Deforming Properly Convex 3-manifolds

Sam Ballas

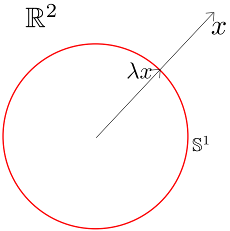
(joint with J. Danciger, G.-S. Lee, D. Cooper, and A. Leitner)

Florida State University
Algebra Seminar
January 14, 2016

The Projective Sphere

Let

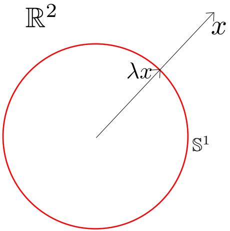
- $\mathbb{S}^n := (\mathbb{R}^{n+1} \setminus \{0\})/(x \sim \lambda x)$, $\lambda > 0$.
- $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R}) = \{A \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid \det(A) = \pm 1\}$



The Projective Sphere

Let

- $\mathbb{S}^n := (\mathbb{R}^{n+1} \backslash \{0\})/(x \sim \lambda x)$, $\lambda > 0$.
- $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R}) = \{A \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid \det(A) = \pm 1\}$

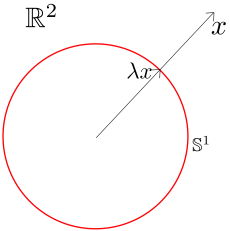


$\mathbb{S}^n$ and $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ double cover $\mathbb{RP}^n$ and $\mathrm{PGL}_{n+1}(\mathbb{R})$, respectively.

The Projective Sphere

Let

- $\mathbb{S}^n := (\mathbb{R}^{n+1} \setminus \{0\})/(x \sim \lambda x)$, $\lambda > 0$.
- $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R}) = \{A \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid \det(A) = \pm 1\}$



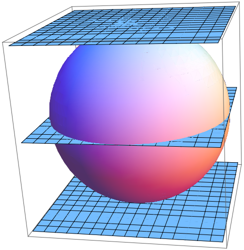
$\mathbb{S}^n$ and $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ double cover $\mathbb{RP}^n$ and $\mathrm{PGL}_{n+1}(\mathbb{R})$, respectively.

We can do projective geometry with simply connected and orientable model space.

Properly Convex Geometry

Affine Patches

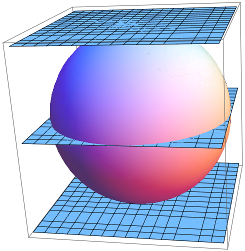
Let H be a hyperplane in $\mathbb{R}^{n+1}$. Then $\mathbb{S}^n \backslash \overline{H}$ decomposes as $\mathbb{R}^n_+ \sqcup \mathbb{S}^{n-1} \sqcup \mathbb{R}^n_-$



Properly Convex Geometry

Affine Patches

Let H be a hyperplane in $\mathbb{R}^{n+1}$. Then $\mathbb{S}^n \backslash \overline{H}$ decomposes as $\mathbb{R}^n_+ \sqcup \mathbb{S}^{n-1} \sqcup \mathbb{R}^n_-$

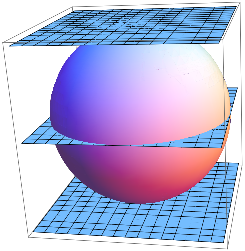


A component of $\mathbb{S}^n \backslash \overline{H}$ is called an *affine patch*

Properly Convex Geometry

Affine Patches

Let H be a hyperplane in $\mathbb{R}^{n+1}$. Then $\mathbb{S}^n \setminus \overline{H}$ decomposes as $\mathbb{R}^n_+ \sqcup \mathbb{S}^{n-1} \sqcup \mathbb{R}^n_-$



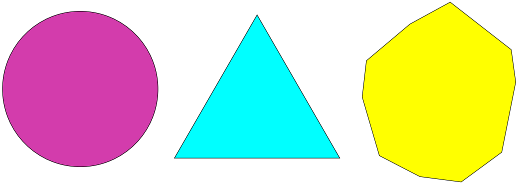
A component of $\mathbb{S}^n \setminus \overline{H}$ is called an *affine patch*

Affine patches inherit a notion of *convexity*

Properly Convex Geometry

Properly Convex Domains

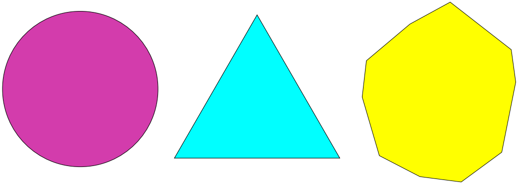
A domain $\Omega \subset \mathbb{S}^n$ is *properly convex* if $cl(\Omega)$ is a convex subset of an affine patch.



Properly Convex Geometry

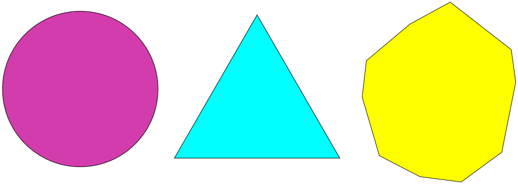
Properly Convex Domains

A domain $\Omega \subset \mathbb{S}^n$ is *properly convex* if $cl(\Omega)$ is a convex subset of an affine patch.



Equivalent to Ω not containing a complete affine line

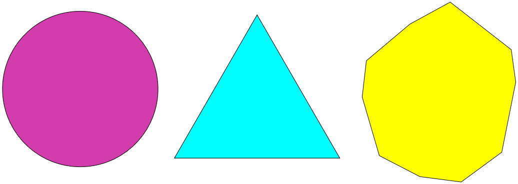
A domain $\Omega \subset \mathbb{S}^n$ is *properly convex* if $cl(\Omega)$ is a convex subset of an affine patch.



Equivalent to Ω not containing a complete affine line

If $\partial\Omega$ contains no line segments then Ω is *strictly convex*

A domain $\Omega \subset \mathbb{S}^n$ is *properly convex* if $cl(\Omega)$ is a convex subset of an affine patch.



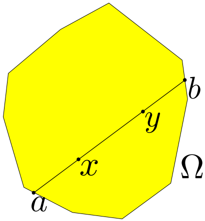
Equivalent to Ω not containing a complete affine line

If $\partial\Omega$ contains no line segments then Ω is *strictly convex*

$\mathrm{Aut}(\Omega) = \{A \in \mathrm{SL}^{\pm}_{n+1}(\mathbb{R}) \mid A(\Omega) = \Omega\}$

We define the *Hilbert metric* on Ω by

$$d_\Omega(x, y) = \frac{1}{2}\log([a : x : y : b])$$

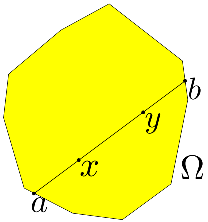


Properly Convex Geometry

Hilbert Metric

We define the *Hilbert metric* on Ω by

$$d_\Omega(x, y) = \frac{1}{2}\log([a : x : y : b])$$



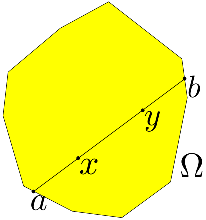
- Straight lines are geodesics (there can be others)
- This metric is usually not Riemannian (only Finsler)

Properly Convex Geometry

Hilbert Metric

We define the *Hilbert metric* on Ω by

$$d_\Omega(x, y) = \frac{1}{2}\log([a : x : y : b])$$

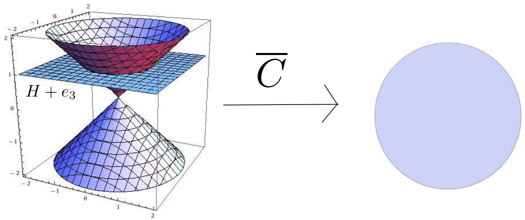


- Straight lines are geodesics (there can be others)
- This metric is usually not Riemannian (only Finsler)
- $\mathrm{Aut}(\Omega) \subset \mathrm{Isom}(\Omega)$

Examples

Hyperbolic Geometry

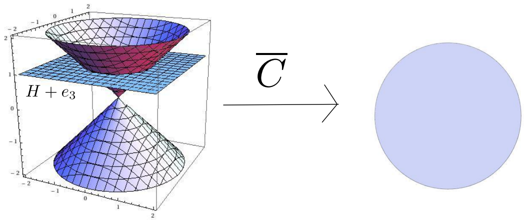
- Let $\langle x, y \rangle = x_1 y_1 + \ldots x_n y_n - x_{n+1} y_{n+1}$ be the standard bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$
- Let $C_+ = \{x \in \mathbb{R}^{n+1} | \langle x, x \rangle < 0, x_{n+1} > 0\}$



Examples

Hyperbolic Geometry

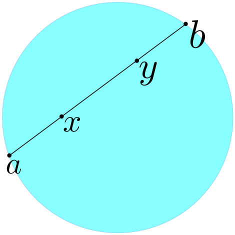
- Let $\langle x, y \rangle = x_1 y_1 + \ldots x_n y_n - x_{n+1} y_{n+1}$ be the standard bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$
- Let $C_+ = \{x \in \mathbb{R}^{n+1} | \langle x, x \rangle < 0, x_{n+1} > 0\}$
- $\overline{C_+} = \mathbb{H}^n$ is the *Klein model* of hyperbolic space.



Examples

Hyperbolic Geometry

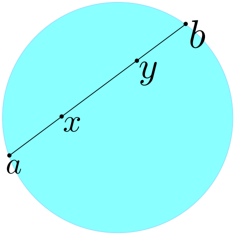
- This Hilbert metric on $\mathbb{H}^n$ is Riemannian and has constant curvature -1



Examples

Hyperbolic Geometry

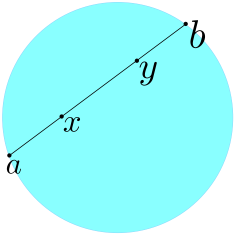
- This Hilbert metric on $\mathbb{H}^n$ is Riemannian and has constant curvature -1
- Straight lines are the only geodesics



Examples

Hyperbolic Geometry

- This Hilbert metric on $\mathbb{H}^n$ is Riemannian and has constant curvature -1
- Straight lines are the only geodesics
- $\text{Isom}(\mathbb{H}^n) \cong \text{Aut}(\mathbb{H}^n) \cong O^+(n,1)$.



Let

- $\Omega \subset \mathbb{S}^n$ be properly convex and
- $\Gamma \subset \mathrm{Aut}(\Omega)$ be a discrete and torsion free subgroup

then $\Gamma \backslash \Omega$ is a *properly convex n-manifold*

Let

- $\Omega \subset \mathbb{S}^n$ be properly convex and
- $\Gamma \subset \mathrm{Aut}(\Omega)$ be a discrete and torsion free subgroup

then $\Gamma \backslash \Omega$ is a *properly convex n-manifold*

Let

- M be an orientable n-manifold,
- $\Gamma \backslash \Omega$ be a properly convex manifold, and
- $f : M \to \Gamma \backslash \Omega$ be a diffeomorphism (called a *marking*)

then $(f, \Gamma \backslash \Omega)$ is a *properly convex structure on M*

Properly Convex Geometry

Properly Convex Manifolds

$$
\begin{array}{ccc}
\widetilde{M} & \xrightarrow[\cong]{\mathrm{Dev}} & \Omega \\
{\scriptstyle \pi_1 M \circlearrowright} \downarrow & & \downarrow {\scriptstyle \circlearrowleft \Gamma} \\
M & \xrightarrow[\cong]{f} & \Gamma \backslash \Omega
\end{array}
$$

By lifting f we get a map $\mathrm{Dev} : \widetilde{M} \to \Omega$ called a *developing map*.

Properly Convex Geometry

Properly Convex Manifolds

$$
\begin{array}{ccc}
\widetilde{M} & \xrightarrow[\cong]{\text{Dev}} & \Omega \\
{\scriptstyle \pi_1 M \circlearrowright}\big\downarrow & & \big\downarrow{\scriptstyle \circlearrowleft \Gamma} \\
M & \xrightarrow[\cong]{f} & \Gamma \backslash \Omega
\end{array}
$$

By lifting f we get a map $\mathrm{Dev} : \widetilde{M} \to \Omega$ called a *developing map*.

f also gives a representation

$$\rho : \pi_1 M \to \Gamma \subset \mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$$

called a *holonomy representation*. Dev is ρ-equivariant.

Equivalence of Structures

Let M be a manifold and consider the equivalence relation on properly convex structures generated by

1. $(f, \Gamma\backslash\Omega) \sim (f', \Gamma\backslash\Omega)$ if f and f' are isotopic and

Equivalence of Structures

Let M be a manifold and consider the equivalence relation on properly convex structures generated by

1. $(f, \Gamma\backslash\Omega) \sim (f', \Gamma\backslash\Omega)$ if f and f' are isotopic and
2. $(f, \Gamma\backslash\Omega) \sim (\overline{g} \circ f, \Gamma'\backslash\Omega')$

$$
\begin{array}{ccccc}
\widetilde{M} & \xrightarrow[\cong]{\text{Dev}} & \Omega & \xrightarrow[\cong]{g \in \mathrm{SL}^{\pm}} & \Omega' \\
\pi_1 M \circlearrowright \big\downarrow & & \big\downarrow \circlearrowleft \Gamma & & \big\downarrow \circlearrowleft \Gamma' \\
M & \xrightarrow[\cong]{f} & \Gamma\backslash\Omega & \xrightarrow[\cong]{\overline{g}} & \Gamma'\backslash\Omega'
\end{array}
$$

Equivalence of Structures

Let M be a manifold and consider the equivalence relation on properly convex structures generated by

1. $(f, \Gamma\backslash\Omega) \sim (f', \Gamma\backslash\Omega)$ if f and f' are isotopic and
2. $(f, \Gamma\backslash\Omega) \sim (\overline{g} \circ f, \Gamma'\backslash\Omega')$

$$\begin{array}{ccccc}
\widetilde{M} & \xrightarrow[\cong]{\text{Dev}} & \Omega & \xrightarrow[\cong]{g \in \mathrm{SL}^{\pm}} & \Omega' \\
{\scriptstyle \pi_1 M \circlearrowright}\big\downarrow & & \big\downarrow{\scriptstyle \circlearrowleft \Gamma} & & \big\downarrow{\scriptstyle \circlearrowleft \Gamma'} \\
M & \xrightarrow[\cong]{f} & \Gamma\backslash\Omega & \xrightarrow[\cong]{\overline{g}} & \Gamma'\backslash\Omega'
\end{array}$$

$\mathfrak{B}(M) = \{[(f, \Gamma\backslash\Omega)] \mid \Gamma\backslash\Omega \text{ properly convex}\}$ is called the *deformation space* of M

Complete Hyperbolic Manifolds

$\mathbb{H}^n$ is a properly convex domain in $\mathbb{S}^n$

Complete Hyperbolic Manifolds

$\mathbb{H}^n$ is a properly convex domain in $\mathbb{S}^n$

Let Γ be a discrete and torsion-free subgroup of $\mathrm{Isom}(\mathbb{H}^n) \cong \mathrm{Aut}(\mathbb{H}^n)$

Complete Hyperbolic Manifolds

$\mathbb{H}^n$ is a properly convex domain in $\mathbb{S}^n$

Let Γ be a discrete and torsion-free subgroup of $\mathrm{Isom}(\mathbb{H}^n) \cong \mathrm{Aut}(\mathbb{H}^n)$

$\Gamma \backslash \mathbb{H}^n$ is a properly convex manifold.

Complete Hyperbolic Manifolds

$\mathbb{H}^n$ is a properly convex domain in $\mathbb{S}^n$

Let Γ be a discrete and torsion-free subgroup of $\mathrm{Isom}(\mathbb{H}^n) \cong \mathrm{Aut}(\mathbb{H}^n)$

$\Gamma \backslash \mathbb{H}^n$ is a properly convex manifold.
(i.e. complete hyperbolic manifolds are properly convex manifolds.)

Examples

Dimension 2

Let Σ be a closed surface of genus $g \geqslant 2$

- $\mathfrak{B}(\Sigma)$ is a cell of dimension $-8\chi(\Sigma) = 16g - 16$ (Goldman)

Examples

Dimension 2

Let Σ be a closed surface of genus $g \geqslant 2$

- $\mathfrak{B}(\Sigma)$ is a cell of dimension $-8\chi(\Sigma) = 16g - 16$ (Goldman)
- Contains Teichmüller space as a $-3\chi(\Sigma) = 6g - 6$ dimensional subcell.

Examples

Dimension 2

Let Σ be a closed surface of genus $g \geqslant 2$

- $\mathfrak{B}(\Sigma)$ is a cell of dimension $-8\chi(\Sigma) = 16g - 16$ (Goldman)
- Contains Teichmüller space as a $-3\chi(\Sigma) = 6g - 6$ dimensional subcell.
- Constructed by gluing properly convex structures on pairs of pants.

Examples

Dimension 2

Let Σ be a closed surface of genus $g \geqslant 2$

- $\mathfrak{B}(\Sigma)$ is a cell of dimension $-8\chi(\Sigma) = 16g - 16$ (Goldman)
- Contains Teichmüller space as a $-3\chi(\Sigma) = 6g - 6$ dimensional subcell.
- Constructed by gluing properly convex structures on pairs of pants.
- Analogous results for finite volume structures if Σ is non-compact (Marquis)

Examples

Dimension 2

Let Σ be a closed surface of genus $g \geqslant 2$

- $\mathfrak{B}(\Sigma)$ is a cell of dimension $-8\chi(\Sigma) = 16g - 16$ (Goldman)
- Contains Teichmüller space as a $-3\chi(\Sigma) = 6g - 6$ dimensional subcell.
- Constructed by gluing properly convex structures on pairs of pants.
- Analogous results for finite volume structures if Σ is non-compact (Marquis)

One of few cases where $\mathfrak{B}(M)$ is understood globally!

Examples

Higher dimensions

- If M is hyperbolic contains a finite volume totally geodesic surface then we can find non-hyperbolic structures by “bending” (Johnson–Millson, Koszul, Marquis).

Examples

Higher dimensions

- If M is hyperbolic contains a finite volume totally geodesic surface then we can find non-hyperbolic structures by "bending" (Johnson–Millson, Koszul, Marquis).
- Most, but not all, closed 2-generator census manifolds are locally rigid near their complete hyperbolic structure (Cooper–Long–Thistlethwaite).

Examples

Higher dimensions

- If M is hyperbolic contains a finite volume totally geodesic surface then we can find non-hyperbolic structures by "bending" (Johnson–Millson, Koszul, Marquis).
- Most, but not all, closed 2-generator census manifolds are locally rigid near their complete hyperbolic structure (Cooper–Long–Thistlethwaite).
- A large class of cusped 3-manifolds admit properly convex deformations near their hyperbolic structure (B–Danciger–Lee)

Examples

Higher dimensions

- If M is hyperbolic contains a finite volume totally geodesic surface then we can find non-hyperbolic structures by “bending” (Johnson–Millson, Koszul, Marquis).
- Most, but not all, closed 2-generator census manifolds are locally rigid near their complete hyperbolic structure (Cooper–Long–Thistlethwaite).
- A large class of cusped 3-manifolds admit properly convex deformations near their hyperbolic structure (B–Danciger–Lee)
- There are strictly convex structures on some non-hyperbolic manifolds in dimension $\geqslant 4$ (Kapovich)

Examples

Higher dimensions

- If M is hyperbolic contains a finite volume totally geodesic surface then we can find non-hyperbolic structures by “bending” (Johnson–Millson, Koszul, Marquis).
- Most, but not all, closed 2-generator census manifolds are locally rigid near their complete hyperbolic structure (Cooper–Long–Thistlethwaite).
- A large class of cusped 3-manifolds admit properly convex deformations near their hyperbolic structure (B–Danciger–Lee)
- There are strictly convex structures on some non-hyperbolic manifolds in dimension $\geqslant 4$ (Kapovich)
- Several two-bridge knot and link complements do not admit strictly convex structures other than the hyperbolic structure (B).

Deforming Structures

Instead of trying to understand $\mathfrak{B}(M)$ globally we will try to understand it locally.

Deforming Structures

Instead of trying to understand $\mathfrak{B}(M)$ globally we will try to understand it locally.

Need a basepoint to look near

Deforming Structures

Instead of trying to understand $\mathfrak{B}(M)$ globally we will try to understand it locally.

Need a basepoint to look near

Let M be a finite volume hyperbolic manifold of dimension $\geqslant 3$

Deforming Structures

Instead of trying to understand $\mathfrak{B}(M)$ globally we will try to understand it locally.

Need a basepoint to look near

Let M be a finite volume hyperbolic manifold of dimension $\geqslant 3$

(Mostow–Prasad rigidity): There is a unique complete hyperbolic structure on M

Deforming Structures

Instead of trying to understand $\mathfrak{B}(M)$ globally we will try to understand it locally.

Need a basepoint to look near

Let M be a finite volume hyperbolic manifold of dimension $\geqslant 3$

(Mostow–Prasad rigidity): There is a unique complete hyperbolic structure on M

This gives a canonical basepoint in $\mathfrak{B}(M)$

Deforming Structures

Closed Case

Let M be an n-manifold

$$\mathcal{X}(M) := \mathrm{Hom}(\pi_1 M, \mathrm{SL}_{n+1}(\mathbb{R}))/\mathrm{SL}_{n+1}(\mathbb{R})$$

be the *"character variety"* of M.

Deforming Structures

Closed Case

Let M be an n-manifold

$$\mathcal{X}(M) := \mathrm{Hom}(\pi_1 M, \mathrm{SL}_{n+1}(\mathbb{R}))/\mathrm{SL}_{n+1}(\mathbb{R})$$

be the *"character variety"* of M.

Equivalent structures have conjugate holonomy, so we get

$$\mathrm{Hol} : \mathfrak{B}(M) \to \mathcal{X}(M)$$

Deforming Structures

Closed Case

Let M be an n-manifold

$$\mathcal{X}(M) := \mathrm{Hom}(\pi_1 M, \mathrm{SL}_{n+1}(\mathbb{R}))/\,\mathrm{SL}_{n+1}(\mathbb{R})$$

be the *"character variety"* of M.

Equivalent structures have conjugate holonomy, so we get

$$\mathrm{Hol} : \mathfrak{B}(M) \to \mathcal{X}(M)$$

(Ehresmann–Thurston, Koszul) When M is closed Hol is a local homeomorphism.

Deforming Structures

Closed Case

Let M be an n-manifold

$$\mathcal{X}(M) := \text{Hom}(\pi_1 M, \text{SL}_{n+1}(\mathbb{R}))/\,\text{SL}_{n+1}(\mathbb{R})$$

be the *"character variety"* of M.

Equivalent structures have conjugate holonomy, so we get

$$\text{Hol} : \mathfrak{B}(M) \to \mathcal{X}(M)$$

(Ehresmann–Thurston, Koszul) When M is closed Hol is a local homeomorphism.

Deforming representations is a necessary and sufficient condition for deforming structures

Deforming Structures

Non-compact Case

When M is non-compact

$$\mathrm{Hol} : \mathfrak{B}(M) \to \mathcal{X}(M)$$

is no longer a local homeomorphism!

Deforming Structures

Non-compact Case

When M is non-compact

$$\mathrm{Hol} : \mathfrak{B}(M) \to \mathcal{X}(M)$$

is no longer a local homeomorphism!

For example the complete finite volume hyperbolic structure on a non-compact 3-manifold admits nearby incomplete structures with indiscrete holonomy.

Deforming Structures

Non-compact Case

When M is non-compact

$$\mathrm{Hol} : \mathfrak{B}(M) \to \mathcal{X}(M)$$

is no longer a local homeomorphism!

For example the complete finite volume hyperbolic structure on a non-compact 3-manifold admits nearby incomplete structures with indiscrete holonomy.

(Cooper–Long–Tillmann) When M is non-compact Hol restricts to a local homeomorphism

$$\mathrm{Hol} : \mathfrak{B}(M)_{ce} \to \mathcal{X}(M)_{rel}$$

Hyperbolic Ends

Let M be a finite volume hyperbolic n-manifold, then

$$M = M_K \sqcup (\sqcup_{i=1}^{k} E_i)$$

Where M_k is compact and each E_i is finitely covered by $T^{n-1} \times [0, \infty)$

Hyperbolic Ends

Let M be a finite volume hyperbolic n-manifold, then

$$M = M_K \sqcup (\sqcup_{i=1}^{k} E_i)$$

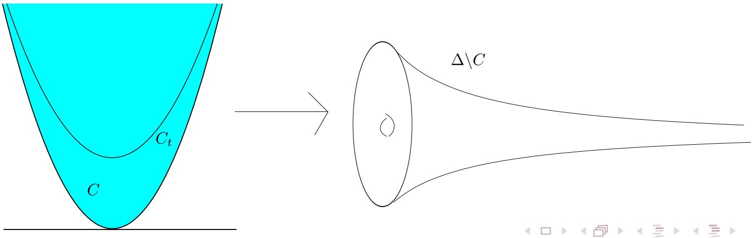
Where M_k is compact and each E_i is finitely covered by $T^{n-1} \times [0, \infty)$

The E_i are called *cusps*

Hyperbolic Ends

Parabolic Cusps

- Let $C = \{(x, v) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x \geqslant \frac{1}{2}|v|^2\} \cong \mathbb{H}^n$
- C is foliated by $C_t = \{(x, v) \in C \mid x = \frac{1}{2}|v|^2 + t\}$ (horospheres)



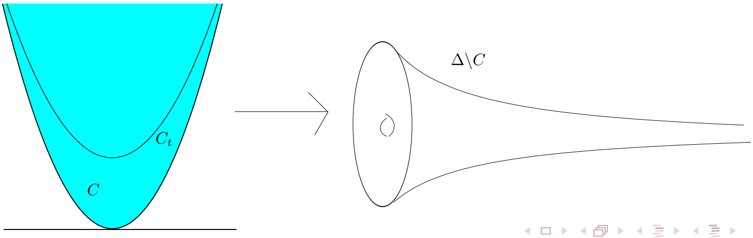
Hyperbolic Ends

Parabolic Cusps

- Let $C = \{(x, v) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x \geqslant \frac{1}{2}|v|^2\} \cong \mathbb{H}^n$
- C is foliated by $C_t = \{(x, v) \in C \mid x = \frac{1}{2}|v|^2 + t\}$ (horospheres)

Then this cover can be realized as $\Delta \backslash C$ where Δ is a lattice in the Lie group (of parabolic translations)

$$\left\{ \begin{pmatrix} 1 & u & \frac{1}{2}|u|^2 \\ 0 & I & u \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid u \in \mathbb{R}^{n-1} \right\}$$



Generalized Cusps

A properly convex manifold $C = \Gamma \backslash \Omega$ is a *generalized cusp*

- $C \cong \partial C \times [0, \infty)$, with ∂C compact,
- $\pi_1 C$ is virtually nilpotent, and
- C has strictly convex boundary.

Generalized Cusps

A properly convex manifold $C = \Gamma \backslash \Omega$ is a *generalized cusp*

- $C \cong \partial C \times [0, \infty)$, with ∂C compact,
- $\pi_1 C$ is virtually nilpotent, and
- C has strictly convex boundary.

Ends of finite volume hyperbolic manifolds are examples

Generalized Cusps

A properly convex manifold $C = \Gamma\backslash\Omega$ is a *generalized cusp*

- $C \cong \partial C \times [0, \infty)$, with ∂C compact,
- $\pi_1 C$ is virtually nilpotent, and
- C has strictly convex boundary.

Ends of finite volume hyperbolic manifolds are examples

Γ virtually preserves a real flag (this property defines $\mathcal{X}_{rel}(M)$)

Generalized Cusps

A properly convex manifold $C = \Gamma \backslash \Omega$ is a *generalized cusp*

- $C \cong \partial C \times [0, \infty)$, with ∂C compact,
- $\pi_1 C$ is virtually nilpotent, and
- C has strictly convex boundary.

Ends of finite volume hyperbolic manifolds are examples

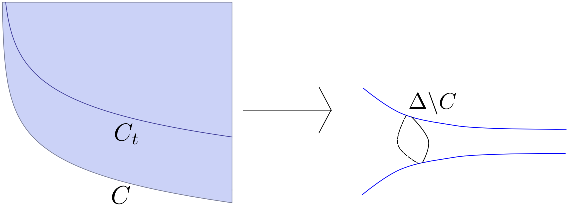
Γ virtually preserves a real flag (this property defines $\mathcal{X}_{rel}(M)$)

Having ends of this type defines $\mathfrak{B}_{ce}(M)$

Generalized Cusps

Quasi Hyperbolic Examples

- Let $C = \{(x, y) \in \mathbb{R} \times \mathbb{R}^+ \mid x \geqslant -\log(y)\}$
- C is foliated by $C_t = \{(x, y) \in C \mid x = -\log(y) + t\}$ (generalized horospheres)



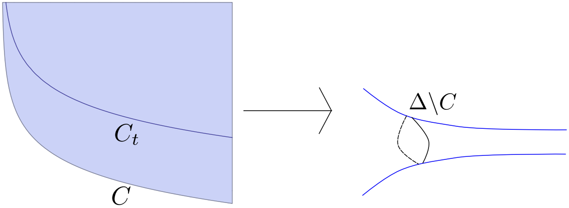
Generalized Cusps

Quasi Hyperbolic Examples

- Let $C = \{(x, y) \in \mathbb{R} \times \mathbb{R}^+ \mid x \geqslant -\log(y)\}$
- C is foliated by $C_t = \{(x, y) \in C \mid x = -\log(y) + t\}$ (generalized horospheres)

Let Δ be a lattice in the Lie group

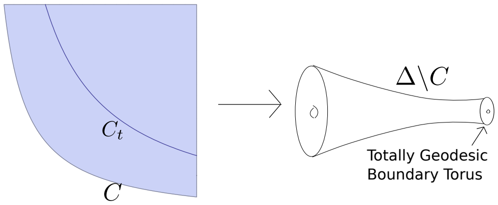
$$\left\{ \begin{pmatrix} 1 & 0 & -\log(u) \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid u \in \mathbb{R}^+ \right\}$$



Generalized Cusps

Hyperbolic Examples

- Let $\lambda_1, \ldots, \lambda_n > 0$,
- Let $C = \{(x_1, \ldots, x_n) \in (\mathbb{R}^+)^n \mid \sum_{i=1}^n \lambda_i \log(x_i) \geqslant 0\}$,
- C is foliated by $C_t = \{(x_1, \ldots, x_n) \in C \mid \sum_{i=1}^n \lambda_i \log(x_i) = t\}$ (generalized horospheres)



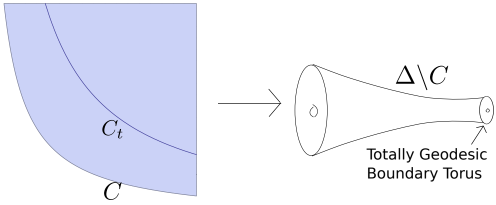
Generalized Cusps

Hyperbolic Examples

- Let $\lambda_1, \dots, \lambda_n > 0$,
- Let $C = \{(x_1, \dots, x_n) \in (\mathbb{R}^+)^n \mid \sum_{i=1}^n \lambda_i \log(x_i) \geqslant 0\}$,
- C is foliated by $C_t = \{(x_1, \dots, x_n) \in C \mid \sum_{i=1}^n \lambda_i \log(x_i) = t\}$ (generalized horospheres)

Let Δ be a lattice in the Lie group

$$\left\{ \begin{pmatrix} u_1 & & & \\ & \ddots & & \\ & & u_n & \\ & & & 1 \end{pmatrix} \mid \sum_{i=1}^{n} \lambda_i \log(u_i) = 0 \right\}$$



Generalized Cusps

Let $O = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0\}$ and let $\mathcal{S}_0^n = O/(x \sim \lambda x)$, $\lambda > 0$.

Generalized Cusps

Let $O = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0\}$ and let $\mathcal{S}_0^n = O/(x \sim \lambda x)$, $\lambda > 0$.

$\mathcal{S}_0^n$ is a $(n-1)$-simplex plus a non-Hausdorff point $[0]$

Generalized Cusps

Let $O = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0\}$ and let $\mathcal{S}_0^n = O/(x \sim \lambda x)$, $\lambda > 0$.

$\mathcal{S}_0^n$ is a $(n-1)$-simplex plus a non-Hausdorff point $[0]$

Each $v \in \mathcal{S}_0^n$ gives rise to an $(n-1)$-dimensional abelian Lie group G_v and a domain C_v foliated by horospheres

Generalized Cusps

Let $O = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0\}$ and let $\mathcal{S}_0^n = O/(x \sim \lambda x)$, $\lambda > 0$.

$\mathcal{S}_0^n$ is a $(n-1)$-simplex plus a non-Hausdorff point $[0]$

Each $v \in \mathcal{S}_0^n$ gives rise to an $(n-1)$-dimensional abelian Lie group G_v and a domain C_v foliated by horospheres

- Hyperbolic examples come from $v \in int(\mathcal{S}_0^n)$

Generalized Cusps

Let $O = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0\}$ and let $\mathcal{S}_0^n = O/(x \sim \lambda x)$, $\lambda > 0$.

$\mathcal{S}_0^n$ is a $(n-1)$-simplex plus a non-Hausdorff point $[0]$

Each $v \in \mathcal{S}_0^n$ gives rise to an $(n-1)$-dimensional abelian Lie group G_v and a domain C_v foliated by horospheres

- Hyperbolic examples come from $v \in int(\mathcal{S}_0^n)$
- Parabolic example comes from 0

Generalized Cusps

Let $O = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0\}$ and let $\mathcal{S}_0^n = O/(x \sim \lambda x)$, $\lambda > 0$.

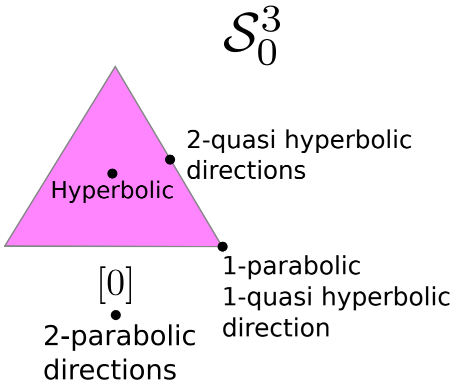
$\mathcal{S}_0^n$ is a $(n-1)$-simplex plus a non-Hausdorff point $[0]$

Each $v \in \mathcal{S}_0^n$ gives rise to an $(n-1)$-dimensional abelian Lie group G_v and a domain C_v foliated by horospheres

- Hyperbolic examples come from $v \in int(\mathcal{S}_0^n)$
- Parabolic example comes from 0
- Points on faces of $\mathcal{S}_0^n$ are “products” of parabolic and quasi hyperbolic examples

Generalized Cusps

3-dimensional Case



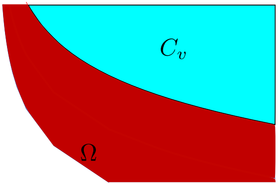
Generalized Cusps

Classification

Theorem 1 (B–Cooper–Leitner)

Let $N = \Gamma \backslash \Omega$ be an n-dimensional generalized cusp. Then there is a finite index subgroup $\mathbb{Z}^{n-1} \cong \Gamma' \subset \Gamma$ and a $v \in \mathcal{S}_0^n$ such that

- *After applying a projective transformation $C_v \subset \Omega$*
- *Γ' is conjugate to a lattice in G_v*
- *$\Gamma' \backslash C_v \subset \Gamma' \backslash \Omega$ and is a deformation retract.*



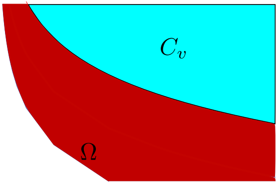
Generalized Cusps

Classification

Theorem 1 (B–Cooper–Leitner)

Let $N = \Gamma\backslash\Omega$ be an n-dimensional generalized cusp. Then there is a finite index subgroup $\mathbb{Z}^{n-1} \cong \Gamma' \subset \Gamma$ and a $v \in \mathcal{S}_0^n$ such that

- *After applying a projective transformation $C_v \subset \Omega$*
- *Γ' is conjugate to a lattice in G_v*
- *$\Gamma'\backslash C_v \subset \Gamma'\backslash\Omega$ and is a deformation retract.*



Generalizes work of Leitner for $n = 3$

Generalized Cusps

Realization Problem

Can we realize generalized cusps as ends of more complicated manifolds?

Generalized Cusps

Realization Problem

Can we realize generalized cusps as ends of more complicated manifolds?

Let $v \in \mathcal{S}_0^n$, is there a properly convex n-manifold N with $\pi_1 N$ not virtually nilpotent such that N has an end that is finitely covered by $\Delta \backslash C_v$ where $\Delta \subset G_v$ is a lattice?

Generalized Cusps

Realization Problem

Can we realize generalized cusps as ends of more complicated manifolds?

Let $v \in \mathcal{S}_0^n$, is there a properly convex n-manifold N with $\pi_1 N$ not virtually nilpotent such that N has an end that is finitely covered by $\Delta \backslash C_v$ where $\Delta \subset G_v$ is a lattice?

- If $v = 0$, yes, let N be a finite volume hyperbolic manifold

Generalized Cusps

Realization Problem

Can we realize generalized cusps as ends of more complicated manifolds?

Let $v \in \mathcal{S}_0^n$, is there a properly convex n-manifold N with $\pi_1 N$ not virtually nilpotent such that N has an end that is finitely covered by $\Delta \backslash C_v$ where $\Delta \subset G_v$ is a lattice?

- If $v = 0$, yes, let N be a finite volume hyperbolic manifold
- If $n = 2$, yes (Goldman, Choi, Marquis)

Generalized Cusps

Realization Problem

Can we realize generalized cusps as ends of more complicated manifolds?

Let $v \in \mathcal{S}_0^n$, is there a properly convex n-manifold N with $\pi_1 N$ not virtually nilpotent such that N has an end that is finitely covered by $\Delta \backslash C_v$ where $\Delta \subset G_v$ is a lattice?

- If $v = 0$, yes, let N be a finite volume hyperbolic manifold
- If $n = 2$, yes (Goldman, Choi, Marquis)
- If $n = 3$

Generalized Cusps

Realization Problem

Can we realize generalized cusps as ends of more complicated manifolds?

Let $v \in \mathcal{S}_0^n$, is there a properly convex n-manifold N with $\pi_1 N$ not virtually nilpotent such that N has an end that is finitely covered by $\Delta\backslash C_v$ where $\Delta \subset G_v$ is a lattice?

- If $v = 0$, yes, let N be a finite volume hyperbolic manifold
- If $n = 2$, yes (Goldman, Choi, Marquis)
- If $n = 3$
  - If v is a "vertex", yes, let M be complement of figure-eight (B)

Can we realize generalized cusps as ends of more complicated manifolds?

Let $v \in \mathcal{S}_0^n$, is there a properly convex n-manifold N with $\pi_1 N$ not virtually nilpotent such that N has an end that is finitely covered by $\Delta \backslash C_v$ where $\Delta \subset G_v$ is a lattice?

- If $v = 0$, yes, let N be a finite volume hyperbolic manifold
- If $n = 2$, yes (Goldman, Choi, Marquis)
- If $n = 3$
  - If v is a “vertex”, yes, let M be complement of figure-eight (B)
  - If v is in interior, yes (B–Danciger–Lee)

Generalized Cusps

Realization Problem

Can we realize generalized cusps as ends of more complicated manifolds?

Let $v \in \mathcal{S}_0^n$, is there a properly convex n-manifold N with $\pi_1 N$ not virtually nilpotent such that N has an end that is finitely covered by $\Delta \backslash C_v$ where $\Delta \subset G_v$ is a lattice?

- If $v = 0$, yes, let N be a finite volume hyperbolic manifold
- If $n = 2$, yes (Goldman, Choi, Marquis)
- If $n = 3$
  - If v is a "vertex", yes, let M be complement of figure-eight (B)
  - If v is in interior, yes (B–Danciger–Lee)
  - If v in a "side", probably yes.

Transitions of Cusps

Pro: We have nice models that allow us to understand the geometry of generalized cusps

Transitions of Cusps

Pro: We have nice models that allow us to understand the geometry of generalized cusps

Con: Models don't account for how cusps can transition to each other

Transitions of Cusps

Pro: We have nice models that allow us to understand the geometry of generalized cusps

Con: Models don't account for how cusps can transition to each other

Need a family of models that account for how the geometry of one type of cusp degenerates to the geometry of another type of cusp

The Slice

Let $\langle \gamma_1, \gamma_2 \rangle \cong \mathbb{Z}^2$, let $v = [1, 1, 1] \in \mathcal{S}_0^3$.

The Slice

Let $\langle \gamma_1, \gamma_2 \rangle \cong \mathbb{Z}^2$, let $v = [1, 1, 1] \in \mathcal{S}_0^3$. For $t > 0$ define

$$x_{t,\theta} = \exp \begin{pmatrix} t\cos\theta & & & \\ & t\cos(\theta + 2\pi/3) & & \\ & & t\cos(\theta + 4\pi/3) & \\ & & & 0 \end{pmatrix} \in G_v$$

$$y_{t,\theta} = \exp \begin{pmatrix} t\sin\theta & & & \\ & t\sin(\theta + 2\pi/3) & & \\ & & t\sin(\theta + 4\pi/3) & \\ & & & 0 \end{pmatrix} \in G_v$$

The Slice

Let $\langle \gamma_1, \gamma_2 \rangle \cong \mathbb{Z}^2$, let $v = [1, 1, 1] \in \mathcal{S}_0^3$. For $t > 0$ define

$$x_{t,\theta} = \exp \begin{pmatrix} t\cos\theta & & & \\ & t\cos(\theta + 2\pi/3) & & \\ & & t\cos(\theta + 4\pi/3) & \\ & & & 0 \end{pmatrix} \in G_v$$

$$y_{t,\theta} = \exp \begin{pmatrix} t\sin\theta & & & \\ & t\sin(\theta + 2\pi/3) & & \\ & & t\sin(\theta + 4\pi/3) & \\ & & & 0 \end{pmatrix} \in G_v$$

Define $\rho_{(t,\theta,a,b)} : \mathbb{Z}^2 \to \mathrm{SL}_4(\mathbb{R})$ by

$$\rho_{(t,\theta,a,b)}(\gamma_1) = \exp(x_{t,\theta}), \rho_{(t,\theta,a,b)}(\gamma_2) = \exp(ax_{t,\theta} + by_{t,\theta}).$$

The Slice

Let $\langle \gamma_1, \gamma_2 \rangle \cong \mathbb{Z}^2$, let $v = [1, 1, 1] \in \mathcal{S}_0^3$. For $t > 0$ define

$$x_{t,\theta} = \exp \begin{pmatrix} t\cos\theta & & & \\ & t\cos(\theta + 2\pi/3) & & \\ & & t\cos(\theta + 4\pi/3) & \\ & & & 0 \end{pmatrix} \in G_v$$

$$y_{t,\theta} = \exp \begin{pmatrix} t\sin\theta & & & \\ & t\sin(\theta + 2\pi/3) & & \\ & & t\sin(\theta + 4\pi/3) & \\ & & & 0 \end{pmatrix} \in G_v$$

Define $\rho_{(t,\theta,a,b)} : \mathbb{Z}^2 \to \mathrm{SL}_4(\mathbb{R})$ by

$$\rho_{(t,\theta,a,b)}(\gamma_1) = \exp(x_{t,\theta}), \rho_{(t,\theta,a,b)}(\gamma_2) = \exp(ax_{t,\theta} + by_{t,\theta}).$$

Let $\Gamma_{t,\theta,a,b} = \rho_{t,\theta,a,b}(\mathbb{Z}^2)$, then $\Gamma_{t,\theta,a,b} \backslash C_v$ is a generalized cusp

The Slice

Let $\langle \gamma_1, \gamma_2 \rangle \cong \mathbb{Z}^2$, let $v = [1,1,1] \in \mathcal{S}_0^3$. For $t > 0$ define

$$x_{t,\theta} = \exp \begin{pmatrix} t\cos\theta & & & \\ & t\cos(\theta + 2\pi/3) & & \\ & & t\cos(\theta + 4\pi/3) & \\ & & & 0 \end{pmatrix} \in G_v$$

$$y_{t,\theta} = \exp \begin{pmatrix} t\sin\theta & & & \\ & t\sin(\theta + 2\pi/3) & & \\ & & t\sin(\theta + 4\pi/3) & \\ & & & 0 \end{pmatrix} \in G_v$$

Define $\rho_{(t,\theta,a,b)} : \mathbb{Z}^2 \to \mathrm{SL}_4(\mathbb{R})$ by

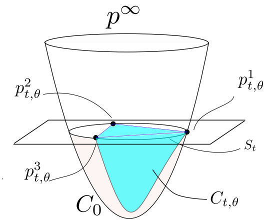
$$\rho_{(t,\theta,a,b)}(\gamma_1) = \exp(x_{t,\theta}), \rho_{(t,\theta,a,b)}(\gamma_2) = \exp(ax_{t,\theta} + by_{t,\theta}).$$

Let $\Gamma_{t,\theta,a,b} = \rho_{t,\theta,a,b}(\mathbb{Z}^2)$, then $\Gamma_{t,\theta,a,b} \backslash C_v$ is a generalized cusp

As $t \to 0$, $\Gamma_{t,\theta,a,b} \backslash C_v$ collapse

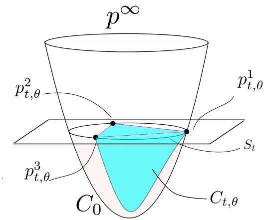
The Slice

Let C_0 be a parabolic cusp domain and let $p^\infty = [1:0:0:0] \in \partial C_0$.
For $t > 0$, let S_t cross section of ∂C_0 at $x_1 = \frac{1}{2t^2}$.



The Slice

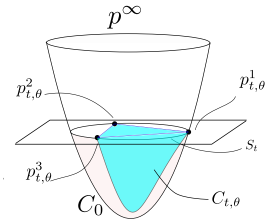
Let C_0 be a parabolic cusp domain and let $p^\infty = [1 : 0 : 0 : 0] \in \partial C_0$.
For $t > 0$, let S_t cross section of ∂C_0 at $x_1 = \frac{1}{2t^2}$.



- Using $x_{t,\theta}$ and $y_{t,\theta}$ we construct three complex numbers $\{z^i_{t,\theta}\}^3_{i=1}$ equally spaced on the circle of radius t.

The Slice

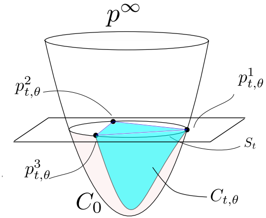
Let C_0 be a parabolic cusp domain and let
$p^\infty = [1 : 0 : 0 : 0] \in \partial C_0$.
For $t > 0$, let S_t cross section of ∂C_0 at $x_1 = \frac{1}{2t^2}$.



- Using $x_{t,\theta}$ and $y_{t,\theta}$ we construct three complex numbers $\{z^i_{t,\theta}\}_{i=1}^3$ equally spaced on the circle of radius t.
- Let $p^i_{t,\theta}$ be the corresponding points on S_t.

The Slice

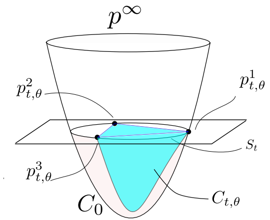
Let C_0 be a parabolic cusp domain and let
$p^\infty = [1 : 0 : 0 : 0] \in \partial C_0$.
For $t > 0$, let S_t cross section of ∂C_0 at $x_1 = \frac{1}{2t^2}$.



- Using $x_{t,\theta}$ and $y_{t,\theta}$ we construct three complex numbers $\{z^i_{t,\theta}\}^3_{i=1}$ equally spaced on the circle of radius t.
- Let $p^i_{t,\theta}$ be the corresponding points on S_t.
- Let $M_{t,\theta} \in \mathrm{SL}_4(\mathbb{R})$ be an element taking the vertices of the standard simplex to $p^1_{t,\theta}, p^2_{t,\theta}, p^3_{t,\theta}$, and p^∞.

The Slice

Let C_0 be a parabolic cusp domain and let $p^\infty = [1:0:0:0] \in \partial C_0$.
For $t > 0$, let S_t cross section of ∂C_0 at $x_1 = \frac{1}{2t^2}$.



- Using $x_{t,\theta}$ and $y_{t,\theta}$ we construct three complex numbers $\{z^i_{t,\theta}\}_{i=1}^3$ equally spaced on the circle of radius t.
- Let $p^i_{t,\theta}$ be the corresponding points on S_t.
- Let $M_{t,\theta} \in \mathrm{SL}_4(\mathbb{R})$ be an element taking the vertices of the standard simplex to $p^1_{t,\theta}, p^2_{t,\theta}, p^3_{t,\theta}$, and p^∞.
- Let $C_{t,\theta}$ be image of C_v under $M_{t,\theta}$

The Transition

The Slice

Let $\rho'_{t,\theta,a,b} = M_{t,\theta}\rho_{(t,\theta,a,b)}M_{t,\theta}^{-1}$

$$\lim_{t\to 0} \rho'_{(t,\theta,a,b)}(\gamma_1) = \begin{pmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\lim_{t\to 0} \rho'_{(t,\theta,a,b)}(\gamma_2) = \begin{pmatrix} 1 & a & b & \frac{1}{2}(a^2+b^2) \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Slice

Let $\rho'_{t,\theta,a,b} = M_{t,\theta}\rho_{(t,\theta,a,b)}M_{t,\theta}^{-1}$

$$\lim_{t\to 0}\rho'_{(t,\theta,a,b)}(\gamma_1) = \begin{pmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\lim_{t\to 0}\rho'_{(t,\theta,a,b)}(\gamma_2) = \begin{pmatrix} 1 & a & b & \frac{1}{2}(a^2+b^2) \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{S} = \{\rho'_{(t,\theta,a,b)} \mid a, b, \theta \in \mathbb{R}, t \in \mathbb{R}^{\geqslant 0}\} \subset \mathrm{Hom}(\mathbb{Z}^2, \mathrm{SL}_4(\mathbb{R}))$$

The Slice

Let $\rho'_{t,\theta,a,b} = M_{t,\theta}\rho_{(t,\theta,a,b)}M_{t,\theta}^{-1}$

$$\lim_{t\to 0}\rho'_{(t,\theta,a,b)}(\gamma_1) = \begin{pmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\lim_{t\to 0}\rho'_{(t,\theta,a,b)}(\gamma_2) = \begin{pmatrix} 1 & a & b & \frac{1}{2}(a^2+b^2) \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{S} = \{\rho'_{(t,\theta,a,b)} \mid a,b,\theta \in \mathbb{R}, t \in \mathbb{R}^{\geqslant 0}\} \subset \mathrm{Hom}(\mathbb{Z}^2, \mathrm{SL}_4(\mathbb{R}))$

$\mathcal{S}$ are holonomies of hyperbolic generalized cusps that converge to parabolic cusps as $t \to 0$

Deforming Hyperbolic Manifolds

Let M be a finite volume hyperbolic 3-manifold with 1-cusp and let ρ_{hyp} the holonomy of its hyperbolic structure

Deforming Hyperbolic Manifolds

Let M be a finite volume hyperbolic 3-manifold with 1-cusp and let ρ_{hyp} the holonomy of its hyperbolic structure

Theorem 2 (B–Danciger–Lee)

Let M be as above. Suppose that M is infinitesimally rigid rel ∂M Then the hyperbolic structure on M can be deformed to a properly convex structure with hyperbolic generalized cusp end.

Deforming Hyperbolic Manifolds

Let M be a finite volume hyperbolic 3-manifold with 1-cusp and let ρ_{hyp} the holonomy of its hyperbolic structure

Theorem 2 (B–Danciger–Lee)

Let M be as above. Suppose that M is infinitesimally rigid rel ∂M Then the hyperbolic structure on M can be deformed to a properly convex structure with hyperbolic generalized cusp end.

M is *infinitesimally rigid rel ∂M* if

$$\mathrm{H}^1_{\rho_{hyp}}(M, \mathfrak{sl}_4) \to \mathrm{H}^1_{\rho_{hyp}}(\partial M, \mathfrak{sl}_4)$$

induced by $\partial M \hookrightarrow M$ is an injection

Deforming Hyperbolic Manifolds

Let M be a finite volume hyperbolic 3-manifold with 1-cusp and let ρ_{hyp} the holonomy of its hyperbolic structure

Theorem 2 (B–Danciger–Lee)

Let M be as above. Suppose that M is infinitesimally rigid rel ∂M Then the hyperbolic structure on M can be deformed to a properly convex structure with hyperbolic generalized cusp end.

M is *infinitesimally rigid rel ∂M* if

$$H^1_{\rho_{hyp}}(M, \mathfrak{sl}_4) \to H^1_{\rho_{hyp}}(\partial M, \mathfrak{sl}_4)$$

induced by $\partial M \hookrightarrow M$ is an injection

The cohomology groups are tangent spaces of “non-trivial” deformations

Deforming Hyperbolic Manifolds

Let M be a finite volume hyperbolic 3-manifold with 1-cusp and let ρ_{hyp} the holonomy of its hyperbolic structure

Theorem 2 (B–Danciger–Lee)

Let M be as above. Suppose that M is infinitesimally rigid rel ∂M Then the hyperbolic structure on M can be deformed to a properly convex structure with hyperbolic generalized cusp end.

M is *infinitesimally rigid rel ∂M* if

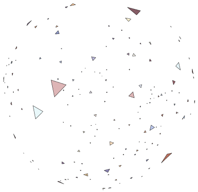
$$\mathrm{H}^1_{\rho_{hyp}}(M, \mathfrak{sl}_4) \to \mathrm{H}^1_{\rho_{hyp}}(\partial M, \mathfrak{sl}_4)$$

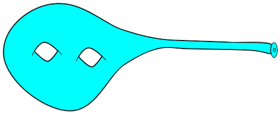
induced by $\partial M \hookrightarrow M$ is an injection

The cohomology groups are tangent spaces of “non-trivial” deformations

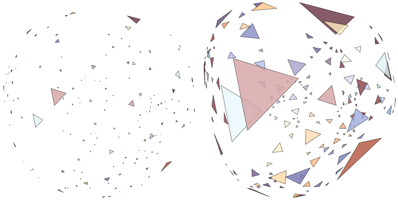
Very common amongst known examples

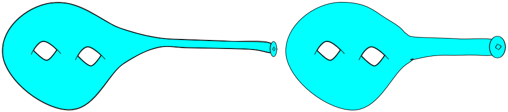
Deforming Hyperbolic Manifolds



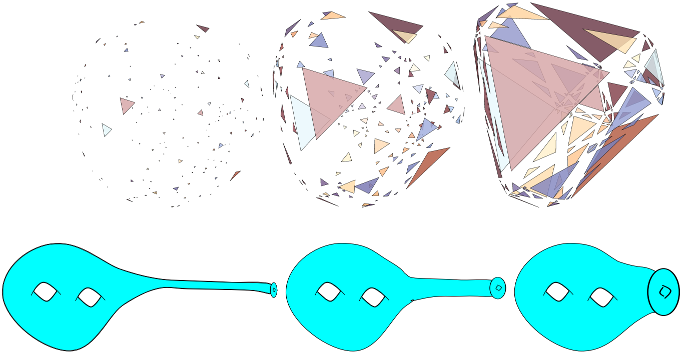


Deforming Hyperbolic Manifolds





Deforming Hyperbolic Manifolds



Deforming Hyperbolic Manifolds

There is a map

$$\text{res} : \text{Hom}(M, \text{SL}_4(\mathbb{R})) \to \text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$$

Deforming Hyperbolic Manifolds

There is a map

$$\text{res} : \text{Hom}(M, \text{SL}_4(\mathbb{R})) \to \text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$$

- $\text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$ is a smooth manifold near $\text{res}(\rho_{hyp})$

Deforming Hyperbolic Manifolds

There is a map

$$\text{res} : \text{Hom}(M, \text{SL}_4(\mathbb{R})) \to \text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$$

- $\text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$ is a smooth manifold near $\text{res}(\rho_{hyp})$
- Image of res has codimension 3 near $\text{res}(\rho_{hyp})$ (Poincare duality)

Deforming Hyperbolic Manifolds

There is a map

$$\text{res} : \text{Hom}(M, \text{SL}_4(\mathbb{R})) \to \text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$$

- $\text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$ is a smooth manifold near $\text{res}(\rho_{hyp})$
- Image of res has codimension 3 near $\text{res}(\rho_{hyp})$ (Poincare duality)
- If M is infinitesimally rigid rel ∂M then $\text{Hom}(M, \text{SL}_4(\mathbb{R}))$ is smooth at ρ_{hyp}.

Deforming Hyperbolic Manifolds

There is a map

$$\text{res} : \text{Hom}(M, \text{SL}_4(\mathbb{R})) \to \text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$$

- $\text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$ is a smooth manifold near $\text{res}(\rho_{hyp})$
- Image of res has codimension 3 near $\text{res}(\rho_{hyp})$ (Poincare duality)
- If M is infinitesimally rigid rel ∂M then $\text{Hom}(M, \text{SL}_4(\mathbb{R}))$ is smooth at ρ_{hyp}.
- $\mathcal{S}$ is a 4-dimensional submanifold of $\text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$ res (generically) consisting of representations diagonalizable over $\mathbb{R}$.

Deforming Hyperbolic Manifolds

There is a map

$$\text{res} : \text{Hom}(M, \text{SL}_4(\mathbb{R})) \to \text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$$

- $\text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$ is a smooth manifold near $\text{res}(\rho_{hyp})$
- Image of res has codimension 3 near $\text{res}(\rho_{hyp})$ (Poincare duality)
- If M is infinitesimally rigid rel ∂M then $\text{Hom}(M, \text{SL}_4(\mathbb{R}))$ is smooth at ρ_{hyp}.
- $\mathcal{S}$ is a 4-dimensional submanifold of $\text{Hom}(\partial M, \text{SL}_4(\mathbb{R}))$ res (generically) consisting of representations diagonalizable over $\mathbb{R}$.
- res is transverse to $\mathcal{S}$ near ρ_{hyp} and so we can deform ρ_{hyp} to be diagonalizable when restricted to $\pi_1 \partial M$.

Remaining Questions

1. Can we find a properly convex manifold with quasi hyperbolic generalized cusp end of form $\Delta\backslash C_v$, where v is in a "side" of $\mathcal{S}_0^3$?

Remaining Questions

1. Can we find a properly convex manifold with quasi hyperbolic generalized cusp end of form $\Delta \backslash C_v$, where v is in a "side" of $\mathcal{S}_0^3$?
    - Probably, by taking limits of examples with hyperbolic generalized cusp ends

Remaining Questions

1. Can we find a properly convex manifold with quasi hyperbolic generalized cusp end of form $\Delta\backslash C_v$, where v is in a "side" of $\mathcal{S}_0^3$?
   - Probably, by taking limits of examples with hyperbolic generalized cusp ends
2. Can we solve the realization problem in higher dimensions?

Remaining Questions

1. Can we find a properly convex manifold with quasi hyperbolic generalized cusp end of form $\Delta\backslash C_v$, where v is in a "side" of $\mathcal{S}_0^3$?
   - Probably, by taking limits of examples with hyperbolic generalized cusp ends
2. Can we solve the realization problem in higher dimensions?

Theorem 3 (B–Marquis)

For each $n \geqslant 3$ and $v \in \mathcal{S}_0^n$ be a vertex there is a finite volume hyperbolic n-manifold whose hyperbolic structure can be deformed to have a generalized cusp of the form $\Delta\backslash C_v$

Thank you