Exotic properly convex manifolds via Dehn filling

Sam Ballas

(joint with J. Danciger, G.-S. Lee, and L. Marquis)

Florida State University

University of Virginia
Geometry Seminar
Oct 26, 2021

Motivation

Let M be a closed hyperbolic manifold.

Let $\mathbb{H}(M)$ be the space of hyperbolic structures on M

Motivation

Let M be a closed hyperbolic manifold.

Let $\mathbb{H}(M)$ be the space of hyperbolic structures on M

Theorem

$\mathbb{H}(M)$ *is connected*

Motivation

Let M be a closed hyperbolic manifold.

Let $\mathbb{H}(M)$ be the space of hyperbolic structures on M

Theorem

$\mathbb{H}(M)$ *is connected*

proof sketch:

- $\dim(M) = 2$: Fenchel-Neilsen coordinates on Teichmüller space
- $\dim(M) > 2$: Mostow rigidity.

$\square$

Motivation

Let M be a closed hyperbolic manifold.

Let $\mathbb{H}(M)$ be the space of hyperbolic structures on M

Theorem

$\mathbb{H}(M)$ *is connected*

proof sketch:

- $\dim(M) = 2$: Fenchel-Neilsen coordinates on Teichmüller space
- $\dim(M) > 2$: Mostow rigidity.

□

Motivating Question: What happens if we look at other geometries?

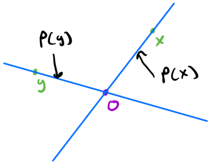
Projective Geometry

Let $\mathbb{RP}^n$ be the space of lines through the origin in $\mathbb{R}^{n+1}$.

Projective Geometry

Let $\mathbb{RP}^n$ be the space of lines through the origin in $\mathbb{R}^{n+1}$.

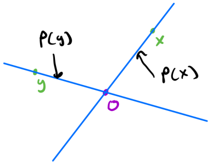
Let $P : \mathbb{R}^{n+1} \backslash \{0\} \rightarrow \mathbb{RP}^n$ be *projectivization*.



Projective Geometry

Let $\mathbb{RP}^n$ be the space of lines through the origin in $\mathbb{R}^{n+1}$.

Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{RP}^n$ be *projectivization*.

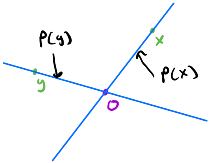


$$G = \mathrm{PGL}_{n+1}(\mathbb{R}) := \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^{\times}\mathbf{I}$$

Projective Geometry

Let $\mathbb{RP}^n$ be the space of lines through the origin in $\mathbb{R}^{n+1}$.

Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{RP}^n$ be *projectivization*.



$$G = \mathrm{PGL}_{n+1}(\mathbb{R}) := \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^{\times}\mathbf{I}$$

$\mathbb{RP}^n$ is a geometry with automorphism group G.

Convex Projective Geometry

Let $\widetilde{H}$ be a hyperplane in $\mathbb{R}^{n+1}$

Let $H = P(\widetilde{H})$ be the corresponding projective hyperplane

Convex Projective Geometry

Let $\widetilde{H}$ be a hyperplane in $\mathbb{R}^{n+1}$

Let $H = P(\widetilde{H})$ be the corresponding projective hyperplane

$A_H := \mathbb{RP}^n \backslash H$ is an *affine patch*

Let $\widetilde{H}$ be a hyperplane in $\mathbb{R}^{n+1}$

Let $H = P(\widetilde{H})$ be the corresponding projective hyperplane

$A_H := \mathbb{RP}^n \backslash H$ is an *affine patch* (i.e. $\mathbb{RP}^n = \mathbb{R}^n \sqcup \mathbb{RP}^{n-1}$)

Convex Projective Geometry

Let $\widetilde{H}$ be a hyperplane in $\mathbb{R}^{n+1}$

Let $H = P(\widetilde{H})$ be the corresponding projective hyperplane

$A_H := \mathbb{RP}^n \backslash H$ is an *affine patch* (i.e. $\mathbb{RP}^n = \mathbb{R}^n \sqcup \mathbb{RP}^{n-1}$)

$\Omega \subset \mathbb{RP}^n$ is *properly convex* if $\overline{\Omega}$ is a convex subset of some affine patch

Convex Projective Geometry

Let $\widetilde{H}$ be a hyperplane in $\mathbb{R}^{n+1}$

Let $H = P(\widetilde{H})$ be the corresponding projective hyperplane

$A_H := \mathbb{RP}^n \backslash H$ is an *affine patch* (i.e. $\mathbb{RP}^n = \mathbb{R}^n \sqcup \mathbb{RP}^{n-1}$)

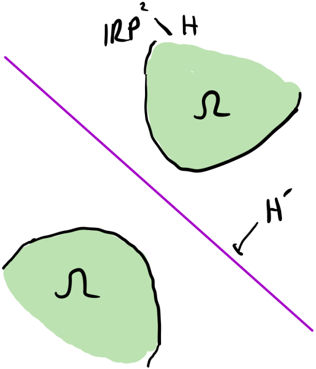
$\Omega \subset \mathbb{RP}^n$ is *properly convex* if $\overline{\Omega}$ is a convex subset of some affine patch

Let Ω be properly convex.
Define

$$\mathrm{PGL}(\Omega) = \{A \in G \mid A(\Omega) = \Omega\}$$

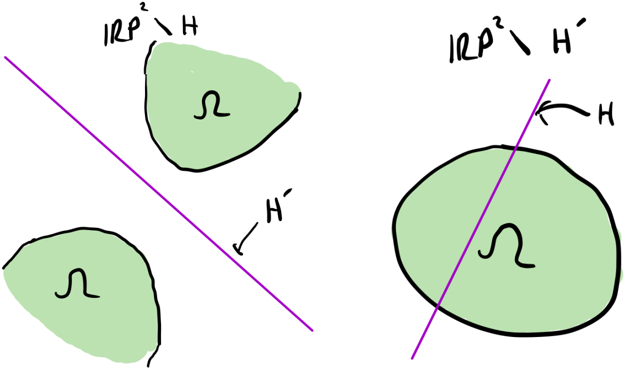
Convex projective geometry

Some examples



Convex projective geometry

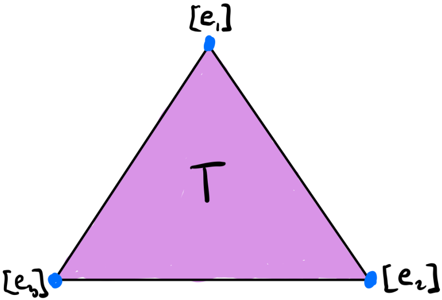
Some examples



Convex Projective Geometry

Some examples

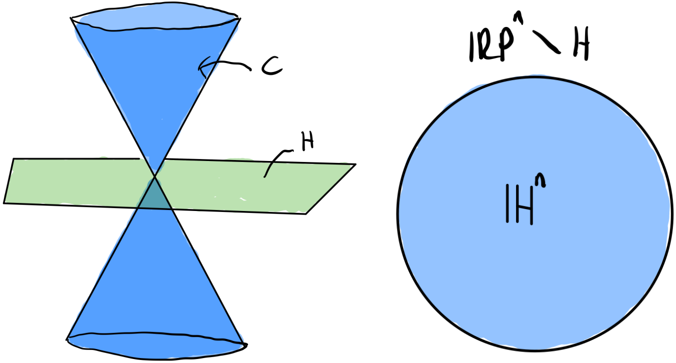
- $\widetilde{T} = \mathbb{R}^3_+$ (positive orthant)
- $T = P(\widetilde{T})$
- $\mathrm{PGL}(T) \cong \mathrm{Diag}_3 \rtimes S_3 \subset \mathrm{PGL}_3(\mathbb{R})$



Convex Projective Geometry

Some Examples

- L a Lorentzian form on $\mathbb{R}^{n+1}$
- $C = \{v \in \mathbb{R}^{n+1} \mid L(v,v) < 0\}$
- $\mathbb{H}^n = P(C)$ (Klein Model)
- $\mathrm{PGL}(\mathbb{H}^n) \cong \mathrm{PO}(L)$



Convex Projective Manifolds

Let Ω be properly convex

Let $\Gamma \subset \mathrm{PGL}(\Omega)$ be discrete

Convex Projective Manifolds

Let Ω be properly convex

Let $\Gamma \subset \mathrm{PGL}(\Omega)$ be discrete

Ω/Γ is a *convex projective manifold*

Some Examples

Complete Hyperbolic Manifolds

- $\Omega \cong \mathbb{H}^n$
- $\Gamma \subset \mathrm{PGL}(\mathbb{H}^n)$ discrete

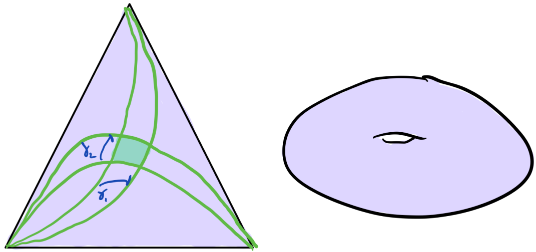
The $\mathbb{H}^n/\Gamma$ is a *complete hyperbolic manifold*

Some Examples

Hex Torus

- $\Omega \cong T$
- $\Delta \cong \langle \gamma_1, \gamma_2 \rangle \subset \mathrm{Diag}_3$

T/Δ is a *hex torus*

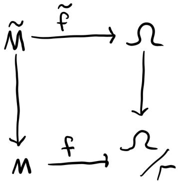


Convex Projective Structures

Let M be a compact manifold

A *convex projective structure* on M is $(f, \Omega/\Gamma)$

- Ω/Γ properly convex
- $f : M \to \Omega/\Gamma$ a diffeomorphism

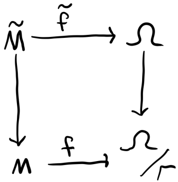


Convex Projective Structures

Let M be a compact manifold

A *convex projective structure* on M is $(f, \Omega/\Gamma)$

- Ω/Γ properly convex
- $f : M \to \Omega/\Gamma$ a diffeomorphism



There is an equivalence relation generated by

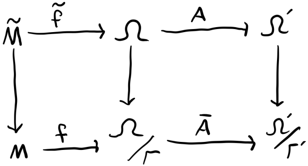
- Isotopy of f
- Replace Ω/Γ with Ω'/Γ' where $\Omega' = A(\Omega)$, $\Gamma' = A\Gamma A^{-1}$ for $A \in G$

Convex Projective Structures

Let M be a compact manifold

A *convex projective structure* on M is $(f, \Omega/\Gamma)$

- Ω/Γ properly convex
- $f : M \to \Omega/\Gamma$ a diffeomorphism



There is an equivalence relation generated by

- Isotopy of f
- Replace Ω/Γ with Ω'/Γ' where $\Omega' = A(\Omega)$, $\Gamma' = A\Gamma A^{-1}$ for $A \in G$

Exotic Convex Projective Structures

Let M be a closed hyperbolic manifold

Let $\mathrm{CP}(M)$ be the set of equivalence classes

Topologize $\mathrm{CP}(M)$ using C^∞ topology on $C^\infty(\widetilde{M}, \mathbb{RP}^n)$

Exotic Convex Projective Structures

Let M be a closed hyperbolic manifold

Let $\mathrm{CP}(M)$ be the set of equivalence classes

Topologize $\mathrm{CP}(M)$ using C^∞ topology on $C^\infty(\tilde{M}, \mathbb{RP}^n)$

Definition

$p \in \mathrm{CP}(M)$ is *exotic* if it is not the same connected component as $\mathbb{H}(M) \subset \mathrm{CP}(M)$.

p is exotic if it cannot be continuously deformed to a hyperbolic structure

Exotic Convex Projective Structures

Existence

When do exotic structures exist?

When do exotic structures exist?

- Dimension 2: No exotic structures (Goldman ’90)

Exotic Convex Projective Structures

Existence

When do exotic structures exist?

- Dimension 2: No exotic structures (Goldman '90)
- Dimension 3: Infinitely many examples (B-Danciger-Lee-Marquis)

Exotic Convex Projective Structures

When do exotic structures exist?

- Dimension 2: No exotic structures (Goldman '90)
- Dimension 3: Infinitely many examples (B-Danciger-Lee-Marquis)
- Dimension $\geqslant 4$: ???

Exotic Convex Projective Structures

Existence

When do exotic structures exist?

- Dimension 2: No exotic structures (Goldman '90)
- Dimension 3: Infinitely many examples (B-Danciger-Lee-Marquis)
- Dimension $\geqslant 4$: ???

Question: Does every closed hyperbolic 3-manifold admit an exotic convex projective structure?

Exotic Convex Projective Structures

When do exotic structures exist?

- Dimension 2: No exotic structures (Goldman '90)
- Dimension 3: Infinitely many examples (B-Danciger-Lee-Marquis)
- Dimension $\geqslant 4$: ???

Question: Does every closed hyperbolic 3-manifold admit an exotic convex projective structure? (maybe yes!)

Some Tools

Let $[(f, \Omega/\Gamma)] \in \mathrm{CP}(M)$.

Define $f_* : \pi_1 M \hookrightarrow \Gamma \subset G$ (*holonomy*)

This is only well defined up to conjugacy in G

Some Tools

Let $[(f, \Omega/\Gamma)] \in \mathrm{CP}(M)$.

Define $f_* : \pi_1 M \hookrightarrow \Gamma \subset G$ (*holonomy*)

This is only well defined up to conjugacy in G

$\mathrm{Hol} : \mathrm{CP}(M) \to \mathrm{Rep}(\pi_1 M, G) := \mathrm{Hom}(\pi_1 M, G)/G$
$[(f, \Omega/\Gamma)] \mapsto [f_*]$ (*holonomy map*)

Some Tools

Let $[(f, \Omega/\Gamma)] \in \mathrm{CP}(M)$.

Define $f_* : \pi_1 M \hookrightarrow \Gamma \subset G$ (*holonomy*)

This is only well defined up to conjugacy in G

$\mathrm{Hol} : \mathrm{CP}(M) \to \mathrm{Rep}(\pi_1 M, G) := \mathrm{Hom}(\pi_1 M, G)/G$
$[(f, \Omega/\Gamma)] \mapsto [f_*]$ (*holonomy map*)

Theorem (Koszul)

Hol *is an open map*

Some Tools

Let $[(f, \Omega/\Gamma)] \in \mathrm{CP}(M)$.

Define $f_* : \pi_1 M \hookrightarrow \Gamma \subset G$ (*holonomy*)

This is only well defined up to conjugacy in G

$\mathrm{Hol} : \mathrm{CP}(M) \to \mathrm{Rep}(\pi_1 M, G) := \mathrm{Hom}(\pi_1 M, G)/G$
$[(f, \Omega/\Gamma)] \mapsto [f_*]$ (*holonomy map*)

Theorem (Koszul)

Hol *is an open map*

Moral: If you can deform the representation you can deform the structure.

Some Tools

- M a closed hyperbolic 3-manifold
- $[(f_{hyp}, \mathbb{H}^n/\Gamma)] \in \mathrm{CP}(M)$ the hyperbolic structure
- $\rho_{hyp} = (f_{hyp})_*$ hyperbolic holonomy
- $\mathfrak{g}$ the Lie algebra of G
- $H^1_{\rho_{hyp}}(\pi_1 M, \mathfrak{g})$ (*twisted cohomology*)

Some Tools

- M a closed hyperbolic 3-manifold
- $[(f_{hyp}, \mathbb{H}^n/\Gamma)] \in \mathrm{CP}(M)$ the hyperbolic structure
- $\rho_{hyp} = (f_{hyp})_*$ hyperbolic holonomy
- $\mathfrak{g}$ the Lie algebra of G
- $H^1_{\rho_{hyp}}(\pi_1 M, \mathfrak{g})$ (*twisted cohomology*)

Fact: $H^1_{\rho_{hyp}}(\pi_1 M, \mathfrak{g})$ is the "Zariski tangent space" to $\mathrm{Rep}(\pi_1 M, G)$ at $[\rho_{hyp}]$

Some Tools

- M a closed hyperbolic 3-manifold
- $[(f_{hyp}, \mathbb{H}^n/\Gamma)] \in \mathrm{CP}(M)$ the hyperbolic structure
- $\rho_{hyp} = (f_{hyp})_*$ hyperbolic holonomy
- $\mathfrak{g}$ the Lie algebra of G
- $H^1_{\rho_{hyp}}(\pi_1 M, \mathfrak{g})$ (*twisted cohomology*)

Fact: $H^1_{\rho_{hyp}}(\pi_1 M, \mathfrak{g})$ is the "Zariski tangent space" to $\mathrm{Rep}(\pi_1 M, G)$ at $[\rho_{hyp}]$

If $H^1_{\rho_{hyp}}(\pi_1 M, \mathfrak{g}) = 0$ then M is *infinitesimally rigid*

Some Tools

- M a closed hyperbolic 3-manifold
- $[(f_{hyp}, \mathbb{H}^n/\Gamma)] \in \mathrm{CP}(M)$ the hyperbolic structure
- $\rho_{hyp} = (f_{hyp})_*$ hyperbolic holonomy
- $\mathfrak{g}$ the Lie algebra of G
- $H^1_{\rho_{hyp}}(\pi_1 M, \mathfrak{g})$ (*twisted cohomology*)

Fact: $H^1_{\rho_{hyp}}(\pi_1 M, \mathfrak{g})$ is the "Zariski tangent space" to $\mathrm{Rep}(\pi_1 M, G)$ at $[\rho_{hyp}]$

If $H^1_{\rho_{hyp}}(\pi_1 M, \mathfrak{g}) = 0$ then M is *infinitesimally rigid*

Fact: Infinitesimally rigid $\Rightarrow$ locally rigid $\Rightarrow$ all non-hyperbolic structures are exotic.

Dehn Filling

Let N be a manifold with $\partial N \cong T^2$.

Let $[\gamma] \in \pi_1(\partial N)$ be simple

Let D be a solid torus with meridian m

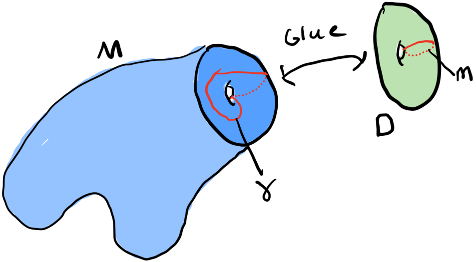
Dehn Filling

Let N be a manifold with $\partial N \cong T^2$.

Let $[\gamma] \in \pi_1(\partial N)$ be simple

Let D be a solid torus with meridian m

Let N_γ be obtained by gluing N and D along boundaries by diffeomorphism mapping γ to m (*Dehn filling of N along* γ)



Dehn Filling

Let N be the complement of the figure-8 knot

Dehn Filling

Let N be the complement of the figure-8 knot

Theorem (Thurston's Dehn Filling Theorem)

All but finitely many Dehn fillings of N admit a hyperbolic structure.

Dehn Filling

Let N be the complement of the figure-8 knot

Theorem (Thurston's Dehn Filling Theorem)

All but finitely many Dehn fillings of N admit a hyperbolic structure.

Theorem (Heusener–Porti)

All but finitely many Dehn fillings of N are infinitesimally rigid.

Dehn Filling

Let N be the complement of the figure-8 knot

Theorem (Thurston's Dehn Filling Theorem)

All but finitely many Dehn fillings of N admit a hyperbolic structure.

Theorem (Heusener–Porti)

All but finitely many Dehn fillings of N are infinitesimally rigid.

Theorem (B-Danciger-Lee-Marquis)

Infinitely many Dehn fillings of N admit exotic convex projective structures.

Dehn Filling

Let N be the complement of the figure-8 knot

Theorem (Thurston's Dehn Filling Theorem)

All but finitely many Dehn fillings of N admit a hyperbolic structure.

Theorem (Heusener–Porti)

All but finitely many Dehn fillings of N are infinitesimally rigid.

Theorem (B-Danciger-Lee-Marquis)

Infinitely many Dehn fillings of N admit exotic convex projective structures.

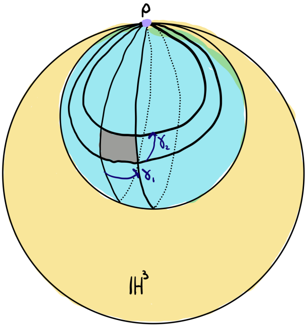
N can be replaced by other 1-cusped hyperbolic manifolds.

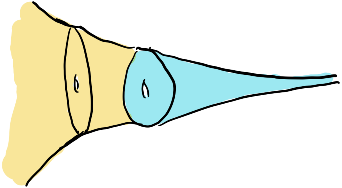
Hyperbolic Dehn Filling

Let $\rho_{hyp} : \pi_1 N \to \mathrm{PSL}(2, \mathbb{C})$ be the hyperbolic holonomy
Let $\Delta = \pi_1 \partial N = \langle \gamma_1, \gamma_2 \rangle \cong \mathbb{Z}^2$.
$\rho_{hyp}(\Delta) \subset G_p \cong \mathbb{R}^2$ (*stabilizer of* $p \in \partial \mathbb{H}^3$)

Hyperbolic Dehn Filling

Let $\rho_{hyp} : \pi_1 N \to \mathrm{PSL}(2, \mathbb{C})$ be the hyperbolic holonomy
Let $\Delta = \pi_1 \partial N = \langle \gamma_1, \gamma_2 \rangle \cong \mathbb{Z}^2$.
$\rho_{hyp}(\Delta) \subset G_p \cong \mathbb{R}^2$ (*stabilizer of* $p \in \partial \mathbb{H}^3$)

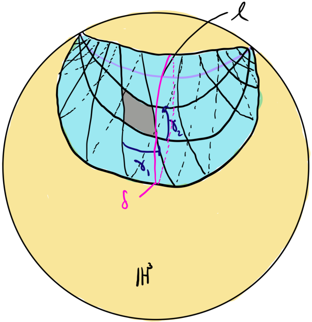


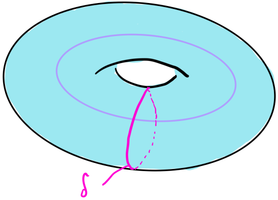


Hyperbolic Dehn Filling

Deform ρ_{hyp} to non-conjugate $\rho' \in \mathrm{Hom}(\pi_1 N, \mathrm{PSL}(2,\mathbb{C}))$

$\rho'(\Delta) \subset G_\ell \cong \mathbb{C}^*$ (*stabilizer of geodesic* ℓ)



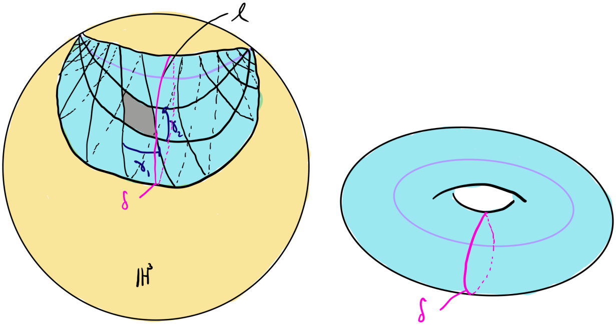


Hyperbolic Dehn Filling

Deform ρ_{hyp} to non-conjugate $\rho' \in \mathrm{Hom}(\pi_1 N, \mathrm{PSL}(2,\mathbb{C}))$

$\rho'(\Delta) \subset G_\ell \cong \mathbb{C}^*$ (*stabilizer of geodesic* ℓ)

ρ' is the holonomy of an *incomplete* hyperbolic structure on N.



Hyperbolic Dehn Filling

Deform ρ_{hyp} to non-conjugate $\rho' \in \mathrm{Hom}(\pi_1 N, \mathrm{PSL}(2,\mathbb{C}))$

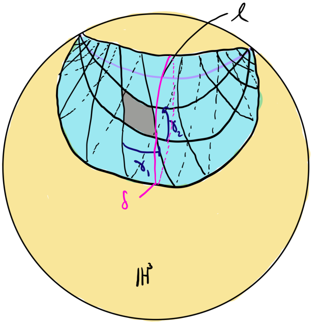
$\rho'(\Delta) \subset G_\ell \cong \mathbb{C}^*$ (*stabilizer of geodesic* ℓ)

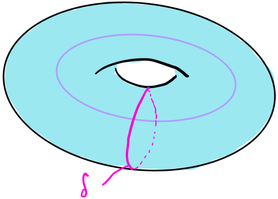
ρ' is the holonomy of an *incomplete* hyperbolic structure on N.

Let $g_1 = \rho'(\gamma_1)$, $g_2 = \rho'(\gamma_2)$

There are unique $(a,b) \in \mathbb{R}^2$ so that *Dehn filling coordinates*

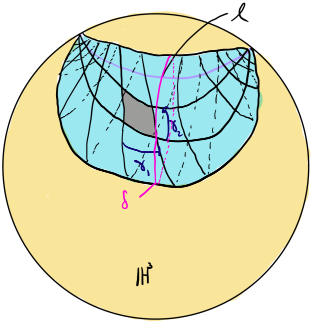
$$a \log(g_1) + b \log(g_2) = 2\pi i$$

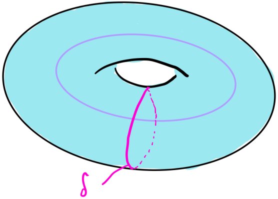




Hyperbolic Dehn filling

Dehn filling coordinates control geometry of the completion

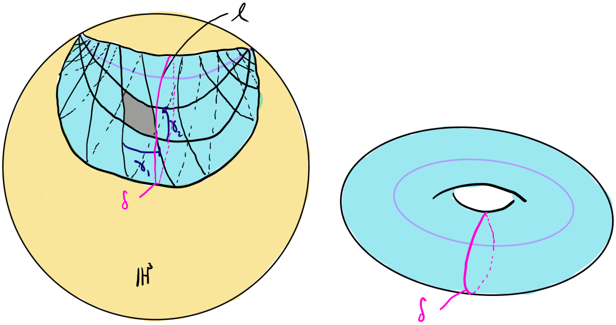




Hyperbolic Dehn filling

Dehn filling coordinates control geometry of the completion

If $(a, b) \in \mathbb{Z}^2$ relatively prime
$\delta = \gamma_1^a \gamma_2^b$ is simple curve in $\ker \rho'$, $\rho'(\Delta) \cong \mathbb{Z}$

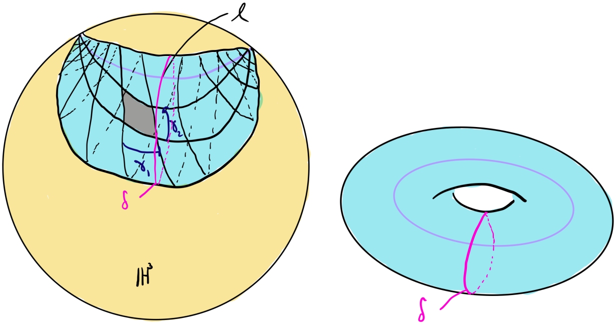


Hyperbolic Dehn filling

Dehn filling coordinates control geometry of the completion

If $(a, b) \in \mathbb{Z}^2$ relatively prime
$\delta = \gamma_1^a \gamma_2^b$ is simple curve in $\ker \rho'$, $\rho'(\Delta) \cong \mathbb{Z}$

The completion of incomplete structure is N_δ
N_δ has a hyperbolic structure!!



Hyperbolic Dehn Filling

Which δ arise from this construction?

Hyperbolic Dehn Filling

Which δ arise from this construction?

Thurston: there is k so that if

- $(a, b) \in \mathbb{Z}^2$
- a, b relatively prime
- $a^2 + b^2 > k^2$

then (a, b) are the Dehn filling coordinates of incomplete structure on N

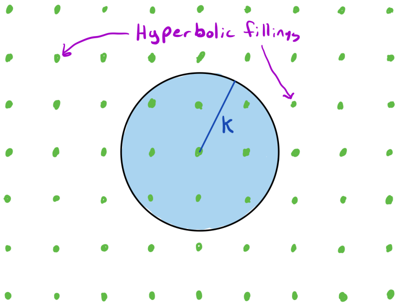
Hyperbolic Dehn Filling

Which δ arise from this construction?

Thurston: there is k so that if

- $(a, b) \in \mathbb{Z}^2$
- a, b relatively prime
- $a^2 + b^2 > k^2$

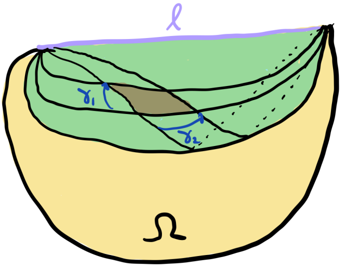
then (a, b) are the Dehn filling coordinates of incomplete structure on N

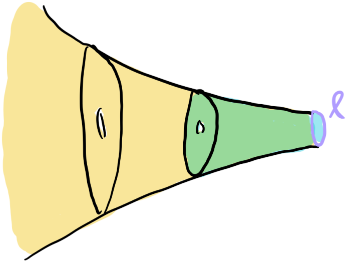


Properly Convex Dehn Filling

Step 1

Deform ρ_{hyp} to $\rho' \in \mathrm{Hom}(\pi_1 N, G)$ where ρ' is holonomy of convex projective structure with "generalized cusp" (*Cooper-Long-Tillmann extension of Koszul Thm*)





Properly Convex Dehn Filling

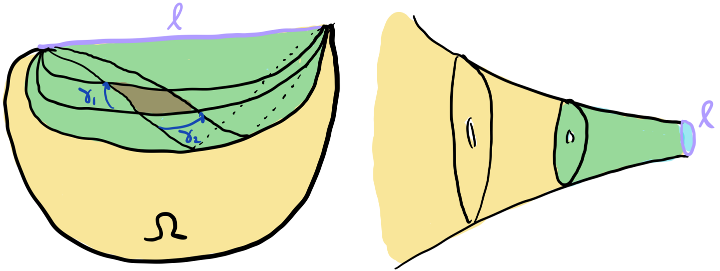
first deformation

$\rho'(\Delta) \subset G_\ell^\Omega \cong \mathbb{R}_{dil} \oplus i\mathbb{R}_{uni} \cong \mathbb{C}$ (*stabilizer of* ℓ *in* $\mathrm{PGL}(\Omega)$)

There is (non-unique) $(a, b) \in \mathbb{R}^2$ so that

$$\rho'(\gamma_1^a \gamma_2^b) \in i\mathbb{R}_{uni}$$

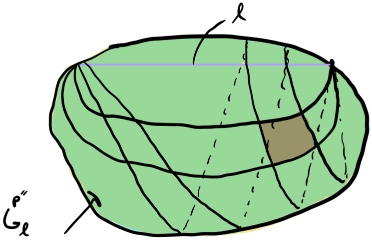
$a/b \in S^1 = \mathbb{R} \cup \{\infty\}$ is well defined (*unipotent slope*)



Properly Convex Dehn Filling

Step 2

Deform ρ' to $\rho'' \in \mathrm{Hom}(\pi_1 M, G)$ so that $\rho''(\Delta) \subset G_\ell^{\rho''} \cong \mathbb{C}^*$ (*stabilizer of convex "nbhd" of ℓ*)



Properly Convex Dehn Filling

Step 2

Let $g_1 = \rho''(\gamma_1)$, $g_2 = \rho''(\gamma_2)$
Get *Dehn filling coordinates* (a, b)

$$a \log(g_1) + b \log(g_2) = 2\pi i$$

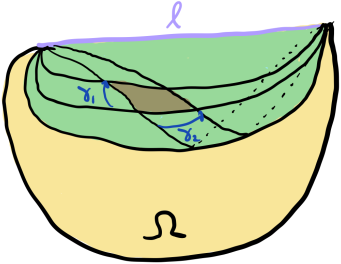
Properly Convex Dehn Filling

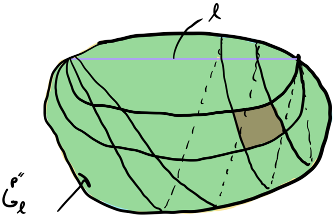
Step 2

Let $g_1 = \rho''(\gamma_1)$, $g_2 = \rho''(\gamma_2)$
Get *Dehn filling coordinates* (a, b)

$$a \log(g_1) + b \log(g_2) = 2\pi i$$

Unipotent elements in $i\mathbb{R}_{uni} \subset G_\ell^\Omega$ deform to rotations in $G_\ell^{\rho''}$ so a/b is close to unipotent slope of ρ'





Properly Convex Dehn Filling

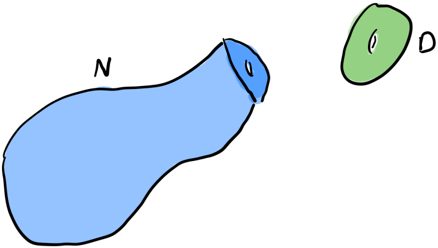
Suppose $(a, b) \in \mathbb{Z}^2$, relatively prime, $\delta = \gamma_1^a \gamma_2^b$

Properly Convex Dehn Filling

Suppose $(a, b) \in \mathbb{Z}^2$, relatively prime, $\delta = \gamma_1^a \gamma_2^b$
$\rho''(\Delta) \cong \mathbb{Z}$
$D \cong G_\ell^{\rho''} / \rho''(\Delta)$ (*properly convex solid torus*)

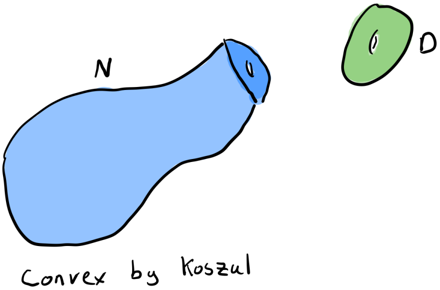
Properly Convex Dehn Filling

Suppose $(a, b) \in \mathbb{Z}^2$, relatively prime, $\delta = \gamma_1^a \gamma_2^b$
$\rho''(\Delta) \cong \mathbb{Z}$
$D \cong G_\ell^{\rho''} / \rho''(\Delta)$ (*properly convex solid torus*)



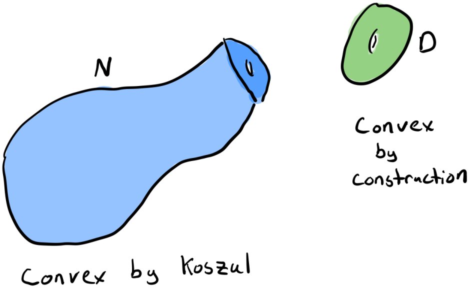
Properly Convex Dehn Filling

Suppose $(a, b) \in \mathbb{Z}^2$, relatively prime, $\delta = \gamma_1^a \gamma_2^b$
$\rho''(\Delta) \cong \mathbb{Z}$
$D \cong G_\ell^{\rho''} / \rho''(\Delta)$ (*properly convex solid torus*)



Properly Convex Dehn Filling

Suppose $(a, b) \in \mathbb{Z}^2$, relatively prime, $\delta = \gamma_1^a \gamma_2^b$
$\rho''(\Delta) \cong \mathbb{Z}$
$D \cong G_\ell^{\rho''} / \rho''(\Delta)$ (*properly convex solid torus*)

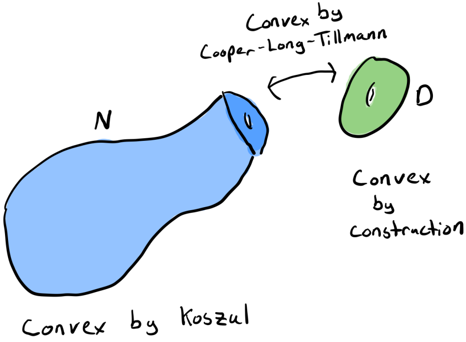


Properly Convex Dehn Filling

Suppose $(a, b) \in \mathbb{Z}^2$, relatively prime, $\delta = \gamma_1^a \gamma_2^b$

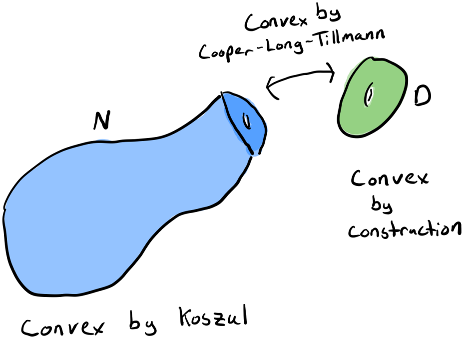
$\rho''(\Delta) \cong \mathbb{Z}$

$D \cong G_\ell^{\rho''} / \rho''(\Delta)$ (*properly convex solid torus*)



Properly Convex Dehn Filling

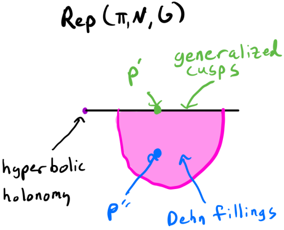
Suppose $(a, b) \in \mathbb{Z}^2$, relatively prime, $\delta = \gamma_1^a \gamma_2^b$
$\rho''(\Delta) \cong \mathbb{Z}$
$D \cong G_\ell^{\rho''} / \rho''(\Delta)$ (*properly convex solid torus*)



N_δ admits a non-hyperbolic properly convex structure

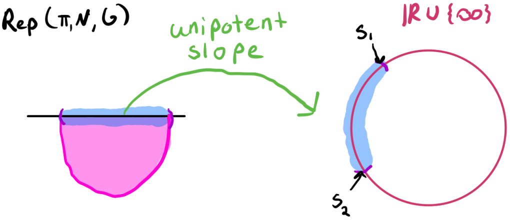
Properly Convex Dehn Filling

Which δ arise



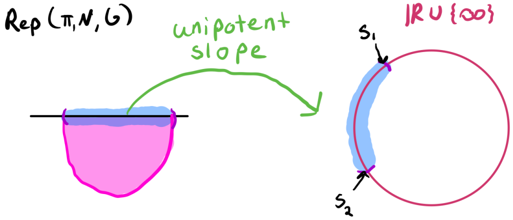
Properly Convex Dehn Filling

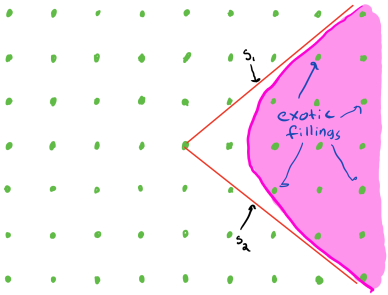
Which δ arise



Properly Convex Dehn Filling

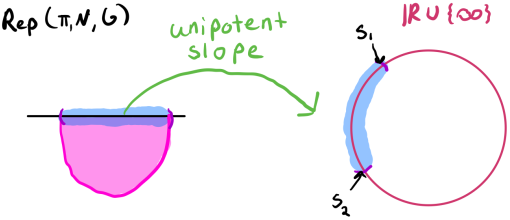
Which δ arise

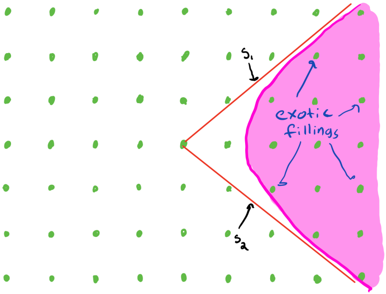




Properly Convex Dehn Filling

Which δ arise





A positive proportion of fillings are exotic!

Properly Convex Dehn Fillings

Constructing the deformations

- $\mathrm{Rep}(\mathbb{Z}^2, G)\text{“} \cong \text{”} \mathbb{R}^6$, $\mathrm{Rep}(\pi_1 N, G)\text{“} \cong \text{”} \mathbb{R}^3$
- There is a 3-dim locus of “pure” reps $P \subset \mathrm{Rep}(\mathbb{Z}^2, G)$ with repeated eigenvalue
- Contains holonomy of with generalized cusps and Dehn fillings
- Examine how P intersects $\mathrm{res} : \mathrm{Rep}(\pi_1 N, G) \to \mathrm{Rep}(\mathbb{Z}^2, G)$

The Real Result

Theorem (B-Danciger-Lee-Marquis)

Let M be a 1-cusped infinitesimally rigid 3-manifold with non-constant unipotent slope then a positive proportion of the Dehn fillings of M admit exotic convex projective structures

The Real Result

Theorem (B-Danciger-Lee-Marquis)

Let M be a 1-cusped infinitesimally rigid 3-manifold with non-constant unipotent slope then a positive proportion of the Dehn fillings of M admit exotic convex projective structures

So far M_{004} (fig-8), M_{003} (fig-8 sister), M_{007}, and M_{019} have been shown to satisfy these hypotheses.

Effective Questions

- Which cusped 3-manifolds are infinitesimally rigid?
- Which cusped 3-manifolds have non-constant unipotent slope?
- For a given M what is the range of the unipotent slope map?

Thank you