Gluing Properly Convex Manifolds

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(joint with J. Danciger and G.-S. Lee)

Higher Teichmüller theory and Higgs bundles
Heidelberg
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Overview

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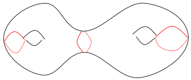

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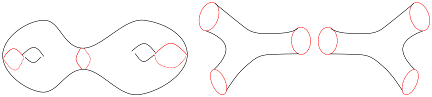

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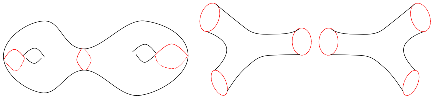

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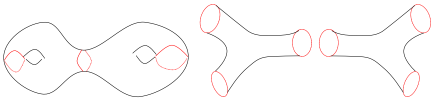

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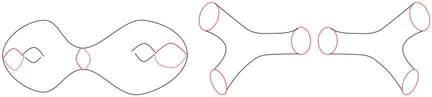

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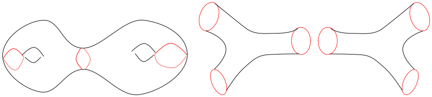

  - Understand hyperbolic structures on pants. (Completely determined by geometry of boundary!)
  - Understand how to glue together pants. (Completely determined by twisting!)

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- Can put a complete finite volume Thurston geometric structure on each of the pieces.
- If there are multiple geometric pieces, we can't glue them to get a Thurston geometric structure on all of M.
- However, if we allow more general geometric structures then this strategy still works (at least some of the time)

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3. Try to glue the pieces together by matching the geometry on the boundary.
4. Analyze the different ways to glue structures with matching boundary geometry.

Projective Space

- $\mathbb{RP}^n$ is the space of lines through origin in $\mathbb{R}^{n+1}$.
- Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{RP}^n$ be the obvious projection.
- The automorphism group of $\mathbb{RP}^n$ is $\mathrm{PGL}_{n+1}(\mathbb{R}) := \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^\times$.

Affine Patches

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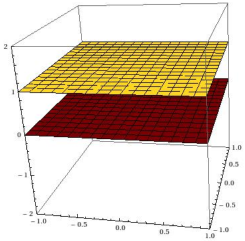

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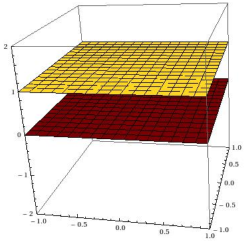
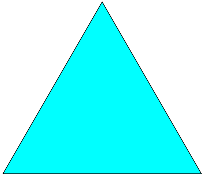


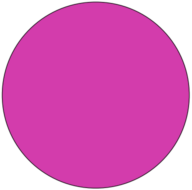
- $\mathbb{RP}^n \backslash P(H)$ is called an *affine patch*.

Convex Projective Domains

- $\Omega \subset \mathbb{RP}^n$ is *properly convex* if it is a bounded convex subset of some affine patch.
- If $\partial\Omega$ contains no non-trivial line segments then Ω is *strictly convex*.



Properly Convex



Strictly Convex

Convex Projecive Structures

- A *convex projective n-manifold* is a manifold of the form $\Gamma\backslash\Omega$, where $\Omega \subset \mathbb{RP}^n$ is properly convex and $\Gamma \subset \mathrm{PGL}(\Omega)$ is a discrete torsion free subgroup.

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- A (marked) *convex projective structure* on a manifold M is an identification of M with a properly convex manifold (up to equivalence).

Convex Projecive Structures

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathrm{Dev}} & \Omega \\ \downarrow & & \downarrow \\ M & \longrightarrow & \Gamma\backslash\Omega \end{array}$$

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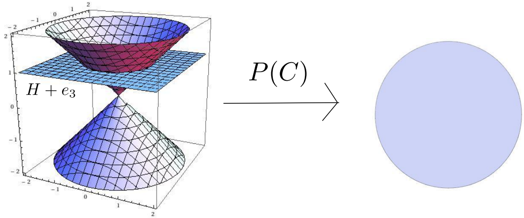
- A marked convex projective structure gives rise to a (conjugacy class of) representation $\rho : \pi_1 M \to \mathrm{PGL}_{n+1}(\mathbb{R})$ called a *holonomy* of the structure and an equivariant map $\mathrm{Dev} : \tilde{M} \to \Omega$ called a *developing map*.

Some Examples

- Let $\langle x, y\rangle = x_1y_1 + \ldots x_ny_n - x_{n+1}y_{n+1}$ be the standard bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$
- Let $C = \{x \in \mathbb{R}^{n+1} | \langle x, x\rangle < 0\}$



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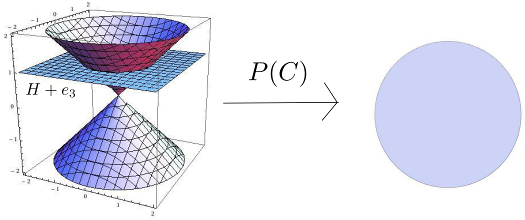
- Let $C = \{x \in \mathbb{R}^{n+1} | \langle x, x\rangle < 0\}$
- $P(C) = \mathbb{H}^n$ is the *Klein model* of hyperbolic space.
- $\mathrm{PGL}(\mathbb{H}^n) \cong PO(n, 1) \leq \mathrm{PGL}_{n+1}(\mathbb{R})$



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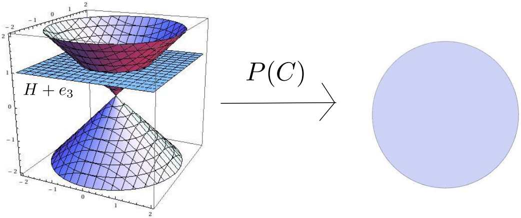
- If Γ is a torsion-free Kleinian group then $\Gamma\backslash\mathbb{H}^n$ is a (strictly) convex projective manifold.



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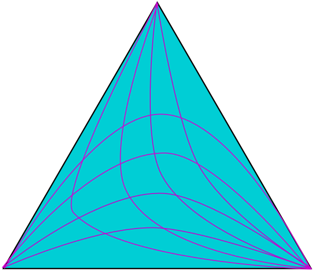
- Let $\Gamma \leq Diag_+ \leq \mathrm{PGL}(\Delta)$ be lattice, then $\Gamma \cong \mathbb{Z}^2$ and $\Gamma \backslash \Delta$ is a torus (really, a Hex Torus)



Hilbert Metric

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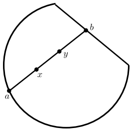

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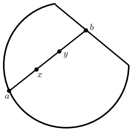

Ω

Every properly convex set Ω admits a Hilbert metric given by

$$d_\Omega(x, y) = \log[a : x : y : b] = \log\left(\frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}\right)$$

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Convex projective structures are like Thurston geometric structures, sans homogeneity

Convex Projective Structure in Dimension 3

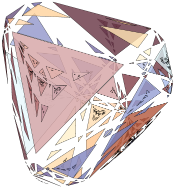
Let $M \cong \Gamma\backslash\Omega$ be a closed indecomposable convex projective 3-manifold.

Theorem (Benoist 2006)

Let M be as above then either

i *M is strictly convex and admits a hyperbolic structure*

ii *M is not strictly convex and contains a finite number of embedded totally geodesic Hex tori. The pieces obtained by cutting along these tori are a JSJ decomposition for M. Furthermore, each piece admits a finite volume hyperbolic structure.*



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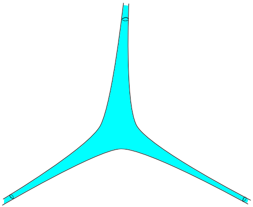

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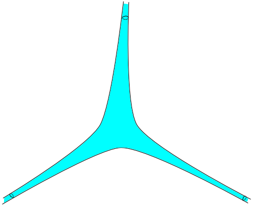

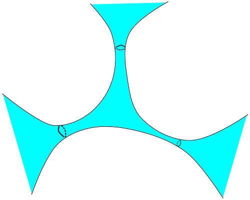

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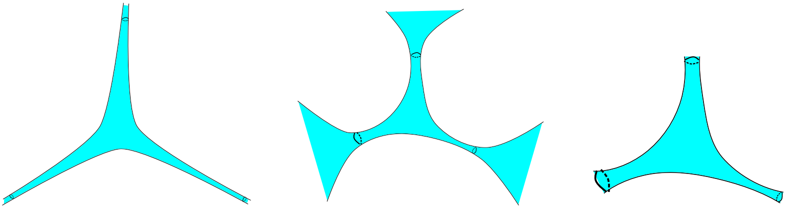
- We can truncate the ends of this infinite volume structure along geodesics to get a structure on a pair of pants $\overline{\mathcal{P}}$.



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- All possible cuff lengths can be realized by this construction.
- We can glue pairs of pants along boundary components whenever the cuff lengths agree.

Deforming Projective Structures

Let N be a finite-volume hyperbolic 3-manifold

- $\mathfrak{B}(N)$ = Space of marked convex projective structures
- $\mathcal{X}(N) = \mathrm{Hom}(\pi_1 N, \mathrm{PGL}_4(\mathbb{R}))/\mathrm{conj}$
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3. When N is non-compact, Hol is a local homeomorphism near $[N_{hyp}]$ onto a subset of $\mathcal{X}(N)$ (Cooper–Long–Tillmann)

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Theorem 1 (B–Danciger–Lee)

Let N be a finite volume hyperbolic 3-manifold which is infinitesimally rigid rel boundary. Then N admits nearby convex projective structures with totally geodesic boundary.

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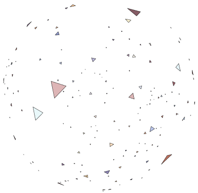

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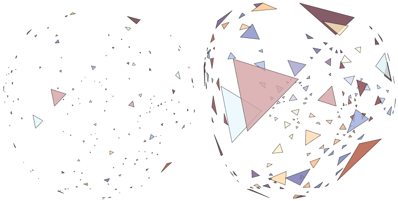

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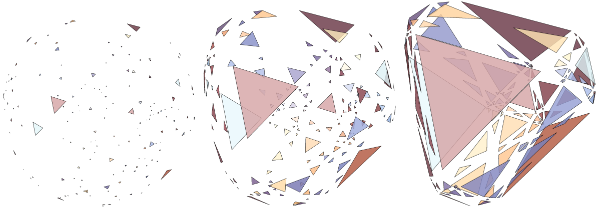

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3. Use a convex hull construction to build a structure with totally geodesic boundary.

More on Step 1

Theorem (B–D–L)

If N is a 1-cusped finite volume hyperbolic 3-manifold that is infinitesimally rigid rel boundary then $[\rho_{hyp}]$ is a smooth point of $\mathcal{X}(N)$. Furthermore, $\mathcal{X}(N)$ is 3-dimensional near $[\rho_{hyp}]$

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- $\text{res} : \mathcal{X}(N) \to \mathcal{X}(\partial N)$ is smooth local embedding near $[\rho_{geo}]$
- There is a 4-dimensional slice $\mathcal{S} \subset \mathcal{X}(\partial N)$ of generically diagonalizable representations transverse to res at $[\rho_{hyp}]$

More on Step 1

Theorem (B–D–L)

If N is a 1-cusped finite volume hyperbolic 3-manifold that is infinitesimally rigid rel boundary then $[\rho_{hyp}]$ is a smooth point of $\mathcal{X}(N)$. Furthermore, $\mathcal{X}(N)$ is 3-dimensional near $[\rho_{hyp}]$

- $\mathcal{X}(\partial N)$ is 6 dimensional (sort of)
- res : $\mathcal{X}(N) \to \mathcal{X}(\partial N)$ is smooth local embedding near $[\rho_{geo}]$
- There is a 4-dimensional slice $\mathcal{S} \subset \mathcal{X}(\partial N)$ of generically diagonalizable representations transverse to res at $[\rho_{hyp}]$
- We get a curve $[\rho_t]$ in $\mathcal{X}(N)$ diagonalizable over $\mathbb{R}$ on $\pi_1 \partial N$.

The Slice

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Define $\rho_{(t,\theta,a,b)} : \pi_1 \partial N \to A = \exp(\mathfrak{a}) \subset \mathrm{PGL}_4(\mathbb{R})$ by

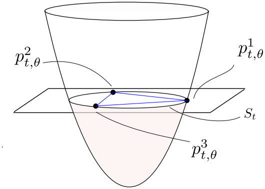
$$\rho_{(t,\theta,a,b)}(\gamma_1) = \exp(x_{t,\theta}), \rho_{(t,\theta,a,b)}(\gamma_2) = \exp(a x_{t,\theta} + b y_{t,\theta}).$$

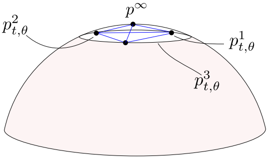
The Slice

Another model for $\mathbb{H}^3$ is

$$\{[x_1 : x_2 : x_3 : 1] \in \mathbb{RP}^3 \mid x_1 > 2(x_2^2 + x_3^2)\}$$

For $t > 0$, let S_t crosssection of $\partial\mathbb{H}^n$ at $x_1 = \frac{1}{4t^2}$.





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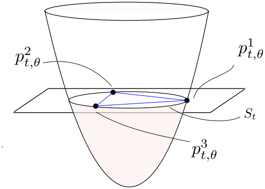

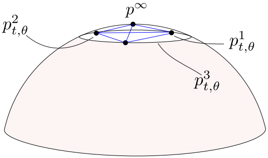

- Using $x_{t,\theta}$ and $y_{t,\theta}$ we construct three complex numbers $\{z^i_{t,\theta}\}_{i=1}^3$ equally spaced on the circle of radius $2t$.

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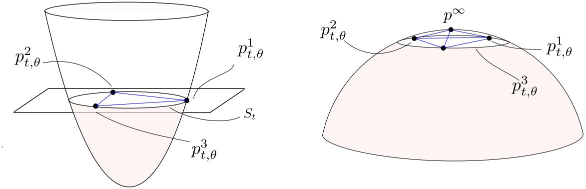

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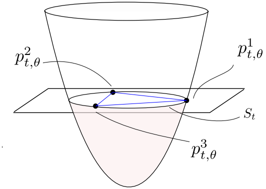

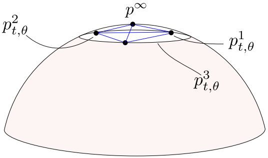

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- Let $C_{t,\theta} \in \mathrm{PGL}_4$ be an element taking the vertices of the standard simplex to $p^1_{t,\theta}, p^2_{t,\theta}, p^3_{t,\theta}$, and p^∞.

The Slice

Let $\rho'_{t,\theta,a,b} = C_{t,\theta}\rho_{(t,\theta,a,b)}C_{t,\theta}^{-1}$

$$\lim_{t\to 0}\rho'_{(t,\theta,a,b)}(\gamma_1) = \begin{pmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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$\mathcal{S} = \{[\rho'_{(t,\theta,a,b)}] \mid a, b, \theta \in \mathbb{R}, t \in \mathbb{R}^{\geq 0}\}$

The Slice

- $\mathcal{S}$ is transverse to $res(\mathcal{X}(N))$ at $[\rho_{hyp}]$ with 1-dimensional intersection $[\rho_s]$.
- $[\rho_s]$ is diagonalizable over $\mathbb{R}$ for $s \neq 0$.
- If $t \neq 0$ then elements of $\mathcal{S}$ are diagonalizable over reals.
- If $z = x + iy$ is the cusp shape of N w.r.t. $\{\gamma_1, \gamma_2\}$ then $res(\rho_{hyp}) = \rho'_{(0,0,x,y)}$.

Gluing Manifolds with Totally Geodesic Boundary

- Let $M_1 \cong \Gamma_1 \backslash \Omega_1$ and $M_2 \cong \Gamma_2 \backslash \Omega_2$ be a properly convex 3-manifolds with principal totally geodesic torus boundary components, ∂_1 and ∂_2

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Theorem (B–D–L)

If there exists $g \in \mathrm{PGL}_4(\mathbb{R})$ such that $f_* : \pi_1 \partial_1 \to \pi_1 \partial_2$ is induced by conjugation by g then there is a properly convex projective structure on M such that the inclusion $M_i \hookrightarrow M$ is a projective embedding.

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Corollary

If N is a 1-cusped hyperbolic 3-manifold that is infinitesimally rigid rel. boundary then $2N$ admits a properly convex projective structure.

The Matching Problem

Let N_1 and N_2 are infinitesimally rigid rel. boundary hyperbolic 3-manifolds and M be obtained by gluing N_1 and N_2 along their boundaries. Can we find a convex projective structure on M?

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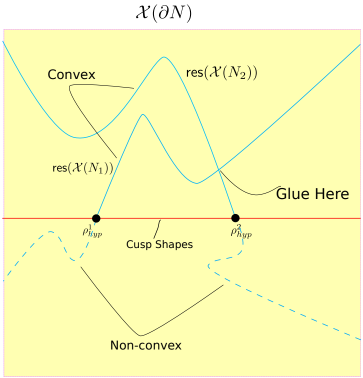

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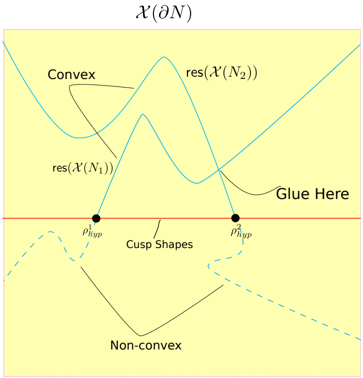

Blue curves $\rightsquigarrow$ Zero locus of A-polynomial

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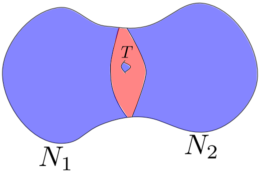

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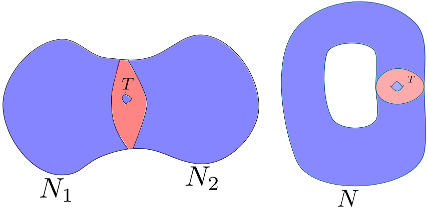

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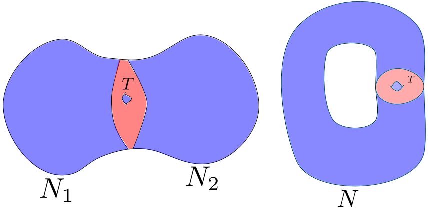

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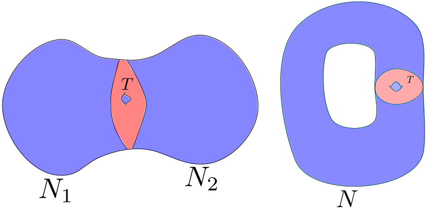

We get “twist coordinates” on $\mathfrak{B}(N)$!

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- If not, what are some obstructions to gluing?
- Are there "Fenchel-Nielsen" coordinates?

Thank you