Complex Projective Structures on Surfaces

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UF Colloquium

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- Correspondences between an analytic object (*ODEs & measured laminations*) and geometric objects (*complex projective structures*)

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- Today: In certain cases we can make these correspondences are explicit

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- There is no $\mathrm{PSL}_2(\mathbb{C})$-invariant metric on $\mathbb{CP}^1$
- Circles are invariant and play the role of geodesics

Hyperbolic surfaces

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Let $\mathcal{T}(\Sigma)$ be the space of hyperbolic structures on Σ

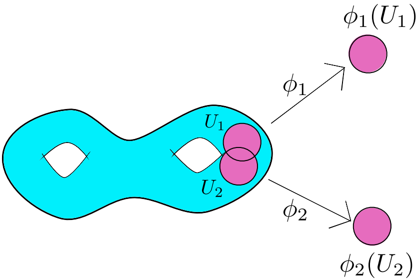
Theorem

The space, $\mathcal{T}(\Sigma) \cong \mathbb{R}^{6g-6}$

Complex projective structures

Definition

Let Σ be a surface. A *complex projective structure* on Σ consists of charts from Σ into $\mathbb{CP}^1$ whose transition functions are elements of $\mathrm{PSL}_2(\mathbb{C})$



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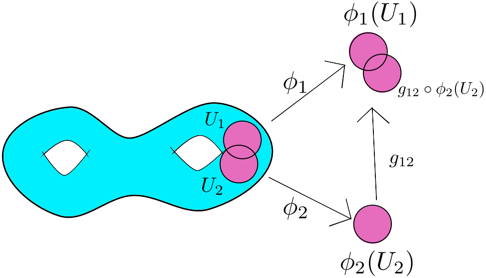
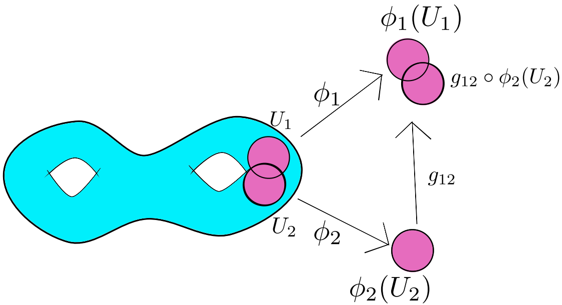


For $z \in U_1 \cap U_2$, $\phi_1(z) = g_{12}\phi_2(z)$

Development and holonomy

A more global approach

Using *analytic continuation* we can attempt to enlarge our charts



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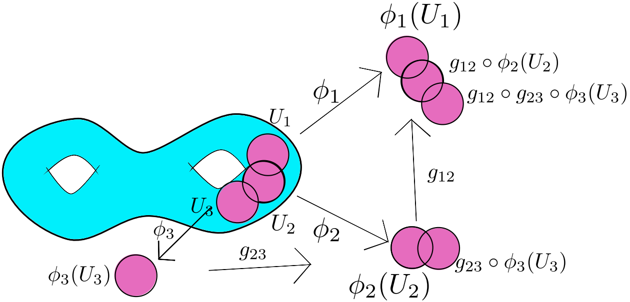

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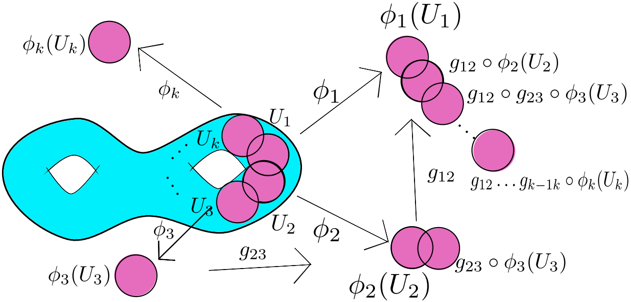

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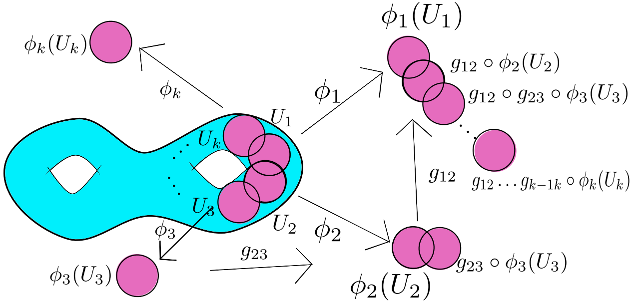

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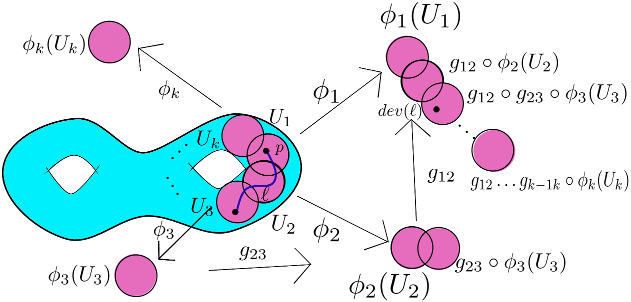

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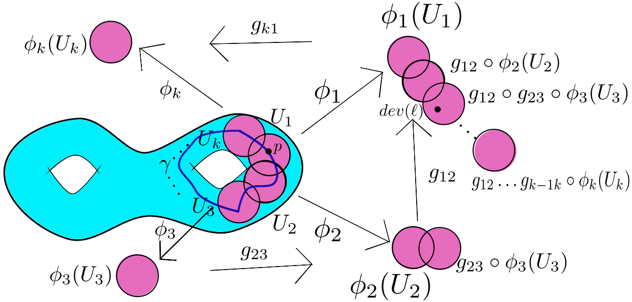

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Simply connected case

Let $\phi : \mathbb{D} \to \mathbb{C}$ be holomorphic and consider the differential equation

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The solutions to (1) form a 2-dimensional vector space

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A local approach

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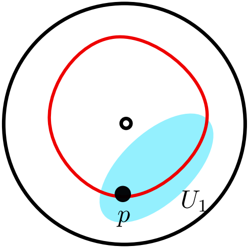
For $p \in U$ there is a basis $\{u_1, u_2\}$ of local solutions to (1)



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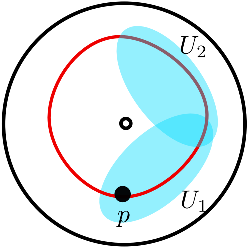

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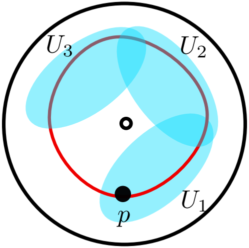

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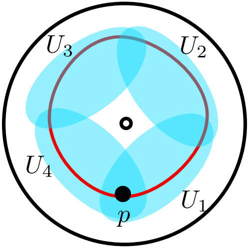
Problem: when we analytically continue around a loop γ we may arrive at new solutions $(v_1, v_2) \neq (u_1, u_2)$.



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Get an equivariant pair:

$$(u_1, u_2) : \widetilde{U} \to \mathbb{C} \qquad\qquad M : \pi_1(\Sigma) \to \mathrm{GL}_2(\mathbb{C})$$

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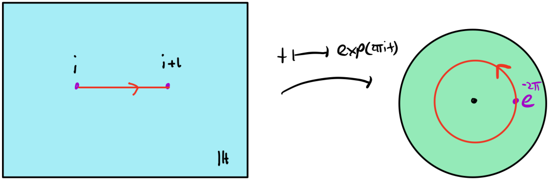

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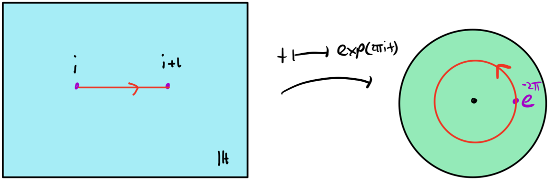

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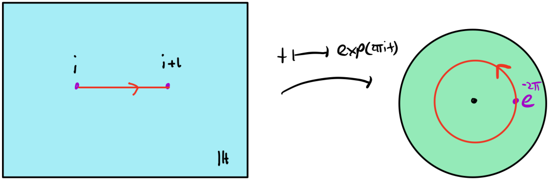

Relation between constructions

Equations give structure

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$(\mathrm{dev}, [M])$ give a complex projective structure on M.

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- Can form the ODE $u'' + \frac{1}{2}\phi u = 0$ on Σ

dev comes from a solution to this equation

Overview

Good News: Have constructions that relate an analytic object (*ODEs*) to a geometric object (*complex projective structures*)

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Bad News: The correspondence is opaque:

$$\text{Analytic properties} \overset{?}{\Longleftrightarrow} \text{Geometric properties}$$

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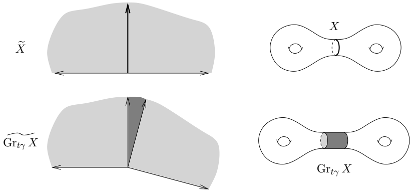

Figure: Picture from Dumas, *Complex Projective Structures*

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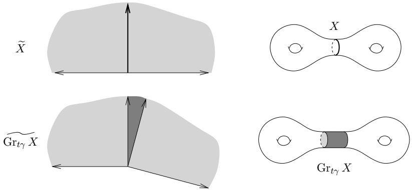

Figure: Picture from Dumas, *Complex Projective Structures*

Let $\mathcal{S}$ be free homotopy class of s.c.c's. Get

$$\mathrm{Gr} : \mathcal{S} \times \mathbb{R}^+ \times \mathcal{T}(\Sigma) \to \mathcal{P}(\Sigma)$$

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Good News: Every complex projective structure arises from grafting a hyperbolic surface.

Bad News: The inverse procedure is fairly non-constructive.

A transparent case

Let $\Sigma = \Sigma_{0,3}$ (*thrice punctured sphere*)
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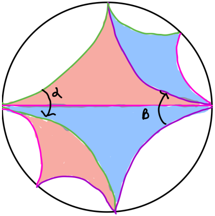
- *non-degenerate* if $\rho(\pi_1 \Sigma)$ has no finite orbits (e.g. no global fixed points)

Let $\mathcal{P}^{\odot}(\Sigma)$ be the space of tame, relatively elliptic, and non-degenerate structures on Σ

Examples

Triangular structures

Given a configuration of 3 circles in $\mathbb{CP}^1$ we can build (several) complex projective structures on Σ. (*triangular structures*)



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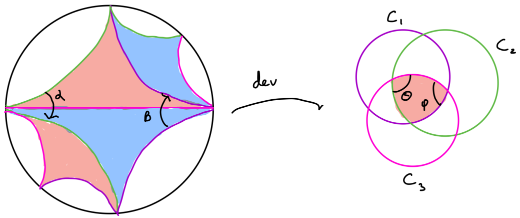

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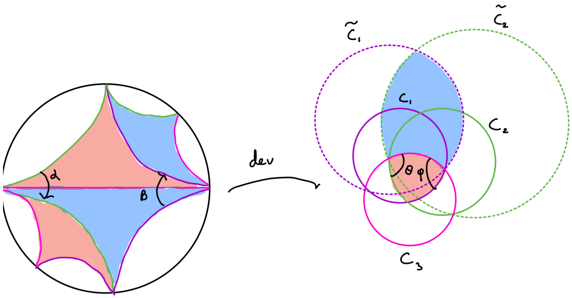

$\pi_1(\Sigma) \cong \langle \alpha, \beta \rangle$,
$\rho(\alpha) = R(C_2)R(C_3) \cong (z \mapsto e^{2i\theta} z)$,
$\rho(\beta) = R(C_3)R(C_1) \cong (z \mapsto e^{2i\phi} z)$

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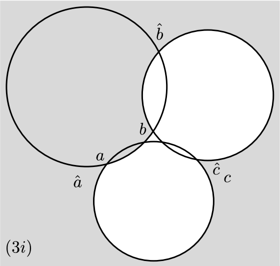

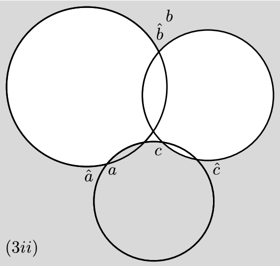

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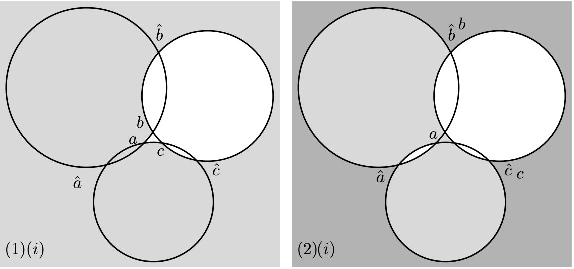

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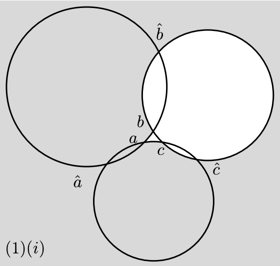

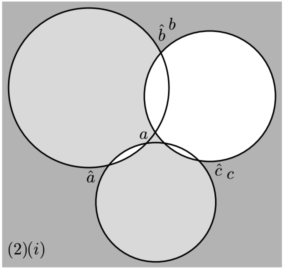

The same circles support several different developing maps.

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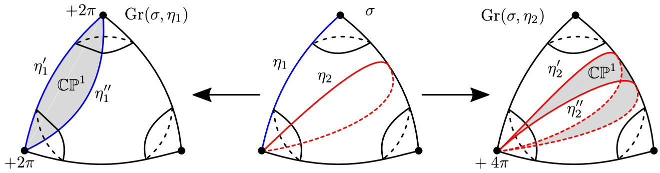

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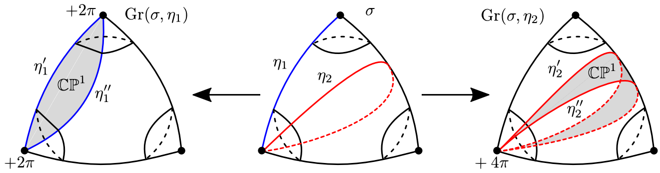

This grafting is discrete, not continuous!

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Edge grafting

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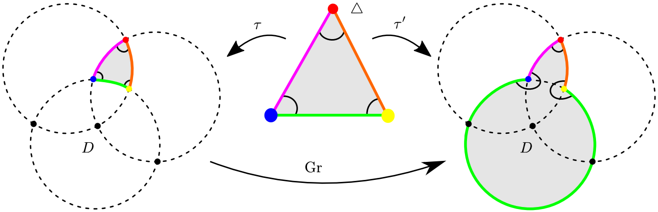

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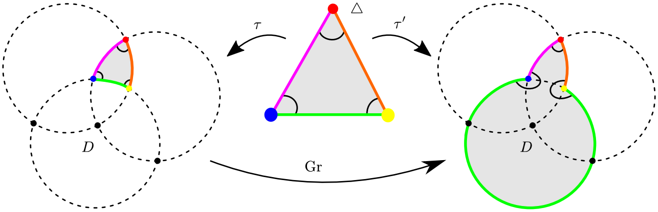

How does grafting change the holonomy?

It doesn't!!

Theorem 1

Theorem 1 (B-Bowers-Casella-Ruffoni)

Let $\Sigma = \Sigma_{0,3}$ *and let* $\tau \in \mathcal{P}^{\odot}(\Sigma)$. *Then* τ *is obtained from a triangular structure by a finite sequence of edge and core graftings.*

The sequence of graftings and the triangular structure can be computed explicitly (*Algorithmic*).

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  (a', b', c') determine triangular structure, (k_a, k_b, k_c) determine grafting.

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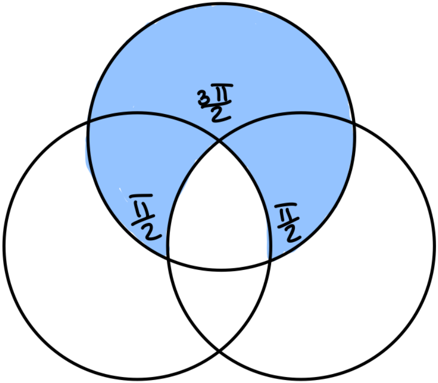

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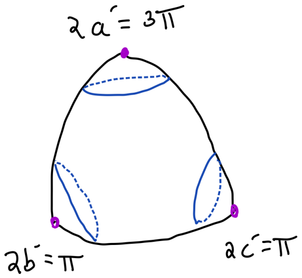

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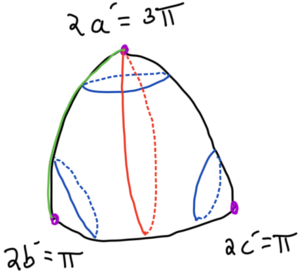

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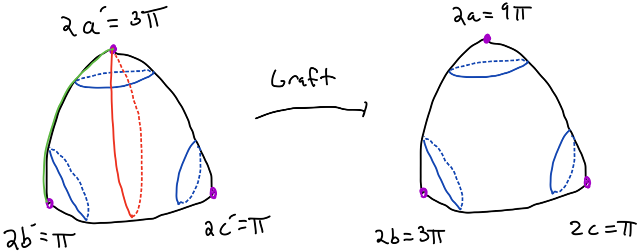

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$\tau \in \mathcal{P}^{\odot}(\Sigma_{0,3})$ *iff* τ *comes from a solution to* $u'' + 1/2\phi u = 0$ *where* $\phi : \mathbb{CP}^1 \to \mathbb{C}$ *is meromorphic with poles of order* $\leqslant 2$ *at* $\{0, 1, \infty\}$.

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We can determine the winding numbers from the poles of ϕ!!

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- $2\pi\theta$ is winding number and $\theta = \pm\sqrt{1-2a}$

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- Not an obvious candidate to replace triangular structures
- Winding numbers don't determine structure (*complex structure not unique*)

Thank you!