}b_{32} & b_{21}b_{32}^2 \end{pmatrix},$$

and we now assume that the entries of A and B satisfy (4.5). If we let x, y, z, and t, be coordinates for $\mathbb{R}^4$ then we see that the quadratic form given by X is

$$x^2 - 2xy + y^2 - b_{21}xt + 4b_{32}^2 yt - 2b_{21}b_{32}zt + 4b_{32}xz - 4b_{32}yz + b_{21}z^2 + b_{21}b_{32}^2 t^2.$$

If we let $D = b_{21} - 4b_{32}^2$, then a simple calculation shows that $\det X < 0$ if and only if $D > 0$. We can now define a new map, ψ', from $\mathbb{R}^4$ to 2×2 hermitian matrices that takes (x, y, z, t) to

$$\begin{pmatrix} x & x - y + 2b_{32}z - 2b_{32}^2 t + i(\sqrt{D}z - b32\sqrt{D}t) \\ x - y + 2b_{32}z - 2b_{32}^2 t - i(\sqrt{D}z - b32\sqrt{D}t) & Dt \end{pmatrix}.$$

It is again easy to see that this map respects the quadratic forms given by X on $\mathbb{R}^4$ and the negative determinant on hermitian matrices, and so as above we get a map ϕ' from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{SO}(3,1)$[1]. Another simple calculation shows that

$$\phi'\left(\begin{pmatrix} 1 & i/\sqrt{d} \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 2 & 1+a_{14} \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

However, by precomposing by a conjugation in $\mathrm{SL}_2(\mathbb{C})$ we get a new map ϕ such that

$$\phi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 2 & 1+a_{14} \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A simple, yet tedious, computation shows that if

$$b_{21} = |\omega|^2 \text{ and } b_{32} = \mathrm{Re}(\omega)/2 \tag{4.6}$$

then ϕ will take elements of the form (4.1) to elements of the form (4.4). With these assumptions on b_{21} and b_{32} we see that

$$D = b_{21} - 4b_{32}^2 = |\omega|^2 - \mathrm{Re}(\omega)^2 = \mathrm{Im}(\omega)^2,$$

and so as long as $\mathrm{Im}(\omega) \neq 0$ we will be able to put a and b into (4.4).

We are now in position to prove that A and B can be put in the normal form (4.4).

Lemma 4.1.2. *Let $f_0 \in \mathrm{Hom}_{scp}(F_2, \mathrm{PGL}_4(\mathbb{R}))$ satisfy the following conditions*

1. *$\langle f_0(\alpha), f_0(\beta)\rangle$ is irreducible and conjugate into* $\mathrm{SO}(3,1)$.

[1]The images of ψ and ϕ' are conjugate in $\mathrm{PGL}_4(\mathbb{R})$, but not equal

2. $\langle f_0(\alpha), f_0(\beta)\rangle$ *is not conjugate into* $\mathrm{SO}(2,1)$*.*

Then for $f \in \mathrm{Hom}_{scp}(F_2, \mathrm{PGL}_4(\mathbb{R}))$ *sufficiently close to* f_0 *there exists a unique (up to* $\pm I$*) element* $G \in \mathrm{SL}_4(\mathbb{R})$ *such that*

$$G^{-1}f(\alpha)G = \begin{pmatrix} 1 & 0 & 2 & 1+a_{14} \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad G^{-1}f(\beta)G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_{21} & 1 & 0 & 0 \\ b_{31}+b_{21}b_{32} & 2b_{32} & 1 & 0 \\ b_{21}+b_{41} & 2 & 0 & 1 \end{pmatrix}.$$

Additionally, the map taking f *to its normal form is continuous.*

Proof. The previous argument combined with properties 1 and 2 ensure that $f_0(\alpha)$ and $f_0(\beta)$ can be put into the form (4.4). Let $A = f(\alpha)$ and $B = f(\beta)$. Let E_A and E_B be the 1-eigenspaces of A and B, respectively. Since both A and B are strictly convex parabolics we know they are each conjugate to (4.2) and so both of these spaces are 2-dimensional. Irreducibility is an open condition and so we can assume that f is also irreducible and so E_A and E_B have trivial intersection. Therefore $\mathbb{R}^4 = E_A \oplus E_B$. If we select a basis with respect to this decomposition then our matrices will be of the following block form.

$$\begin{pmatrix} I & A_U \\ 0 & A_L \end{pmatrix}, \qquad \begin{pmatrix} B_U & 0 \\ B_L & I \end{pmatrix}. \tag{4.7}$$

Observe that 1 is the only eigenvalue of A_L (resp. B_U) and that neither of these matrices is diagonalizable, otherwise $(A-I)^2 = 0$ (resp. $(B-I)^2 = 0$) and so A (resp. B) would not have the right Jordan form. Thus we can further conjugate E_A and E_B so that

$$A_L = \begin{pmatrix} 1 & a_{34} \\ 0 & 1 \end{pmatrix}, \qquad B_U = \begin{pmatrix} 1 & 0 \\ b_{21} & 1 \end{pmatrix},$$

where $a_{34} \neq 0 \neq b_{21}$. Conjugacies that preserve the block form (4.7) are all of the form

$$\begin{pmatrix} u_{11} & 0 & 0 & 0 \\ u_{21} & u_{22} & 0 & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}.$$

Finally, a tedious computation[2] allows us to determine that there exist unique values of the $u_{ij}s$ that will finish putting our matrices in the desired normal form. Note that the existence of solutions depends on the fact that the entries of A and B are close to the entries of $f_0(\alpha)$ and $f_0(\beta)$, which live in $\mathrm{SO}(3,1)$. Finally, observe that the entries of G continuously depend on the entries of A and B and so taking f to its normal form is a continuous operation. □

Proof of Proposition 4.1.1. By Lemma 4.1.2 we can assume that A and B are in the normal form (4.4), then by conjugation by the matrix

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2-b_{21}-b_{42}}{4} & \frac{2+b_{21}+b_{41}}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{2b_{32}+b_{21}b_{32}+b_{32}b_{41}-1}{2} \\ 0 & 0 & 0 & \frac{2+b_{21}+b_{41}}{2} \end{pmatrix}$$

we can put A and B into the desired form. Since the entries of V are continuous functions of the entries of A and B we see that this operation is continuous. □

[2] This computation is greatly expedited by using Mathematica.

Remark 4.1.1. Conjugation by V makes sense because A and B are close to matrices satisfying equations (4.5) and (4.6). Thus $2 + b_{21} + b_{41} \neq 0$ and we conclude that V is non-singular.

4.2 Two-Bridge Examples

In this section we will prove that several two-bridge knots and links do not admit strictly convex deformations near their complete hyperbolic structures. Two-bridge knots and links are an ideal class of examples to study because, as we will see, they have relatively simple two generator one relator presentations where the generators can be taken as meridians. Furthermore they do not contain any closed totally geodesic surfaces [25]. We will analyze representations of the fundamental groups of two-bridge knots and links by using the normal forms from the previous section to show that these two-bridge examples are locally projectively rigid with respect to the boundary at $[\rho_{\text{geo}}]$.

We begin with some background information about two-bridge knots and links (see [12, 36] for details). A knot (or link) L in $\mathbb{S}^3$ is *two-bridge* if it can be isotoped so that the natural height function coming from its embedding into $\mathbb{R}^3$ has exactly 2 minima and 2 maxima as its critical points. Two-bridge knots and links are determined by a pair of relatively prime integers p/q where $0 < q < p$ and q is odd and the parity of p determines whether we have a knot or a link. Two pairs p/q and p'/q' give rise to the same knot if and only if they satisfy the following relationship (see [12] for details).

$$p = p' \text{ and } qq' = \pm 1 \pmod{p}. \tag{4.8}$$

If $q = 1$ then then the resulting knot is a torus knot and hence not hyperbolic, however if $q > 1$ then the knots are hyperbolic by work of Thurston [47]

We say that L has *tunnel number 1* if there exists an arc ℓ in its complement with endpoints on the L such that the complement of a regular neighborhood of $L \cup \ell$ is homeomorphic to a genus 2 handlebody. The arc ℓ is called an *unknotting tunnel*. Two-bridge knots and links are all of tunnel number 1 which can be seen by taking an unknotted arc connecting the maxima (or minima). The existence of an unknotting tunnel gives a 2 generator 1 relator presentation of $\pi_1(\mathbb{S}^3 \backslash L)$, and when L is two-bridge these generators can be chosen to be meridians. Given the data of p/q an explicit presentation can be written down [36, §4.5] which takes the following form.

$$\pi_1(\mathbb{S}^3 \backslash L) = \langle a, b | aw = wb \rangle, \tag{4.9}$$

where w a word in a and b that depends explicitly on p/q. Therefore we can apply the techniques of the previous section to analyze fundamental groups of two-bridge knot and link complements and spend the rest of this chapter proving the following theorem.

Theorem 4.2.1. *The two bridge knots and links with rational number $\frac{5}{3}$ (figure-eight) , $\frac{7}{3}$, $\frac{9}{5}$, and $\frac{8}{3}$ (Whitehead link) do not admit strictly convex deformations near their complete hyperbolic structures.*

Remark 4.2.1. The knots and links mentioned in Theorem 4.2.1 correspond to the 4_1, 5_2, 6_1, and 5_1^2 in Rolfsen's table of knots and links [44]

The idea of the proof is as follows: by Corollary 3.2.5 it suffices to show that these knot complements are locally projectively rigid relative to the boundary at ρ_{geo}. Using Proposition 4.1.1 we can reduce the problem to finding representations of $\pi_1(\mathbb{S}^3 \backslash L)$ into $\mathrm{PGL}_4(\mathbb{R})$ sending peripheral elements to parabolics to finding generators A and B that are in the normal form (4.3) and satisfy the matrix equation

$$AW - WB = 0 \text{ or } AW - WA = 0, \tag{4.10}$$

depending on whether L is a knot or link, respectively. Any knot or link for which this solution set is a discrete subset of $\mathbb{R}^5$ (coming from the 5 unknown entries of (4.3)) will satisfy the conclusion of Theorem 4.2.1.

4.2.1 The Figure-Eight Knot

In this section we prove Theorem 4.2.1 for the figure-eight knot (see Figure 4.1). The exact computations described in this section can be found in Appendix 1. The figure-eight knot is a two bridge knot with rational number 5/3 and in this case the word w from the presentation (4.9) is given by $w = ba^{-1}b^{-1}a$. Using Mathematica we solve Equation (4.10) and find that there is the following 1-dimensional solution set

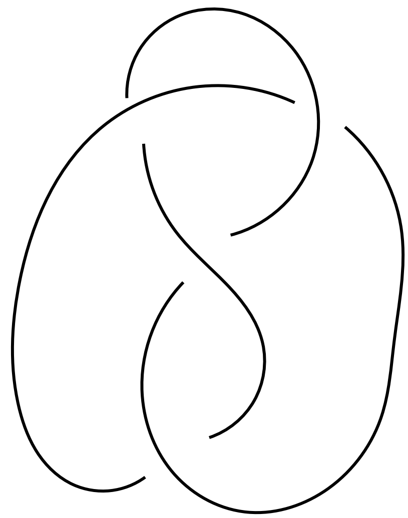

Figure 4.1: The figure-eight knot

$$A = \begin{pmatrix} 1 & 0 & 1 & \frac{3-t}{t-2} \\ 0 & 1 & 1 & \frac{1}{2(t-2)} \\ 0 & 0 & 1 & \frac{t}{2(t-2)} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \tag{4.11}$$

with ρ_{geo} occurring at $t = 4$. However, most of these representations do not correspond to *strictly convex* deformations, as they map non-meridional peripheral elements to hyperbolic elements. For example, the element $L = BA^{-1}B^{-1}A^2B^{-1}A^{-1}B$ is a longitude of the knot complement and hence a peripheral element. A simple calculation shows that $\text{tr}(L) = \frac{48+(t-2)^4}{8(t-2)}$, and so L is parabolic if and only if $t = 4$. Thus there are no strictly convex deformations of the complete hyperbolic structure on the figure-eight knot, proving Theorem 4.2.1 for this knot.

4.2.2 The Whitehead Link

The Whitehead link (see Figure 3.5) is a two-bridge link with rational number 8/3 and the word w from the presentation (4.9) is given by $w = bab^{-1}a^{-1}b^{-1}ab$. Again using Mathematica we see that (4.10) has a unique solution given by

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}. \tag{4.12}$$

The explicit computation can again be found in Appendix 1. Thus there are no strictly convex deformations of the complete hyperbolic structure on the Whitehead link, proving Theorem 4.2.1 for this link. In this case it was not necessary to place any restriction on the trace of any other peripheral element in order to get a unique solution. We have previously seen in our discussion of bending that the Whitehead link contains totally geodesic surfaces that give rise to non-trivial deformations of $[\rho_{\text{geo}}]$, however Theorem 3.2.7 tells us that these deformations do not send peripheral elements to parabolics. This explains why our previous calculation did not detect these deformations.

Remark 4.2.2. Details of similar calculations for the two-bridge knots with rational numbers 7/3 and 9/5 can be found in Appendix 1. For these two knots the solution sets form discrete sets, but contain multiple points. This is to be expected, as it is known that there are multiple representations of $\pi_1(\mathbb{S}^3 \backslash L)$ into $\mathrm{SL}_2(\mathbb{C})$ for these knots that send peripheral elements to parabolics, but only one of them is discrete and faithful. These representations correspond to solutions to a 1-variable polynomial [36, §4.5]. These computations complete the proof of Theorem 4.2.1 for these knots.

Chapter 5

Deformations Coming from Symmetry

In this chapter we will discuss certain relationships between symmetries of infinitesimally rigid knot complements and deformations of certain surgeries on the knot.

5.1 Decomposing $H^*(\Gamma, \mathfrak{sl}(4)_{\rho_{\text{geo}}})$

As we have previously mentioned, the cohomology group $H^1(\Gamma, \mathfrak{sl}(4)_{\rho_{\text{geo}}})$ can be thought of as the space of infinitesimal deformations of representations of Γ into $\mathrm{SL}_4(\mathbb{R})$, up to conjugacy. Therefore, understanding this group can help us to understand various rigidity and flexibility phenomena. To simplify the situation we will exploit a splitting of $H^1(\Gamma, \mathfrak{sl}(4)_{\rho_{\text{geo}}})$ coming from a splitting of $\mathfrak{sl}(4)$ introduced in [30].

The standard quadratic form used to define $\mathbb{H}^3$ is given by the following matrix

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and we realize the group $\mathrm{SO}(3,1)$ as $\{A \in \mathrm{SL}_4(\mathbb{R}) \mid A^T J A = J\}$ with Lie algebra $\mathfrak{so}(3,1) = \{a \in \mathfrak{sl}(4) \mid a^T J = -Ja\}$. Via the adjoint action we see that $\mathfrak{sl}(4)$ is an $\mathrm{SO}(3,1)$-module and that J gives rise to a module isomorphism given by $a \mapsto -Ja^TJ$. This map is clearly an involution and so we can decompose $\mathfrak{sl}(4)$ into submodules coming from the ± 1-eigenspaces of this involution. A simple computation shows that $\mathfrak{so}(3,1)$ is the 1-eigenspace and the -1-eigenspace is given by

$$\mathfrak{v} = \{a \in \mathfrak{sl}(4) \mid a^tJ = Ja\}.$$

In this way we get the following splitting as $\mathrm{SO}(3,1)$ modules:

$$\mathfrak{sl}(4) = \mathfrak{so}(3,1) \oplus \mathfrak{v}. \tag{5.1}$$

It should be noted that this is not a splitting as Lie algebras since $\mathfrak{v}$ is not closed under the Lie bracket.

The *Killing form*, denoted $B(X,Y)$, is a non-degenerate bilinear form defined on $\mathfrak{sl}(4)$ by $B(X,Y) = 8\mathrm{tr}(XY)$. Invariance of trace function under conjugation tells us that B is invariant under the adjoint action of $\mathrm{SL}_4(\mathbb{R})$ on $\mathfrak{sl}(4)$. Another simple computation shows that B is non-degenerate when restricted to $\mathfrak{so}(3,1)$ and that $\mathfrak{v}$ is the orthogonal complement of $\mathfrak{so}(3,1)$ and so the splitting (5.1) is an orthogonal decomposition. Finally, given a representation $\rho : \Gamma \to \mathrm{SO}(3,1)$, the splitting (5.1) gives rise to a splitting

$$H^*(\Gamma, \mathfrak{sl}(4)_\rho) = H^*(\Gamma, \mathfrak{so}(3,1)_\rho) \oplus H^*(\Gamma, \mathfrak{v}_\rho), \tag{5.2}$$

in the obvious way.

5.2 Cuspidal Cohomology and Poincaré Duality

Henceforth, let M be a finite volume hyperbolic 3-manifold with $\Gamma = \pi_1(M)$, and let ρ_{geo} be the geometric representation of M. Assume that M has k cusps which we denote $\partial_i M$ for $1 \leq i \leq k$. Recall that since M is aspherical there is a natural isomorphism between $H^*(\pi_1(M), \mathfrak{sl}(4)_{\rho_{\text{geo}}})$ and $H^*(M, \mathfrak{sl}(4)_{\rho_{\text{geo}}})$. Let ∂M be the (possibly empty) boundary of M and let $\iota : \partial M \to M$ be the map induced by inclusion. The map ι induces a map

$$\iota^*_{\mathfrak{g}_{\rho_{\text{geo}}}} : H^*(M, \mathfrak{g}_{\rho_{\text{geo}}}) \to H^*(\partial M, \mathfrak{g}_{\rho_{\text{geo}}}), \tag{5.3}$$

where $\mathfrak{g}$ denotes $\mathfrak{sl}(4)$, $\mathfrak{so}(3,1)$, or $\mathfrak{v}$. When no confusion will arise we refer to $\iota^*_{\mathfrak{g}_{\rho_{\text{geo}}}}$ as ι^*. For any such system of coefficients we refer to the kernel of $\iota^*_{\mathfrak{g}_{\rho_{\text{geo}}}}$ as $\mathfrak{g}_{\rho_{\text{geo}}}$-*cuspidal cohomology*, and we say that M (or Γ) is *infinitesimally projectively rigid relative to the boundary* if it has trivial $\mathfrak{sl}(4)_{\rho_{\text{geo}}}$-cuspidal cohomology. Being infinitesimally projectively rigid relative to the boundary means that any deformation of ρ_{geo} in G is infinitesimally induced by a deformation of ρ_{geo} restricted to $\pi_1(\partial M)$. This definition is an extension of the notion of infinitesimal projective rigidity to non-compact manifolds, and when M is closed the two definitions coincide.

Next, we will describe a twisted cohomology version of Poincaré duality and how it is used to analyze $\iota^*_{\mathfrak{v}_{\rho_{\text{geo}}}}$. The Killing form gives rise to a perfect pairing which we call the *cup product* (see [28] for details of this construction)

$$\cup : H^i(M, \mathfrak{g}_{\rho_{\text{geo}}}) \times H^{3-i}(M, \partial M, \mathfrak{g}_{\rho_{\text{geo}}}) \to \mathbb{R}. \tag{5.4}$$

Using this pairing we get the natural identification

$$H^i(M, \mathfrak{g}_{\rho_{\text{geo}}}) \cong H^{3-i}(M, \partial M, \mathfrak{g}_{\rho_{\text{geo}}})^*,$$

where in this setting $*$ refers to the vector space dual. This identification is commonly called *Poincaré duality.* Combining Poincaré duality with the long exact sequences coming from the pair $(M, \partial M)$ we get the following diagram:

$$\begin{array}{ccccc} H^1(M, \mathfrak{g}_{\rho_{\text{geo}}}) & \xrightarrow{\iota^*} & H^1(\partial, \mathfrak{g}_{\rho_{\text{geo}}}) & \xrightarrow{\beta} & H^2(M, \partial M, \mathfrak{g}_{\rho_{\text{geo}}}) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(M, \partial M, \mathfrak{g}_{\rho_{\text{geo}}})^* & \xrightarrow{\beta^*} & H^1(\partial M, \mathfrak{g}_{\rho_{\text{geo}}})^* & \xrightarrow{(\iota^*)^*} & H^1(M, \mathfrak{g}_{\rho_{\text{geo}}})^* \end{array} \quad (5.5)$$

Using (5.5) and elementary linear algebra we arrive at the following result.

Lemma 5.2.1. *Let M be a finite volume hyperbolic 3 manifold then*

$$\dim \iota^*_{\mathfrak{g}_{\rho_{\text{geo}}}}(H^1(M, \mathfrak{g}_{\rho_{\text{geo}}})) = \frac{1}{2} \dim H^1(\partial M, \mathfrak{g}_{\rho_{\text{geo}}}).$$

This result admits several generalizations, including to arbitrary compact 3-manifolds with untwisted $\mathbb{Q}$ coefficients [27] as well as twisted coefficients in a semisimple Lie algebra [29]. Results of this type are often referred to as half-lives/half-dies.

Due to its relevance in hyperbolic geometry, the map

$$\iota^*_{\mathfrak{so}(3,1)_{\rho_{\text{geo}}}} : H^1(M, \mathfrak{so}(3,1)_{\rho_{\text{geo}}}) \to H^1(\partial M, \mathfrak{so}(3,1)_{\rho_{\text{geo}}})$$

has been well studied. By work of Garland, [21] $\iota^*_{\mathfrak{so}(3,1)_{\rho_{\text{geo}}}}$ is an injection (this includes the case where $\partial M = \emptyset$ where it is known as Weil rigidity [49]). Additionally,

$$H^0(\partial M, \mathfrak{so}(3,1)_{\rho_{\text{geo}}}) \cong \oplus_{i=1}^k H^0(\partial M_i, \mathfrak{so}(3,1)_{\rho_{\text{geo}}}),$$

where ∂M_i is the ith boundary component. For each i, $H^0(\partial M_i, \mathfrak{so}(3,1)_{\rho_{\text{geo}}}) \cong H^0(\mathbb{Z}^2, \mathfrak{so}(3,1)_{\rho_{\text{geo}}})$, the latter of which consists of the elements of $\mathfrak{so}(3,1)$ on which that adjoint action of $\mathbb{Z}^2$ is trivial. Therefore the dimension of $H^0(\partial M, \mathfrak{so}(3,1)_{\rho_{\text{geo}}})$ is $2k$ dimensional. Combining this with Poincaré duality and the fact that the Euler characteristic of ∂M is zero we find that $H^1(\partial M, \mathfrak{so}(3,1)_{\rho_{\text{geo}}})$ is $4k$ dimensional. Finally, by half-lives/half-dies we see that

$$\dim H^1(M, \mathfrak{so}(3,1)_{\rho_{\text{geo}}}) = \dim \iota^*_{\mathfrak{so}(3,1)_{\rho_{\text{geo}}}}(H^1(M, \mathfrak{so}(3,1)_{\rho_{\text{geo}}})) = 2k.$$

We close this section by mentioning two lemmas from [28] about the cohomology of ∂M with coefficients in $\mathfrak{v}_{\rho_{\text{geo}}}$. If $\partial_i M$ is a component of ∂M and m and l are generators of $\pi_1(\partial M)$ then $\rho_{\text{geo}}(m)$ and $\rho_{\text{geo}}(l)$ are affine translations and can thus be identified with vectors in $\mathbb{R}^2$.

Lemma 5.2.2 ([28, Lem 5.5])**.** *Let m and l be generators of $\pi_1(\partial_i M)$ and let ι_m^* and ι_l^* be the maps induced by inclusion. If the angle between m and l is not an integral multiple of $\pi/3$ then the map*

$$H^1(\partial_i M, \mathfrak{v}_{\rho_{\text{geo}}}) \overset{\iota_m^* \oplus \iota_l^*}{\to} H^1(m, \mathfrak{v}_{\rho_{\text{geo}}}) \oplus H^1(l, \mathfrak{v}_{\rho_{\text{geo}}})$$

is an injection. Furthermore, ι_m^* and ι_l^* both have rank 1.

We previously mentioned that there are representations near ρ_{geo} that correspond to incomplete structures on M. Let ρ_u be such a representation, then we have the following lemma

Lemma 5.2.3 ([28, Lem 5.3]). *If ρ_u is the holonomy of an incomplete hyperbolic structure on M then there is a natural isomorphism*

$$H^*(\partial_i M, \mathfrak{v}_{\rho_{\text{geo}}}) \cong H^*(\partial_i M, \mathbb{R}) \otimes \mathfrak{v}^{\rho_u(\pi_1(\partial_i(M)))},$$

where $\mathfrak{v}^{\rho_u(\pi_1(\partial_i M))}$ are elements of $\mathfrak{v}$ on which the adjoint action is trivial.

5.3 Deformations Coming From Surgery

In this section we describe how to build deformations near the geometric representation of certain hyperbolic manifolds (and orbifolds) resulting from surgery on certain knot complements. We begin with a discussion of hyperbolic Dehn filling (see [41, 47] for more details). Let M be a finite volume hyperbolic 3-manifold with a single cusp. A *slope* α of ∂M is a homotopy class of a simple closed curve on ∂M. Since $\pi_1(\partial M) \cong \mathbb{Z}^2$ we see that after identifying the generators of $\mathbb{Z}^2$ with m and l that $\alpha = pm + ql$ where $(p, q) = 1$. It then follows that slopes are in bijective correspondence with $\mathbb{Q} \cup \{\infty\}$, and we write a slope $\alpha = p/q$. Given a slope α we can form a new manifold $M(\alpha)$ by gluing a solid torus to M along their boundaries so that the meridian of the solid torus is mapped to α. The manifold $M(\alpha)$ is called the *Dehn filling of M along α*. A simple Van-Kampen argument shows that

$$\pi_1(M(\alpha)) \cong \pi_1(M)/\langle\langle\alpha\rangle\rangle, \tag{5.6}$$

where $\langle\langle\alpha\rangle\rangle$ is the normal closure of α in $\pi_1(M)$.

We now relate this construction to hyperbolic geometry. By work of Thurston [47] there is a parameterization of hyperbolic structures on M by a neighborhood of ∞ in $\mathbb{R}^2 \cup \{\infty\}$ such that ∞ corresponds to the complete hyperbolic structure and points in $\mathbb{R}^2$ correspond to incomplete structures. The coordinates of a point $(p,q) \in \mathbb{R}^2$ help us to understand the metric completion of these incomplete structures on M, which we refer to as $\hat{M}$. These completions come in the following three flavors.

1. If $p/q \in \mathbb{R}\backslash\mathbb{Q}$ then $\hat{M}$ is the one point compactification of M. Neighborhoods of the added point are solid tori and thus $\hat{M}$ is not a manifold. We will not be interested in completions of this type.

2. If p and q are a pair of relatively prime integers then $\hat{M}$ is the Dehn filling $M(p/q)$.

3. If $p/q = p'/q'$ where p' and q' are relatively prime integers then $\hat{M}$ is a *cone manifold*. This case is the singular version of type 2. Geometrically, this manifold can be realized by gluing a solid torus with longitudinal cone singularity with cone angle $2\pi p'/p$ so that the meridian of this singular torus is mapped to the slope p'/q' and we denote the completion by $M(p/q)$. When $p/p' = n$ is an integer then $M(p/q)$ is an orbifold with orbifold fundamental group $\pi_1^{orb}(M(p/q)) = \pi_1(M)/\langle\langle\alpha^n\rangle\rangle$, where α is the slope corresponding to p'/q' (see [31, §6] for definitions concerning orbifolds and $\pi_1^{orb}(M(p/q))$).

Suppose now that $M(\alpha)$ is hyperbolic and that $\rho_0 : \pi_1(M(\alpha)) \to \mathrm{PGL_{n+1}}(\mathbb{R})$ is its geometric representation. Let ρ_t be a non-trivial deformation of ρ_0 in $\mathrm{PGL}_4(\mathbb{R})$. From (5.6) we see that we are able to pull ρ_t back to a curve of representations $\tilde{\rho}_t : \Gamma \to \mathrm{PGL_{n+1}}(\mathbb{R})$ such that $\rho_t(\alpha) = 1$ for all t. In terms of cohomology we see that if $\omega \in H^1(M, \mathfrak{sl}(4)_{\tilde{\rho}_0)})$ is the cohomology class coming from $\tilde{\rho}_t$ then ω has trivial image in $H^1(\alpha, \mathfrak{sl}(4)_{\tilde{\rho}_0})$. Thus we would like to know given a cohomology class $\theta \in H^1(M, \mathfrak{sl}(4)_{\tilde{\rho}_0})$ whose image in $H^1(\alpha, \mathfrak{sl}(4)_{\tilde{\rho}_0})$ is trivial when does this cohomology class come from a deformation of $M(\alpha)$.

In order to begin answering this question we need to carefully analyze the image of the map

$$H^1(M, \mathfrak{v}_{\rho_u}) \to H^1(\partial M, \mathfrak{v}_{\rho_u}),$$

when ρ_u is the holonomy of an incomplete structure on M. Combining Lemma 5.2.3 with half-lives/half-dies we see that this map has rank 1. With this in mind we say that a slope α is *rigid* if the map $H^1(M, \mathfrak{v}_{\rho_{\mathrm{geo}}}) \to H^1(\alpha, \mathfrak{v}_{\rho_{\mathrm{geo}}})$ is not the zero map. In Section 4.2.2 we saw that bending along the appropriate totally geodesic thrice punctured spheres we were able to deform the meridian of each of the cusps of the Whitehead link, and thus we conclude that both meridians are rigid slopes. Additionally, in [28] it is shown that the figure-eight knot has trivial $\mathfrak{sl}(4)_{\rho_{\mathrm{geo}}}$-cuspidal cohomology. Combining this with the calculations from Section 4.2.1 exhibiting a family of deformations that do not deform the meridian shows that the meridian of the figure-eight is not a rigid slope. Since the longitude non-trivially deforms under the deformation (4.11) this same computation shows that the longitude of the figure-eight is a rigid slope.

We now examine how certain symmetries of M constrain the way this image sits inside of $H^1(\partial M, \mathfrak{v}_{\rho_u})$. For the remainder of this chapter assume that M is the complement of a hyperbolic amphicheiral knot complement, $\Gamma = \pi_1(M)$, and m and l are a meridian/longitude pair generating $\pi_1(\partial M)$. From amphicheirality we know that M admits an orientation reversing symmetry ϕ such that $\phi(l) = l$ and $\phi(m) = m^{-1}$. The map ϕ also induces an isomorphism, which we will also denote by ϕ, on Γ. The existence of this symmetry places strong restrictions on the holonomy of ∂M. Let ρ_u be the holonomy of some (possibly incomplete) hyperbolic structure on M.

If we view ρ_u as a representation into $\mathrm{PSL}_2(\mathbb{C})$ then after conjugation we can arrange so that

$$\rho_u(m) = \begin{pmatrix} e^{a_u/2} & 1 \\ 0 & e^{-a_u/2} \end{pmatrix}, \qquad \rho_u(l) = \begin{pmatrix} e^{b_u/2} & \tau_u \\ 0 & e^{-b_u/2} \end{pmatrix}, \tag{5.7}$$

where a_u, b_u, and τ_u are complex numbers and we call τ_u the *τ-invariant* of ρ_u (see also [9, App B]). The values of a_u, b_u and τ_u are all invariants of the conjugacy class of ρ_u and thus the form (5.7) contains geometric information about ρ_u. Specifically, $\rho_u(m)$ and $\rho_u(l)$ preserve a common geodesic and a_u and b_u represent the complex translation length of the isometry. When a_u (resp. b_u) is real then we say that $\rho_u(m)$ (resp. $\rho_u(l)$) is a *pure translation*, which is equivalent to $\mathrm{tr}^2\rho_u(m)$ (resp. $\mathrm{tr}^2\rho_u(l)$) being real and greater than 4. Similarly if a_u (resp. b_u) is imaginary then we say that $\rho_u(m)$ (resp. $\rho_u(l)$) is a *pure rotation*, which is equivalent to $\mathrm{tr}^2\rho_u(m)$ (resp. $\mathrm{tr}^2\rho_u(l)$) being real and between 0 and 4. Additionally, if ρ_u is the geometric representation of M then

$a_u = b_u = 0$ and τ_u coincides with the cusp shape and identifies the Euclidean structure on the cusp of M.

Suppose now that ρ_u is a representation such that the completion of the corresponding hyperbolic structure is $M(\alpha/0)$, that is to say the completion is realized by gluing a singular solid torus with longitudinal singularity of angle $2\pi/\alpha$ along the meridian of M. We will be most interested in the case where M is an amphicheiral hyperbolic knot complement and $\alpha = n$ is an integer. In this case the representation ρ_u naturally factors through a representation $\Gamma_n := \pi_1^{orb}(M(n/0)) \to \mathrm{PSL}_2(\mathbb{C})$, which we also call ρ_u. In this case, ρ_u is the geometric representation of the hyperbolic orbifold $M(n/0)$ and to avoid confusion with the geometric representation of M we will henceforth refer to ρ_u as ρ_n. In this setting we can prove the following theorem whose proof will comprise the remainder of this chapter.

Theorem 5.3.1. *Let M be the complement of a hyperbolic, amphicheiral knot, and suppose that M is infinitesimally projectively rigid relative to the boundary at ρ_{geo} and the longitude is a rigid slope. Then for sufficiently large n, $M(n/0)$ has a one dimensional space of strictly convex projective deformations near the complete hyperbolic structure.*

Remark 5.3.1. This result generalizes work of Heusener and Porti [28, Thm 1.8] who proved Theorem 5.3.1 for the figure-eight knot, and many of our arguments closely parallel theirs.

The idea behind the proof will be to build a curve of representations tangent to a cohomology class in $H^1(\Gamma_n, \mathfrak{sl}(4)_{\rho_n})$ which by the orbifold generalizations of Theorem 3.2.1 and Corollary 3.2.2 give rise to strictly convex projective deformations. We begin by proving the following lemma about the holonomy ρ_u.

Lemma 5.3.2. *Let M be an amphicheiral hyperbolic knot complement and let ρ_u be the holonomy of an incomplete hyperbolic structure on M such that the completion of M is $M(\alpha/0)$. If $\alpha \geq 2$ then $\rho_u(m)$ is a pure rotation and $\rho_u(l)$ is a pure translation. Furthermore, the cusp shape of M is imaginary.*

Proof. Since the completion associated to ρ_u is $M(\alpha/0)$ we know that $\rho_u(m)$ is elliptic and thus fixes its axis pointwise and acts as a pure rotation. We now show that $\mathrm{tr}^2\rho_u(l)$ is real and greater than 4. Since $\alpha \geq 2$ we can apply rigidity results of cone manifolds from [29, 32] to show that there exists and element $A_u \in \mathrm{PSL}_2(\mathbb{C})$ such that $\rho_u(\phi(\gamma)) = \overline{A_u\rho_u(\gamma)A_u^{-1}}$, where $\overline{M}$ denotes complex conjugation of the entries of the matrix M. Since the τ-invariant is independent of conjugacy we find that $\tau_{\rho_u\circ\phi} = \overline{\tau_u}$. On the other hand ϕ preserves l and sends m to its inverse and so we see that $\tau_{\rho_u\circ\phi} = -\tau_u$, and thus τ_u is imaginary. Since $\rho_u \to \rho_{\mathrm{geo}}$ as $\alpha \to \infty$ and τ coincides with the cusp shape for ρ_{geo} this proves the second part of the lemma. It should be noted that this fact about the cusp shape was originally observed by Riley [43] and that the proof that τ is imaginary is similar in spirit to Riley's proof for the cusp shape.

The fact that $\rho_u(m)$ and $\rho_u(l)$ commute gives rise to the following relationship between a_u, b_u and τ_u:

$$\tau_u \sinh(a_u/2) = \sinh(b_u/2). \tag{5.8}$$

Using the fact that $\mathrm{tr}\rho_u(m)/2 = \cosh(a/2)$ and the analogous relationship for l we see that when we square both sides of (5.8) we get

$$\tau_u^2(\mathrm{tr}^2\rho_u(m) - 4) + 4 = \mathrm{tr}^2\rho_u(l).$$

Finally, we have seen already that $\mathrm{tr}^2\rho_u(m)$ is real and between 0 and 4, and combining this with the fact that τ_u is imaginary forces $\mathrm{tr}^2\rho_u(l)$ to be real and greater than 4. □

The map ϕ induces a map $\phi^* : H^1(\partial M, \mathbb{R}) \to H^1(\partial M, \mathbb{R})$ which can be easily understood. The cohomology group $H^1(\partial M, \mathbb{R}) \cong H^1(\mathbb{Z}^2, \mathbb{R}) \cong \mathbb{R}^2$ and ϕ^* is an involution with m and l serving as an eigenbasis for ϕ^*. Since ϕ send m to m^{-1} we see that ϕ descends to a map $\phi_n : \Gamma_n \to \Gamma_n$ and in [28] it is shown that ϕ_n induces a map

$$\phi_n^* : H^1(\Gamma_n, \mathfrak{sl}(4)_{\rho_n}) \to H^1(\Gamma_n, \mathfrak{sl}(4)_{\rho_n}).$$

The rough idea is that ϕ induces a map from $H^1(\Gamma_n, \mathfrak{sl}(4)_{\rho_n})$ to $H^1(\Gamma_n, \mathfrak{sl}(4)_{\rho_n \circ \phi})$, however by Mostow-Prasad rigidity we know that there exists $A_n \in \mathrm{O}(3,1)$ such that $\rho_n \circ \phi = \mathrm{Adj}\ A_n \circ \rho_n$, which allows to identify $\mathfrak{sl}(4)_{\rho_n \circ \phi}$ with $\mathfrak{sl}(4)_{\rho_n}$. As $n \to \infty$ the representations $\rho_n \to \rho_{\mathrm{geo}}$ so we also get a map

$$\phi_\infty^* : H^1(\Gamma, \mathfrak{sl}(4)_{\rho_{\mathrm{geo}}}) \to H^1(\Gamma, \mathfrak{sl}(4)_{\rho_{\mathrm{geo}}}).$$

By Weil rigidity [49] we know that the first factor in the splitting (5.2) is trivial and so we focus our attention on the maps

$$\phi_n^* : H^1(\Gamma_n, \mathfrak{v}_{\rho_n}) \to H^1(\Gamma_n, \mathfrak{v}_{\rho_n}),$$

which in turn gives rise to a maps

$$\phi_n^* : H^1(\partial M, \mathfrak{v}_{\rho_n}) \to H^1(\partial M, \mathfrak{v}_{\rho_n}),$$

implicit in which is the identification between group and singular cohomology. In [28] it is shown how understanding the action of ϕ_n^* on $H^1(\partial M, \mathfrak{v}_{\rho_n})$ can help to understand its action on $H^1(\Gamma_n, \mathfrak{v}_{\rho_n})$. With this in mind we prove the following two lemmas that generalize [28, Lem 8.2].

Lemma 5.3.3. *Let M be an amphicheiral hyperbolic knot complement, then $H^*(\partial, \mathfrak{v}_{\rho_n}) \cong H^*(\partial M, \mathbb{R}) \otimes \mathfrak{v}^{\rho_n(\pi_1(\partial M))}$ and $\phi_n^* = \phi^* \otimes Id$.*

Proof. By Lemma 5.2.3 we know that there is a natural identification $H^*(\partial, \mathfrak{v}_{\rho_n}) \cong H^*(\partial M, \mathbb{R}) \otimes \mathfrak{v}^{\rho_n(\pi_1(\partial M))}$ and so $\phi_n^* = \phi^* \otimes \psi$, and we would like to show that $\psi = Id$. Since M is amphicheiral Lemma 5.3.2 tells us that after conjugating the standard embedding of $\mathrm{PSL}_2(\mathbb{C})$ into $\mathrm{SO}(3,1)$ that

$$\rho_n(m) = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) & 0 & 0 \\ \sin(2\pi/n) & \cos(2\pi/n) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\rho_n(l) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(\lambda_n) & \sinh(\lambda_n) \\ 0 & 0 & \sinh(\lambda_n) & \cosh(\lambda_n) \end{pmatrix},$$

where λ_n is the translation length of $\rho_n(l)$. After this observation the proof that $\psi = Id$ is the identical to the proof found in [28, Lem 8.2], to which we refer the reader. □

The next lemma examines the action of ϕ_∞^*. When M is amphicheiral Lemma 5.3.2 tells us that $\rho_{\text{geo}}(m)$ and $\rho_{\text{geo}}(l)$ are Euclidean translations by orthogonal vectors (when viewed in the upper half space model). Therefore by Lemma 5.2.2 we know that

$$H^1(\partial M, \mathfrak{v}_{\rho_{\text{geo}}}) \overset{\iota_m^* \oplus \iota_l^*}{\to} H^1(m, \mathfrak{v}_{\rho_{\text{geo}}}) \oplus H^1(l, \mathfrak{v}_{\rho_{\text{geo}}})$$

is injective.

Lemma 5.3.4. *Let M be an amphicheiral hyperbolic knot complement then*

$$\iota_l^* \circ \phi_\infty^* = \iota_l^* \text{ and } \iota_m^* \circ \phi_\infty^* = -\iota_m^*$$

Proof. Since M is amphicheiral Lemma 5.3.2 tells us that the cusp shape is imaginary and thus after conjugation

$$\rho_{\text{geo}}(m) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \rho_{\text{geo}}(l) = \begin{pmatrix} 1 & ic \\ 0 & 1 \end{pmatrix},$$

where c is a positive real number. By post composing with the standard embedding of $\text{PSL}_2(\mathbb{C})$ into $\text{SO}(3,1)$ we see that

$$\rho_{\text{geo}}(m) = \exp \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } \rho_{\text{geo}}(l) = \exp \begin{pmatrix} 0 & 0 & -c & c \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix},$$

where exp denotes the matrix exponential. By Mostow-Prasad rigidity we know that there exists an element $A_\infty \in \mathrm{O}(3,1)$ such that $\rho_{\mathrm{geo}} \circ \phi = \mathrm{Adj}\ A_\infty \circ \rho_{\mathrm{geo}}$. Since ϕ takes preserves l and sends m to its inverse we see that

$$A_\infty = T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where T is a parabolic element of $\mathrm{SO}(3,1)$ that fixes the vector $(0,0,1,1)$ (which corresponds to the common fixed point of $\rho_{\mathrm{geo}}(m)$ and $\rho_{\mathrm{geo}}(l)$). Such a T will be of the form

$$T = \exp \begin{pmatrix} 0 & 0 & -a & a \\ 0 & 0 & -b & b \\ a & b & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix},$$

where a and b are real numbers which can be thought of as the real and imaginary parts of the complex number that determines the parabolic translation. From arguments in the proof of [28, Lem 5.5] we know that $H^1(\partial M, \mathfrak{v}_{\rho_{\mathrm{geo}}})$ is generated by cocycles z_m and z_l, where z_m is given by $z_m(l) = 0$ and $z_m(m) = a_l$, where

$$a_l = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and z_l is given by $z_l(m) = 0$ and $z_l(l) = a_m$, where

$$a_m = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Next, observe that $\phi_\infty(z_l(m)) = 0 = z_l(m)$ and that

$$\phi_\infty(z_m(m)) = \text{Ad } A_\infty^{-1} \circ z_m(m^{-1}) = -\text{Ad } A_\infty^{-1} \rho_{\text{geo}}(m^{-1}) \circ a_l.$$

We have previously seen that the Killing form on $\mathfrak{v}$ gives rise to a cup product on cohomology, and we now use this cup product to show that $\iota_m^* \circ \phi_\infty^*(z_m(m))$ is cohomologous to $-\iota_m^*(z_m(m))$. The cup product yields a map

$$\cup : H^1(m, \mathfrak{v}_{\rho_{\text{geo}}}) \otimes H^0(m, \mathfrak{v}_{\rho_{\text{geo}}}) \to H^1(m, \mathbb{R}) \cong \mathbb{R}$$

via the formula

$$(a \cup b)(m) = B(a(m), b) = 8\text{tr}(a(m)b), \tag{5.9}$$

where B is the killing form and we think of $H^1(m, \mathbb{R})$ as homomorphisms from $\mathbb{Z}$ to $\mathbb{R}$ where m is the generator of $\mathbb{Z}$. A simple matrix calculation reveals that

$$(\iota_m^* \circ \phi_\infty^*(z_m) \cup a_m)(m) = B(\phi_\infty^*(z_m(m)), a_m) = 32$$
$$= -B(a_l, a_m) = -(\iota_m^*(z_m) \cup a_m)(m).$$

By Lemma 5.2.2 we know that $[\iota_m^* \circ \phi_\infty^*(z_m)]$ and $[-\iota_m^*(z_m)]$ are in the same 1-dimensional subspace and thus $\iota_m^* \circ \phi_\infty^*(z_m)$ and $-\iota_m^*(z_m)$ are cohomologous and so $\iota_m^* \circ \phi_\infty^* = -\iota_m^*$.

Finally, a similar computation shows that

$$(\iota_l^* \circ \phi_\infty^*(z_l) \cup a_l)(l) = B(\phi_\infty^*(z_l(l)), a_l) = -32$$
$$= B(a_m, a_l) = (\iota_l^*(z_l) \cup z_l)(l),$$

and a similar argument shows that $\iota_l^* \circ \phi_\infty^* = \iota_l^*$. □

The previous two lemmas can be thought of as saying that ϕ_n^* and ϕ_∞^* act the way we would expect them to on cohomology, namely they act as involutions with eigenvalues coming from m and l respectively. Another consequence of these lemmas is to help us understand the image of $H^1(M, \mathfrak{v}_{\rho_n})$ in $H^1(\partial M, \mathfrak{v}_{\rho_n})$ under the map ι^*. In particular, we see that $\pi_1(\partial M)$ is invariant under ϕ and thus the image of $H^1(M, \mathfrak{v}_{\rho_n})$ under ι^* will be an invariant subspace for the action of ϕ_∞^* and must therefore be equal to either the ± 1-eigenspace of ϕ_∞^*. By Lemmas 5.3.3 and 5.3.4 we see that these eigenspaces sit inside of $H^1(l, \mathfrak{v}_{\rho_n})$ and $H^1(m, \mathfrak{v}_{\rho_n})$, respectively. The following lemma (similar to [28, Cor 8.3]) tells us that under the hypotheses of Theorem 5.3.1 that the image will live in the 1-eigenspace.

Lemma 5.3.5. *Let M be the complement of a hyperbolic, amphicheiral knot, and suppose that M is infinitesimally projectively rigid relative to the boundary at ρ_{geo} and the longitude is a rigid slope. For sufficiently large n the image under ι^* of $H^1(M, \mathfrak{v}_{\rho_n})$ is contained in the 1-eigenspace of ϕ_n^* inside $H^1(\partial M, \mathfrak{v}_{\rho_n})$*

Proof. Since $\rho_\infty = \rho_{\text{geo}}$ the lemma also covers the case where $n = \infty$. Since the longitude is a rigid slope and the maps ϕ_∞^* and ι^* commute we see that Lemma 5.2.2 implies that ϕ_∞^* acts as the identity on $H^1(M, \mathfrak{v}_{\rho_{\text{geo}}})$. Combining this fact with the infinitesimal projective rigidity of M proves the lemma for $n = \infty$.

Furthermore, that fact that M is infinitesimally projectively relative to the boundary combined with [28, Cor 6.6] implies that $H^1(M, \mathfrak{v}_{\rho_n})$ injects into $H^1(\partial M, \mathfrak{v}_{\rho_n})$ as a 1-dimensional invariant subspace for ϕ_n^* provided that $M(n/0)$ is hyperbolic. Thus the image is contained in either ± 1-eigenspace of ϕ_n^*. Continuity implies that for sufficiently large n the image must be in the 1-eigenspace. $\square$

We will soon see how this fact can help us build representations, but before we proceed we need to relate the group cohomology of Γ_n to a variant of simplicial cohomology for orbifolds.

Henceforth, we will refer to $M(n/0)$ as O_n. The theory of twisted cohomology for orbifolds is similar to that of manifolds and is detailed in [28]. The rough idea is to choose a CW structure on the underlying manifold of the orbifold ($\mathbb{S}^3$ in our case) for which the singular locus is a subset of the 1-skeleton of the CW structure. For sufficiently large n, O_n is finitely covered by a hyperbolic (and hence aspherical) manifold. Combining this fact with a transfer argument [28] shows that $H^*(O_n, \mathfrak{sl}(4)_{\rho_n}) \cong H^*(\Gamma_n, \mathfrak{sl}(4)_{\rho_n})$[1]. The utility of this fact is that orbifold cohomology provides us with Mayer-Vietoris sequences to calculate $H^*(O_n, \mathfrak{sl}(4)_{\rho_n})$ and thus gain information about $H^*(\Gamma_n, \mathfrak{sl}(4)_{\rho_n})$ which can help us build representations.

We now discuss a technique to build representations. Suppose that we are given a curve of representations $\rho_t : \Gamma \to \mathrm{SL}_4(\mathbb{R})$, then for small values of t and $\gamma \in \Gamma$ we can use power series to write

$$\rho_t(\gamma) = (I + u_1(\gamma)t + u_2(\gamma)t^2 + \ldots)\rho_0(\gamma),$$

[1]This remains true if we replace $\mathfrak{sl}(4)_{\rho_n}$ with either $\mathfrak{so}(3,1)_{\rho_n}$ or $\mathfrak{v}_{\rho_n}$

where $u_i : \Gamma \to \mathfrak{sl}(4)$ are 1-cochains. A direct calculation shows that ρ_t is a homomorphism if and only if for all $k \in \mathbb{Z}^+$ we have

$$\delta u_k + \sum_{i=1}^{k-1} u_i \cup u_{k-i} = 0, \tag{5.10}$$

where $(a \cup b)(c,d) = a(c)c \cdot b(d)$, where a, b are 1-cochains, $c, d \in \Gamma$, the action is by conjugation and the multiplication is matrix multiplication in $\mathfrak{sl}(4)$. The map δ is the differential from group cohomology and is given by

$$\delta a(b,c) = a(b) + \rho_0(b) \cdot a(c) - a(bc),$$

where a is a 1-cochain and $b, c \in \Gamma$. In particular (5.10) tells us that u_1 is a 1-cocyle and thus corresponds to an element of $H^1(\Gamma, \mathfrak{sl}(4)_{\rho_0})$. Conversely, a deep result of Artin [2] tells us that given a representation ρ_0, a cohomology class $[u_1]$, and a collection $\{u_i\}_{i=2}^{\infty}$ of 1-cochains satisfying (5.10) that there is another collection of 1-cochains $\{u_i'\}_{i=2}^{\infty}$ and a positive real number T such that for all $\gamma \in \Gamma$ and $0 \leq t < T$ the series

$$(I + u_1(\gamma)t + u_2'(\gamma)t^2 + \ldots)\rho_0(\gamma)$$

converges. The takeaway from this result is that if we are able to build a formal representation near ρ_0 (i.e. find 1-cochains satisfying (5.10)) then we are able to find actual representation. If a cohomology class $[\omega]$ is tangent to a curve of representations ρ_t then we say that $[\omega]$ is *integrable*. We can now prove Theorem 5.3.1.

Proof of Theorem 5.3.1. Throughout this proof we will denote O_n as O and for the purpose of a Mayer-Vietoris argument we decompose O as $M \cup N$

where N is a solid torus with cone singularities along the longitude of cone angle $2\pi/n$. Observe that $M \cap N \cong \partial M$. We now examine the cohomology $H^*(\Gamma_n, \mathfrak{sl}(4)_{\rho_n}) \cong H^*(O_n, \mathfrak{sl}(4)_{\rho_n})$. As O_n is 3 dimensional $H^k(O_n, \mathfrak{sl}(4)_{\rho_n}) = 0$ if $k > 3$. Since ρ_n is irreducible we see that $H^0(\Gamma_n, \mathfrak{sl}(4)_{\rho_n}) = 0$ and thus by Poincaré duality $H^3(\Gamma_n, \mathfrak{sl}(4)_{\rho_n}) = 0$ as well. Combining Weil rigidity with the decomposition (5.2) we see that $H^1(\Gamma_n, \mathfrak{sl}(4)_{\rho_n}) \cong H^1(\Gamma_n, \mathfrak{v}_{\rho_n})$. Finally, another application of Poincaré duality tells us that $H^2(\Gamma_n, \mathfrak{sl}(4)_{\rho_n}) \cong H^2(\Gamma_n, \mathfrak{v}_{\rho_n})$.

We now use a Mayer-Vietoris sequence to analyze $H^i(O_n, \mathfrak{v}_{\rho_n})$ for $i = 1, 2$ as well as the action of ι_n^*. To simplify notation $H^*(\_, \mathfrak{v}_{\rho_n})$ will now be denoted $H^*(\_)$. Consider the following section of the Mayer-Vietoris sequence:

$$H^0(M) \oplus H^0(N) \to H^0(\partial M) \to H^1(O) \to H^1(M) \oplus H^1(N) \to H^1(\partial M). \tag{5.11}$$

First, we will determine the cohomology of N. Since N has the homotopy type of S^1 it will only have cohomology in dimension 0 and 1. Since $\rho_n(\partial M) = \rho_n(N)$ we see that $H^0(N) \cong H^0(\partial M)$ (both are 1-dimensional). By duality we see that $H^1(N)$ is also 1-dimensional. Combining these facts, we see that the first arrow is an isomorphism and thus the penultimate arrow of (5.11) is injective. We also learn that $H^1(O)$ injects into $H^1(M)$, because if a cohomology class from $H^1(O)$ dies in $H^1(M)$ then exactness tells us that it must also die when mapped into $H^1(N)$ (since $H^1(N)$ injects into $H^1(\partial M)$).

However, this contradicts the fact that $H^1(O)$ injects into $H^1(M) \oplus H^1(N)$.

Since $H^1(O)$ injects into $H^1(M)$ Lemma 5.3.2 implies that ϕ_n^* acts as the identity on $H^1(O)$. Since $H^1(M)$ and $H^1(N)$ both have the 1-eigenspace of ϕ_n^* as their image in $H^1(\partial M)$, and so we see that the last arrow of (5.11) is not a surjection, and so $H^1(O)$ is 1-dimensional.

Duality tells us that $H^2(O)$ is also 1-dimensional and we will now show that ϕ_n^* act as multiplication by -1. As $H^3(O) = 0$ we see that the Mayer-Vietoris sequence also contains the following piece.

$$H^1(M) \oplus H^1(N) \to H^1(\partial M) \to H^2(O) \to H^2(M) \oplus H^2(N) \to H^2(\partial M) \to 0. \tag{5.12}$$

Since $H^2(O)$ is 1-dimensional, the second arrow of (5.12) is either trivial or surjective. If this arrow is trivial then the third arrow is an injection and thus an isomorphism for dimensional reasons, but this is a contradiction since the penultimate arrow is a surjection and $H^2(\partial M)$ is non-trivial. Since the first arrow of (5.12) has the 1-eigenspace of ϕ_n^* as its image we see that the -1-eigenspace of ϕ_n^* surjects $H^2(O)$. However, since the Mayer-Vietoris sequence is natural and ϕ respects the splitting of O into $M \cup N$ we see that ϕ^* acts on $H^2(O)$ as -1.

By the previous arguments we know that $H^1(O)$ is 1-dimensional and thus there is at most 1 dimensions worth of strictly convex deformations. Let $[u_1]$ be a generator of $H^1(O)$. Since ϕ is an isometry of a finite volume hyperbolic manifold we know that it has finite order when viewed as an element of $\mathrm{Out}(\pi_1(M))$ [47], and so there exists a finite order map ψ that is conjugate to ϕ, and we let L be the order of ψ. Because ϕ and ψ are conjugate, they have the same action on the cohomology groups [11]. Since ψ acts as the identity on $H^1(O)$ we know that the cocycle

$$u_1^* = \frac{1}{L}\left(u_1 + \psi_n^*(u_1) + \ldots (\psi_n^*)^{L-1}(u_1)\right)$$

is both invariant under ψ_n^* and cohomologous to u_1. By replacing u_1 with u_1^* we can assume that u_1 is invariant under ψ_n^{*2}. Next, observe that

$$-u_1 \cup u_1 \sim \psi_n^*(u_1 \cup u_1) = \psi_n^*(u_1) \cup \psi_n^*(u_1) = u_1 \cup u_1,$$

and so $[u_1 \cup u_1] = 0$ and there exist a 1-cochain u_2 such that $\delta u_2 + u_1 \cup u_1 = 0$. Using the same averaging trick as before we can replace u_2 with a ψ_n^* invariant cochain u_2^*. By invariance of u_1 we see that u_2^* has the same coboundary as u_2 and so this replacement does not affect the first part of our construction. Again we see that

$$-(u_1 \cup u_2 + u_2 \cup u_2) \sim \psi_n^*(u_1 \cup u_2 + u_2 \cup u_1) = u_1 \cup u_2 + u_2 \cup u_1,$$

and so there exists u_3 such that (5.10) is satisfied. Repeating this process indefinitely, we can construct a sequence u_i that satisfies (5.10), and thus by Artin's theorem we can construct our curve of representations. □

[2] As a cocycle and not just as a cohomology class.

Other than the figure-eight knot, we cannot yet prove that there exist other knots satisfying the hypotheses of Theorem 5.3.1. However, there is strong numerical evidence that the two bridge knot with rational number $\frac{13}{5}$ satisfies these conditions. Also, in light of Theorem 4.2.1 and the rarity of deformations in the closed examples computed in [16], it seems that two bridge knots that are infinitesimally rigid relative to the boundary may be quite abundant. Additionally, there are infinitely many amphicheiral two-bridge knots and the condition of the longitude being a rigid slope is quite general, and so there is hope that there are many situations in which Theorem 5.3.1 applies.

Chapter 6

Future Directions

The results from the previous chapter suggest several future directions of research which we briefly outline in this chapter.

6.1 Smoothness of Character Varieties and Geometry of Representations

As previously discussed, if M is a finite volume hyperbolic 3-manifold with $\pi_1(M) = \Gamma$ then the dimension of $H^1(M, \mathfrak{sl}(4)_{\rho_{\text{geo}}})$ gives an upper bound for the dimension of $\mathfrak{X}(\Gamma, \mathrm{SL}_4(\mathbb{R}))$ near $[\rho_{\text{geo}}]$. It is well known [29, 47] that the dimension $H^1(M, \mathfrak{so}(3,1)_{\rho_{\text{geo}}})$ is equal to the dimension of $\mathfrak{X}(\Gamma, \mathrm{SO}(3,1))$ near $[\rho_{\text{geo}}]$. Thus a natural question is whether this is true for $\mathfrak{X}(\Gamma, \mathrm{SL}_4(\mathbb{R}))$. However, using obstruction theoretic techniques, the authors in [16] found examples of closed M for which there are non-trivial elements of $H^1(\Gamma, \mathfrak{sl}(4)_{\rho_{\text{geo}}})$ that cannot be integrated to actual representations and so in general the answer to this question is no. For this reason we restrict our attention to the following question.

Question 2. *Let M be a finite volume hyperbolic 3-manifold. Suppose that M is infinitesimally projectively rigid relative to the boundary. Then is*

$\mathfrak{X}(\Gamma, \mathrm{SL}_4(\mathbb{R}))$ *smooth near* $[\rho_{\mathrm{geo}}]$*?*

There is some evidence that the answer to Question 2 may be yes for both the figure-eight knot complement and the Whitehead link complement. For the figure-eight knot complement $H^1(M, \mathfrak{sl}(4)_{\rho_{\mathrm{geo}}}) \cong H^1(M, \mathfrak{so}(3,1)_{\rho_{\mathrm{geo}}}) \oplus H^1(M, \mathfrak{v}_{\rho_{\mathrm{geo}}})$ has dimension 3 by half-lives/half-dies. As previously mentioned, any cohomology class in $H^1(M, \mathfrak{so}(3,1)_{\rho_{\mathrm{geo}}})$ is integrable. Additionally, $H^1(M, \mathfrak{v}_{\rho_{\mathrm{geo}}})$ is 1-dimensional and Theorem 5.3.1 showed that any cohomology class in this factor is integrable. Thus we have found deformations in 3 linearly independent directions, but we currently do not know if linear combinations of these cohomology classes give rise to integrable cohomology classes. In a similar fashion we can find 6 linearly independent deformations for the Whitehead link (the cohomology classes of $H^1(M, \mathfrak{v}_{\rho_{\mathrm{geo}}})$ arise from the bending construction and are thus integrable).

Another related question deals with the geometry of representations near ρ_{geo}. Using hyperbolic Dehn surgery we know that there are hyperbolic representations near ρ_{geo} that do not correspond to convex structures on M. However, in (4.11) we constructed a curve ρ_t of representations passing through ρ_{geo} that does not correspond to a strictly convex structure on M. However, in future work with D. Cooper and D. Long it will be shown that this curve of representations actually corresponds to a properly convex deformation of M. This situation prompts the following question.

Question 3. *Let* M *be a finite volume hyperbolic 3-manifold and let* ρ_t *be* *a curve of representations passing through ρ_{geo}. What conditions on ρ_t will guarantee that they correspond to properly convex structures on M?*

The representations coming from hyperbolic Dehn surgery all have peripheral elements whose holonomy have complex eigenvalues, whereas the properly convex representations on the figure-eight send all peripheral elements to matrices with real eigenvalues. Thus the answer to Question 3 probably involves information about eigenvalues.

6.2 Cuspidal Cohomology and Totally Geodesic Surfaces

The Bianchi groups, $\Gamma_d := \mathrm{PSL}_2(\mathcal{O}_d)$, where d is a positive, square free integer and $\mathcal{O}_d$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, provide a 3-dimensional analogue of the classic modular group, $\mathrm{PSL}_2(\mathbb{Z})$. These groups are discrete inside of $\mathrm{PSL}_2(\mathbb{C})$, and the cusped hyperbolic orbifolds, $O_d := \mathbb{H}^3/\Gamma_d$, have been well studied from this perspective [36].

The $\mathbb{Z}$-cuspidal cohomology of the Bianchi groups is well known. Specifically, the cuspidal cohomology of O_d is trivial if and only if

$$d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}$$

[48]. One consequence of an infinitesimal analogue of theorem 4.2.1 (see [28] for these computations) is that the complements of the figure-eight and the Whitehead link have trivial $\mathfrak{sl}(4)_{\rho_{\text{geo}}}$-cuspidal cohomology. The figure-eight

knot and Whitehead link are both arithmetic (see [36]) and thus cover the Bianchi orbifolds O_3 and O_1, respectively. Since cuspidal cohomology is inherited by finite covers, we see that this implies that O_3 and O_1 also have trivial $\mathfrak{sl}(4)_{\rho_{\text{geo}}}$-cuspidal cohomology. Additionally, there is strong numerical evidence that the two bridge link with rational number 12/5 is also locally projectively rigid relative to the boundary near ρ_{geo}, which suggests that O_7 also has trivial $\mathfrak{sl}(4)_{\rho_{\text{geo}}}$-cuspidal cohomology. This result complements the vanishing results of for $\mathfrak{so}(4,1)$-cuspidal cohomology found in [4, Cor 4.2]. This observation prompts the following question:

Question 4. *For which values of d is the $\mathfrak{sl}(4)_{\rho_{\text{geo}}}$-cuspidal cohomology of O_d trivial?*

Another motivation for investigating rigidity results for hyperbolic knot complements is that the previously mentioned cuspidal cohomology techniques can also be used to address the Menasco-Reid conjecture. This conjecture (see [38]) states that a hyperbolic knot complement cannot contain a closed, totally geodesic surface. The vanishing of $\mathbb{R}$-cuspidal cohomology is enough to obstruct the existence of closed, *non-separating*, totally geodesic surfaces, but says nothing about separating surfaces. The following corollary of Proposition 3.2.6 tells us that $\mathfrak{sl}(4)_{\rho_{\text{geo}}}$-cuspidal cohomology is sensitive enough to detect separating surfaces as well.

Corollary 6.2.1. *Let M be a finite volume hyperbolic manifold. If M has trivial $\mathfrak{sl}(4)_{\rho_{\text{geo}}}$-cuspidal cohomology then M does not contain a closed totally geodesic hypersurface.*

This implies that the vanishing of $\mathfrak{sl}(4)_{\rho_{\text{geo}}}$-cuspidal cohomology provides an obstruction to the existence of closed totally geodesic hypersurfaces. Therefore, any knot complement whose cuspidal cohomology vanishes will satisfy the conclusion of the Menasco-Reid conjecture, prompting the following question:

Question 5. *Do there exist hyperbolic knot complements with non-trivial $\mathfrak{sl}(4)_{\rho_{\text{geo}}}$-cuspidal cohomology?*

6.3 Thin Subgroups of $\mathrm{SL}_4(\mathbb{R})$

A finitely generated, non-free subgroup, Δ, of a semi-simple Lie group G is *thin* if it is Zariski dense and has infinite index inside of a lattice of G. These subgroups have become an active area of research due to recently discovered connections with other areas of mathematics, including lattice point counting problems and dynamics on Teichmüller space (see [45] for more details).

Recent work of Darren Long and Alan Reid [34] has produced examples of thin subgroups inside of $\mathrm{SL}_4(\mathbb{R})$. A key element of their construction of thin subgroups relies on flexing projective structures on closed manifolds and understanding the geometry of such flexings in order to produce subgroups that are Zariski dense. Understanding the geometry of the space of properly convex structures on non-compact manifolds would be the first step towards extending this construction to produce thin subgroups isomorphic to non-compact 3-manifolds. For example, the representations in (4.11) are faithful with discrete image inside of $SL_4(\mathbb{R})$ and are thus good candidates for thin subgroups.

Appendix

Appendix 1

Computations

In this appendix we include the calculations showing that the two-bridge knots and links in Theorem 4.2.1 are rigid. What follows is the raw Mathematica code and so we explain some of the following notation. Logical conjunction in Mathematica is written as either "&&" or "$\wedge$", both of which should be read as "and". Logical disjunction in Mathematica is written as "$\vee$", and should be read as "or".

Several computations in the code refer to the roots of polynomials using the Mathematica command "Root". In this command the symbol "#1" is a place holder for the variable of the polynomial whose roots we are considering. For example,

$$\text{Root}\left[\#1^3 + \#1^2 + 2\#1 + 1\&, 1\right]$$

refers to the first root of the polynomial $x^3 + x^2 + 2x + 1$. The convention for ordering of roots is somewhat arbitrary, but real roots are ordered before complex roots and conjugate pairs of roots are ordered consecutively.

1.1 The Code

"SO(3,1) parabolics could be put into the following form";

$A = \{\{1,0,1,\mathrm{a14}\},\{0,1,1,\mathrm{a24}\},\{0,0,1,\mathrm{a34}\},\{0,0,0,1\}\};$

$B = \{\{1,0,0,0\},\{\mathrm{b21},1,0,0\},\{\mathrm{b31},1,1,0\},\{1,1,0,1\}\};$

Map[MatrixForm, {MatrixPower[A − IdentityMatrix[4], 2],

MatrixPower[B − IdentityMatrix[4], 2]}]

$$\left\{\begin{pmatrix} 0 & 0 & 0 & \mathrm{a34} \\ 0 & 0 & 0 & \mathrm{a34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{b21} & 0 & 0 & 0 \\ \mathrm{b21} & 0 & 0 & 0 \end{pmatrix}\right\}$$

"So we see that in order to preserve the same Jordan type we must insist that a34≠0 and b21≠0";

“Now we can compute some examples”;

“First the figure 8 knot”;

$W = B$.Inverse[A].Inverse[B].A;

$L = W.A$.Inverse[B].Inverse[A].B;

Reduce[$A.W - W.B$ == 0&&a34 ≠ 0&&b21 ≠ 0]

$$\mathrm{b31} = 2 \land \mathrm{b21} - 2 \neq 0 \land \mathrm{a34} = \frac{\mathrm{b21}}{2(\mathrm{b21}-2)} \land \mathrm{a24} = \frac{1}{2}(2\mathrm{a34} - 1) \land \mathrm{a14} = \frac{1}{2}(2\mathrm{a34} - 3) \land \mathrm{b21} \neq 0$$

"The fact that there is a 1 parameter family of deformations comes from the fact that for the figure 8 the meridian is not a rigid slope";

"If we also insist that the trace of the longitude stay fixed then we do not have any local deformations";

Reduce[$A.W - W.B$ == 0&&Tr[L] == 4&&a34 ≠ 0&&b21 ≠ 0]

$b31 = 2 \wedge \left(b21 = 4 \vee b21 = -2i\sqrt{2} \vee b21 = 2i\sqrt{2}\right) \wedge$
$a34 = \frac{1}{48}\left(-b21^3 + 6b21^2 - 12b21 + 64\right) \wedge$
$a24 = \frac{1}{48}\left(-b21^3 + 6b21^2 - 12b21 + 40\right) \wedge$
$a14 = \frac{1}{48}\left(-b21^3 + 6b21^2 - 12b21 - 8\right)$

"These solutions form a discrete set and so we see that the hyperbolic structure is locally rigid.";

"The discrete faithful representation occurs when b21=2";

"Now 5_2";

$W = B.A$.Inverse[B].Inverse[A].$B.A$;

list = Reduce[$A.W - W.B$ == 0&&a34 $\neq$ 0&&b21 $\neq$ 0]//TraditionalForm

$\left(b31 = \frac{1}{4}\left(1 - i\sqrt{7}\right) \vee b31 = \frac{1}{4}\left(1 + i\sqrt{7}\right) \vee\right.$
$b31 = \text{Root}\left[\#1^3 - \#1^2 + 2\#1 - 1\&, 1\right] \vee$
$b31 = \text{Root}\left[\#1^3 - \#1^2 + 2\#1 - 1\&, 2\right] \vee$
$\left. b31 = \text{Root}\left[\#1^3 - \#1^2 + 2\#1 - 1\&, 3\right]\right) \wedge$
$b21 = 4 \wedge a34 = \frac{1}{2}\left(2b31^4 - b31^3 + 5b31^2 - 2b31 + 3\right) \wedge$
$a24 = \frac{1}{4}\left(6b31^4 - 11b31^3 + 19b31^2 - 14b31 + 1\right) \wedge$
$a14 = \frac{1}{4}\left(2b31^4 - b31^3 + 5b31^2 - 2b31 - 1\right)$

"Next we do 6_1";

$W = B$.Inverse[A].Inverse[B].$A.B$.Inverse[A].Inverse[B].A;

list = Reduce[$A.W - W.B$ == 0&&a34 $\neq$ 0&&b21 $\neq$ 0];

list[[2]]//TraditionalForm

$$\left(\text{b31}=\frac{1}{2}\left(3-\sqrt{4\text{Root}\left[\#1^3+12\#1^2+43\#1+47\&,1\right]+9}\right)\vee\right.$$
$$\text{b31}=\frac{1}{2}\left(\sqrt{4\text{Root}\left[\#1^3+12\#1^2+43\#1+47\&,1\right]+9}+3\right)\vee$$
$$\text{b31}=\frac{1}{2}\left(3-\sqrt{4\text{Root}\left[\#1^3+12\#1^2+43\#1+47\&,2\right]+9}\right)\vee$$
$$\text{b31}=\frac{1}{2}\left(\sqrt{4\text{Root}\left[\#1^3+12\#1^2+43\#1+47\&,2\right]+9}+3\right)\vee$$
$$\text{b31}=\frac{1}{2}\left(3-\sqrt{4\text{Root}\left[\#1^3+12\#1^2+43\#1+47\&,3\right]+9}\right)\vee$$
$$\text{b31}=\frac{1}{2}\left(\sqrt{4\text{Root}\left[\#1^3+12\#1^2+43\#1+47\&,3\right]+9}+3\right)\vee$$
$$\text{b31}=\text{Root}\left[16\#1^8-200\#1^7+1156\#1^6-4002\#1^5+9021\#1^4\right.$$
$$\left.-13496\#1^3+13042\#1^2-7422\#1+1901\&,1\right]\vee$$
$$\text{b31}=\text{Root}\left[16\#1^8-200\#1^7+1156\#1^6-4002\#1^5+9021\#1^4\right.$$
$$\left.-13496\#1^3+13042\#1^2-7422\#1+1901\&,2\right]\vee$$
$$\text{b31}=\text{Root}\left[16\#1^8-200\#1^7+1156\#1^6-4002\#1^5+9021\#1^4\right.$$
$$\left.-13496\#1^3+13042\#1^2-7422\#1+1901\&,3\right]\vee$$
$$\text{b31}=\text{Root}\left[16\#1^8-200\#1^7+1156\#1^6-4002\#1^5+9021\#1^4\right.$$
$$\left.-13496\#1^3+13042\#1^2-7422\#1+1901\&,4\right]\vee$$
$$\text{b31}=\text{Root}\left[16\#1^8-200\#1^7+1156\#1^6-4002\#1^5+9021\#1^4\right.$$
$$\left.-13496\#1^3+13042\#1^2-7422\#1+1901\&,5\right]\vee$$
$$\text{b31}=\text{Root}\left[16\#1^8-200\#1^7+1156\#1^6-4002\#1^5+9021\#1^4\right.$$
$$\left.-13496\#1^3+13042\#1^2-7422\#1+1901\&,6\right]\vee$$
$$\text{b31}=\text{Root}\left[16\#1^8-200\#1^7+1156\#1^6-4002\#1^5+9021\#1^4\right.$$
$$\left.-13496\#1^3+13042\#1^2-7422\#1+1901\&,7\right]\vee$$
$$\text{b31}=\text{Root}\left[16\#1^8-200\#1^7+1156\#1^6-4002\#1^5+9021\#1^4\right.$$
$$\left.\left.-13496\#1^3+13042\#1^2-7422\#1+1901\&,8\right]\right)\wedge$$
$$\text{b21}=4\wedge$$

$\mathrm{a34} = \frac{1}{31956192720}\left(-179796413616\mathrm{b31}^{15} + 3733995033256\mathrm{b31}^{14}\right.$

$-37452502661204\mathrm{b31}^{13} + 238886942932090\mathrm{b31}^{12}$

$-1075958795921889\mathrm{b31}^{11} + 3597489474076549\mathrm{b31}^{10}$

$-9140675598222262\mathrm{b31}^{9} + 17750454679395481\mathrm{b31}^{8}$

$-26045001320077501\mathrm{b31}^{7} + 27816300271275448\mathrm{b31}^{6}$

$-19526453475128180\mathrm{b31}^{5} + 5698801365850447\mathrm{b31}^{4}$

$+4263650630430149\mathrm{b31}^{3} - 5854678735677884\mathrm{b31}^{2}$

$\left.+2868426513794067\mathrm{b31} - 557399676045503\right) \wedge$

$\mathrm{a24} = \frac{1}{63912385440}\left(722812004784\mathrm{b31}^{15} - 14964003858344\mathrm{b31}^{14}\right.$

$+149418614136916\mathrm{b31}^{13} - 947978297826170\mathrm{b31}^{12}$

$+4244298734746401\mathrm{b31}^{11} - 14098782932080061\mathrm{b31}^{10}$

$+35573539006408718\mathrm{b31}^{9} - 68570468324339849\mathrm{b31}^{8}$

$+99825608272345469\mathrm{b31}^{7} - 105716578643568632\mathrm{b31}^{6}$

$+73461399550231780\mathrm{b31}^{5} - 20924114085212303\mathrm{b31}^{4} -$

$16388753271093661\mathrm{b31}^{3} + 22092444941813956\mathrm{b31}^{2}$

$\left.-10785037897132083\mathrm{b31} + 2100160181211727\right) \wedge$

$\mathrm{a14} = \frac{1}{63912385440}\left(574264051664\mathrm{b31}^{15} - 12012269104664\mathrm{b31}^{14}\right.$

$+121248755334796\mathrm{b31}^{13} - 778031186453110\mathrm{b31}^{12}$

$+3525504039482231\mathrm{b31}^{11} - 11862708281527731\mathrm{b31}^{10}$

$+30352296559023618\mathrm{b31}^{9} - 59414766026089839\mathrm{b31}^{8}$

$+88023829428938259\mathrm{b31}^{7} - 95205648706611272\mathrm{b31}^{6}$

$+68172037326861820\mathrm{b31}^{5} - 21191049139962633\mathrm{b31}^{4}$

$-13801598627577411\mathrm{b31}^{3} + 20250630941780876\mathrm{b31}^{2}$

$-10224174988365973\mathrm{b31} + 2044968066804257)$

"Again the solutions form a discrete set and so the hyperbolic structure is locally rigid";

“Finally, we try the Whitehead link”;

$W = B.A.$Inverse$[B]$.Inverse$[A]$.Inverse$[B].A.B$;

Reduce$[A.W - W.A ==$ 0&&a34 $\neq$ 0&&b21 $\neq$ 0]

$\mathrm{b31} = -1 \land \mathrm{b21} = 4 \land \mathrm{a34} = 2 \land \mathrm{a24} = -2 \land \mathrm{a14} = 0$

“So again we see that the hyperbolic structure is rigid”;

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