Frame theory on vector bundles

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(joint with T. Needham & C. Shonkwiler)

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Frames (through a geometric lens)

Vector Bundles

Frames Fields on Vector Bundles

Frames in vector spaces

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(Frame inequality)

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Henceforth V is finite dimensional and $\mathcal{C} = \{v_1, \dots v_k\}$

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A subset $\mathcal{C} = \{v_1, \ldots v_k\}$ of V gives rise to an *Analysis operator* $A_{\mathcal{C}} : V \to \mathbb{R}^k$

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$$\sum_i \langle v, v_i \rangle^2 = ||A_{\mathcal{C}}(v)||^2 \leqslant ||A_{\mathcal{C}}||_{op}^2 \, ||v||^2 < \infty$$

so upper frame inequality is automatic for finite subsets

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There is also a *Synthesis operator* $S_{\mathcal{C}} : \mathbb{R}^k \to V$ given by

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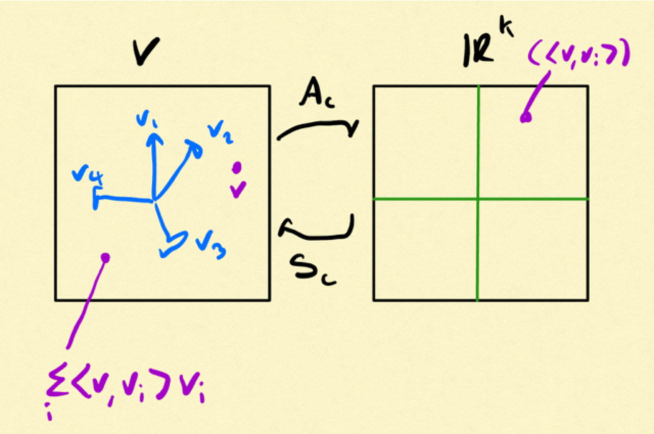
Moral: Finite frames in V $\Leftrightarrow$ Spanning sets of V $\Leftrightarrow$ surjective $M : \mathbb{R}^k \to V$

Frame operator

Given a frame $\mathcal{C}$, we can define the frame operator $F_{\mathcal{C}} = S_{\mathcal{C}} \circ A_{\mathcal{F}} : V \to V$



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The map $F_{\mathcal{C}}^{-1} S_{\mathcal{C}} : \mathbb{R}^k \to V$ is the Moore-Penrose Pseudoinverse, $A_{\mathcal{C}}^{\dagger}$, of $A_{\mathcal{C}}$

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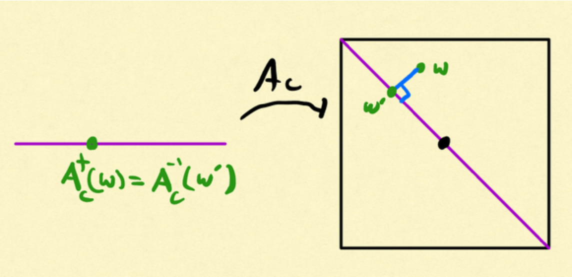
Given $w \in \mathbb{R}^k$ we...

- Orthogonally project w to $\mathrm{Im}(A_{\mathcal{C}})$
- Take inverse of $A_{\mathcal{C}}$



Parseval frames

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These generalize notion of orthonormal bases

Parseval frames

Alternate characterizations

Parseval frames can be described in a variety of equivalent ways

- $F_{\mathcal{C}} = S_{\mathcal{C}} A_{\mathcal{C}} = A_{\mathcal{C}}^t A_{\mathcal{C}} = I$
- $a = b = 1$ in frame inequality
- $A_{\mathcal{C}} : V \to \mathbb{R}^k$ is an isometric embedding (preserves inner products)

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Question: How does size of $v' - v$ compare to $w' - w = \eta$?

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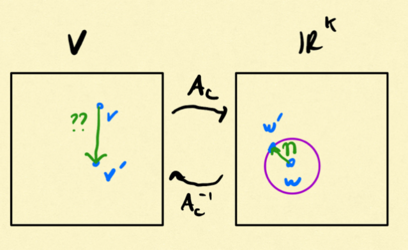

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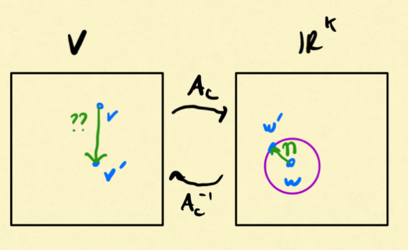

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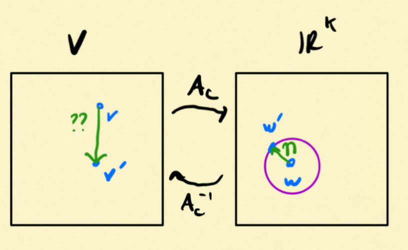

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If the basis is orthonormal $A_{\mathcal{C}}$ is maximally well conditioned

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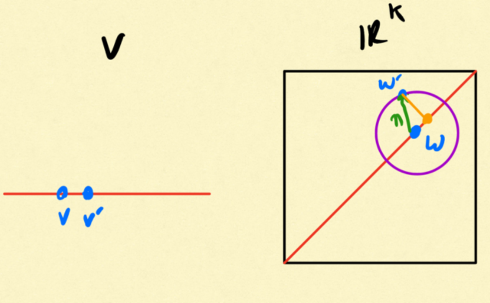
When reconstructing with a frame the reconstruction error depends on the distance between w and the *projection of w' !*



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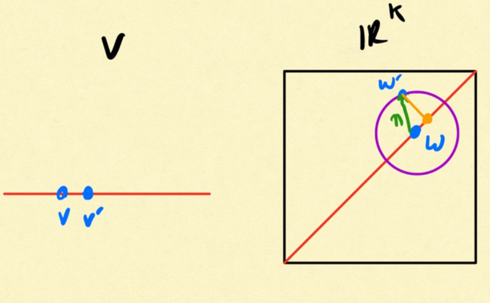

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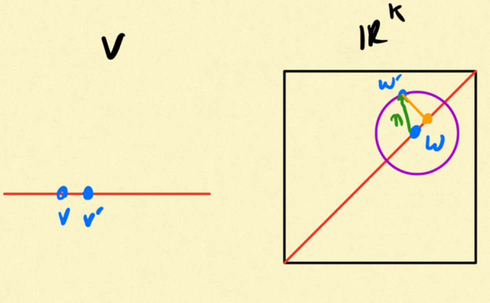

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If the frame is Parseval the A_C is again maximally well conditioned

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Moral: The larger the frame, the more robust it is to noise!

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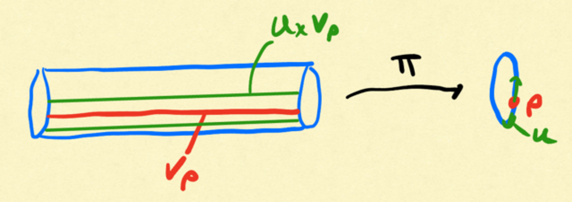
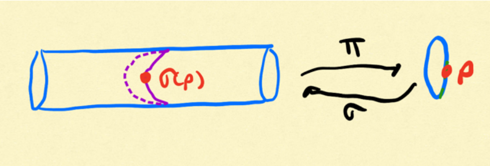
- $\pi^{-1}(p) = V_p$ (fiber)
- There is a nbhd $p \in U$ of each p so that $\pi^{-1}(U) \cong U \times V_p$ (locally a product)



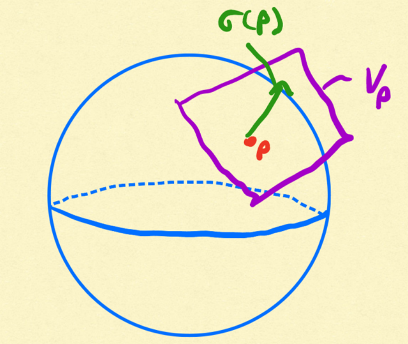
Vector fields

A *vector field* is a (continuous) choice of a vector in each fiber
More precisely, a vector field is a continuous $\sigma : M \to E$ so that
$\pi \circ \sigma = Id$ ($\sigma(p)$ lives in V_p)



A toy example

- $M = S^2$ (surface of earth)
- V_p =tangent vectors at p
- E =tangent bundle to S^2
- σ = wind velocity



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Yes iff $\{\mu_1(p), \ldots, \mu_k(p)\}$ is a frame in V_p for each p.

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- Can we always find a Parseval frame field?

Toy Example

revisited

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Global bases don't usually exist!

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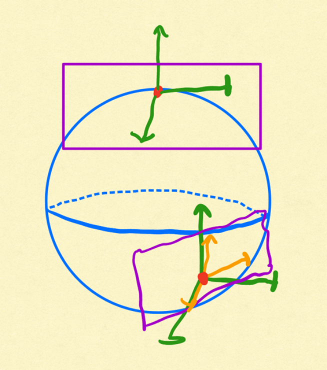

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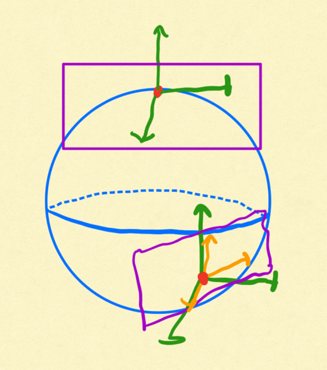

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The previous example shows that this is not sharp in general

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The following theorem answers all three questions for vector bundles over manifolds

Theorem 1 (B-Needham-Shonkwiler)

Let $E \to M$ be a rank n vector bundle over a d-dimensional manifold. Then there is Parseval frame of size k on E for $k \geqslant n + d$.

The previous example shows that this is not sharp in general

This generalizes previous work by Freeman-Poore-Wei-Wyse

Ideas behind proof

- Build a (non-vector) bundle $\mathcal{P}_k(V) \to M$ whose fiber at p is the "space of all Parseval frames" of size k in V_p

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- If $k \geqslant d+n$ then all the obstructions vanish and we can find a section.

Future Directions

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- Analyze robustness properties of Parseval frames for various types of noise
- Study minimal frame dimension (i.e. Given $E \to M$ find smallest $n \leqslant k \leqslant n + d$ so that a k-frame field exists

Have Hammer, Seeking Nail

Please let me know if you have potential applications

Thank you!